**MOORE FLOWS**

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- The category of multipointed $d$-spaces $\mathcal{G}\mathbf{dTop}$
- The category of flows $\mathbf{Flow}$
- The functor $cat : \mathcal{G}\mathbf{dTop} \to \mathbf{Flow}$
- The category of Moore flows $\mathcal{G}\mathbf{Flow}$
- The zig-zag $\mathcal{G}\mathbf{dTop} \stackrel{\mathbb{M}^{\mathcal{G}}}{\to} \mathcal{G}\mathbf{Flow} \stackrel{\mathbb{M}}{\leftarrow} \mathbf{Flow}$
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# The category of multipointed $d$-spaces $\mathcal{G}\mathbf{dTop}$

- A **multipointed $d$-space** $X$ (Gaucher, 2009) is a triple $(|X|, X^0, (\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X)_{(\alpha,\beta)\in X^0\times X^0})$ where
  - $(|X|, X^0)$ is a multipointed space ($X^0$ is an arbitrary subset of the *underlying space* $|X|$).
  - $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X$ is a set of continuous paths $\phi$ from $[0,1]$ to $|X|$ such that $\phi(0) = \alpha \in X^0$ and $\phi(1) = \beta \in X^0$ which is closed under strictly increasing reparametrization
  - For every $(\phi,\psi) \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \times \mathbb{P}^{\mathcal{G}}_{\beta,\gamma}X$, $t \mapsto \begin{cases} \phi(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \psi(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases} \in \mathbb{P}^{\mathcal{G}}_{\alpha,\gamma}X$
  - Note that **the composition of paths is associative up to homotopy**
- Many relevant model structures, *even if they are not perfect*, to understand directed homotopy: the **q-model structure** (q for Quillen), the h-model structure and the H-model structure (both h for Hurewicz), the m-model structure (m for mixed), the Dwyer-Kan model structure, and many others...

# The category of flows **Flow**

- A **flow** $X$ (Gaucher, 2003) is a small semicategory (also called non-unital category) enriched over topological spaces
  - The set of objects (the *states*) is denoted by $X^0$
  - The space of morphisms (the *execution paths*) from $\alpha$ to $\beta$ is denoted by $\mathbb{P}_{\alpha,\beta}X$
  - Note that **the composition of morphisms is strictly associative**
- Many relevant model structures, *even if they are not perfect*, to understand directed homotopy: the **q-model structure** (q for Quillen), the h-model structure and the H-model structure (both h for Hurewicz), the m-model structure (m for mixed), the Dwyer-Kan model structure, and many others...

# The functor $cat : \mathcal{G}\mathbf{dTop} \to \mathbf{Flow}$

- There exists a functor $cat : \mathcal{G}\mathbf{dTop} \to \mathbf{Flow}$ such that
  - $cat(X)^0 = X^0$
  - $\mathbb{P}cat(X) = \mathbb{P}^{\mathcal{G}} X/$strictly increasing reparametrization
  - Obvious definition on morphisms
- There is a categorical equivalence (Gaucher, 2005) and (Gaucher, 2009)

$$\begin{tikzcd}
\mathcal{G}\mathbf{dTop} \arrow[r, "(-)^{cof}"] \arrow[d] & \mathcal{G}\mathbf{dTop} \arrow[d, "cat"] \\
\mathbf{Ho}(\mathcal{G}\mathbf{dTop}) \arrow[r, "\simeq"] & \mathbf{Ho}(\mathbf{Flow})
\end{tikzcd}$$

  between the homotopy categories of multipointed $d$-spaces and of flows for their q-model structures
- **This functor is not a left adjoint (or a right adjoint)**

# The category of Moore flows $\mathcal{G}$**Flow**

- $\mathcal{G}$ denotes the **enriched small category** such that
  - The objects are the segments $[0,\ell]$ with $\ell > 0$
  - The morphisms are the nondecreasing homeomorphisms $\phi \in [0,\ell] \cong^+ [0,\ell']$
- A **Moore flow** $X$ consists of
  - A set of states $X^0$
  - For each pair of states $(\alpha,\beta) \in X^0 \times X^0$, an **enriched presheaf** of topological spaces $\mathbb{P}_{\alpha,\beta}X$ over $\mathcal{G}$: $\mathbb{P}^{\ell}_{\alpha,\beta}X$ is called *the space of paths of length $\ell$ of $X$*
  - A composition law natural with respect to the morphisms of $\mathcal{G}$

$$\begin{tikzcd}
\mathbb{P}^{\ell_1}_{\alpha,\beta}X \times \mathbb{P}^{\ell_2}_{\beta,\gamma}X \arrow[r, "*"] \arrow[d] & \mathbb{P}^{\ell_1+\ell_2}_{\alpha,\gamma}X \arrow[d] \\
\mathbb{P}^{\ell'_1}_{\alpha,\beta}X \times \mathbb{P}^{\ell'_2}_{\beta,\gamma}X \arrow[r, "*"] & \mathbb{P}^{\ell'_1+\ell'_2}_{\alpha,\gamma}X
\end{tikzcd}$$

  which satisfies

$$\forall(\ell_1,\ell_2,\ell_3)\forall(\alpha,\beta,\gamma,\delta)\forall(x,y,z) \in \mathbb{P}^{\ell_1}_{\alpha,\beta}X \times \mathbb{P}^{\ell_2}_{\beta,\gamma}X \times \mathbb{P}^{\ell_3}_{\gamma,\delta}X, (x*y)*z = x*(y*z)$$

## The zig-zag $\mathcal{G}\mathbf{dTop} \xrightarrow{\mathbb{M}^{\mathcal{G}}} \mathcal{G}\mathbf{Flow} \xleftarrow{\mathbb{M}} \mathbf{Flow}$

- There exists a model structure on Moore flows and a zig-zag of right Quillen equivalences $\mathcal{G}\mathbf{dTop} \xrightarrow{\mathbb{M}^{\mathcal{G}}} \mathcal{G}\mathbf{Flow} \xleftarrow{\mathbb{M}} \mathbf{Flow}$
- $\mathbb{M}^{\mathcal{G}}$ and $\mathbb{M}$ preserve the set of states
- The enriched presheaf $\mathbb{P}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X)$ is defined by

$$\mathbb{P}^{\ell}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X) = \{t \mapsto \gamma(t/\ell) \mid \gamma \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X\} \text{ for } \ell > 0$$
$$f \mapsto f\phi \text{ for } f \in \mathbb{P}^{\ell'}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X) \text{ and } \phi \in [0,\ell] \cong^{+} [0,\ell']$$

  for all multipointed $d$-spaces $X$
- $\mathbb{P}_{\alpha,\beta}\mathbb{M}(Y)$ is defined as the **constant diagram functor** $\mathbb{P}^{\ell}_{\alpha,\beta}\mathbb{M}(Y) = \mathbb{P}_{\alpha,\beta}Y$ for all flows $Y$

# An inverse up to homotopy for *cat*

- $cat \cong \mathbb{M}_! \mathbb{M}^{\mathcal{G}}$ where $\mathbb{M}_! \dashv \mathbb{M}$
- Consider

$$(\mathbf{L}cat) : \mathcal{G}\mathbf{dTop} \xrightarrow{(-)^{cof}} \mathcal{G}\mathbf{dTop} \xrightarrow{cat} \mathbf{Flow}$$

$$(\mathbf{L}cat)^{-1} : \mathbf{Flow} \xrightarrow{\mathbb{M}} \mathcal{G}\mathbf{Flow} \xrightarrow{(-)^{cof}} \mathcal{G}\mathbf{Flow} \xrightarrow{\mathbb{M}_!^{\mathcal{G}}} \mathcal{G}\mathbf{dTop}$$

  where $\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$
- $(\mathbf{L}cat)^{-1}(\mathbf{L}cat)(X) \simeq X$ for all multipointed $d$-spaces $X$
- $(\mathbf{L}cat)(\mathbf{L}cat)^{-1}(Y) \simeq Y$ for all flows $Y$
- The unit and the counit of $\mathbb{M}_!^{\mathcal{G}} \dashv \mathbb{M}^{\mathcal{G}}$ are isomorphisms on q-cofibrant objects (all objects are q-fibrant)

# Bibliography


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