ABOUT LOCALLY FINITE CELLULAR MULTIPOINTED d-SPACES

PHILIPPE GAUCHER

Abstract. It is proved that the set of execution paths of a locally finite cellular multipointed d-space equipped with the compact-open topology is Δ-generated. Thus the space of execution paths of a locally finite cellular multipointed d-space is metrizable with the distance of the uniform convergence. Note: this note will be never published as it is; the part about locally finite cellular spaces must be considered as well known and the part about locally finite multipointed d-spaces does not have yet any application.

Contents

1. Introduction

Presentation of the main result. Multipointed d-spaces are introduced in [5] as a variant of Grandis' notion of d-space [12]. They are sufficient to model concurrency in computer science. The model structure constructed in [5], which is now called the q-model structure after [11], is known to be Quillen equivalent to its (semi)categorical analogue introduced on the category of topologically enriched small semicategories (a.k.a. flows) [8] [9].

Recall that a *cellular object* in a cofibrantly generated model category with a set of generating cofibrations I is an object X such that the canonical map $\varnothing \to X$ is a transfinite composition of pushouts of maps of I. In this note, the terminology of *cellular space* refers to the cellular objects of the q-model category of a convenient category of topological spaces (so it is a little bit more general than the notion of CW-complex) and the terminology of *cellular multipointed d-spaces* refers to the cellular objects of the q-model category of multipointed d-spaces.

All computer scientific examples are cellular objects of multipointed d-spaces or of flows because they are basically nothing else than pastings of cubes of various dimension, a n-cube representing the concurrent execution of n actions. For example, the papers [4] [6] provide a method to associate a labelled precubical set to any process algebra of

Date: November 15, 2021.
Key words and phrases. cellular multipointed d-space, Δ-generated space, compact-open topology, metrizable topology.

any synchronization algebra describing the space of execution paths and the concurrent structure. It is the reason why cellular multipointed d-spaces are important to study.

In [9, Proposition 2.5, Proposition 6.3 and Lemma 6.9], it is proved, as a technical tool for a proof, that the space of execution paths from the initial state to the final state of a chain of globes (cf. Definition 3.2) of the form $\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_1}) * \cdots * \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_p})$ where $\mathbf{D}^n$ is the n-dimensional closed disk is homeomorphic to the product calculated in the category of general topological spaces $\mathcal{G}(1,p) \times \mathbf{D}^{n_1} \times \ldots \times \mathbf{D}^{n_p}$ where $\mathcal{G}(1,p)$ is the set of nondecreasing homeomorphisms from $[0,1]$ to $[0,p]$ equipped with the compact-open topology and where the disks are equipped with the topology induced by the Euclidian metric. The important fact is that this topology is already Δ-generated. In particular, this space is metrizable. It is the key remark to conclude the proof of [9, Theorem 6.11] which is recalled in Theorem 3.10.

A chain of globes is a very particular example of a finite cellular multipointed d-space. It turns out that the result above can be generalized as follows:

Theorem. *(Theorem 4.7 and Corollary 4.8) Let X be a locally finite cellular multipointed d-space, i.e. such that the underlying space is locally finite as a cellular space. Then the topology of the space of execution paths $\mathbb{P}^{\mathcal{G}}X$ is equal to the compact-open topology and it is therefore metrizable.*

The striking fact is that the compact-open topology is already Δ-generated in the locally finite case: the Δ-kelleyfication is not required. Note that we already know by [11, Theorem 8.6] that $\mathbb{P}^{\mathcal{G}}X$ has the homotopy type of a CW-complex for any q-cofibrant multipointed d-space X, and in particular for any cellular (and not necessarily locally finite) multipointed d-space X. Another way to see that is to observe that for any q-cofibrant multipointed d-space X, $\mathbb{M}^{\mathcal{G}}(X)$ is a q-cofibrant Moore flow by [9, Corollary 7.5], and then that the $\mathcal{G}$-space of execution paths $\mathbb{P}\mathbb{M}^{\mathcal{G}}(X)$ is projective q-cofibrant by [8, Theorem 9.11], and then by [7, Proposition 7.1] we deduce that $\mathbb{P}\mathbb{M}^{\mathcal{G}}(X)$ is injective m-cofibrant, i.e. objectwise m-cofibrant. Since $\mathbb{P}^1\mathbb{M}^{\mathcal{G}}(X) = \mathbb{P}^{\mathcal{G}}X$ by definition of $\mathbb{P}\mathbb{M}^{\mathcal{G}}(X)$ (see [9, Theorem 4.12]), we deduce that the space $\mathbb{P}^{\mathcal{G}}X$ is m-cofibrant, i.e. homotopy equivalent to a CW-complex.

Organization of the note. Section 2 generalizes to cellular spaces the well-known metrizability condition for CW-complexes (Theorem 2.17). We follow the proofs of [3, Proposition 1.5.10 and Proposition 1.5.17]. They are adapted to the transfinite setting. Some arguments of Fritsch and Piccinini's book are also corrected and simplified. Some material of the proof like Proposition 2.12 is used in the proof of Theorem 4.3. Section 3 expounds the definition of a locally finite cellular multipointed d-space and fixes some notations used in Section 4. Finally, Section 4 proves the main result of the note in Theorem 4.7 and Corollary 4.8 after analyzing the compact-open topology in the locally finite case. The technical core of this note is Theorem 4.3 which is a variant of [9, Theorem 5.18].

Prerequisites and notations. This note uses very little homotopy theory. It is only required to know that a q-cofibration of topological spaces is a closed inclusion. The techniques of this note belong to general topology. Some relevant bibliographical references are given throughout the paper. This note is written with the French convention: a compact space is a quasicompact Hausdorff space. We work with the category, denoted by **Top**, either of Δ-*generated spaces* or of Δ-*Hausdorff* Δ-*generated spaces* (cf. [10, Section 2 and Appendix B]). The internal hom functor of **Top** is denoted by $\mathbf{TOP}(-,-)$.

2. Locally finite cellular topological space

Let λ be an ordinal. We work with a colimit-preserving functor $X : \lambda \longrightarrow \mathbf{Top}$ such that

- $X_0 = \varnothing$.
- For all $\nu < \lambda$, there is a pushout diagram of topological spaces

$$\begin{array}{ccc} \mathbf{S}^{n_\nu - 1} & \xrightarrow{g_\nu} & X_\nu \\ \downarrow & & \downarrow \\ \mathbf{D}^{n_\nu} & \xrightarrow{\widehat{g_\nu}} & X_{\nu+1} \end{array}$$

  with $n_\nu \geqslant 0$ where $\mathbf{S}^{n_\nu - 1}$ is the $n_\nu - 1$-dimensional sphere and $\mathbf{D}^{n_\nu}$ is the n_ν-dimensional disk.

  Let

$$X_\lambda = \varinjlim_{\nu<\lambda} X_\nu.$$

A *cellular topological space* is a topological space of the form X_λ: it is a cellular object for the q-model structure of **Top**. A cellular space is *finite* (*countable* resp.) if λ is finite (countable resp.). All cellular spaces are weakly Hausdorff. All cellular spaces are Δ-generated and therefore sequential. It implies that a cellular space is connected if and only if it is path-connected by [5, Proposition 2.8] and that a cellular space is homeomorphic to the disjoint sum of its path-connected components. The following proposition is a consequence of [2, Proposition 3.4] and [2, Proposition 3.10]. It can be easily proved without using diffeological spaces.

2.1. **Proposition.** *Every Δ-generated space, and in particular every cellular space, is locally path-connected.*

Proof. Let U be an open subset of a Δ-generated space X. Then U equipped with the relative topology is Δ-generated by [10, Proposition 2.4]. Therefore U equipped with the relative topology is homeomorphic to the disjoint sum of its path-connected components by [5, Proposition 2.8]. Thus X is locally path-connected. □

2.2. **Proposition.** *Every cellular space is normal (i.e. it separates disjoint closed subsets) and Hausdorff.*

Proof. Consider the first ordinal $\nu_0 \leqslant \lambda$ such that X_{ν_0} is not normal. We have $\nu_0 > 0$ since $X_0 = \varnothing$ is normal. Adding one cell preserves normality by [3, Proposition 1.1.2 (ii)]. Thus ν_0 is a limit ordinal. Each X_μ for $\mu < \nu_0$ is normal and weakly Hausdorff, and hence Hausdorff since the points are closed. Let F_0 and F_1 be two disjoint closed subsets of X_{ν_0}. Consider the continuous map $f : F \to \mathbb{R}$ defined as follows: $F = F_0 \cup F_1$, $f = 0$ on F_0 and $f = 1$ on F_1. By induction on $\mu \geqslant 0$, we construct a continuous map $f_\mu : X_\mu \to \mathbb{R}$ such that for all $\mu' \leqslant \mu$, $f_{\mu'} = f_\mu \restriction_{X_{\mu'}}$ and such that $f_\mu \restriction_{F \cap X_\mu} = f \restriction_{F \cap X_\mu}$ as follows. If μ is a limit ordinal, we just take the colimit. Otherwise $\mu - 1$ exists. There exists a unique continuous map $f_\mu : (F \cap X_\mu) \cup X_{\mu-1} \to \mathbb{R}$ such that $f_\mu \upharpoonright_{F\cap X_\mu} = f \upharpoonright_{F\cap X_\mu}$ and $f_\mu \upharpoonright_{X_{\mu-1}} = f_{\mu-1}$. Indeed, $F \cap X_\mu$ and $X_{\mu-1}$ are two closed subsets of X_μ and $(F \cap X_\mu) \cap X_{\mu-1} = F \cap X_{\mu-1}$. Since $(F \cap X_\mu) \cup X_{\mu-1}$ is a closed subset of X_μ, we can extend $f_\mu : X_\mu \to \mathbb{R}$ by Tietze's extension theorem [1, Chapter IX Section 4.2 Theorem 2] since X_μ is normal Hausdorff. By taking the colimit of all maps f_μ for $\mu < \nu_0$, we obtain a map $f_{\nu_0} : X_{\nu_0} \to \mathbb{R}$ which extends $f : F \to \mathbb{R}$. Then $F_0 \subset f_{\nu_0}^{-1}(]-1/2, 1/2[)$, $F_1 \subset f_{\nu_0}^{-1}(]1/2, 3/2[)$, which means that F_0 and F_1 are separated by two disjoint open subsets of X_{ν_0}. Thus X_{ν_0} is normal, and also Hausdorff since it is weakly Hausdorff. Contradiction. Hence ν_0 does not exist. □

Denote by

$$c_\nu = \mathbf{D}^{n_\nu} \backslash \mathbf{S}^{n_\nu - 1}$$

the ν-th *cell* of X_λ. Like in the usual setting of CW-complexes, $\widehat{g_\nu}$ induces a homeomorphism from c_ν to $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology which will be therefore denoted in the same way. It also means that $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology is Δ-generated. The *closure* of c_ν in X_λ is denoted by

$$\widehat{c_\nu} = \widehat{g_\nu}(\mathbf{D}^{n_\nu}).$$

A point y belongs to a unique open cell c_{ν_y} for some unique $\nu_y < \lambda$. Let $L \subset X_\lambda$. The smallest cellular subspace containing L is denoted by $X_\lambda(L)$. Since a cellular subspace is closed (the inclusion into X_λ is a q-cofibration, and therefore a closed inclusion), we have $X_\lambda(L) = X_\lambda(\widehat{L})$ where $\widehat{L}$ is the closure of L in X_λ.

2.3. **Definition.** *Let $V \subset X_\nu$ for some $\nu < \lambda$. The* collar *of V is by definition the subset of $X_{\nu+1}$*

$$C(V) = V \cup \widehat{g_\nu}\left(\left\{t\mathbf{s} \mid g_\nu(\mathbf{s}) \in V \text{ and } \frac{1}{2} < t \leqslant 1\right\}\right).$$

2.4. **Proposition.** *Let $V \subset X_\nu$ for some $\nu < \lambda$. Let $\mu < \lambda$. If $c_\mu \cap C(V) \neq \varnothing$, then $\widehat{c_\mu} \cap V \neq \varnothing$.*

Proof. For all $\mu > \nu$, $c_\mu \cap X_{\nu+1} = \varnothing$. Since $C(V) \subset X_{\nu+1}$, it means that either $\mu = \nu$ and the proof is complete, or $\mu < \nu$. In the latter case, let $y \in c_\mu \cap C(V)$. If $y \notin V$, then $y = \widehat{g_\nu}(\mathbf{s}) \in V$ since $\widehat{c_\mu} \subset X_\nu$. Thus $y \in V$. □

We can repeat the construction above and define $C_\mu(V)$ for all $\mu \geqslant \nu$ as follows:

(1) By convention, $C_\nu(V) = V$.
(2) For all $\mu \geqslant \nu$, $C_{\mu+1}(V) = C(C_\mu(V))$.
(3) For a limit ordinal $\mu \geqslant \nu$, let $C_\mu(V) = \varinjlim_{\mu' < \mu} C_{\mu'}(V)$.
(4) If $\mu > \lambda$, then $C_\mu(V) = C_\lambda(V)$.

2.5. **Definition.** *Let $V \subset X_\nu$ for some $\nu < \lambda$. The* infinite collar *is the subset $C_\lambda(V)$ of X_λ. It is denoted by $C_\infty(V)$ according to the usual notation for CW-complexes. Note that the construction of the collar implies that $X_\mu \cap C_\infty(V) = C_\mu(V)$ for all $\mu \geqslant \nu$.*

2.6. **Proposition.** *Let $V \subset X_\nu$ for some $\nu < \lambda$. Suppose that V is an open subset of X_ν. Then $C_\infty(V)$ is an open subset of X_λ.*

Proof. By [3, Lemma 1.1.7], $C(V)$ is an open subset of $X_{\nu+1}$. The transfinite sequence of spaces $(C_\mu(V))_{\mu \geqslant \nu}$ is equipped with the final topology because it is a tower of one-to-one maps (see [10, Proposition B.16]). We complete the proof by an easy transfinite induction. Let μ be a limit ordinal such that for all $\mu' < \mu$, $C_{\mu'}(V)$ is an open subset of $X_{\mu'}$. Since the inverse image of $C_\mu(V)$ in any $X_{\mu'}$ is equal to $C_{\mu'}(V)$, and since X_μ is equipped with the final topology, we obtain than $C_\mu(V)$ is an open subset of X_μ. $\square$

2.7. **Proposition.** *Let K be a compact subspace of X_λ. Then K intersects finitely many c_ν.*

Proof. It is an adaptation of [13, Proposition A.1]. Assume that there exists an infinite set $S = \{m_j \mid j \geqslant 0\}$ with $m_j \in K \cap c_{\nu_j}$. Then by transfinite induction on $\nu \geqslant 0$, we prove that S is closed in X_ν for all $0 \leqslant \nu \leqslant \lambda$. The same argument proves that every subset of S is closed in X_λ. Thus S has the discrete topology. But it is compact, and therefore finite. Contradiction. $\square$

2.8. **Proposition.** *Let $\nu < \lambda$. Then $X_\lambda(c_\nu)$ is a finite cellular space. In particular, it is compact.*

Proof. The proof is by induction on $\nu \geqslant 0$. If $\nu = 0$, then $X_\lambda(c_\nu)$ is a point. Suppose that $X_\lambda(c_\mu)$ is a finite cellular space for any $\mu < \nu$ with $\nu \geqslant 1$. Then $\widehat{c_\nu} \backslash c_\nu = \widehat{g_\nu}(\mathbf{S}^{n_\nu - 1})$ is a compact of X_λ which, by Proposition 2.7, intersects finitely many c_μ for $\mu < \nu$, say $c_{\mu_1}, \dots, c_{\mu_p}$. By induction hypothesis, $X(\mu_i)$ is a finite cellular space for $i = 1, \dots, p$. We deduce that $X_\lambda(c_\nu)$ is a finite cellular space. Note that the argument holds whether ν is a limit ordinal or not. Consider a sequence of $X_\lambda(c_\nu)$. Since $X_\lambda(c_\nu)$ is finite, there exists a subsequence included in one closed cell. Since the cell is compact, this subsequence has a convergent subsequence. We deduce that $X_\lambda(c_\nu)$ is sequentially compact, and therefore compact since it is sequential. $\square$

2.9. **Definition.** *The cellular space X_λ is* locally finite *if for all $\nu < \lambda$, the set*

$$\Omega_\nu = \{\nu' < \lambda \mid \widehat{c_{\nu'}} \cap c_\nu \neq \varnothing\}$$

is finite. Note that every finite cellular space is locally finite.

2.10. **Definition.** *The* star *of a subset $L \subset X_\lambda$ is*

$$\mathrm{St}(L) = \bigcup_{\substack{\nu<\lambda \\ \widehat{c_\nu}\cap L\neq\varnothing}} X_\lambda(c_\nu).$$

One has $\mathrm{St}(L) = \mathrm{St}(\widehat{L})$ where $\widehat{L}$ is the closure of L in X_λ.

2.11. **Proposition.** *Let c_ν be an open cell of X_λ. If X_λ is locally finite, then $\mathrm{St}(c_\nu)$ is a finite cellular subspace. In particular, it is compact.*

Proof. It is a consequence of Proposition 2.8 and Proposition 2.7. $\square$

2.12. **Proposition.** *If X_λ is locally finite, then for all $\nu < \lambda$, $\mathrm{St}(c_\nu)$ is a compact neighborhood of c_ν for the topology of X_λ.*

Proof. We follow the proof of [3, Lemma 1.5.9] by fixing some mistakes. Choose $\nu < \lambda$. Each closed cell is a compact of the weakly Hausdorff space X_λ. Since X_λ is locally finite, the set

$$W = \bigcup_{\nu' \in \Omega_\nu} \widehat{c_{\nu'}}$$

is therefore closed being a finite union of closed subsets. Every sequence of W has a subsequence living in one closed cell $\widehat{c_{\nu_0}}$ for some $\nu_0 \in \Omega_\nu$. Therefore W is sequentially compact. Since X_λ is sequential, we deduce that W is compact. Let

$$\Omega' = \{\nu' < \lambda \mid \widehat{c_{\nu'}} \cap W \neq \varnothing \text{ and } \widehat{c_{\nu'}} \cap c_\nu = \varnothing\}.$$

and

$$C = \bigcup_{\nu' \in \Omega'} \widehat{c_{\nu'}}.$$

By Proposition 2.7, Ω' is finite because W is compact. For the same reason as for W, C is compact, and therefore closed in X_λ. By Proposition 2.6, $C_\infty(c_\nu)$ is an open subset of X_λ. Thus $C_\infty(c_\nu)\backslash C$ is an open neighborhood of c_ν. Let

$$V_\mu = X_\mu \cap (C_\infty(c_\nu)\backslash C)$$

for $\mu \geqslant \nu$. There is the equality $V_\nu = c_\nu \subset W$. Let $\mu > \nu$ be an ordinal such that $V_{\mu'} \subset W$ for all $\nu \leqslant \mu' < \mu$. If μ is a limit ordinal, then $V_\mu = \bigcup_{\mu'<\mu} V_{\mu'} \subset W$. It remains the case $\mu = \mu' + 1$. There is nothing to do if $V_\mu = V_{\mu'}$. Assume that there exists $y \in V_\mu \backslash V_{\mu'}$. Let $y \in c_{\mu_y}$ for some unique μ_y (which is not necessarily μ'). Since $y \in C(V_{\mu'})$, we obtain $\widehat{c_{\mu_y}} \cap V_{\mu'} \neq \varnothing$ by Proposition 2.4. We deduce that $\widehat{c_{\mu_y}} \cap W \neq \varnothing$ by induction hypothesis. If $\mu_y \notin \Omega_\nu$, then $\mu_y \in \Omega'$. It implies that $\widehat{c_{\mu_y}} \subset C$, which is a contradiction since $y \notin C$. We deduce that $\mu_y \in \Omega_\nu$ and that $V_\mu \subset W$. We have proved by transfinite induction the inclusion $V_\mu \subset W$ for all $\nu \leqslant \mu \leqslant \lambda$. We deduce that $C_\infty(c_\nu)\backslash C \subset W$. Therefore W is a compact neighborhood of c_ν. It remains to observe that $\mathrm{St}(c_\nu)$ is compact by Proposition 2.11 and that $W \subset \mathrm{St}(c_\nu)$ to complete the proof. □

2.13. **Proposition.** *A cellular space X_λ is locally finite if and only if it is locally compact.*

Proof. If X_λ is locally finite, then it is locally compact by Proposition 2.12. Now suppose that X_λ is locally compact. We follow the proof of [3, Proposition 1.5.10]. Let c_ν be an open cell of X_λ. Every point of the closed cell $\widehat{c_\nu}$ has a compact neighborhood. Since $\widehat{c_\nu}$ is compact, c_ν is covered by finitely many of these compact neighborhoods. Therefore c_ν has a compact neighborhood V in X_λ. Now observe that, on one hand, c_ν does not intersect the closure of any cell of X_λ contained in $X_\lambda\backslash V$ because V is a neighborhood of c_ν and, on the other hand, V intersects only finitely many open cells of X_λ by Proposition 2.7, V being compact. These observations prove that the open cell c_ν intersects only finitely many closed cells of X_λ. □

2.14. **Proposition.** *If X_λ is locally finite and connected (or equivalently path-connected), then λ is countable.*

Proof. We adapt the proof of [3, Proposition 1.5.12]. Let $\nu_0 < \lambda$. For each $n \geqslant 0$, let

$$A_n = \{(\nu_0, \nu_1, \dots, \nu_n) \in \lambda^{n+1} \mid \widehat{c_{\nu_i}} \cap \widehat{c_{\nu_{i+1}}} \neq \varnothing, i = 1, \dots, n\}.$$

Since X_λ is locally finite, each $\mathrm{St}(c_\nu) = \mathrm{St}(\widehat{c_\nu})$ is a finite cellular subspace for each $\nu < \lambda$ by Proposition 2.11. Thus each A_n is finite since for $i = 1, \dots, n$, c_{ν_i} belongs to $\mathrm{St}(\widehat{c_{\nu_{i-1}}})$. Now consider

$$A = \bigcup_{n \geqslant 0} A_n$$

and consider the map

$$\alpha : A \longrightarrow \lambda$$

which takes $(\nu_0, \nu_1, \dots, \nu_n)$ to ν_n. Since X_λ is connected, it is path-connected being a Δ-generated space. Each continuous path intersects finitely many open cells by Proposition 2.7. Thus, the map α is onto. Therefore λ is countable. □

2.15. **Proposition.** *Suppose that λ is a countable ordinal and that X_λ is locally finite. Then there exists an increasing sequence of finite cellular subspaces $(Y_n)_{n\geqslant 0}$ of X_λ such that for all $n \geqslant 0$, the inclusion $Y_n \subset Y_{n+1}$ is a q-cofibration, such that Y_n is contained in the interior $\overset{\circ}{Y}_{n+1}$ of Y_{n+1} for the topology of X_λ and such that*

$$\bigcup_{n\geqslant 0} \overset{\circ}{Y}_n = \bigcup_{n\geqslant 0} Y_n = X_\lambda.$$

In other terms, for any countable locally finite cellular space X_λ, one can suppose without lack of generality that $\lambda = \aleph_0$ and that each X_n for $n < \aleph_0$ is a finite cellular space. Thanks to Proposition 2.15, it is possible to get rid of the transfinite construction in the locally finite countable case.

Proof. It is an adaptation of [3, Proposition 1.5.13]. Since λ is a countable ordinal, write

$$\lambda = \{\nu_i \mid i \in \mathbb{N}\}.$$

Define Y_0 as the empty set and assume that Y_n has been constructed. Consider the integer i defined by

$$i = \min\{j \mid c_{\nu_j} \not\subseteq Y_n\},$$

and also the finite set Ω_n of all cells contained in Y_n. Then let

$$Y_{n+1} = \mathrm{St}(c_{\nu_i}) \cup \bigcup_{c\in\Omega_n} \mathrm{St}(c).$$

Because of Proposition 2.11, Y_{n+1} is a finite cellular subspace of X_λ. By Proposition 2.12, Y_n is in the interior of Y_{n+1} for the topology of X_λ. Finally, by construction, $\bigcup_{n\geqslant 0} Y_n$ contains all cells of X_λ. □

2.16. **Proposition.** *Suppose that λ is a countable ordinal and that X_λ is locally finite. Then X_λ is metrizable and it can be embedded in the Hilbert cube.*

Proof. We adapt the proof of [3, Theorem 1.5.16] by also fixing the argument [1]. By Proposition 2.15, one can suppose that $\lambda = \aleph_0$ and that each X_n is a finite cellular space. Each X_n can then be embedded by a closed inclusion into a finitely dimensional Euclidian space by [13, Corollary A.10]. We deduce the existence of a countable basis $\mathcal{B}_n$ of open subsets of the interior $\overset{\circ}{X}_n$ of X_n in X_λ. Let

$$x \in X_\lambda = \bigcup_{n\geqslant 0} \overset{\circ}{X}_n.$$

Then $x \in \overset{\circ}{X}_n$ for some $n \geqslant 0$. Let Ω be an open neighborhood of X in X_λ. Then $\Omega \cap \overset{\circ}{X}_n$ is an open neighborhood of x in X_λ. There exists $B \in \mathcal{B}_n$ such that $x \in B \subset \Omega \cap \overset{\circ}{X}_n \subset \Omega$.

[1]I thank Joao Faria Martins and Tim Porter for helping me clarifying the argument of the book which seems to be incorrect. The book invokes the Cantor diagonalization argument for an unknown reason.

Thus the countable union

$$\bigcup_{n\geqslant 0} \mathcal{B}_n$$

is a countable basis of open subsets of X_λ. Because X_λ is normal and Hausdorff by Proposition 2.2, it is regular. Urysohn's metrization theorem then implies the metrizability of X_λ. Thus, X_λ is metrizable and has a countable open basis. Consequently it can be embedded in the Hilbert cube by [1, Chapter IX Section 2.8 Proposition 12]. □

We can conclude the section by the generalization to cellular spaces of a well-known statement about CW-complexes:

2.17. **Theorem.** *For a cellular topological space X_λ, the following properties are equivalent:*
(1) locally finite
(2) locally compact
(3) metrizable
(4) first countable.

Proof. We follow the proof of [3, Proposition 1.5.17] by fixing some arguments. The equivalence (1) $\Leftrightarrow$ (2) is Proposition 2.13. Suppose that X_λ is locally compact. Being a Δ-generated space, it is homeomorphic to the disjoint sum of its path-connected components, which are countable cellular subcomplexes by Proposition 2.14. So, by Proposition 2.16, X_λ is homeomorphic to the disjoint sum of subsets of the Hilbert cube equipped with the ℓ^2 metric. The latter is bounded: the maximal distance is $\frac{\pi}{\sqrt{6}}$. By setting $d(x, y) = \frac{\pi}{\sqrt{6}}$ if x and y are two elements of X_λ not belonging to the same path-connected component, we obtain a metric on X_λ. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) is obvious. The implication (4) $\Rightarrow$ (1) goes as in the proof of [3, Proposition 1.5.17]. Assume that X_λ is not locally finite. There there is an open cell c_ν which meets the closure of infinitely many closed cells $\{\widehat{c_{\nu_i}} \mid i \in \mathbb{N}\}$. Choose $x_i \in \widehat{c_{\nu_i}} \cap c_\nu$. Because c_ν is relatively sequentially compact, one can assume without lack of generality that the sequence $(x_i)_{i\geqslant 0}$ converges to a point $x \in \widehat{c_\nu}$. Take a countable basis $U_0 \supset U_1 \supset U_2 \ldots$ of open neighborhoods of x. Each U_i meets infinitely many open cells c_{ν_i}. Define a sequence of natural numbers $\{j_i \mid i \in \mathbb{N}\}$ by taking

- $j_0 = \min\{j \in \mathbb{N} \mid U_0 \cap c_{\nu_j} \neq \varnothing\}$,
- $j_{i+1} = \min\{j \in \mathbb{N} \mid j > j_i \text{ and } U_{i+1} \cap c_{\nu_j} \neq \varnothing\}$.

For every $i \in \mathbb{N}$, choose $z_i \in U_i \cap c_{\nu_{j_i}}$. Any open cell of X_λ contains at most one point of $Z = \{z_i \mid i \in \mathbb{N}\}$. Thus any closed cell, which intersects finitely many open cells by Proposition 2.7 contains at most finitely many points of Z. Therefore Z is discrete in X_λ. It is a contradiction since the sequence $(z_i)_{i\geqslant 0}$ converges to x. □

3. Locally finite cellular multipointed d-space

A *multipointed d-space* X is a triple $(|X|, X^0, \mathbb{P}^{\mathcal{G}}X)$ where

- The pair $(|X|, X^0)$ is a multipointed space. The space $|X|$ is called the *underlying space* of X and the set X^0 the *set of states* of X.
- The set $\mathbb{P}^{\mathcal{G}}X$ is a set of continous maps from $[0, 1]$ to $|X|$ called the *execution paths*, satisfying the following axioms:
  - For any execution path γ, one has $\gamma(0), \gamma(1) \in X^0$.

– Let γ be an execution path of X. Then any composite $\gamma\phi$ with $\phi \in \mathcal{G}(1,1)$ is an execution path of X where $\mathcal{G}(1,1)$ is the set of nondecreasing homeomorphisms from $[0,1]$ to itself.
– Let γ_1 and γ_2 be two execution paths of X with $\gamma_1(1) = \gamma_2(0)$; then the normalized composition $\gamma_1 *_N \gamma_2$ defined by

$$t \mapsto \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \gamma_2(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

is an execution path of X.

A map $f : X \to Y$ of multipointed d-spaces is a map of multipointed spaces from $(|X|, X^0)$ to $(|Y|, Y^0)$ such that for any execution path γ of X, the map

$$\mathbb{P}^{\mathcal{G}} f : \gamma \mapsto f.\gamma$$

is an execution path of Y. The following examples are used in this note.

(1) Any set E will be identified with the multipointed d-space $(E, E, \varnothing)$.
(2) The *topological globe of* Z, which is denoted by $\mathrm{Glob}^{\mathcal{G}}(Z)$, is the multipointed d-space defined as follows
   - the underlying topological space is the quotient space (it is the suspension of Z)
   $$\frac{\{0,1\} \sqcup (Z \times [0,1])}{(z,0) = (z',0) = 0, (z,1) = (z',1) = 1}$$
   - the set of states is $\{0,1\}$
   - the set of execution paths is the set of continuous maps
   $$\{\delta_z\phi \mid \phi \in \mathcal{G}(1,1), z \in Z\}$$
   with $\delta_z(t) = (z,t)$. It is equal to the underlying set of $\mathcal{G}(1,1) \times Z$.
   - The *directed segment* is the multipointed d-space $\overrightarrow{I}^{\mathcal{G}} = \mathrm{Glob}^{\mathcal{G}}(\{0\})$.

   In particular, $\mathrm{Glob}^{\mathcal{G}}_{\ell}(\varnothing)$ is the multipointed d-space $\{0,1\} = (\{0,1\}, \{0,1\}, \varnothing)$.

The category of multipointed d-spaces is denoted by $\mathcal{G}\mathbf{dTop}$. The subset of execution paths from α to β is the set of $\gamma \in \mathbb{P}^{\mathcal{G}}X$ such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$; it is denoted by $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X$: α is called the *initial state* and β the *final state*. It is equipped with the Δ-kelleyfication of the relative topology induced by the inclusion $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \subset \mathbf{TOP}([0,1], |X|)$. In other terms, a set map $U \to \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X$ is continuous if and only if the composite set map $U \to \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \subset \mathbf{TOP}([0,1], |X|)$ is continuous. The category $\mathcal{G}\mathbf{dTop}$ is locally presentable by [5, Theorem 3.5]. Therefore, it is bicomplete.

3.1. **Definition.** *Let X be a multipointed d-space X. Denote again by $\mathbb{P}^{\mathcal{G}}X$ the topological space*

$$\mathbb{P}^{\mathcal{G}}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X.$$

3.2. **Definition.** *Let $Z_1, \dots, Z_p$ be p nonempty topological spaces with $p \geqslant 1$. The multipointed d-space*

$$X = \mathrm{Glob}^{\mathcal{G}}(Z_1) * \cdots * \mathrm{Glob}^{\mathcal{G}}(Z_p).$$

with $p \geqslant 1$ means that the final state of a globe is identified with the initial state of the next one by reading from the left to the right.

Let λ be an ordinal. We work with a colimit-preserving functor

$$X : \lambda \longrightarrow \mathcal{G}\mathbf{dTop}$$

such that

- The multipointed d-space X_0 is a set, in other terms $X_0 = (X^0, X^0, \varnothing)$ for some *finite* set X^0.
- For all $\nu < \lambda$, there is a pushout diagram of multipointed d-spaces

$$\begin{array}{ccc} \mathrm{Glob}^{\mathcal{G}}(\mathbf{S}^{n_\nu - 1}) & \xrightarrow{g_\nu} & X_\nu \\ \big\downarrow & & \big\downarrow \\ \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_\nu}) & \xrightarrow{\widehat{g_\nu}} & X_{\nu+1} \end{array}$$

  with $n_\nu \geqslant 0$ where $\mathbf{S}^{n_\nu - 1}$ is the $n_\nu - 1$-dimensional sphere and $\mathbf{D}^{n_\nu}$ is the n_ν-dimensional disk.

A *cellular multipointed d-space* is a multipointed d-space of the form X_λ: it is a cellular object for the q-model structure of [11]. Let $X_\lambda = \varinjlim_{\nu<\lambda} X_\nu$. For all $\nu \leqslant \lambda$, there is the equality $X^0_\nu = X^0$. Denote by

$$c_\nu = |\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_\nu})\backslash|\mathrm{Glob}^{\mathcal{G}}(\mathbf{S}^{n_\nu - 1})|$$

the ν-th cell of X_λ. It is called a *globular cell*. Like in the usual setting of CW-complexes, $\widehat{g_\nu}$ induces a homeomorphism from c_ν to $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology which will be therefore denoted in the same way. It also means that $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology is Δ-generated. The closure of c_ν in $|X_\lambda|$ is denoted by

$$\widehat{c_\nu} = \widehat{g_\nu}(|\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_\nu})|).$$

The state $\widehat{g_\nu}(0) \in X^0$ ($\widehat{g_\nu}(1) \in X^0$ resp.) is called the *initial (final resp.) state* of c_ν. We want to recall

3.3. **Proposition.** [9, Proposition 5.2] *The space $|X_\lambda|$ is a cellular space. It contains X^0 as a discrete closed subspace. For every $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \lambda$, the continuous map $|X_{\nu_1}| \to |X_{\nu_2}|$ is a q-cofibration of spaces, and in particular a closed T_1-inclusion.*

Hence the following definition makes sense:

3.4. **Definition.** *A multipointed d-space is* locally finite *(*finite *resp.) if it is a cellular multipointed d-space of the form X_λ such that the underlying cellular space $|X_\lambda|$ is locally finite (finite resp.).*

3.5. **Notation.** *In the sequel, α and β are two elements of X^0.*

3.6. **Notation.** *The set of execution paths $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda$ equipped with the compact-open topology is denoted by $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$.*

3.7. **Definition.** *An execution path γ of a multipointed d-space X is* minimal *if*

$$\gamma(]0,1[) \cap X^0 = \varnothing.$$

3.8. **Theorem.** [9, Theorem 5.9] *Let γ be an execution path of X_λ. Then there exist minimal execution paths $\gamma_1, \dots, \gamma_n$ and $\ell_1, \dots, \ell_n > 0$ with $\sum_i \ell_i = 1$ such that*

$$\gamma = (\gamma_1 \mu_{\ell_1}) * \cdots * (\gamma_n \mu_{\ell_n}).$$

Moreover, if there is the equality

$$\gamma = (\gamma_1 \mu_{\ell_1}) * \cdots * (\gamma_n \mu_{\ell_n}) = (\gamma'_1 \mu_{\ell'_1}) * \cdots * (\gamma_{n'} \mu_{\ell'_{n'}})$$

such that all γ'_j are also minimal and with $\ell'_1, \dots, \ell'_{n'} > 0$, then $n = n'$ and $\gamma_i = \gamma'_i$ and $\ell_i = \ell'_i$ for all $1 \leqslant i \leqslant n$.

Let γ be an execution path of X_λ. Consider the normal form

$$\gamma = (\gamma_1 \mu_{\ell_1}) * \cdots * (\gamma_n \mu_{\ell_n}).$$

of Theorem 3.8. There exists a unique sequence $[c_{\nu_1}, \dots, c_{\nu_n}]$ of globular cells such that for all $1 \leqslant i \leqslant n$, $\gamma_i(]0,1[) \subset c_{\nu_i}$, $\gamma_i(0) = \widehat{g_{\nu_i}}(0)$ and $\gamma_i(1) = \widehat{g_{\nu_i}}(1)$. This leads to the following notion:

3.9. **Definition.** *With the notations above. The sequence of globular cells*

$$\mathrm{Carrier}(\gamma) = [c_{\nu_1}, \dots, c_{\nu_n}]$$

is called the carrier *of γ. The integer n is called the* length *of the carrier.*

Each carrier $\underline{c} = [c_{\nu_1}, \dots, c_{\nu_n}]$ gives rise to a map of multipointed d-spaces from a chain of globes to X_λ

$$\widehat{g_{\underline{c}}} : \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_n}}) \longrightarrow X_\lambda$$

by "concatenating" the attaching maps of the cells $c_{\nu_1}, \dots, c_{\nu_n}$. Let α_{i-1} (α_i resp.) be the initial state (the final state resp.) of $\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_i}})$ for $1 \leqslant i \leqslant n$ in

$$\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_n}}).$$

It induces a continuous map

$$\mathbb{P}^{\mathcal{G}}\widehat{g_{\underline{c}}} : X_{\underline{c}} \longrightarrow \mathbb{P}^{\mathcal{G}} X_\lambda.$$

with

$$X_{\underline{c}} = \mathbb{P}^{\mathcal{G}}_{\alpha_0,\alpha_n}(\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_{\nu_n}})).$$

We have the important (even if $X_{\underline{c}}$ is not relatively compact, as explained in [9]):

3.10. **Theorem.** [9, Theorem 6.11] *Let $\underline{c}$ be the carrier of some execution path of X_λ.*

(1) Consider a sequence $(\gamma_k)_{k \geqslant 0}$ of the image of $\mathbb{P}^{\mathcal{G}}\widehat{g_{\underline{c}}}$ which converges pointwise to γ_∞ in $\mathbb{P}^{\mathcal{G}} X_\lambda$. Let

$$\gamma_k = (\mathbb{P}^{\mathcal{G}}\widehat{g_{\underline{c}}})(\phi_k, z_k^1, \dots, z_k^n)$$

with $\phi_k \in \mathcal{G}(1,n)$ and $z_k^i \in \mathbf{D}^{n_{\nu_i}}$ for $1 \leqslant i \leqslant n$ and $k \geqslant 0$. Then there exist $\phi_\infty \in \mathcal{G}(1,n)$ and $z_\infty^i \in \mathbf{D}^{n_{\nu_i}}$ for $1 \leqslant i \leqslant n$ such that

$$\gamma_\infty = (\mathbb{P}^{\mathcal{G}}\widehat{g_{\underline{c}}})(\phi_\infty, z_\infty^1, \dots, z_\infty^n)$$

and such that $(\phi_\infty, z_\infty^1, \dots, z_\infty^n)$ is a limit point of the sequence $((\phi_k, z_k^1, \dots, z_k^n))_{k \geqslant 0}$.

(2) The image of $\mathbb{P}^{\mathcal{G}}\widehat{g_{\underline{c}}}$ is closed in $\mathbb{P}^{\mathcal{G}} X_\lambda$.

4. Space of execution paths in the locally finite case

4.1. **Proposition.** *Suppose that X_λ is locally finite. Then the space $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$ is metrizable.*

Proof. By Theorem 2.17, $|X_\lambda|$ is metrizable. By [13, Proposition A.13], the compact-open topology is then metrizable with the distance of the uniform convergence. □

We want to recall:

4.2. **Proposition.** *([9, Proposition 5.16 and Proposition 5.17]) Let c_ν be a globular cell of X_λ. Let $0 < h < 1$. Let*

$$\widehat{c_\nu}[h] = \left\{ \widehat{g_\nu}(z,h) \mid (z,h) \in |\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_\nu})| \right\}$$

For any minimal execution path γ and any $h \in]0,1[$, the cardinal of the set

$$\left\{ t \in]0,1[\mid \gamma(t) \in \widehat{c_\nu}[h] \right\}$$

is at most one. Moreover, there exists $h_\nu \in]0,1]$ such that for all $h \in]0,h_\nu]$, one has $\widehat{c_\nu}[h] \cap X^0 = \varnothing$.

4.3. **Theorem.** *Suppose X_λ locally finite. Let γ_∞ be an execution path of X_λ with*

$$\mathrm{Carrier}(\gamma_\infty) = [c_{\nu_1}, \dots, c_{\nu_n}].$$

There exists an open neighborhood $\overline{\Omega}$ of γ_∞ in $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$ and a finite set of nonzero dimensional globular cells

$$\{c_{\nu_j} \mid \nu_j \in J\}$$

such that for all execution paths $\gamma \in \overline{\Omega}$, the length of $\mathrm{Carrier}(\gamma)$ *does not exceed n and such that* $\mathrm{Carrier}(\gamma)$ *contains only globular cells of $\{c_{\nu_j} \mid \nu_j \in J\}$.*

Proof. Consider the decomposition of Theorem 3.8

$$\gamma_\infty = (\gamma^1_\infty \mu_{\ell_1}) * \cdots * (\gamma^n_\infty \mu_{\ell_n})$$

with $\sum_i \ell_i = 1$ and all execution paths γ^i_∞ minimal for $i = 1, \dots, n$. For $1 \leqslant i \leqslant n$, let $\nu_i < \lambda$, $\phi_i \in \mathcal{G}(1,1)$ and $z_i \in \mathbf{D}^{n_{\nu_i}} \backslash \mathbf{S}^{n_{\nu_i}-1}$ such that

$$\begin{aligned} &\mathrm{Carrier}(\gamma^i_\infty) = [c_{\nu_i}], \\ &\gamma^i_\infty(]0,1[) \subset c_{\nu_i}, \\ &\gamma^i_\infty = \delta_{z_i}\phi_i. \end{aligned}$$

Since X_λ is locally finite, the set of ordinals

$$K = \{\nu < \lambda \mid \exists i \in \{1,\dots,n\}, c_{\nu_i} \cap \widehat{c_\nu} \neq \varnothing\}$$

is finite. By Proposition 2.12, the finite cellular subspace of $|X_\lambda|$

$$\left(\bigcup_{\nu \in K} \mathrm{St}(c_\nu) \right) \cup \bigcup_{\alpha \in \gamma_\infty([0,1]) \cap X^0} \mathrm{St}(\{\alpha\})$$

is a compact neighborhood of $\gamma_\infty([0,1])$ in $|X_\lambda|$. Let

$$\{c_{\nu_j} \mid \nu_j \in J\}$$

be its finite set of nonzero dimensional globular cells. There exists an open subset Ω of $|X_\lambda|$ such that

$$\gamma_\infty([0,1]) \subset \Omega \subset \left(\bigcup_{\nu\in K} \mathrm{St}(c_\nu)\right) \cup \bigcup_{\alpha\in\gamma_\infty([0,1])\cap X^0} \mathrm{St}(\{\alpha\}).$$

Since the space $|\mathrm{Glob}^{\mathcal{G}}(\mathbf{D}^{n_\nu})|$ is compact for all $\nu < \lambda$, the subset $\widehat{c_\nu}$ is a compact subspace of the Hausdorff space $|X_\lambda|$ for all $\nu < \lambda$. The set $\widehat{c_\nu} \cap X^0$ is therefore finite because X^0 is discrete in $|X_\lambda|$. Consequently, the set

$$\left(\bigcup_{\nu\in J} \widehat{c_\nu}\right) \cap X^0 = \bigcup_{\nu\in J} \left(\widehat{c_\nu} \cap X^0\right)$$

is finite as well since J is finite. Using Proposition 4.2, consider

$$h = \min\{h_\nu \mid \nu \in J\} > 0$$

Then, for all $\nu \in J$, $\widehat{c_\nu}[h]$ does not intersect X^0. For all $1 \leqslant i \leqslant n$ and all $\nu \in J$, the set

$$\left\{t \in]0,1[\mid \gamma_\infty^i(t) \in \widehat{c_\nu}[h]\right\}$$

contains at most one point $t_{i,\nu}$ by Proposition 4.2; if the set above is empty, let $t_{i,\nu} = 1/2$. For all $1 \leqslant i \leqslant n$, let L_i and L'_i be two real numbers such that

$$0 < L_i < \min\{t_{i,\nu} \mid \nu \in J\} \leqslant \max\{t_{i,\nu} \mid \nu \in J\} < L'_i < 1.$$

For $1 \leqslant i \leqslant n$, consider the covering of the segment $[\sum_{j<i} \ell_j, \sum_{j\leqslant i} \ell_j]$ in three contiguous segments of strictly positive length:

$$K_i^- = \left[\sum_{j<i} \ell_j, \sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L_i)\right],$$

$$K_i^m = \left[\sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L_i), \sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L'_i)\right],$$

$$K_i^+ = \left[\sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L'_i), \sum_{j\leqslant i} \ell_j\right].$$

The restriction $\gamma_\infty \restriction_{[\sum_{j<i} \ell_j, \sum_{j\leqslant i} \ell_j]}$ goes from the initial state of the globular cell c_{ν_i} to its final state. We have therefore

$$\gamma_\infty(K_i^m) \subset c_{\nu_i}.$$

We deduce

$$\gamma_\infty(K_i^m) \cap X^0 = \varnothing.$$

We also have for all $\nu \in J$

$$\begin{aligned}
\gamma_\infty(K_i^-) \cap \widehat{c_\nu}[h] &= \Big(\gamma_\infty(\{\sum_{j<i} \ell_j\}) \cup \gamma_\infty(]\sum_{j<i} \ell_j, \sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L_i)])\Big) \cap \widehat{c_\nu}[h] \\
&= \Big(\{\widehat{g_{\nu_i}}(0)\} \cup \gamma_\infty(]\sum_{j<i} \ell_j, \sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L_i)])\Big) \cap \widehat{c_\nu}[h] \\
&\subset \Big(\{\widehat{g_{\nu_i}}(0)\} \cup \gamma_\infty(]\sum_{j<i} \ell_j, \sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(t_i)[)\Big) \cap \widehat{c_\nu}[h] \\
&= \varnothing,
\end{aligned}$$

the first equality by formal set identities, the second equality by definition of $\widehat{g_{\nu_i}}(0)$, the inclusion because $L_i < t_i$, and the last equality because $\widehat{g_{\nu_i}}(0) \in X^0$ and by definition of t_i. In the same way, we also have for all $\nu \in J$

$$\begin{aligned}
\gamma_\infty(K_i^+) \cap \widehat{c_\nu}[h] &= \Big(\gamma_\infty([\sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L'_i), \sum_{j\leqslant i} \ell_j[) \cup \gamma_\infty(\{\sum_{j\leqslant i} \ell_j\})\Big) \cap \widehat{c_\nu}[h] \\
&= \Big(\gamma_\infty([\sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(L'_i), \sum_{j\leqslant i} \ell_j[) \cup \{\widehat{g_{\nu_i}}(1)\}\Big) \cap \widehat{c_\nu}[h] \\
&\subset \Big(\gamma_\infty(]\sum_{j<i} \ell_j + \mu_{\ell_i}^{-1}\phi_i^{-1}(t_i), \sum_{j\leqslant i} \ell_j[) \cup \{\widehat{g_{\nu_i}}(1)\}\Big) \cap \widehat{c_\nu}[h] \\
&= \varnothing,
\end{aligned}$$

the first equality by formal set identities, the second equality by definition of $\widehat{g_{\nu_i}}(1)$, the inclusion because $t_i < L'_i$, and the last equality because $\widehat{g_{\nu_i}}(1) \in X^0$ and by definition of h_i. Since $|X_\lambda|$ is Hausdorff, the set $\widehat{c_\nu}[h]$ is a closed subset of $|X_\lambda|$ for all $\nu < \lambda$. Therefore

$$F = \bigcup_{\nu \in J} \widehat{c_\nu}[h]$$

is a closed subset of $|X_\lambda|$. Moreover, X^0 is a closed subset of the space $|X_\lambda|$ as well by Proposition 3.3. Consequently, the set

$$\overline{\Omega} = W([0,1], \Omega) \cap \bigcap_{i=1}^{i=n} \Big(W\Big(K_i^-, |X_\lambda|\backslash F\Big) \cap W\Big(K_i^m, |X_\lambda|\backslash X^0\Big) \cap W\Big(K_i^+, |X_\lambda|\backslash F\Big)\Big)$$

where

$$W([a,b], U) = \{f \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda \mid f([a,b]) \subset U\}$$

is an open neighborhood of γ_∞ in $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$. For all $\gamma \in \Omega$, one has

$$\gamma(K_i^m) \cap X^0 = \varnothing$$

and, for all $\nu \in J$, one has

$$\gamma(K_i^-) \cap \widehat{c_\nu}[h] = \gamma(K_i^+) \cap \widehat{c_\nu}[h] = \varnothing.$$

It turns out that the segments of strictly positive length K_i^-, K_i^m, K_i^+ for $1 \leqslant i \leqslant n$ are a finite partition of $[0,1]$ into nonoverlapping segments because we have by definition of the

K_i^-, K_i^m, K_i^+ for $1 \leqslant i \leqslant n$:

$$[0,1] = \bigcup_{i=1}^{i=n} \left[\sum_{j<i} \ell_j, \sum_{j\leqslant i} \ell_j\right] = \bigcup_{i=1}^{i=n} \left(K_i^- \cup K_i^m \cup K_i^+\right).$$

Since $\overline{\Omega} \subset W([0,1],\Omega)$, the only possible globular cells appearing in Carrier(γ) for $\gamma \in \overline{\Omega}$ are the globular cells of $\{c_{\nu_j} \mid \nu_j \in J\}$. Each c_ν for $\nu \in J$ appearing in the carrier Carrier(γ) corresponds to a minimal execution path from $\widehat{g_\nu}(0)$ to $\widehat{g_\nu}(1)$ of the decomposition of γ obtained using Theorem 3.8. It necessarily intersects $\widehat{c_\nu}[h]$ and therefore F. Thus, the length of the carrier of γ cannot exceed the number of K_i^m. □

4.4. **Definition.** *A topological space X is* weakly locally path-connected *if for every $x \in X$ and every neighborhood W of x, there exists a path-connected neighborhood (not necessarily open) W' of x such that $W' \subset W$.*

4.5. **Lemma.** *Every weakly locally path-connected space is locally path-connected.*

Proof. Let W be a neighborhood of $x \in X$. Then there exists a path-connected neighborhood W' of x such that $W' \subset W$. It means that W' is included in the path-connected component C of x in W. Therefore x is in the interior of C. Thus C is open and X is locally path-connected. □

4.6. **Theorem.** *Assume X_λ locally finite. The space $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$ is locally path-connected.*

Proof. By Lemma 4.5, it suffices to prove that $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$ is weakly locally path-connected. Consider an execution path γ_∞ of $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda$. Let Ω be an open subset of $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$ containing γ_∞. We want to construct a path-connected neighborhood of γ_∞ included in Ω. We can suppose that $\Omega \subset \overline{\Omega}$ where $\overline{\Omega}$ is the open subset constructed in Theorem 4.3. Then the set of carriers

$$\mathcal{T} = \{\text{Carrier}(\gamma) \mid \gamma \in \Omega\}$$

is finite by Theorem 4.3. For each $\underline{c} \in \mathcal{T}$ and for each $\Gamma \in (\mathbb{P}\widehat{g_{\underline{c}}})^{-1}(\gamma_\infty)$, there exists an open neighborhood Ω_Γ of Γ such that $(\mathbb{P}\widehat{g_{\underline{c}}})(\Omega_\Gamma) \subset \Omega$. Since $X_{\underline{c}}$ is Δ-generated, it is locally path-connected by Proposition 2.1. We can therefore suppose that Ω_Γ is path-connected. Consider

$$U = \bigcup_{\underline{c}\in\mathcal{T}} \bigcup_{\Gamma\in(\mathbb{P}\widehat{g_{\underline{c}}})^{-1}(\gamma_\infty)} (\mathbb{P}\widehat{g_{\underline{c}}})(\Omega_\Gamma).$$

Then U is path-connected and $U \subset \Omega$. Suppose that γ_∞ is not in the interior of U in $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$. The space $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$ is metrizable by Corollary 4.1. Therefore it is sequential. It means that there exists a sequence $(\gamma_k)_{k\geqslant 0}$ of execution paths not belonging to U converging to γ_∞ in $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta} X_\lambda)_{co}$. There exists $N \geqslant 0$ such that for all $k \geqslant N$, $\gamma_k \in \Omega$. Since the set $\{\text{Carrier}(\gamma_k) \mid k \geqslant N\} \subset \mathcal{T}$ is finite, we can always suppose that the sequence of carriers $(\text{Carrier}(\gamma_k))_{k\geqslant 0}$ is constant and e.g. equal to some $\underline{c} \in \mathcal{T}$ by extracting a subsequence. Therefore we can write $\gamma_k = (\mathbb{P}\widehat{g_{\underline{c}}})(\Gamma_k)$ with $\Gamma_k \in X_{\underline{c}}$. By Theorem 3.10, we can suppose that the sequence $(\Gamma_k)_{k\geqslant 0}$ converges to $\Gamma_\infty \in X_{\underline{c}}$. By continuity, we obtain the equality $\gamma_\infty = (\mathbb{P}\widehat{g_{\underline{c}}})(\Gamma_\infty)$. There exists $N \geqslant 0$ such that for all $k \geqslant N$, $\Gamma_k \in \Omega_{\Gamma_\infty}$. Contradiction. Thus γ_∞ is in the interior of U. □

4.7. **Theorem.** *Suppose* X_λ *locally finite. The topological space* $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$ *equipped with the compact-open topology is* Δ*-generated. In other terms, the canonical map*

$$\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda \longrightarrow (\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$$

is a homeomorphism.

Proof. The space $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$ is locally path-connected by Theorem 4.6. It is first countable by Corollary 4.1. The proof is complete thanks to [2, Proposition 3.11]. □

4.8. **Corollary.** *Suppose* X_λ *locally finite. The topological space*

$$\mathbb{P}^{\mathcal{G}}X_\lambda = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda$$

is metrizable with the distance of the uniform convergence.

Proof. The category of metric spaces is not cocomplete. Instead, we observe that the set of all execution paths equipped with the compact-open topology $(\mathbb{P}^{\mathcal{G}}X_\lambda)_{co}$ satisfies

$$(\mathbb{P}^{\mathcal{G}}X_\lambda)_{co} \cong \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} (\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X_\lambda)_{co}$$

because X^0 is a discrete subspace of $|X_\lambda|$. Hence $(\mathbb{P}^{\mathcal{G}}X_\lambda)_{co}$ is Δ-generated by Theorem 4.7 and metrizable by Proposition 4.1. □

References

[1] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 5 à 10.* Berlin: Springer, 2007. 4, 8

[2] D. Christensen, G. Sinnamon, and E. Wu. The *D*-topology for diffeological spaces. *Pacific Journal of Mathematics*, 272(1):87–110, 2014. `https://doi.org/10.2140/pjm.2014.272.87`. 3, 16

[3] R. Fritsch and R. A. Piccinini. *Cellular structures in topology*, volume 19 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. `https://doi.org/10.1017/cbo9780511983948`. 2, 3, 4, 5, 6, 7, 8

[4] P. Gaucher. Towards a homotopy theory of process algebra. *Homology, Homotopy and Applications*, 10(1):353–388, 2008. `https://doi.org/10.4310/HHA.2008.v10.n1.a16`. 1

[5] P. Gaucher. Homotopical interpretation of globular complex by multipointed d-space. *Theory and Applications of Categories*, 22(22):588–621, 2009. 1, 3, 9

[6] P. Gaucher. Combinatorics of labelling in higher dimensional automata. 411(11-13):1452–1483, 2010. `https://doi.org/10.1016/j.tcs.2009.11.013`. 1

[7] P. Gaucher. Enriched diagrams of topological spaces over locally contractible enriched categories. *New York Journal of Mathematics*, 25:1485–1510, 2019. 2

[8] P. Gaucher. Homotopy theory of Moore flows (I). *Compositionality*, 3(3), 2021. `https://doi.org/10.32408/compositionality-3-3`. 1, 2

[9] P. Gaucher. Homotopy theory of Moore flows (II). *Extracta Mathematicae*, 36(2):157–239, 2021. `https://doi.org/10.17398/2605-5686.36.2.157`. 1, 2, 10, 11, 12

[10] P. Gaucher. Left properness of flows. *Theory and Applications of Categories*, 37(19):562–612, 2021. 3, 5

[11] P. Gaucher. Six model categories for directed homotopy. *Categories and General Algebraic Structures with Applications*, 15(1):145–181, 2021. `https://doi.org/10.52547/cgasa.15.1.145`. 1, 2, 10

[12] M. Grandis. Directed homotopy theory. I. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 44(4):281–316, 2003. 1

[13] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002. 5, 7, 12

Université de Paris, CNRS, IRIF, F-75006, Paris, France
URL: http://www.irif.fr/~gaucher