ERRATUM TO "HOMOTOPY THEORY OF MOORE FLOWS I"

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Abstract. The notion of reparametrization category is incorrectly axiomatized and it must be adjusted. It is proved that for a general reparametrization category $\mathcal{P}$, the tensor product of $\mathcal{P}$-spaces yields a biclosed semimonoidal structure. It is also described some kind of objectwise braiding for $\mathcal{G}$-spaces.

Contents

1. Introduction

Presentation. The notion of reparametrization category introduced in [1] is incorrectly axiomatized. The reparametrization categories $(\mathcal{G},+)$ and $(\mathcal{M},+)$ are not symmetric indeed. Moreover, the third axiom of reparametrization category is slightly modified to obtain the expected result for the tensor product of two constant $\mathcal{P}$-spaces in full generality. It also enables us to write a short proof of the pentagon axiom. The main theorem is:

Theorem. *(Proposition 3.4 and Theorem 3.5) For any reparametrization category $\mathcal{P}$, the tensor product of $\mathcal{P}$-spaces yields a biclosed semimonoidal structure.*

The semimonoidal category of $\mathcal{G}$-spaces still has some kind of objectwise braiding which is formalized in Theorem 4.9. This fact is specific to $\mathcal{G}$-spaces. It is used nowhere in [1,2].

Theorem. *(Theorem 4.9) There is a homeomorphism*

$$B : (D \otimes E)(L) \longrightarrow (E \otimes D)(L)$$

for all $L > 0$ and all $\mathcal{G}$-spaces D and E which is not natural with respect to $L > 0$.

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Outline of the note. In Section 2, the notion of reparametrization category is adjusted. In Section 3, the corrections are listed. The absence of braiding forces us to relocate some parameters ℓ in the calculations, and also to replace the shift operator s_ℓ either by the *left shift* s^L_ℓ (see Proposition 2.6) or by the *right shift* s^R_ℓ (see Proposition 2.7). Finally, Section 4 gives an explicit description of a homeomorphism $(D \otimes E)(L) \cong (E \otimes D)(L)$ for all $L > 0$ and for all $\mathcal{G}$-spaces D and E which is not natural with respect to $L > 0$.

Prerequisites and notations. We refer to [1] for the notations and for the full categorical argumentations. We refer to [2] for the full topological argumentations.

2. Adjustment

2.1. **Definition.** *A* semimonoidal category $(\mathcal{K}, \otimes)$ *is a category* $\mathcal{K}$ *equipped with a functor* $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ *together with a natural isomorphism* $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ *called the* associator *satisfying the pentagon axiom.*

2.2. **Definition.** *A semimonoidal category* $(\mathcal{K}, \otimes)$ *is* enriched *(all enriched categories are enriched over* **Top***) if the category* $\mathcal{K}$ *is enriched and if the set map*

$$\mathcal{K}(a, b) \times \mathcal{K}(c, d) \longrightarrow \mathcal{K}(a \otimes c, b \otimes d)$$

is continuous for all objects $a, b, c, d \in \mathrm{Obj}(\mathcal{K})$*.*

2.3. **Definition.** *A* reparametrization category $(\mathcal{P}, \otimes)$ *is a small enriched semimonoidal category satisfying the following additional properties:*

1. *The semimonoidal structure is strict, i.e. the associator is the identity.*
2. *All spaces of maps* $\mathcal{P}(\ell, \ell')$ *for all objects* ℓ *and* ℓ' *of* $\mathcal{P}$ *are contractible.*
3. *For all maps* $\phi : \ell \to \ell'$ *of* $\mathcal{P}$*, for all* $\ell'_1, \ell'_2 \in \mathrm{Obj}(\mathcal{P})$ *such that* $\ell'_1 \otimes \ell'_2 = \ell'$*, there exist two maps* $\phi_1 : \ell_1 \to \ell'_1$ *and* $\phi_2 : \ell_2 \to \ell'_2$ *of* $\mathcal{P}$ *such that* $\phi = \phi_1 \otimes \phi_2 : \ell_1 \otimes \ell_2 \to \ell'_1 \otimes \ell'_2$ *(which implies that* $\ell_1 \otimes \ell_2 = \ell$*).*

2.4. **Notation.** *The notations* $\ell, \ell', \ell_i, L, \ldots$ *mean objects of a reparametrization category* $\mathcal{P}$*.*

2.5. **Notation.** *To stick to the intuition, we set* $\ell + \ell' := \ell \otimes \ell'$ *for all* $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$*. Indeed, morally speaking,* ℓ *is the length of a path.*

The enriched categories $(\mathcal{G}, +)$ (Proposition 4.4), $(\mathcal{M}, +)$ [1, Proposition 4.11] as well as the terminal category are examples of reparametrization categories. In the cases of $(\mathcal{G}, +)$ and $(\mathcal{M}, +)$, the functors $(\ell, \ell') \mapsto \ell + \ell'$ and $(\ell, \ell') \mapsto \ell' + \ell$ coincide on objects, but not on morphisms. The terminal category is a symmetric reparametrization category. We do not know if there exist symmetric reparametrization categories not equivalent to the terminal category. [1, Proposition 5.8] must be replaced by the two propositions:

2.6. **Proposition.** *(The left shift functor) The following data assemble to an enriched functor* $s^L_\ell : \mathcal{P} \to \mathcal{P}$*:*

$$\begin{cases} s^L_\ell(\ell') = \ell + \ell' \\ s^L_\ell(\phi) = \mathrm{Id}_\ell \otimes \phi \quad \text{for a map } \phi : \ell' \to \ell''. \end{cases}$$

2.7. **Proposition.** *(The right shift functor) The following data assemble to an enriched functor* $s_\ell^R : \mathcal{P} \to \mathcal{P}$*:*

$$\begin{cases} s_\ell^R(\ell') = \ell' + \ell \\ s_\ell^R(\phi) = \phi \otimes \mathrm{Id}_\ell \quad \textit{for a map } \phi : \ell' \to \ell''. \end{cases}$$

For the convenience of the reader, we recall the

2.8. **Definition.** *[1, Definition 5.1] An object of* $[\mathcal{P}^{op}, \mathbf{Top}]_0$ *is called a* $\mathcal{P}$-space. *Let* D *be a* $\mathcal{P}$*-space. Let* $\phi : \ell \to \ell'$ *be a map of* $\mathcal{P}$*. Let* $x \in D(\ell')$*. We will use the notation*

$$x.\phi := D(\phi)(x).$$

2.9. **Notation.** *The two enriched functors* $(s_\ell^L)^*$ *and* $(s_\ell^R)^*$ *take a* $\mathcal{P}$*-space* D *to* Ds_ℓ^L *and* Ds_ℓ^R *respectively.*

3. Corrections

3.1. **Lemma.** *(First replacement for [1, Lemma 5.10]) For all* $\ell', \ell'' \in \mathrm{Obj}(\mathcal{P})$*, there is the isomorphism of* $\mathcal{P}$*-spaces (natural with respect to* ℓ' *and* ℓ''*)*

$$\int^\ell \mathcal{P}(-, \ell + \ell') \times \mathcal{P}(\ell, \ell'') \cong \mathcal{P}(-, \ell'' + \ell').$$

The isomorphism takes the equivalence class of $(\psi, \phi) \in \mathcal{P}(-, \ell+\ell') \times \mathcal{P}(\ell, \ell'')$ *to* $(s_{\ell'}^R)^*(\phi)\psi = (\phi \otimes \mathrm{Id}_{\ell'})\psi$*.*

Proof. Pick a $\mathcal{P}$-space D. Then there is the sequence of homeomorphisms

$$\begin{aligned}
[\mathcal{P}^{op}, \mathbf{Top}]\Big(\int^\ell \mathcal{P}(-, \ell + \ell') \times \mathcal{P}(\ell, \ell''), D\Big) &\cong \int_\ell [\mathcal{P}^{op}, \mathbf{Top}]\big(\mathcal{P}(-, \ell + \ell') \times \mathcal{P}(\ell, \ell''), D\big) \\
&\cong \int_\ell \mathbf{TOP}(\mathcal{P}(\ell, \ell''), D(\ell + \ell')) \\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](\mathcal{P}(-, \ell''), (s_{\ell'}^R)^* D) \\
&\cong D(\ell'' + \ell') \\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](\mathcal{P}(-, \ell'' + \ell'), D).
\end{aligned}$$

The proof is complete thanks to the Yoneda lemma. □

There is the following variation of Lemma 3.1 which is also used below:

3.2. **Lemma.** *(Second replacement for [1, Lemma 5.10]) For all* $\ell', \ell'' \in \mathrm{Obj}(\mathcal{P})$*, there is the isomorphism of* $\mathcal{P}$*-spaces (natural with respect to* ℓ' *and* ℓ''*)*

$$\int^\ell \mathcal{P}(-, \ell' + \ell) \times \mathcal{P}(\ell, \ell'') \cong \mathcal{P}(-, \ell' + \ell'').$$

The isomorphism takes the equivalence class of $(\psi, \phi) \in \mathcal{P}(-, \ell'+\ell) \times \mathcal{P}(\ell, \ell'')$ *to* $(s_{\ell'}^L)^*(\phi)\psi = (\mathrm{Id}_{\ell'} \otimes \phi)\psi$*.*

Proof. Pick a $\mathcal{P}$-space D. Then there is the sequence of homeomorphisms

$$\begin{aligned}
[\mathcal{P}^{op}, \mathbf{Top}]\left(\int^{\ell} \mathcal{P}(-,\ell'+\ell)\times\mathcal{P}(\ell,\ell''), D\right) &\cong \int_{\ell}[\mathcal{P}^{op}, \mathbf{Top}]\big(\mathcal{P}(-,\ell'+\ell)\times\mathcal{P}(\ell,\ell''), D\big)\\
&\cong \int_{\ell}\mathbf{TOP}(\mathcal{P}(\ell,\ell''), D(\ell'+\ell))\\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](\mathcal{P}(-,\ell''), (s^{L}_{\ell'})^{*}D)\\
&\cong D(\ell'+\ell'')\\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](\mathcal{P}(-,\ell'+\ell''), D).
\end{aligned}$$

The proof is complete thanks to the Yoneda lemma. □

3.3. **Proposition.** *Let D_1 and D_2 be two $\mathcal{P}$-spaces and $L \in \mathrm{Obj}(\mathcal{P})$. Then the mapping $(x,y)\mapsto(\mathrm{Id},x,y)$ yields a surjective continuous map*

$$\bigsqcup_{\substack{(\ell_1,\ell_2)\\ \ell_1+\ell_2=L}} D_1(\ell_1)\times D_2(\ell_2) \longrightarrow (D_1\otimes D_2)(L).$$

Proof. Let $(\psi,x_1,x_2)\in\mathcal{P}(L,\ell_1+\ell_2)\times D_1(\ell_1)\times D_2(\ell_2)$ be a representative of an element of $(D_1\otimes D_2)(L)$. Then there exist two maps $\psi_i:\ell'_i\to\ell_i$ for $i=1,2$ such that $\psi=\psi_1\otimes\psi_2$. By [1, Corollary 5.13], one has $(\psi,x_1,x_2)\sim(\mathrm{Id}_L,x_1\psi_1,x_2\psi_2)$ in $(D_1\otimes D_2)(L)$ and the proof is complete. □

3.4. **Proposition.** *(Replacement for [1, Proposition 5.11]) Let D and E be two $\mathcal{P}$-spaces. Let*

$$D\otimes E=\int^{(\ell_1,\ell_2)}\mathcal{P}(-,\ell_1+\ell_2)\times D(\ell_1)\times E(\ell_2).$$

The pair $([\mathcal{P}^{op},\mathbf{Top}]_0,\otimes)$ is a semimonoidal category.

Proof. Let D_1,D_2,D_3 be three $\mathcal{P}$-spaces. Let $a_{D_1,D_2,D_3}:(D_1\otimes D_2)\otimes D_3\to D_1\otimes(D_2\otimes D_3)$ be the composite of the isomorphisms (by using Lemma 3.1 and Lemma 3.2)

$$\begin{aligned}
&(D_1\otimes D_2)\otimes D_3\\
&\quad\cong\int^{(\ell_1,\ell_2,\ell_3)}\left(\int^{\ell}\mathcal{P}(-,\ell+\ell_3)\times\mathcal{P}(\ell,\ell_1+\ell_2)\right)\times D_1(\ell_1)\times D_2(\ell_2)\times D_3(\ell_3)\\
&\quad\cong\int^{(\ell_1,\ell_2,\ell_3)}\mathcal{P}(-,\ell_1+\ell_2+\ell_3)\times D_1(\ell_1)\times D_2(\ell_2)\times D_3(\ell_3)\\
&\quad\cong\int^{(\ell_1,\ell_2,\ell_3)}\left(\int^{\ell}\mathcal{P}(-,\ell_1+\ell)\times\mathcal{P}(\ell,\ell_2+\ell_3)\right)\times D_1(\ell_1)\times D_2(\ell_2)\times D_3(\ell_3)\\
&\quad\cong D_1\otimes(D_2\otimes D_3).
\end{aligned}$$

Let $(\psi,(\phi,x_1,x_2),x_3)\in((D\otimes E)\otimes F)(L)$ with $x_i\in D_i(\ell_i)$ for $i=1,2,3$ and $L\in\mathrm{Obj}(\mathcal{P})$. Write $\phi=\phi_1\otimes\phi_2$ with $\phi_i:\ell'_i\to\ell_i$ for $i=1,2$ and $\psi=\psi_1\otimes\psi_2\otimes\psi_3$ with $\psi_i:\ell''_i\to\ell'_i$ for $i=1,2,3$ with $\ell'_3=\ell_3$. In particular, $L=\ell''_1+\ell''_2+\ell''_3$. We obtain $(\psi,(\phi,x_1,x_2),x_3)\sim(\mathrm{Id}_L,(\mathrm{Id}_{\ell''_1+\ell''_2},x_1\phi_1\psi_1,x_2\phi_2\psi_2),x_3\psi_3)$ in $((D\otimes E)\otimes F)(L)$. The above sequence of isomorphisms takes the equivalence class of $(\psi,(\phi,x_1,x_2),x_3)$ at first to the equivalence class of $((\mathrm{Id}_{\ell''_1+\ell''_2}\otimes\mathrm{Id}_{\ell''_3})\,\mathrm{Id}_L,x_1\phi_1\psi_1,x_2\phi_2\psi_2,x_3\psi_3)$ by Lemma 3.1, and, since $(\mathrm{Id}_{\ell''_1+\ell''_2}\otimes\mathrm{Id}_{\ell''_3})\,\mathrm{Id}_L=(\mathrm{Id}_{\ell''_1}\otimes\mathrm{Id}_{\ell''_2+\ell''_3})\,\mathrm{Id}_L$ and by Lemma 3.2, to the equivalence class of $(\mathrm{Id}_L, x_1\phi_1\psi_1, (\mathrm{Id}_{\ell_2''+\ell_3''}, x_2\phi_2\psi_2, x_3\psi_3))$. We deduce that the associator $a_{D,E,F}$: $(D\otimes E)\otimes F \to D\otimes(E\otimes F)$ satisfies the pentagon axiom using Proposition 3.3. □

3.5. **Theorem.** *(Replacement for [1, Theorem 5.14]) Let D, E and F be three $\mathcal{P}$-spaces. Let*

$$\{E,F\}_L := \ell \mapsto [\mathcal{P}^{op}, \mathbf{Top}](E, (s_\ell^L)^* F),$$
$$\{E,F\}_R := \ell \mapsto [\mathcal{P}^{op}, \mathbf{Top}](E, (s_\ell^R)^* F).$$

These yield two $\mathcal{P}$-spaces and there are the natural homeomorphisms

$$[\mathcal{P}^{op}, \mathbf{Top}](D, \{E,F\}_L) \cong [\mathcal{P}^{op}, \mathbf{Top}](D\otimes E, F),$$
$$[\mathcal{P}^{op}, \mathbf{Top}](E, \{D,F\}_R) \cong [\mathcal{P}^{op}, \mathbf{Top}](D\otimes E, F).$$

Consequently, the functor

$$\otimes : [\mathcal{P}^{op}, \mathbf{Top}]_0 \times [\mathcal{P}^{op}, \mathbf{Top}]_0 \to [\mathcal{P}^{op}, \mathbf{Top}]_0$$

induces a structure of biclosed semimonoidal structure on $[\mathcal{P}^{op}, \mathbf{Top}]_0$.

Proof. There are the sequences of natural homeomorphisms

$$\begin{aligned}
[\mathcal{P}^{op}, \mathbf{Top}](D, \{E,F\}_L) &\cong \int_\ell \mathbf{TOP}\big(D(\ell), [\mathcal{P}^{op}, \mathbf{Top}](E, (s_\ell^L)^*F)\big)\\
&\cong \int_{(\ell,\ell')} \mathbf{TOP}\big(D(\ell), \mathbf{TOP}(E(\ell'), F(\ell+\ell'))\big)\\
&\cong \int_{(\ell,\ell')} \mathbf{TOP}\big(D(\ell)\times E(\ell'), F(\ell+\ell')\big)\\
&\cong \int_{(\ell,\ell')} [\mathcal{P}^{op}, \mathbf{Top}]\big(\mathcal{P}(-,\ell+\ell')\times D(\ell)\times E(\ell'), F\big)\\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](D\otimes E, F)
\end{aligned}$$

and

$$\begin{aligned}
[\mathcal{P}^{op}, \mathbf{Top}](E, \{D,F\}_R) &\cong \int_{\ell'} \mathbf{TOP}\big(E(\ell'), [\mathcal{P}^{op}, \mathbf{Top}](D, (s_{\ell'}^R)^*F)\big)\\
&\cong \int_{(\ell,\ell')} \mathbf{TOP}\big(E(\ell'), \mathbf{TOP}(D(\ell), F(\ell+\ell'))\big)\\
&\cong \int_{(\ell,\ell')} \mathbf{TOP}\big(D(\ell)\times E(\ell'), F(\ell+\ell')\big)\\
&\cong \int_{(\ell,\ell')} [\mathcal{P}^{op}, \mathbf{Top}]\big(\mathcal{P}(-,\ell+\ell')\times D(\ell)\times E(\ell'), F\big)\\
&\cong [\mathcal{P}^{op}, \mathbf{Top}](D\otimes E, F).
\end{aligned}$$

□

3.6. **Notation.** *Let*

$$\mathbb{F}_\ell^{\mathcal{P}^{op}} U = \mathcal{P}(-,\ell)\times U \in [\mathcal{P}^{op}, \mathbf{Top}]_0$$

where U is a topological space and where ℓ is an object of $\mathcal{P}$.

3.7. **Proposition.** *(Replacement for [1, Proposition 5.16]) Let U, U' be two topological spaces. Let $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$. There is the natural isomorphism of $\mathcal{P}$-spaces*

$$\mathbb{F}_{\ell}^{\mathcal{P}^{op}} U \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}} U' \cong \mathbb{F}_{\ell+\ell'}^{\mathcal{P}^{op}} (U \times U').$$

Proof. One has

$$\mathbb{F}_{\ell}^{\mathcal{P}^{op}} U \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}} U' = \int^{(\ell_1,\ell_2)} \mathcal{P}(-, \ell_1 + \ell_2) \times \mathcal{P}(\ell_1, \ell) \times \mathcal{P}(\ell_2, \ell') \times U \times U'.$$

Using Lemma 3.2, we obtain

$$\mathbb{F}_{\ell}^{\mathcal{P}^{op}} U \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}} U' = \int^{\ell_1} \mathcal{P}(\ell_1, \ell) \times \mathcal{P}(-, \ell_1 + \ell') \times U \times U'.$$

Using Lemma 3.1, we obtain

$$\mathbb{F}_{\ell}^{\mathcal{P}^{op}} U \otimes \mathbb{F}_{\ell'}^{\mathcal{P}^{op}} U' = \mathcal{P}(-, \ell + \ell') \times U \times U'.$$

□

3.8. **Notation.** *Let U be a topological space. The constant $\mathcal{P}$-space U is denoted by $\Delta_{\mathcal{P}^{op}} U$.*

3.9. **Proposition.** *(Replacement for [1, Proposition 5.17]) Let U and U' be two topological spaces. There is the natural isomorphism of $\mathcal{P}$-spaces*

$$\Delta_{\mathcal{P}^{op}} U \otimes \Delta_{\mathcal{P}^{op}} U' \cong \Delta_{\mathcal{P}^{op}} (U \times U').$$

Proof. Since **Top** is cartesian closed, it suffices to consider the case where $U = U'$ is a singleton. In that case, the topological space $(\Delta_{\mathcal{P}^{op}} U \otimes \Delta_{\mathcal{P}^{op}} U')(L)$ is the quotient of the space

$$\coprod_{(\ell,\ell')} \mathcal{P}(L, \ell + \ell')$$

by the identifications $(\phi_1 \otimes \phi_2).\phi \sim \phi$. Let $\psi \in \mathcal{P}(L, \ell + \ell')$ for some $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$. By definition of a reparametrization category, write $\psi = \psi_1 \otimes \psi_2$ with $\psi_1 : \ell_1 \to \ell$ and $\psi_2 : \ell_2 \to \ell'$. Then we obtain $\psi = (\psi_1 \otimes \psi_2). \mathrm{Id}_L$. We deduce that $\psi \sim \mathrm{Id}_L$ in $(\Delta_{\mathcal{P}^{op}} U \otimes \Delta_{\mathcal{P}^{op}} U')(L)$. □

3.10. **Proposition.** *(Replacement for [1, Proposition 5.18]) Let D and E be two $\mathcal{P}$-spaces. Then there is a natural homeomorphism*

$$\varinjlim (D \otimes E) \cong \varinjlim D \times \varinjlim E.$$

Proof. Let Z be a topological space. There is the sequence of natural homeomorphisms

$$\begin{aligned}
\mathbf{TOP}\big(\varinjlim (D \otimes E), Z\big) &\cong [\mathcal{P}^{op}, \mathbf{Top}]\big(D \otimes E, \Delta_{\mathcal{P}^{op}} Z\big) \\
&\cong [\mathcal{P}^{op}, \mathbf{Top}]\Big(D, \ell \mapsto [\mathcal{P}^{op}, \mathbf{Top}](E, (s_\ell^L)^* \Delta_{\mathcal{P}^{op}}(Z))\Big) \\
&\cong [\mathcal{P}^{op}, \mathbf{Top}]\Big(D, \Delta_{\mathcal{P}^{op}}\big([\mathcal{P}^{op}, \mathbf{Top}](E, \Delta_{\mathcal{P}^{op}}(Z))\big)\Big) \\
&\cong \mathbf{TOP}\big(\varinjlim D, [\mathcal{P}^{op}, \mathbf{Top}](E, \Delta_{\mathcal{P}^{op}}(Z))\big) \\
&\cong \mathbf{TOP}\big(\varinjlim D, \mathbf{TOP}(\varinjlim E, Z)\big) \\
&\cong \mathbf{TOP}\big((\varinjlim D) \times (\varinjlim E), Z\big).
\end{aligned}$$

The proof is complete thanks to the Yoneda lemma. □

Note that in [2, Theorem 4.3], the words "closed symmetric semimonoidal category" must be replaced by "biclosed semimonoidal category".

4. The case of $\mathcal{G}$-spaces

4.1. **Notation.** *In this section, the notations $\ell, \ell', \ell_i, L, \dots$ mean a strictly positive real number.*

For the convenience of the reader, the definition of the reparametrization category $\mathcal{G}$ is recalled:

4.2. **Definition.** *Let $\phi_i : [0, \ell_i] \to [0, \ell'_i]$ for $i = 1, 2$ be two continuous maps preserving the extrema where a notation like $[0, \ell]$ means a segment of the real line. Then the map*

$$\phi_1 \otimes \phi_2 : [0, \ell_1 + \ell_2] \to [0, \ell'_1 + \ell'_2]$$

denotes the continuous map defined by

$$(\phi_1 \otimes \phi_2)(t) = \begin{cases} \phi_1(t) & \text{if } 0 \leqslant t \leqslant \ell_1 \\ \phi_2(t - \ell_1) + \ell'_1 & \text{if } \ell_1 \leqslant t \leqslant \ell_1 + \ell_2 \end{cases}$$

4.3. **Notation.** *The notation $[0, \ell_1] \cong^+ [0, \ell_2]$ means a nondecreasing homeomorphism from $[0, \ell_1]$ to $[0, \ell_2]$. It takes 0 to 0 and ℓ_1 to ℓ_2.*

4.4. **Proposition.** *[1, Proposition 4.9] There exists a reparametrization category, denoted by $\mathcal{G}$, such that the semigroup of objects is the open interval $]0, +\infty[$ equipped with the addition and such that for every $\ell_1, \ell_2 > 0$, there is the equality*

$$\mathcal{G}(\ell_1, \ell_2) = \{[0, \ell_1] \cong^+ [0, \ell_2]\}$$

where the topology is the compact-open topology (which is Δ-generated by [2, Proposition 2.5]) and such that for every $\ell_1, \ell_2, \ell_3 > 0$, the composition map

$$\mathcal{G}(\ell_1, \ell_2) \times \mathcal{G}(\ell_2, \ell_3) \to \mathcal{G}(\ell_1, \ell_3)$$

is induced by the composition of continuous maps.

4.5. **Notation.** *Let $\ell > 0$. Let $\mu_\ell : [0, \ell] \to [0, 1]$ be the homeomorphism defined by $\mu_\ell(t) = t/\ell$. We have $\mu_\ell \in \mathcal{G}(\ell, 1)$.*

Recall again that this reparametrization category is not symmetric as a semimonoidal category because the functors $(\ell, \ell') \mapsto \ell + \ell'$ and $(\ell, \ell') \mapsto \ell' + \ell$ coincide on objects, but not on morphisms

4.6. **Proposition.** *Fix L, ℓ_1, ℓ_2. The mapping $(\phi_1, \phi_2) \mapsto \phi_1 \otimes \phi_2$ induces a continuous bijection which is not a homeomorphism*

$$\bigsqcup_{\substack{\ell'_1 > 0, \ell'_2 > 0 \\ \ell'_1 + \ell'_2 = L}} \mathcal{G}(\ell'_1, \ell_1) \times \mathcal{G}(\ell'_2, \ell_2) \longrightarrow \mathcal{G}(L, \ell_1 + \ell_2)$$

Proof. The mapping is a bijection by [2, Proposition 3.2]. It is continuous since $\mathcal{G}$ is an enriched semimonoidal category. It is not a homeomorphism since the right-hand space is contractible whereas the left-hand one is not. □

4.7. **Proposition.** *Fix L'. The set map*

$$B_2 : \mathcal{G}([0,2],[0,L']) \longrightarrow \mathcal{G}([0,2],[0,L'])$$

which takes $\phi = \phi_1 \otimes \phi_2$ to $\phi_2 \otimes \phi_1$ where $\phi_i \in \mathcal{G}([0,1],[0,L'_i])$ with $L'_1 = \phi(1)$ and $L'_2 = L' - L'_1$ is a idempotent homeomorphism.

Proof. It is bijective since $B_2 B_2 = \mathrm{Id}_{\mathcal{G}([0,2],[0,L'])}$. It remains to prove that B_2 is continuous. It suffices to prove that B_2 is sequentially continuous since the space $\mathcal{G}([0,2],[0,L'])$ is sequential, being metrizable. Let $(\phi^n)_{n\geqslant 0} = (\phi_1^n \otimes \phi_2^n)_{n\geqslant 0}$ be a sequence of $\mathcal{G}([0,2],[0,L'])$ which converges to $\phi = \phi_1 \otimes \phi_2$. Then the sequence $(\phi_i^n)_{n\geqslant 0}$ converges pointwise to ϕ_i for $i = 1,2$. Therefore, the sequence $(B_2(\phi^n))_{n\geqslant 0}$ converges pointwise to $B_2(\phi)$. The proof is complete thanks to [2, Proposition 2.5]. ☐

4.8. **Proposition.** *Fix L, ℓ_1, ℓ_2. There is a unique set map*

$$B_L^{\ell_1,\ell_2} : \mathcal{G}(L, \ell_1 + \ell_2) \to \mathcal{G}(L, \ell_1 + \ell_2)$$

such that the following diagram of spaces is commutative:

$$\begin{array}{ccc}
\displaystyle\bigsqcup_{\substack{\ell'_1>0,\ell'_2>0\\ \ell'_1+\ell'_2=L}} \mathcal{G}(\ell'_1,\ell_1)\times\mathcal{G}(\ell'_2,\ell_2) & \xrightarrow{(\phi_1,\phi_2)\mapsto(\phi_2,\phi_1)} & \displaystyle\bigsqcup_{\substack{\ell'_1>0,\ell'_2>0\\ \ell'_1+\ell'_2=L}} \mathcal{G}(\ell'_2,\ell_2)\times\mathcal{G}(\ell'_1,\ell_1) \\
\downarrow{\scriptstyle (\phi_1,\phi_2)\mapsto\phi_1\otimes\phi_2} & & \downarrow{\scriptstyle (\phi_2,\phi_1)\mapsto\phi_2\otimes\phi_1} \\
\mathcal{G}(L,\ell_1+\ell_2) & \xrightarrow{B_L^{\ell_1,\ell_2}} & \mathcal{G}(L,\ell_1+\ell_2)
\end{array}$$

Moreover, the set map $B_L^{\ell_1,\ell_2}$ is a homeomorphism.

Proof. The existence and the uniqueness of the set map $B_L^{\ell_1,\ell_2}$ is a consequence of Proposition 4.6. It is bijective because all other arrows are bijective. Since $B_L^{\ell_2,\ell_1} B_L^{\ell_1,\ell_2} = \mathrm{Id}_{\mathcal{G}(L,\ell_1+\ell_2)}$, it remains to prove that $B_L^{\ell_1,\ell_2}$ is continuous. Observe that

$$B_L^{\ell_1,\ell_2}(\psi) = B_2\Big(\psi\big(\mu^{-1}_{\psi^{-1}(\ell_1)} \otimes \mu^{-1}_{L-\psi^{-1}(\ell_1)}\big)\Big)\big(\mu_{L-\psi^{-1}(\ell_1)} \otimes \mu_{\psi^{-1}(\ell_1)}\big).$$

By [2, Lemma 6.2], the mapping $\psi \mapsto \psi^{-1} \mapsto \psi^{-1}(\ell_1)$ is continuous. Thus, the continuity of $B_L^{\ell_1,\ell_2}$ is a consequence of the continuity of $\otimes$ proved in Proposition 4.6 and of the continuity of B_2 proved in Proposition 4.7. ☐

Let D and E be two $\mathcal{G}$-spaces. The $\mathcal{G}$-space $D \otimes E$ is the quotient of

$$\bigsqcup_{(\ell_1,\ell_2)} \mathcal{G}(-, \ell_1+\ell_2) \times D(\ell_1) \times E(\ell_2)$$

by the identifications $(\psi, x_1\phi_1, x_2\phi_2) \sim ((\phi_1 \otimes \phi_2)\psi, x_1, x_2)$ by [1, Corollary 5.13]. Consider the set map

$$\bigsqcup_{(\ell_1,\ell_2)} \mathcal{G}(L, \ell_1+\ell_2) \times D(\ell_1) \times E(\ell_2) \longrightarrow (E \otimes D)(L)$$

defined by taking

$$(\psi, x_1, x_2) \in \mathcal{G}(L, \ell_1+\ell_2) \times D(\ell_1) \times E(\ell_2)$$

to the equivalence class of

$$(B_L^{\ell_1,\ell_2}(\psi), x_2, x_1) = (\psi_2 \otimes \psi_1, x_2, x_1)$$

where $\psi = \psi_1 \otimes \psi_2$ is the unique decomposition of ψ such that $\psi_i \in \mathcal{G}(\ell'_i, \ell_i)$ with $\ell'_1 + \ell'_2 = L$. It is continuous by Proposition 4.8.

The triple $(\psi, x_1\phi_1, x_2\phi_2)$ is taken to the equivalence class of $(\psi_2 \otimes \psi_1, x_2\phi_2, x_1\phi_1)$. One has $(\phi_1 \otimes \phi_2)\psi = (\phi_1\psi_1) \otimes (\phi_2\psi_2)$. Therefore, the triple $((\phi_1 \otimes \phi_2)\psi, x_1, x_2)$ is taken to the equivalence class of $((\phi_2 \otimes \phi_1)(\psi_2 \otimes \psi_1), x_2, x_1) \sim (\psi_2 \otimes \psi_1, x_2\phi_2, x_1\phi_1)$. Consequently, we obtain the

4.9. **Theorem.** *This mapping yields a continuous map*

$$B : (D \otimes E)(L) \longrightarrow (E \otimes D)(L)$$

for all $L > 0$ and all $\mathcal{G}$-spaces D and E which is a homeomorphism. It is not natural with respect to $L > 0$.

Proof. The map $(D \otimes E)(L) \to (E \otimes D)(L)$ is not natural with respect to $L \in \mathrm{Obj}(\mathcal{G})$. Indeed, take $(\psi, x_1, x_2) \in (D \otimes E)(L)$. Then $(\psi, x_1, x_2) \sim (\mathrm{Id}_L, x_1\psi_1, x_2\psi_2)$ in $(D \otimes E)(L)$ with $\psi = \psi_1 \otimes \psi_2$. Consider $\omega : L' \to L$ a map of $\mathcal{G}$. Then $(\mathrm{Id}_L, x_1\psi_1, x_2\psi_2) \in (D \otimes E)(L)$ is taken to $(\omega, x_1\psi_1, x_2\psi_2) \sim (\mathrm{Id}_{L'}, x_1\psi_1\omega_1, x_2\psi_2\omega_2) \in (D \otimes E)(L')$ with $\omega = \omega_1 \otimes \omega_2$. On the other hand, $(\mathrm{Id}_L, x_2\psi_2, x_1\psi_1) \in (E \otimes D)(L)$ is taken to $(\omega, x_2\psi_2, x_1\psi_1) \in (E \otimes D)(L')$, and not to $(\omega_2 \otimes \omega_1, x_2\psi_2, x_1\psi_1)$. □

Note that the mapping $(\psi, x_1, x_2) \mapsto (\psi, x_2, x_1)$ does not induce a map from $(D \otimes E)(L)$ to $(E \otimes D)(L)$. Indeed, $(\psi, x_1\phi_1, x_2\phi_2)$ is taken to $(\psi, x_2\phi_2, x_1\phi_1)$ whereas $((\phi_1 \otimes \phi_2)\psi, x_1, x_2)$ is taken to $((\phi_1 \otimes \phi_2)\psi, x_2, x_1)$ and $(\psi, x_2\phi_2, x_1\phi_1) \sim ((\phi_2 \otimes \phi_1)\psi, x_2, x_1)$ which is not equal to $((\phi_1 \otimes \phi_2)\psi, x_2, x_1)$ in $(E \otimes D)(L)$ in general.

References

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