ENRICHED DIAGRAMS OF TOPOLOGICAL SPACES OVER LOCALLY CONTRACTIBLE ENRICHED CATEGORIES

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Abstract. It is proved that the projective model structure of the category of topologically enriched diagrams of topological spaces over a topologically enriched locally contractible small category is Quillen equivalent to the standard Quillen model structure of topological spaces. We give a geometric interpretation of this fact in directed homotopy.

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1. Introduction

Presentation. The model categories of flows [Gau03] (with some updated proofs in [Gau18]) and of multipointed d-spaces [Gau09] were introduced for studying some properties of concurrent systems from a topological point of view. The two model categories look very similar. However, the only known functor from multipointed d-spaces to flows [Gau09, Theorem 7.2] is neither a left adjoint nor a right adjoint. Therefore it cannot give rise to a Quillen equivalence.

The geometric reason of this problem is that the composition of paths is only associative up to homotopy in multipointed d-spaces and, on the contrary, is strictly associative in flows. This situation is not specific to directed homotopy of course: the composition of paths in topological spaces is also associative only up to homotopy in non-directed algebraic topology whereas the composition of morphisms are strictly associative in small categories.

This paper belongs to a research program which must lead to a proof that these two model categories are related by a zig-zag of Quillen equivalences. The intermediate model
category, the category of Moore flows, will be introduced later. We want to focus here on a Quillen equivalence which has a geometric interpretation in directed homotopy. This theorem and the material expounded here will be used in a different way in the future papers.

The starting point is the following geometric observation. Let $\mathcal{G}$ be the topological group of nondecreasing homeomorphisms of $[0, 1]$. It can be viewed as a one-object topologically enriched category such that $\mathcal{G}$ is the unique space of maps. Consider a topological space $X$ and a set of paths $dX$ closed under the action of $\mathcal{G}$. Typical examples are $d$-spaces in Grandis’ sense [Gra03] or multipointed $d$-spaces in the sense of [Gau09]. In each case, a part of the structure is forgotten. We suppose $dX$ equipped with its natural topology making the evaluation maps continuous. Such a set of data gives rise to a contravariant diagram of topological spaces over $\mathcal{G}$ denoted by $\mathcal{D}^\mathcal{G}(X, dX)$ with the only vertex $dX$ and taking $\phi \in \mathcal{G}$ to the mapping $\phi^* : \gamma \mapsto \gamma \phi$. The limit $\lim \mathcal{D}^\mathcal{G}(X, dX)$ is the space of paths of $dX$ invariant by the action of $\mathcal{G}$. It is equal to the subspace of constant paths of $dX$. The colimit $\lim \mathcal{D}^\mathcal{G}(X, dX)$ is nothing else but the quotient of the space of paths $dX$ by the action of $\mathcal{G}$. So far, the homotopy type of $BG \mathcal{G}$ does not interfere. It turns out that it is not the case for $\mathrm{holim} \mathcal{D}^\mathcal{G}(X, dX)$. For example, if $dX$ is a singleton (i.e. a constant path), then $\mathrm{holim} \mathcal{D}^\mathcal{G}(X, dX)$ has the homotopy type of $BG \mathcal{G}$ by [Hir03, Proposition 14.1.6 and Proposition 18.1.6] which is not contractible, although both $\mathcal{G}$ and $dX$ (which is supposed to be here a singleton) are contractible. The main result of this paper is that one possible way to overcome this problem is to work in the enriched setting. The main theorem of this paper is stated now:

**Theorem.** (Theorem 7.4) Let $\mathcal{P}$ be a topologically enriched small category. Suppose that $\mathcal{P}$ is locally contractible (i.e all spaces of maps $\mathcal{P}(\ell, \ell')$ are contractible). Let $\textbf{Top}$ be the category of $\Delta$-generated spaces. Then the colimit functor from the category $[\mathcal{P}, \textbf{Top}]_0$ of topologically enriched functors and natural transformations to $\textbf{Top}$ induces a left Quillen equivalence between the projective model structure and the Quillen model structure.

Note that the particular case where $\mathcal{P}$ has exactly one map between each pair of objects (i.e. each space $\mathcal{P}(\ell, \ell')$ is a singleton) is trivial. In this case, $[\mathcal{P}, \textbf{Top}]_0$ is equivalent to $\textbf{Top}$ as a category indeed.

Using this theorem, our example can now be reinterpreted in the enriched setting. The diagram $\mathcal{D}^\mathcal{G}(X, dX)$ belongs to $[\mathcal{G}^{op}, \textbf{Top}]_0$ because the mapping $\phi \mapsto \phi^*$ from $\mathcal{G}$ to $\textbf{TOP}(dX, dX)$ where $\textbf{TOP}(dX, dX)$ is the space of continuous maps from $dX$ to itself is continuous. Since the inclusion functor $[\mathcal{G}^{op}, \textbf{Top}]_0 \subset [\mathcal{G}^{op}, \textbf{Top}]$ into the category of all contravariant functors from $\mathcal{G}$ to $\textbf{Top}$ is colimit-preserving and limit-preserving, nothing changes concerning the interpretations of $\lim \mathcal{D}^\mathcal{G}(X, dX)$ and $\lim \mathcal{D}^\mathcal{G}(X, dX)$. On the contrary, the behavior of the homotopy colimit is completely different. There exists a cofibrant replacement $\mathcal{D}^\mathcal{G}(X, dX)^{cof}$ of $\mathcal{D}^\mathcal{G}(X, dX)$ in $[\mathcal{P}, \textbf{Top}]_0$ together with a pointwise weak homotopy equivalence $\mathcal{D}^\mathcal{G}(X, dX)^{cof} \rightarrow \Delta_{\mathcal{G}^{op}}(dX)$ (the constant diagram $\Delta_{\mathcal{G}^{op}}(dX)$ belongs to $[\mathcal{G}^{op}, \textbf{Top}]_0$). Since $dX$ is fibrant, we deduce that the canonical map $\lim \mathcal{D}^\mathcal{G}(X, dX)^{cof} \rightarrow dX$ is a weak homotopy equivalence because the colimit functor is a left Quillen equivalence. It means that in the enriched setting, $\mathrm{holim} \mathcal{D}^\mathcal{G}(X, dX)$ always has the homotopy type of $dX$. Everything behaves as if we were in the trivial case above.
Outline of the paper. Section 2 collects the notations and some useful facts about locally presentable categories and model categories which are used in this paper. Section 3 establishes the fact that a locally presentable category which is cartesian closed is a locally presentable base in the sense of [BQR98, Definition 1.1] for the closed monoidal structure induced by the binary product (Theorem 3.12). To the best of our knowledge, the proof of this fact was not yet known. The proof is due to Tim Campion with a minor correction which forces us to introduce and to study succinctly the category of all small diagrams over all small categories. Section 4 recalls some facts about the category of $\Delta$-generated spaces, the Quillen model structure, the Cole-Strøm model structure and the so-called mixed model structure. It culminates in the proof that the mixed model structure is accessible. Section 5 introduces the material of enriched diagrams. Some elementary facts which are used in the next sections are proved or recalled. Section 6 introduces two model structures, the projective one, which is combinatorial, on the category of enriched diagrams using the Quillen model structure, and the injective one, which is only accessible, on the category of enriched diagrams using the mixed model structure. This section discusses the interactions between the two model structures. The existence of the projective model structure on $[\mathcal{P}, \text{Top}]_0$ is a straightforward consequence of Moser’s work [Mos18]. We are also able to prove that this model structure is left proper in Theorem 6.7 (it is right proper because all objects are fibrant). The latter result is not a consequence of Moser’s work. It is a consequence of the way the Quillen model structure and the Cole-Strøm model structure on $\Delta$-generated spaces interact with each other. Section 7 proves the main theorem of the paper. Finally, Section 8 adds a comment based on a surprising answer made by Tyler Lawson in MathOverflow about the monoid of nondecreasing continuous maps from $[0, 1]$ to itself preserving the extremities.

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2. Notations, conventions and prerequisites

We refer to [AR94] for locally presentable categories, to [Ros09] for combinatorial model categories. We refer to [Hov99] and to [Hir03] for more general model categories. We refer to [Kel05] and to [Bor94a, Chapter 6] for enriched categories. All enriched categories are topologically enriched categories: the word topologically is therefore omitted.

- All categories are locally small.
- $\text{Set}$ is the category of sets.
- $\mathcal{K}(X, Y)$ is the set of maps in a category $\mathcal{K}$.
- $\mathcal{K}_0$ denotes sometimes the underlying category of an enriched category $\mathcal{K}$. Since we will be working in a cartesian closed category of topological spaces, $\mathcal{K}_0$ is nothing else but the category $\mathcal{K}$ with the topology of the space of maps forgotten.
- $\text{Cat}$ is the category of all small categories and functors between them.
- $\mathcal{P}$ denotes a nonempty enriched small category.
- $\mathcal{K}^{op}$ denotes the opposite category of $\mathcal{K}$.
- $\text{Obj}(\mathcal{K})$ is the class of objects of $\mathcal{K}$.
- $\text{Mor}(\mathcal{K})$ is the category of morphisms of $\mathcal{K}$ with the commutative squares for the morphisms.
• $\mathcal{K}^I$ is the category of functors and natural transformations from a small category $I$ to $\mathcal{K}$.
• $[I, \mathcal{K}]$ is the enriched category of enriched functors from an enriched small category to an enriched category $\mathcal{K}$, and $[I, \mathcal{K}]_0$ is the underlying category.
• $\Delta_I : \mathcal{K} \to \mathcal{K}^I$ is the constant diagram functor.
• $\varnothing$ is the initial object.
• $1$ is the final object.
• $\text{Id}_X$ is the identity of $X$.
• $g.f$ is the composite of two maps $f : A \to B$ and $g : B \to C$; the composite of two functors is denoted in the same way.
• If $f : \mathcal{I} \to \mathcal{J}$ is a functor between small categories and if $F : \mathcal{I} \to \mathcal{K}$ is a functor, then $\text{Lan}_f F$ denotes the left Kan extension of $F$ along $f$.
• $F \Rightarrow G$ denotes a natural transformation from a functor $F$ to a functor $G$.
• The composite of two natural transformations $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow H$ is denoted by $\nu \circ \mu$ to make the distinction with the composition of maps.
• A subcategory is always isomorphism-closed (replete).
• $f \Box g$ means that $f$ satisfies the left lifting property (LLP) with respect to $g$, or equivalently when $g$ satisfies the right lifting property (RLP) with respect to $f$.
• $\text{inj}(C) = \{g \in \mathcal{K}, \forall f \in C, f \Box g\}$.
• $\text{cof}(C) = \{f \mid \forall g \in \text{inj}(C), f \Box g\}$.
• $\text{cell}(C)$ is the class of transfinite compositions of pushouts of elements of $C$.
• A cellular object $X$ of a combinatorial model category is an object such that the canonical map $\varnothing \to X$ belongs to $\text{cell}(I)$ where $I$ is the set of generating cofibrations.
• A model structure $(C, W, F)$ means that the class of cofibrations is $C$, that the class of weak equivalences is $W$ and that the class of fibrations is $F$ in this order.
• $(-)^{\text{cof}}$ denotes a cofibrant replacement, $(-)^{\text{fib}}$ denotes a fibrant replacement.
• $F \dashv G$ denotes an adjunction where $F$ is the left adjoint and $G$ the right adjoint.

We will use the following known facts:

• A functor $F : \mathcal{K} \to \mathcal{L}$ between locally presentable categories is a left adjoint if and only if it is colimit-preserving; indeed, any left adjoint is colimit-preserving; conversely, if $F$ is colimit-preserving, $F^{\text{op}}$ is limit-preserving; since every locally presentable category is well-copowered by [AR94, Theorem 1.58] and has a generator, the opposite category $\mathcal{K}^{\text{op}}$ is well-powered and has a cogenerator; Hence the Special Adjoint Functor theorem [Bor94a, Theorem 3.3.4] states that $F^{\text{op}}$ is a right adjoint.
• A functor $F : \mathcal{K} \to \mathcal{L}$ between locally presentable categories is a right adjoint if and only if it is limit-preserving and accessible by [AR94, Theorem 1.66].
• In any model category, a colimit of a transfinite tower of cofibrations between cofibrant objects is a homotopy colimit (it is due to the fact that the transfinite tower is a diagram over a direct Reedy category and that, in this case, the tower is Reedy cofibrant).
A weak factorization system \((\mathcal{L}, \mathcal{R})\) of a locally presentable category \(\mathcal{K}\) is accessible if there is a functorial factorization
\[
(A \xrightarrow{f} B) \xrightarrow{(A \xrightarrow{L f} E f \xrightarrow{R f} B)}
\]
with \(Lf \in \mathcal{L}\), \(Rf \in \mathcal{R}\) such that the functor \(E : \text{Mor}(\mathcal{K}) \to \mathcal{K}\) is accessible. In particular, every small weak factorization system (i.e. of the form \((\text{cof}(I), \text{inj}(I))\) for a set \(I\)) is accessible. It can be proved that a weak factorization system is accessible if and only if it is small in Garner’s sense [Ros17]. A model structure \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) on a locally presentable category is accessible if the two weak factorization systems \((\mathcal{C}, \mathcal{W} \cap \mathcal{F})\) and \((\mathcal{C} \cap \mathcal{W}, \mathcal{F})\) are accessible. Every combinatorial model category is an accessible model category.

2.1. Theorem. (Rezk) Let \((\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)\) be two model structures on the same underlying category such that \(\mathcal{W}_1 = \mathcal{W}_2\). Then the model structure \((\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)\) is left proper (right proper resp.) if and only if the model structure \((\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)\) is left proper (right proper resp.).

Proof. This amazing result is a consequence of [Rez02, Proposition 2.5].

We will be using the following characterization of a Quillen equivalence. A Quillen adjunction \(F \dashv G : \mathcal{C} \rightleftarrows \mathcal{D}\) is a Quillen equivalence if and only if for all objects fibrant \(X\) of \(\mathcal{D}\), the natural map \(F(G(X)^\omega f) \to X\) is a weak equivalence of \(\mathcal{D}\) (the functor \(F\) is then said homotopically surjective) and if for all cofibrant objects \(Y\) of \(\mathcal{C}\), the unit of the adjunction \(Y \to G(F(Y))\) is a weak equivalence of \(\mathcal{C}\).

3. ON LOCALLY PRESENTABLE BASES

We recall the terminology of [Roi94]. Let \(p : \mathcal{E} \to \mathcal{B}\) be a functor. Let \(X\) be an object of \(\mathcal{B}\). The fibre of \(p\) over \(X\) consists of the subcategory of \(\mathcal{E}\) generated by the vertical maps \(f\), i.e. the maps \(f\) such that \(F(f) = \text{Id}_X\). A map \(\alpha : a \to b\) of \(\mathcal{E}\) is cartesian if every map \(\omega : a' \to b\) of \(\mathcal{E}\) with \(p(a) = p(a')\) factors uniquely as a composite \(\omega = \alpha.\phi\) with \(\phi\) vertical (see Figure 1). A map \(\alpha : a \to b\) of \(\mathcal{E}\) is cocartesian if every map \(\omega : a \to b'\) of \(\mathcal{E}\) with \(p(b) = p(b')\) factors uniquely as a composite \(\omega = \psi.\alpha\) with \(\psi\) vertical (see Figure 2). If for a given morphism \(f : x \to y\) of \(\mathcal{B}\) and an object \(b \in \mathcal{E}_y\), there exists a cartesian morphism \(b^f : a \to b\) with \(p(b^f) = f\), then \(a\) is determined in \(\mathcal{E}_x\) up to a unique isomorphism. It is called the reciprocal image of \(b\) by \(f\) and it is denoted by \(f^*b\). If for a given morphism \(f : x \to y\) of \(\mathcal{B}\) and an object \(a \in \mathcal{E}_x\), there exists a cartesian morphism \(a_f : a \to b\) with \(p(a_f) = f\), then \(b\) is determined in \(\mathcal{E}_y\) up to a unique isomorphism. It is called the direct image of \(a\) by \(f\) and it is denoted by \(f_*a\).

3.1. Definition. A functor \(p : \mathcal{E} \to \mathcal{B}\) is a bifibration if for every map \(f : x \to y\) of \(\mathcal{B}\), there exists a reciprocal image functor \(f^* : \mathcal{E}_y \to \mathcal{E}_x\) and a direct image functor \(f_* : \mathcal{E}_x \to \mathcal{E}_y\)
and if the canonical morphisms of functors $f^* \odot g^* \Rightarrow (g.f)^*$ and $(g.f)_* \Rightarrow g_* \odot f_*$ are isomorphisms.

Let $\mathcal{K}$ be a locally presentable category. We introduce the category $\mathcal{D}\mathcal{K}$ of all small diagrams over all small categories defined as follows. An object is a functor $F : I \to \mathcal{K}$ from a small category $I$ to $\mathcal{K}$. A morphism from $F : I_1 \to \mathcal{K}$ to $G : I_2 \to \mathcal{K}$ is a pair $(f : I_1 \to I_2, \mu : F \Rightarrow G.f)$ where $f$ is a functor and $\mu$ is a natural transformation.

If $(g, \nu)$ is a map from $G : I_2 \to \mathcal{K}$ to $H : \mathcal{K} \to \mathcal{K}$, then the composite $(g, \nu) \cdot (f, \mu)$ is defined by $(g.f, (\nu.f) \odot \mu)$. The identity of $F : I_1 \to \mathcal{K}$ is the pair $(\text{Id}_{I_1}, \text{Id}_F)$.

Thus the composition law is associative and the category $\mathcal{D}\mathcal{K}$ is well-defined.

3.2. Proposition. The forgetful functor $p : \mathcal{D}\mathcal{K} \to \mathcal{C}\mathcal{A}\mathcal{T}$ is a bifibration.

Proof. Let $f : I \to J$ be a functor between small categories. For a functor $G : J \to \mathcal{K}$, let $f^*(G) = G.f$. There is a canonical map $f^*(G) \to G$ in $\mathcal{D}\mathcal{K}$ defined by the pair $(f, \text{Id}_{G,f} : G.f \Rightarrow G.f)$. For a functor $F : I \to \mathcal{K}$, let $f_*(F) = \text{Lan}_f F$. Since $\mathcal{K}$ is locally presentable and hence bicomplete, there is an adjunction $f_* \dashv f^*$ between $\mathcal{K}^L$ and $\mathcal{K}^L$. There is a canonical map $F \to f_*(F)$ in $\mathcal{D}\mathcal{K}$ defined by the pair $(f, \eta_f : F \Rightarrow (\text{Lan}_f F).f)$ where $\eta_f : \text{Id} \Rightarrow f^*.f_*$ is the unit of the adjunction. Let $F : I_1 \to \mathcal{K}$ and $G : I_2 \to \mathcal{K}$ be two objects of $\mathcal{D}\mathcal{K}$. Every map $\omega = (f, \mu) : F \to G$ of $\mathcal{D}\mathcal{K}$ factors uniquely as a composite

$$
\begin{array}{c}
F \xrightarrow{\omega_f} f^*(G) \xrightarrow{\mu} G
\end{array}
$$

with

$$
\omega_f = (\text{Id}_{I_1}, \mu : F \Rightarrow G.f)
$$

and as a composite

$$
\begin{array}{c}
F \xrightarrow{f_*} f_*(F) \xrightarrow{\omega_f} G
\end{array}
$$

with

$$
\omega_f = (\text{Id}_{I_1}, \epsilon_f \odot f_*(\mu) : f_*(F) \Rightarrow f_*(G.f) \Rightarrow G)
$$

where $\epsilon_f$ is the counit of the adjunction.
It means that $f^*$ and $f_*$ satisfies the properties of the inverse and the direct image respectively. Let $g : \mathcal{L}_2 \rightarrow \mathcal{L}_3$ be another map of $\text{Cat}$. One has

$$f^*(g^*(G)) = G.g.f = (g.f)^*(G).$$

One also has the sequence of isomorphisms

$$\mathcal{K}^{\mathcal{L}_3}( (g.f)_*( F), G) \cong \mathcal{K}^{\mathcal{L}_1}( F, (g.f)^*(G)) \quad \text{by adjunction}$$

$$\cong \mathcal{K}^{\mathcal{L}_1}( F, f^*(g^*(G))) \quad \text{by the calculation above}$$

$$\cong \mathcal{K}^{\mathcal{L}_1}(f_*(F), g^*(G)) \quad \text{by adjunction}$$

$$\cong \mathcal{K}^{\mathcal{L}_1}( (g_*f_*) (F), G) \quad \text{by adjunction again}.$$ 

Thanks to the Yoneda lemma, we conclude that there is a natural isomorphism of functors

$$(g.f)_*(F) = (g_*.f_*)(F); \text{ in other terms we have } (g.f)_* = g_* f_*.$$

3.3. Proposition. The category $\mathcal{D}\mathcal{K}$ is cocomplete. The colimit functor induces a well-defined functor $\varprojlim : \mathcal{D}\mathcal{K} \rightarrow \mathcal{K}$ which is a left adjoint.

It is actually possible to prove that $\mathcal{D}\mathcal{K}$ is locally presentable but we will not need this fact in this paper.

Proof. Every fibre over a small category is a category of diagrams over a fixed small category: therefore all fibres of the bifibration $p : \mathcal{D}\mathcal{K} \rightarrow \text{Cat}$ are cocomplete. Moreover, the category of small categories is locally presentable, and therefore cocomplete as well. Using [Roi94 Proposition 3.3], we deduce that $\mathcal{D}\mathcal{K}$ is cocomplete. Consider the functor $\mathcal{I} : \mathcal{K} \rightarrow \mathcal{D}\mathcal{K}$ which takes an object $X$ of $\mathcal{K}$ to the constant diagram $\Delta_1 X$ over the terminal small category $1$. Then we have the sequence of natural isomorphisms (where $F : \mathcal{L} \rightarrow \mathcal{K}$ is an object of $\mathcal{D}\mathcal{K}$)

$$\mathcal{D}\mathcal{K}(F, \mathcal{I}(X)) \cong \mathcal{D}\mathcal{K}(F, \Delta_1 X) \quad \text{by definition of } \mathcal{I}$$

$$\cong \mathcal{K}^{\mathcal{L}}(F, f^*(\Delta_1 X)) \quad \text{where } f : \mathcal{L} \rightarrow 1 \text{ is the canonical functor}$$

$$\cong \mathcal{K}^{\mathcal{L}}(F, \Delta_1 X) \quad \text{by definition of } f$$

$$\cong \mathcal{K}(\varprojlim F, X) \quad \text{by adjunction}.$$ 

This sequence of natural isomorphisms implies that the mapping $\varprojlim : \mathcal{D}\mathcal{K} \rightarrow \mathcal{K}$ yields a well-defined functor and that it is a left adjoint. \hfill \Box

Let $F : \mathcal{L} \rightarrow \mathcal{K}$ and $G : \mathcal{J} \rightarrow \mathcal{K}$ be two objects of $\mathcal{D}\mathcal{K}$. Then a functor $f : \mathcal{L} \rightarrow \mathcal{J}$ induces a restriction set map

$$f^! : \mathcal{K}^{L}(F, G) \rightarrow \mathcal{K}^{L}(F, f, G, f)$$

by taking a natural transformation $\mu : F \Rightarrow G$ to the natural transformation $\mu.f : F.f \Rightarrow G.f$. We have $(f.g)^! = g^!f^!$.

Let $(\mathcal{L}_k)_{k \in \mathbb{K}}$ be a small diagram of small categories. Let $\mathcal{L} = \varprojlim \mathcal{L}_k$. Let $\iota_k : \mathcal{L}_k \rightarrow \mathcal{L}$ be the canonical functor. Let $u : k \rightarrow k'$ be a map of $\mathbb{K}$. If $\phi_u : \mathcal{L}_k \rightarrow \mathcal{L}_{k'}$ is the corresponding functor in the diagram of small categories, then $\iota_{k'} \phi_u = \iota_k$ and there is a restriction set map $(\phi_u)^! : \mathcal{K}^{L'}(F, k', G, t_{k'}) \rightarrow \mathcal{K}^{L}(F, t_k, G, t_k).$ We obtain a diagram of sets $\mathbb{K}^{op} \rightarrow \text{Set}$ taking $k$ to $\mathcal{K}^{L}(F, t_k, G, t_k)$. The set map $(\phi_u)^!$ yields a set map

$$\mathcal{K}^{L} (\text{lan}_{\phi_u}(F, t_{k'}), G) \cong \mathcal{K}^{L'}(F, t_{k'}, G, t_{k'}) \rightarrow \mathcal{K}^{L}(F, t_k, G, t_k) \cong \mathcal{K}^{L}(\text{lan}_{\iota_k}(F, t_k), G).$$
Let $\mu_u = (\phi_u)^!(\Id_{\Lan_{k'}(F.t_k)}) : \Lan_{k'}(F.t_k) \Rightarrow \Lan_{k'}(F.t_{k'})$. Then the map above is the precomposition by $\mu_u$.

3.4. **Proposition.** The natural transformation $\mu_u$ is the unique natural transformation from $\Lan_{k'}(F.t_k)$ to $\Lan_{k'}(F.t_{k'})$ making the following diagram commute:

$$
\begin{array}{c}
\Lan_{k'}(F.t_k) \\
\downarrow \mu_u \\
\Lan_{k'}(F.t_{k'})
\end{array}
\xymatrix{F \\
\ar_{\mu_u}[l]}
$$

where the horizontal natural transformations are the counits of the adjunction characterizing left Kan extensions.

**Proof.** We have $\iota_{k'} \cdot \phi_u = \iota_k$. Therefore, by adjunction, there is a bijection between the set of commutative squares as in the statement of the proposition and the set of commutative squares of the form

$$
\begin{array}{c}
\Lan_{\phi_u}(F.t_k) \\
\downarrow \\
\Lan_{\phi_u}(F.t_{k'})
\end{array}
\xymatrix{F.t_k \\
\ar_{\mu_u}[l]}
$$

since $\Lan_{\phi_u} : \Lan_{\phi_u} = \Lan_{k'}$.

3.5. **Corollary.** Let $u : k \to k'$ and $v : k' \to k''$ be two maps of $\mathbb{K}$. Then $\mu_{v,w} = \mu_v \circ \mu_w$.

Corollary 3.5 yields a well-defined diagram $\mathbb{K} \to \mathbb{K}^2$ defined on objects by the mapping $k \mapsto \Lan_k(F.t_k)$.

3.6. **Proposition.** With the notations above. Suppose that the diagram of small categories $(I_k)_{k \in \mathbb{K}}$ is $\lambda$-directed. Let $F, G : I \to \mathcal{K}$ be two objects of $\mathcal{D}\mathcal{K}$. Then there is the natural bijection $\mathcal{K}^\mathcal{L}(F, G) \cong \varprojlim \mathcal{K}^\mathcal{L}(F.t_k, G.t_k)$.

**Proof.** The family of maps $(\iota_k)^! : \mathcal{K}^\mathcal{L}(F, G) \to \mathcal{K}^\mathcal{L}(F.t_k, G.t_k)_{k \in \mathbb{K}}$ yields a well-defined cone because of the equality $(h.k)^! = k'.h'$. Let $(f_k)_{k \in \mathbb{K}} : S \xrightarrow{\bullet} \mathcal{K}^\mathcal{L}(F.t_k, G.t_k)_{k \in \mathbb{K}}$ be another cone of maps. We have to prove that it factors uniquely as a composite

$$
S \xrightarrow{f} \mathcal{K}^\mathcal{L}(F, G) \xrightarrow{\bullet} \mathcal{K}^\mathcal{L}(F.t_k, G.t_k)_{k \in \mathbb{K}}.
$$

Every object $c \in I$ can be written $c = \iota_k(d)$ for some $k$ and some $d$. Let $f(s)_c$ be defined by $f_k(s)_d : F(c) \to G(c)$. If $c = \iota_{k'}(d')$ is another possible choice, then there exists a cospan of maps $k \to k'' \leftarrow k'$ of $\mathbb{K}$ by hypothesis and $f_{k''}(s)_{d''} = f_k(s)_{d'}$, since $(f_k)_{k \in \mathbb{K}}$ is a cone of maps. Thus the definition $f(s)_c = f_k(s)_d$ is independent of the choice of $k$ and $d$ in the equality $c = \iota_k(d)$. We want to prove now that the map $f(s)_c : F(c) \to G(c)$ is natural with respect to $c$. Let $c \to d$ be a map of $I$. Then it is equal to a finite composite $\iota_{k_1}(u_1) \ldots \iota_{k_n}(u_n)$ where $u_i$ is a map of $I_k$. The source of $\iota_{k_n}(u_n)$ is $c$, which implies that $c = \iota_{k_n}(c')$ where $c'$ is the source of $u_n$. The target of $\iota_{k_1}(u_1)$ is $d$, which implies that $d = \iota_{k_1}(d')$ where $d'$ is the target of $u_1$. Since $\mathbb{K}$ is $\lambda$-directed, there exists an object $k$ of $\mathbb{K}$ and maps $\phi_1 : k_1 \to k, \ldots, \phi_n : k_n \to k$ of $\mathbb{K}$. We write $\iota_{k_1}(u_1) \ldots \iota_{k_n}(u_n) = \iota_k(\phi_1(u_1) \ldots \phi_n(u_n))$. And we have $c = \iota_k(\phi_n(c'))$ and
The map $\phi_1(u_1) \ldots \phi_n(u_n) : \phi_n(c') \to \phi_1(d')$ gives rise to the commutative square of $\mathcal{K}$:

$$
\begin{array}{ccc}
F(c) = F(t_k(\phi_n(c'))) & \xrightarrow{f(s)_c} & G(c) = G(t_k(\phi_n(c'))) \\
F(t_k(\phi_1(u_1) \ldots \phi_n(u_n))) & \quad & G(t_k(\phi_1(u_1) \ldots \phi_n(u_n))) \\
F(d) = F(t_k(\phi_1(d'))) & \xrightarrow{f(s)_d} & G(d) = G(t_k(\phi_1(d'))). 
\end{array}
$$

We deduce that $f$ induces a well-defined set map $f : S \to \mathcal{K}^{\lambda}(F,G)$ and it is clearly the unique choice. □

3.7. Proposition. With the notations above. Suppose that the diagram of small categories $(I_k)_{k \in \mathbb{K}}$ is $\lambda$-directed. Then $F = \underleftarrow{\lim}_I \text{Lan}_{I_k}(F.t_k)$, the colimit being calculated in the functor category $\mathcal{K}^I$.

Proof. Let $F, G : I \to \mathcal{K}$ be two objects of $\mathcal{D}\mathcal{K}$. Then

$$
\mathcal{K}^I(F, G) \cong \underleftarrow{\lim}_I \mathcal{K}^I(F.t_k, G.t_k)
\cong \underleftarrow{\lim}_I \mathcal{K}^I(\text{Lan}_{I_k}(F.t_k), G)
\cong \mathcal{K}^I(\underleftarrow{\lim}_I \text{Lan}_{I_k}(F.t_k), G)
$$

by adjunction

by definition of the (co)limit.

The proof is complete by the Yoneda lemma. □

3.8. Corollary. With the notations above. Suppose that the diagram of small categories $(I_k)_{k \in \mathbb{K}}$ is $\lambda$-directed. Then $F = \underleftarrow{\lim}_I F.t_k$, the colimit being calculated in $\mathcal{D}\mathcal{K}$.

Proof. The formula $F = \underleftarrow{\lim}_I \text{Lan}_{I_k}(F.t_k)$ of Proposition 3.7 is exactly the description of the colimit in $\mathcal{D}\mathcal{K}$ as given in [Roi94 Proposition 3.3*]. □

3.9. Definition. [BQR98, Definition 1.1] Let $\lambda$ be a regular cardinal. A locally $\lambda$-presentable base is a symmetric monoidal closed category which is locally $\lambda$-presentable and such that

- The unit of the tensor product is $\lambda$-presentable
- The tensor product of two $\lambda$-presentable objects of $\mathcal{V}$ is $\lambda$-presentable.

A locally presentable base is a locally $\lambda$-presentable base for some regular cardinal $\lambda$.

3.10. Proposition. [Kar18] Let $\lambda$ be a regular cardinal. Then there exist arbitrarily large regular cardinals $\mu$ such that $\mu^\lambda = \mu$.

Proof. Let $\kappa > \lambda$. We want to find a regular cardinal $\mu > \kappa$ such that $\mu^\lambda = \mu$. Suppose that $2^\kappa = \aleph_\alpha$. Take $\mu = \aleph_{\alpha+1}$. Then $\mu$ is regular since it is the successor of $\aleph_{\alpha}$. We write $\mu^\lambda = \aleph_{\alpha+1} = \aleph_{\alpha}^{\aleph_{\alpha+1}} = (2^\kappa)^\lambda \aleph_{\alpha+1} = 2^{\kappa^\lambda} \aleph_{\alpha+1} = \aleph_{\alpha}^{\aleph_{\alpha}+1} = \mu$,

the second equality being due to Hausdorff’s formula for exponentiation [HJ99 Chapter 9 Theorem 3.11]. □

3.11. Proposition. Let $\mathcal{K}$ be a locally $\lambda$-presentable category. Let $\mu$ be a regular cardinal such that $\mu^\lambda = \mu$. Then every $\mu$-presentable object is a $\mu$-small $\lambda$-directed colimit of $\lambda$-presentable objects.
Proof. Consider a $\mu$-presentable object $A$ of $\mathcal{K}$. Using [AR94, Remark 1.30(2)], write $A = \lim F$ where $F : \mathcal{K} \rightarrow \mathcal{K}$ is a $\lambda$-small diagram of $\lambda$-presentable objects. The point is that $\mathcal{K}$ is not necessarily $\lambda$-filtered or $\lambda$-directed (as far as we understand [AR94, Remark 1.30(2)]). Consider the set $\mathcal{S}$ of $\lambda$-small subcategories of $\mathcal{K}$ ordered by the inclusion. The poset $(\mathcal{S}, \subset)$ is $\lambda$-directed because $\lambda^2 = \lambda$ and because the smallest subcategory generated by a union of subcategories is generated by the finite compositions of morphisms of this union. For $I \in \mathcal{S}$, let $F_I$ be the composite functor $I \subset \mathcal{K} \overset{F}{\rightarrow} \mathcal{K}$. Every inclusion $I \subset J$ gives rise to a map $F_I \rightarrow F_J$ in $\mathcal{DK}$. By Corollary 3.3 we have

$$\lim_{I \in \mathcal{S}} F_I = F$$

in $\mathcal{DK}$. The latter colimit is $\lambda$-directed since the poset $(\mathcal{S}, \subset)$ is $\lambda$-directed. The poset $(\mathcal{S}, \subset)$ contains a minor correction which forces us to use Proposition 3.10 and Proposition 3.11 above. Since we have the isomorphism

$$\lim_{I \in \mathcal{S}} (\lim_{I \in \mathcal{S}} F_I) \cong \lim_{I \in \mathcal{S}} (\lim_{I \in \mathcal{S}} F_I) \cong \lim_{I \in \mathcal{S}} F \cong A,$$

the first isomorphism being due to Proposition 3.3. Each $\lim F_I$ is a $\lambda$-small colimit of $\lambda$-presentable objects. Using [AR94, Proposition 1.16], we deduce that each $\lim F_I$ is $\lambda$-presentable. We have rewritten $A$ as a $\mu$-small $\lambda$-directed colimit of $\lambda$-presentable objects.

3.12. Theorem. (T. Campion) [Cam18] Let $\mathcal{K}$ be a locally presentable category which is cartesian closed. Then it is a locally presentable base for the closed monoidal structure induced by the binary product.

Proof. The proof is reproduced here for the convenience of the reader and because it contains a minor correction which forces us to use Proposition 3.10 and Proposition 3.11 above. Since we have the isomorphism

$$\mathcal{K}(Z, X \times Y) \cong \mathcal{K}(Z, X) \times \mathcal{K}(Z, Y) \cong (\mathcal{K} \times \mathcal{K})(Z, (Z, (X, Y))),$$

the functor $(X, Y) \mapsto X \times Y$ is a right adjoint. It is therefore accessible. We choose a big enough regular cardinal $\lambda$ such that the functor $(X, Y) \mapsto X \times Y$ is $\lambda$-accessible, the category $\mathcal{K}$ is locally $\lambda$-presentable and the terminal object (i.e. the unit of the binary product) is $\lambda$-presentable. We choose a regular cardinal $\mu > \lambda$ such that the binary product of two $\lambda$-presentable objects is $\mu$-presentable. Using Proposition 3.10 we can suppose that $\mu^\lambda = \mu$ (it is the correction). We are going to prove that the class of $\mu$-presentable objects is closed under the binary product to complete the proof. Let $A$ and $B$ be two $\mu$-presentable objects. Using Proposition 3.11 write $A = \lim_{\gamma \in \mathcal{K}} A_\gamma$ and $B = \lim_{\gamma \in \mathcal{L}} B_\gamma$ as $\mu$-small $\lambda$-directed (and therefore $\lambda$-filtered) colimits of $\lambda$-presentable objects. Let $I = K \times L$ which is $\mu$-small and $\lambda$-filtered. The projections $\pi_1 : K \times L \rightarrow K$ and $\pi_2 : K \times L \rightarrow L$ are right cofinal. Indeed, for $k \in K$, $k \downarrow \pi_1 = (k \downarrow K) \times L$ is a product of filtered categories, and so filtered itself and therefore nonempty and connected. This implies that $\pi_1$ (and also $\pi_2$) is right cofinal. We obtain the isomorphisms $A \cong \lim_{j \in I} A_{\pi_1(i)}$.
and $B \cong \lim_{i \in I} B_{\pi_2(i)}$ by \cite[Theorem 14.2.5(1)]{Hir03}. We deduce the isomorphism of $\mathcal{K} \times \mathcal{K}$
\[(A, B) \cong \lim_{i \in I} (A_{\pi_1(i)} \times B_{\pi_2(i)}) \]
Since $I$ is $\lambda$-filtered and since the functor $(X, Y) \mapsto X \times Y$ is supposed to be $\lambda$-accessible, we obtain the isomorphism
\[A \times B \cong \lim_{i \in I} (A_{\pi_1(i)} \times B_{\pi_2(i)}) \]
Let $C = \lim_{j \in J} C_j$ be a $\mu$-filtered colimit. Then we have
\[\mathcal{K}(A \times B, C) \cong \lim_{i \in I} \mathcal{K} \left( \lim_{j \in J} \left( A_{\pi_1(i)} \times B_{\pi_2(i)} \times \lim_{j \in J} C_j \right) \right) \]
by $A \times B = \lim_{i \in I} \left( A_{\pi_1(i)} \times B_{\pi_2(i)} \right)$
\[\cong \lim_{j \in J} \left( \lim_{i \in I} \mathcal{K}(A_{\pi_1(i)} \times B_{\pi_2(i)}, C_j) \right) \]
since $A_{\pi_1(i)} \times B_{\pi_2(i)}$ is $\mu$-presentable
\[\cong \lim_{j \in J} \mathcal{K}(A \times B, C_j) \]
by $A \times B = \lim_{i \in I} \left( A_{\pi_1(i)} \times B_{\pi_2(i)} \right)$.
\[\square\]

4. Quillen and mixed model structures of topological spaces

The category $\textbf{Top}$ denotes the category of $\Delta$-generated spaces, i.e. the colimits of simplices. For a tutorial about these topological spaces, see for example \cite[Section 2]{Gau09}. The category $\textbf{Top}$ is locally presentable (see \cite[Corollary 3.7]{FR08}), cartesian closed and it contains all CW-complexes. The internal hom functor is denoted by $\textbf{TOP}(\cdot, \cdot)$ (see also \cite[Corollary 7.3]{Kel82}). It is tensored and cotensored over itself because $\textbf{Top}$ is cartesian closed: the tensor product is the binary product and the unit is the singleton. A category enriched over $\textbf{Top}$ to $\textbf{Set}$ is fibre-small and topological. The category $\textbf{Top}$ is a full coreflective subcategory of the category $\textbf{TOP}$ of general topological spaces.

The category $\textbf{Top}$ can be viewed as a category enriched over itself. It is also locally presentable in the enriched sense by \cite[Proposition 2.4]{Mos18} (see also \cite[Corollary 7.3]{Kel82}). It is tensored and cotensored over itself because $\textbf{Top}$ is cartesian closed: the tensor product is the binary product and the unit is the singleton. A category enriched over $\textbf{Top}$ is called an enriched category. As already said in Section 2 the adjective "topologically" is omitted because all enrichments in this paper are over $\textbf{Top}$.

4.1. Proposition. (Cole \cite[Theorem 2.1]{Co06} except the part about the accessibility of the model structure) Let $\mathcal{K}$ be a locally presentable category. Let $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ and $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be two model structures on $\mathcal{K}$ with $\mathcal{W}_1 \subset \mathcal{W}_2$ and with $\mathcal{F}_1 \subset \mathcal{F}_2$. Suppose that the weak factorization system $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1)$ is accessible and that the model structure $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ is combinatorial (i.e. cofibrantly generated). Then the mixed model structure $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$ defined by $\mathcal{W}_m = \mathcal{W}_2$, $\mathcal{F}_m = \mathcal{F}_1$ and $\mathcal{C}_m$ as the map having the LLP with respect to the maps of $\mathcal{W}_m \cap \mathcal{F}_m$ exists and is accessible. Moreover, we have $\mathcal{C}_1 \cap \mathcal{W}_1 = \mathcal{C}_m \cap \mathcal{W}_m$ and $\mathcal{C}_2 \subset \mathcal{C}_m$.

Sketch of proof. By \cite[Theorem 2.1]{Co06}, this model structure exists and the proof of the theorem also establishes the equality $\mathcal{C}_1 \cap \mathcal{W}_1 = \mathcal{C}_m \cap \mathcal{W}_m$. The inclusion $\mathcal{C}_2 \subset \mathcal{C}_m$ comes from the inclusion $\mathcal{W}_2 \cap \mathcal{F}_1 \subset \mathcal{W}_2 \cap \mathcal{F}_2$. There is the equality of weak factorization
systems \((C_1 \cap W_1, F_1) = (C_m \cap W_m, F_m)\). Thus the right-hand weak factorization system is accessible because the left-hand one is accessible by hypothesis. The other factorization is obtained as follows: first \(f\) factors as a composite \(f = R_2(f). L_2(f)\) with \(L_2(f) \in C_2\) and \(R_2(f) \in W_2 \cap F_2\). Since \(C_2 \subset C_m\), \(L_2(f) \in C_m\). Then \(R_2(f)\) factors as a composite \(R_2(f) = \ell.k\) with \(k \in C_1 \cap W_1 = C_m \cap W_m\) and \(\ell \in F_1 = F_m\). By the 2-out-of-3 property, \(\ell \in W_2 = W_m\). Thus the second factorization is \(f = \ell.(k.L_2(f))\). The functor \(R_2 : \text{Mor}(K) \to \text{Mor}(K)\) is accessible by [Dug01, Proposition 7.1] since the model structure \((C_2, W_2, F_2)\) is combinatorial by hypothesis. Since \((C_1 \cap W_1, F_1)\) is accessible by hypothesis, we deduce that the weak factorization system \((C_m, W_m \cap F_m)\) is accessible. □

The model category \(\text{Top}\) can be equipped with the standard Quillen model structure in which the weak equivalences are the weak homotopy equivalences [Hov99, Section 2.4]. There is another well-known model structure on \(\text{Top}\) called the Cole-Ström model structure. The weak equivalences are the homotopy equivalences; the fibrations are the Hurewicz fibrations; the cofibrations are the strong Hurewicz cofibrations. A general proof of its existence can be found in [BR13, Corollary 5.23]; The monomorphism hypothesis is automatically satisfied because \(\text{Top}\) is locally presentable [BR13, Remark 5.20]. All topological spaces are fibrant and cofibrant for this model structure.

By using Proposition 4.1, we obtain the mixed model structure: the weak equivalences are the weak homotopy equivalences and the fibrations are the Hurewicz fibrations. All topological spaces are fibrant for this model structure. The cofibrant objects are the topological spaces homotopy equivalent to a cofibrant object of the Quillen model structure [Col06, Corollary 3.7]. The cofibrations (the cofibrant objects resp.) of the mixed model structure are called the mixed cofibrations (the mixed cofibrant objects resp.). All Quillen cofibrations are mixed cofibrations.

4.2. Notation. By convention, \(\text{Top}_Q\) denotes the category of \(\Delta\)-generated spaces equipped with the Quillen model structure and \(\text{Top}_m\) denotes the category of \(\Delta\)-generated spaces equipped with the mixed model structure.

Convention. The words cofibration and cofibrant without further precision mean cofibration and cofibrant in \(\text{Top}_Q\). The words mixed cofibration and mixed cofibrant mean cofibration and cofibrant in \(\text{Top}_m\).

4.3. Corollary. The model category \(\text{Top}_m\) is accessible.

It is unlikely that the mixed model category \(\text{Top}_m\) be combinatorial. But we are not aware of any proof of that fact.

Sketch of proof. It suffices to check that the factorization of a map by a strong cofibration which is a homotopy equivalence followed by a Hurewicz fibration is accessible. We can use the construction of [BR13, Definition 3.2]. The middle space is given by an accessible functor as soon as the underlying category is locally presentable. □

Note that \(\text{Top}_m\) is proper. Indeed, it has the same class of weak equivalences as the model category \(\text{Top}_Q\). And the latter is known to be proper by [Hir03, Theorem 13.1.11]. Thus the former is proper as well by Theorem 4.1. The mixed model structure \(\text{Top}_m\) is also monoidal closed for the binary product by [Col06, Proposition 6.6].

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Let $\mathcal{P}$ be a nonempty enriched small category. Denote by $\mathcal{P}(\ell, \ell')$ the space of maps from $\ell$ to $\ell'$. The underlying category is denoted by $\mathcal{P}_0$ and we have

$$\mathcal{P}_0(\ell, \ell') = \text{Top}(\{0\}, \mathcal{P}(\ell, \ell'))$$

for all objects $\ell$ and $\ell'$ of $\mathcal{P}$.

An enriched functor from $\mathcal{P}$ to $\text{Top}$ is a functor of $\text{Top}^{\mathcal{P}_0}$ such that the set map

$$\mathcal{P}_0(\ell_1, \ell_2) \rightarrow \text{Top}(F(\ell_1), F(\ell_2))$$

induces a continuous map

$$F_{\ell_1, \ell_2} : \mathcal{P}(\ell_1, \ell_2) \rightarrow \text{TOP}(F(\ell_1), F(\ell_2)).$$

An enriched natural transformation $\eta : F \Rightarrow G$ from an enriched functor $F$ to an enriched functor $G$ is, by definition [Bor94b Proposition 6.13], a family of continuous maps

$$\eta : \{0\} \rightarrow \text{TOP}(F(\ell), G(\ell))$$

such that the diagram of $\text{Top}$ depicted in Figure 3 commutes for all $\ell_1, \ell_2 \in \text{Obj}(\mathcal{P})$. Since $\text{Top}$ is cartesian closed, we have

$$\text{Top}(F(\ell), G(\ell)) = \text{Top}(\{0\}, \text{TOP}(F(\ell), G(\ell))).$$

Therefore $\eta$ is just an ordinary natural transformation from $F$ to $G$ in $\text{Top}^{\mathcal{P}_0}$. The underlying category $[\mathcal{P}, \text{Top}]_0$ of the enriched category of enriched functors $[\mathcal{P}, \text{Top}]$ can then be identified with a full subcategory of the category $\text{Top}^{\mathcal{P}_0}$ of functors $F : \mathcal{P} \rightarrow \text{Top}$ such that the set map $\mathcal{P}_0(\ell_1, \ell_2) \rightarrow \text{Top}(F(\ell_1), F(\ell_2))$ induces a continuous map $\mathcal{P}(\ell_1, \ell_2) \rightarrow \text{TOP}(F(\ell_1), F(\ell_2))$ for all $\ell_1, \ell_2 \in \text{Obj}(\mathcal{P})$.

It is well-known that the enriched category $[\mathcal{P}, \text{Top}]$ is tensored and cotensored (e.g. see [Mos18 Lemma 5.2]). For an enriched diagram $F : \mathcal{P} \rightarrow \text{Top}$, and a topological space $U$, the enriched diagram $F \otimes U : \mathcal{P} \rightarrow \text{Top}$ is defined by $F \otimes U = F(-) \times U$ and $F^U : \mathcal{P} \rightarrow \text{Top}$ is defined by $F^U = \text{TOP}(U, F(-))$.

5.1. Proposition. The category $[\mathcal{P}, \text{Top}]_0$ is locally presentable.

Proof. By Theorem 3.12 the category $\text{Top}$ together with the binary product is a locally presentable base. Using [BQR98 Example 6.2], we deduce that the enriched category $[\mathcal{P}, \text{Top}]$ is enriched locally presentable. The proof is complete using [BQR98 Proposition 6.6].

5.2. Proposition. A functor $F$ of $\text{Top}^{\mathcal{P}_0}$ belongs to the full subcategory $[\mathcal{P}, \text{Top}]_0$ if and only if for all $\ell, \ell' \in \text{Obj}(\mathcal{P})$, the set map $\mathcal{P}(\ell, \ell') \times F(\ell) \rightarrow F(\ell')$ defined by the mapping $(\phi, \gamma) \mapsto F(\phi)(\gamma)$ is continuous.
Note that $\Delta_{\mathcal{P}} \emptyset$ belongs to $[\mathcal{P}, \text{Top}]_0$ just because $\text{Id}_\emptyset$ is continuous.

**Proof.** This comes from the bijection of sets
\[
\text{Top}(\mathcal{P}(\ell, \ell'), \text{TOP}(F(\ell'), F(\ell))) \cong \text{Top}(\mathcal{P}(\ell, \ell') \times F(\ell'), F(\ell)).
\]

5.3. **Proposition.** The inclusion functor $[\mathcal{P}, \text{Top}]_0 \subset \text{Top}^\mathcal{P}_0$ is colimit-preserving and limit-preserving.

**Proof.** Since the category $[\mathcal{P}, \text{Top}]_0$ is a full subcategory of $\text{Top}^\mathcal{P}_0$, it suffices to prove that $[\mathcal{P}, \text{Top}]_0$ is closed under the colimits and the limits of $\text{Top}^\mathcal{P}_0$. Let $(F_i)_{i \in I}$ be a small diagram of functors of $[\mathcal{P}, \text{Top}]_0$. The case of colimits comes from the fact that the colimit of the maps
\[
\mathcal{P}(\ell, \ell') \times F_i(\ell) \to F_i(\ell')
\]
in the category of diagrams $\text{Top}^{0 \to 1}$ is
\[
\mathcal{P}(\ell, \ell') \times (\lim_{\to} F_i(\ell)) \to \lim_{\to} F_i(\ell')
\]
because $\text{Top}$ is cartesian closed and because colimits in $\text{Top}^{0 \to 1}$ are calculated pointwise. The case of limits comes from the fact that the limit of the maps
\[
\mathcal{P}(\ell, \ell') \times F_i(\ell) \to F_i(\ell')
\]
in the category of diagrams $\text{Top}^{0 \to 1}$ is
\[
\mathcal{P}(\ell, \ell') \times \lim_{\leftarrow} F_i(\ell) \to \lim_{\leftarrow} F_i(\ell')
\]
because the functor $\lim_{\leftarrow}$ commutes with binary products as any right adjoint and because limits in $\text{Top}^{0 \to 1}$ are calculated pointwise. □

5.4. **Notation.** Let $\mathbb{F}_\ell^\mathcal{P} X = \mathcal{P}(\ell, -) \times X \in [\mathcal{P}, \text{Top}]_0$ where $X$ is a topological space and where $\ell$ is an object of $\mathcal{P}$.

5.5. **Proposition.** For every enriched functor $F : \mathcal{P} \to \text{Top}$, every $\ell \in \text{Obj}(\mathcal{P})$ and every topological space $X$, we have the natural bijection of sets
\[
[\mathcal{P}, \text{Top}]_0(\mathbb{F}_\ell^\mathcal{P} X, F) \cong \text{Top}(X, F(\ell)).
\]
In particular, the functor $\mathbb{F}_\ell^\mathcal{P} : \text{Top} \to [\mathcal{P}, \text{Top}]_0$ is colimit-preserving for all $\ell \in \text{Obj}(\mathcal{P})$.

**Proof.** We have the sequence of natural homeomorphisms
\[
(1) \quad [\mathcal{P}, \text{Top}](\mathbb{F}_\ell^\mathcal{P} X, F) \cong [\mathcal{P}, \text{Top}](\mathcal{P}(\ell, -), \text{TOP}(X, F(-)))
\]
\[
(2) \quad \cong \text{TOP}(X, F(\ell)) ,
\]
(1) because the enriched category $[\mathcal{P}, \text{Top}]$ is tensored and cotensored, (2) by the enriched Yoneda lemma. By applying the functor $\text{Top} \{0\} -$, we obtain the desired bijection. □

5.6. **Corollary.** Let $f : X \to Y$ be a map of $\text{Top}$. The map of enriched diagrams $\mathbb{F}_\ell^\mathcal{P} X \to \mathbb{F}_\ell^\mathcal{P} Y$ induced by $f$ satisfies the LLP with respect to a map of diagrams $D \to E$ of $[\mathcal{P}, \text{Top}]_0$ if and only if $f$ satisfies the LLP with respect to the continuous map $D_{\ell} \to E_{\ell}$.

5.7. **Theorem.** The inclusion functor $i : [\mathcal{P}, \text{Top}]_0 \subset \text{Top}^\mathcal{P}_0$ has both a left adjoint and a right adjoint. In other terms, the category $[\mathcal{P}, \text{Top}]_0$ is both a reflective and a
coreflective subcategory of $\mathbf{Top}^{P_0}$. If $i^P_\ell : \mathbf{Top}^{P_0} \to [P, \mathbf{Top}]_0$ is the left adjoint, then for all $\ell \in \text{Obj}(P)$ and all topological spaces $U$, $i^P_\ell (P_0(\ell, -) \times U) = P(\ell, -) \times U$.

**Proof.** Since the inclusion functor is colimit-preserving, it is in particular accessible and it is also a left adjoint because both the categories $[P, \mathbf{Top}]_0$ and $\mathbf{Top}^{P_0}$ are locally presentable. Since it is moreover limit-preserving, it is a right adjoint. We have the sequence of bijections

\[(3) \quad [P, \mathbf{Top}]_0(i^P_\ell (P_0(\ell, -) \times U), Y) \cong \mathbf{Top}^{P_0}(P_0(\ell, -) \times U, Y)\]

\[(4) \quad \cong \int_{\ell'} \mathbf{Top}(P_0(\ell, \ell') \times U, Y(\ell'))\]

\[(5) \quad \cong \int_{\ell'} \mathbf{Set}(P_0(\ell, \ell'), \mathbf{Top}(U, Y(\ell'))))\]

\[(6) \quad \cong \mathbf{Set}^{P_0}(P_0(\ell, -), \mathbf{Top}(U, Y(-)))\]

\[(7) \quad \cong \mathbf{Top}(U, Y(\ell))\]

\[(8) \quad \cong [P, \mathbf{Top}]_0(P(\ell, -) \times U, Y),\]

because $[P, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{P_0}$ by Proposition 5.2 and by adjunction, \[(9) \quad \text{by definition of the colimit,}\]

because $P_0(\ell, \ell')$ is a set, \[(10) \quad \text{by \cite{ML98} page 219 (2)},\]

by Yoneda, and finally \[(11) \quad \text{by Proposition 5.5}\] The proof is complete thanks to Yoneda. \hfill $\square$

5.8. **Proposition.** Let $\ell \in \text{Obj}(P)$. Let $U$ be a topological space. Then there is the natural homeomorphism $\lim \mathbb{F}_\ell^P U \cong U$.

**Proof.** There is the sequence of bijections ($V$ being another topological space):

\[(9) \quad \mathbf{Top}(\lim \mathbb{F}_\ell^P U, V) \cong \mathbf{Top}^{P_0}(\mathbb{F}_\ell^P U, \Delta_P V)\]

\[(10) \quad \cong [P, \mathbf{Top}]_0(\mathbb{F}_\ell^P U, \Delta_P V)\]

\[(11) \quad \cong \mathbf{Top} (\{0\}, [P, \mathbf{Top}](P(\ell, -) \times U, \Delta_P V))\]

\[(12) \quad \cong \mathbf{Top} (\{0\}, [P, \mathbf{Top}](P(\ell, -), \mathbf{TOP}(U, \Delta_P V(-))))\]

\[(13) \quad \cong \mathbf{Top} (\{0\}, \mathbf{TOP}(U, V))\]

\[(14) \quad \cong \mathbf{Top}(U, V),\]

by definition of the colimit, \[(15) \quad \text{because $[P, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{P_0}$}\]

and because the constant diagram functor belongs to $[P, \mathbf{Top}]_0$, \[(16) \quad \text{by definition of the enriched category $[P, \mathbf{Top}]$}\]

since the enriched category $[P, \mathbf{Top}]$ is tensored and cotensored, \[(17) \quad \text{by the enriched Yoneda lemma, and finally}\]

by definition of the enrichment of $\mathbf{Top}$. The proof is complete thanks to the (ordinary) Yoneda lemma. \hfill $\square$

6. **The homotopy theory of enriched diagrams of topological spaces**

6.1. **Notation.** Let $n \geq 1$. Denote by $D^n = \{b \in \mathbb{R}^n, |b| \leq 1\}$ the $n$-dimensional disk, and by $S^{n-1} = \{b \in \mathbb{R}^n, |b| = 1\}$ the $(n-1)$-dimensional sphere. By convention, let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

6.2. **Theorem.** The category $[P, \mathbf{Top}]_0$ can be endowed with a structure of combinatorial model category as follows:
• The set of generating cofibrations is the set of maps
\[ \{ F^\ell_P S^{n-1} \to F^\ell_P D^n \mid n \geq 0, \ell \in \text{Obj}(P) \}. \]

• The set of generating trivial cofibrations is the set of maps
\[ \{ F^\ell_P D^n \to F^{\ell+1}_P D^n \mid n \geq 0, \ell \in \text{Obj}(P) \}. \]

• A map \( F \to G \) is a weak equivalence if and only if for all \( \ell \in \text{Obj}(P) \), the continuous map \( F(\ell) \to G(\ell) \) is a weak equivalence of \( \text{Top}_Q \), i.e. the weak equivalences are the pointwise weak homotopy equivalences.

• A map \( F \to G \) is a fibration if and only if for all \( \ell \in \text{Obj}(P) \), the continuous map \( F(\ell) \to G(\ell) \) is a fibration of \( \text{Top}_Q \), i.e. the fibrations are the pointwise Serre fibrations.

This model structure, denoted by \([P, \text{Top}_m]_0^{\text{proj}}\), is called the projective model structure.

The cofibrations are called the projective cofibrations.

Proof. The existence of an accessible model structure is a consequence of Theorem 3.12 [Mos18, Theorem 6.5(ii)] and of the fact that all objects of \( \text{Top}_Q \) are fibrant (it suffices to use the adjunction \([P, \text{Top}]_0 \Rightarrow \text{Top}_Q^\text{Obj}(P)\) and the Quillen Path Object Argument, which implies the acyclicity condition). It is cofibrantly generated thanks to Proposition 5.5 and because the set of inclusions \( \{ S^{n-1} \subset D^n \mid n \geq 0 \} \) (\( \{ D^n \subset D^{n+1} \mid n \geq 0 \} \) resp.) is a set of generating (trivial resp.) cofibrations of \( \text{Top}_Q \).

□

In [Pia91], Piacenza proves a similar result by working in the category of Hausdorff k-spaces in the sense of [Vog71]. We did not read his proof in detail (which is much longer). We do not know if Piacenza’s proof can be adapted to \( \Delta \)-generated spaces, especially because Piacenza works with Hausdorff spaces. It is known that Hausdorff k-spaces do not behave very well for algebraic topology problems and that weak Hausdorff k-spaces are much better (see the end of the introduction of [Gau09] for some bibliographical research about this problem).

6.3. Corollary. The adjunction \( i^P_! \dashv i^P \) of Theorem 5.7 is a Quillen adjunction between the projective model structures of \( \text{Top}_0^P \) and \( [P, \text{Top}]_0 \).

6.4. Theorem. Suppose that all topological spaces \( \mathcal{P}(\ell, \ell') \) are homotopy equivalent to a cofibrant space. The category \([P, \text{Top}_m]_0\) can be endowed with a structure of accessible model category as follows:

• A map \( F \to G \) is a cofibration if and only if for all \( \ell \in \text{Obj}(P) \), the continuous map \( F(\ell) \to G(\ell) \) is a cofibration of \( \text{Top}_m \), i.e. the cofibrations are the pointwise mixed cofibrations.

• A map \( F \to G \) is a weak equivalence if and only if for all \( \ell \in \text{Obj}(P) \), the continuous map \( F(\ell) \to G(\ell) \) is a weak equivalence of \( \text{Top}_m \), i.e. the weak equivalences are the pointwise weak homotopy equivalences.

This model structure, denoted by \([P, \text{Top}_m]_0^{\text{inj}}\), is called the injective mixed model structure. The fibrations are called the injective fibrations.

Proof. By [Col06, Proposition 6.4], the cartesian closed category \( \text{Top} \) equipped with the mixed model structure is a monoidal model category. The proof is complete using Theorem 3.12 Corollary 4.3 and [Mos18, Theorem 6.5(i)]

□
6.5. **Corollary.** Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. Then the projective model structure $[\mathcal{P}, \text{Top}_{Q}]^{\text{proj}}_0$ is proper.

**Proof.** The injective model structure $[\mathcal{P}, \text{Top}_{m}]^{\text{inj}}_0$ is left proper by [Mos18, Proposition 8.1(i)] because all topological spaces $\mathcal{P}(\ell, \ell')$ are mixed cofibrant. We deduce that the functors $- \times \mathcal{P}(\ell, \ell') : \text{Top}_m \to \text{Top}_m$ preserve mixed cofibrations for all $\ell, \ell' \in \text{Obj}(\mathcal{P})$. By Theorem 2.1, we deduce that $[\mathcal{P}, \text{Top}_{Q}]^{\text{proj}}_0$ is left proper.

We can actually remove the hypothesis of Corollary 6.5 but the proof is a little bit more involved.

6.6. **Theorem.** Let $\lambda$ be a ordinal. Let $M : \lambda \to \text{Top}$ and $N : \lambda \to \text{Top}$ be two $\lambda$-sequences of topological spaces. Let $f : M \to N$ be a map of $\lambda$-sequences. Suppose that for each ordinal $\alpha$ with $\alpha + 1 \leq \lambda$, the map $M_\alpha \to M_{\alpha+1}$ and $N_\alpha \to N_{\alpha+1}$ are pushouts of a map of the form $\text{Id} \times \ldots \times \text{Id} \times g \times \text{Id} \times \ldots \times \text{Id}$ where $g$ is a cofibration of Top. Suppose moreover that the map $M_0 \to N_0$ is a weak homotopy equivalence. Then the map $\lim f : \lim M \to \lim N$ is a weak homotopy equivalence as well.

**Proof.** All maps of the form $\text{Id} \times \ldots \times \text{Id} \times g \times \text{Id} \times \ldots \times \text{Id}$ where $g$ is a cofibration of Top are cofibrations of the Cole-Ström model structure of Top. We deduce that the colimit is a homotopy colimit in Top equipped with the Cole-Ström model structure. By [DI04, Theorem A.7], this colimit is therefore a homotopy colimit in Top equipped with the Quillen model structure ([DI04, Theorem A.7] is written in the category of general topological spaces $\text{TOP}$ but the argument remains valid in Top because Top is a coreflective subcategory of $\text{TOP}$). We can also prove this theorem by using [Gau07, Proposition 7.2] and [Gau07, Proposition 7.3]. The proofs of these two propositions are written down in the category of weakly Hausdorff $k$-spaces but they remain valid in Top. The key point is that the cofibrations of the Cole-Ström model structure of Top are closed $T_1$-inclusions and that compact spaces are $\aleph_0$-small relative to closed $T_1$-inclusions by [Hov99, Proposition 2.4.2]. In both arguments, the Cole-Ström model structure plays a key role.

6.7. **Theorem.** The projective model structure $[\mathcal{P}, \text{Top}_{Q}]^{\text{proj}}_0$ is proper.

**Proof.** It suffices to prove that it is left proper because all objects are fibrant. In a model category, weak equivalences are closed under retract. Therefore it suffices to prove that the pushout of a weak equivalence along a transfinite composition of pushouts of maps of the form $F \mathbb{F}_\ell S^{n-1} \to F \mathbb{F}_\ell D^n$ is still a weak equivalence. Consider first the following situation:

\[
\begin{array}{ccc}
F \mathbb{F}_\ell S^{n-1} & \xrightarrow{f} & F \mathbb{F}_\ell D^n \\
\downarrow & & \downarrow \text{H} \\
G & \xrightarrow{G} & K.
\end{array}
\]

We cannot apply [Mos18, Proposition 8.1(i)] directly to $[\mathcal{P}, \text{Top}_{Q}]^{\text{proj}}_0$ because the spaces of maps of $\mathcal{P}$ are not supposed to be cofibrant in the Quillen model structure of Top.
For all \( \ell > 0 \), we obtain the diagram of \( \text{Top} \)
\[
\begin{tikzcd}
\mathcal{P}(\ell, \ell') \times S^{n-1} \ar{r}{F(\ell')} \ar{d} & F(\ell') \ar{d}{\tilde{f}_\nu} \ar{r}{G(\ell')} & G(\ell'). \\
\mathcal{P}(\ell, \ell') \times D^n \ar{r}{H(\ell')} \ar{d} & H(\ell') \ar{d}{\tilde{f}_\nu} \ar{r}{K(\ell')} & K(\ell').
\end{tikzcd}
\]

If \( f_\nu \) is a weak homotopy equivalence, then \( \tilde{f}_\nu \) is weak homotopy equivalence by Theorem 6.6. Thus if \( f \) is a pointwise weak equivalence, then \( \tilde{f} \) is a pointwise weak equivalence. Again by Theorem 6.6, this process can be iterated transfinitely since colimits in \([\mathcal{P}, \text{Top}]_0\) are calculated pointwise by Proposition 5.3.

6.8. Proposition. For all (trivial resp.) cofibrations \( f : U \rightarrow V \) of \( \text{Top}_Q \) and all \( \ell \in \text{Obj}(\mathcal{P}) \), the map of diagrams \( F^\ell_U \rightarrow F^\ell_V \) is a (trivial resp.) projective cofibration.

Proof. The map of diagrams \( F^\ell_U \rightarrow F^\ell_V \) satisfies the LLP with respect to a (trivial) fibration \( D \rightarrow E \) if and only if \( f \) satisfies the LLP with respect to the continuous map \( D_\ell \rightarrow E_\ell \) by Proposition 5.5. The proof is complete because the fibrations and the trivial fibrations are the pointwise ones.

6.9. Corollary. The combinatorial model category \( [\mathcal{P}, \text{Top}_Q]_0^{proj} \) is tractable.

Proof. It is a consequence of the fact that the maps \( \emptyset \subset S^{n-1} \) and \( \emptyset \subset D^n \) are cofibrations for all \( n \geq 0 \) and that \( \mathbb{F}^\ell_\emptyset = \Delta_p \emptyset \) is the initial object of \([\mathcal{P}, \text{Top}]_0\) for all \( \ell \in \text{Obj}(\mathcal{P}) \). The proof is complete thanks to Proposition 6.8.

7. The main theorem

7.1. Proposition. Suppose that all topological spaces \( \mathcal{P}(\ell, \ell') \) are homotopy equivalent to a cofibrant space. Then any cofibration of \([\mathcal{P}, \text{Top}_Q]_0^{proj}\) is a cofibration of \([\mathcal{P}, \text{Top}_m]_0^{inj}\). In other terms, the identity functor induces a left Quillen adjoint \( \text{Id} : [\mathcal{P}, \text{Top}_Q]_0^{proj} \rightarrow [\mathcal{P}, \text{Top}_m]_0^{inj} \).

Another way to formulate this proposition is that any projective cofibration of the model category \([\mathcal{P}, \text{Top}_Q]_0^{proj}\) is a pointwise mixed cofibration of \( \text{Top}_m^{proj} \). Note that a projective cofibration of \([\mathcal{P}, \text{Top}_Q]_0^{proj}\) is not necessarily a pointwise cofibration of \( \text{Top}_m^{proj} \). Indeed the diagram \( \mathbb{F}^\ell_U \) is projective cofibrant for any cofibrant space \( U \) by Proposition 6.8. But the vertices of this diagram are only homotopy equivalent to a cofibrant space.

Proof. Every cofibration of \([\mathcal{P}, \text{Top}_Q]_0^{proj}\) is a retract of a transfinite composition of pushouts of maps of \( \{ \mathbb{F}^\ell S^{n-1} \subset \mathbb{F}^\ell D^n \mid n > 0, \ell \in \text{Obj}(\mathcal{P}) \} \). Therefore it suffices to prove that the maps \( \mathbb{F}^\ell S^{n-1} \subset \mathbb{F}^\ell D^n \) are pointwise mixed cofibrations of \( \text{Top}_m^{proj} \) for all \( n > 0, \ell \in \text{Obj}(\mathcal{P}) \). It suffices to prove that for all \( \ell, \ell' \in \text{Obj}(\mathcal{P}) \) and all \( n > 0 \), the map \( \mathcal{P}(\ell, \ell') \times S^{n-1} \subset \mathcal{P}(\ell, \ell') \times D^n \) is a mixed cofibration. The latter fact comes from the facts that \( \mathcal{P}(\ell, \ell') \) is cofibrant in \( \text{Top}_m \), that any cofibration is a mixed cofibration and that \( (\text{Top}_m, \times) \) is a monoidal model structure.
7.2. Corollary. Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. Then the identity functor induces a left Quillen equivalence

$$\text{Id} : [\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}} \to [\mathcal{P}, \text{Top}_m]_{0}^{\text{inj}}.$$ 

7.3. Proposition. There is a Quillen adjunction $\lim_{\to} \Delta_{\mathcal{P}}$ between the model categories $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$ and $\text{Top}_Q$.

Proof. There is the sequence of bijections ($X$ being an object of $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$ and $U$ being a topological space)

$$\text{Top}(\lim_{\to} X, U) \cong \text{Top}^{\mathcal{P}_0}(X, \Delta_{\mathcal{P}} U) \cong [\mathcal{P}, \text{Top}]_{0}(X, \Delta_{\mathcal{P}} U),$$

the left-hand bijection by definition of the colimit, the right-hand bijection because the category $[\mathcal{P}, \text{Top}]_{0}$ is a full subcategory of $\text{Top}^{\mathcal{P}_0}$ and because the constant diagram functor belongs to $[\mathcal{P}, \text{Top}]_{0}$.

7.4. Theorem. Suppose that all spaces $\mathcal{P}(\ell, \ell')$ are contractible. The Quillen adjunction $\lim_{\to} \Delta_{\mathcal{P}}$ is a Quillen equivalence between the model categories $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$ and $\text{Top}_Q$.

Proof. Since all spaces $\mathcal{P}(\ell, \ell')$ are contractible by hypothesis, i.e. homotopy equivalent to a point, they are cofibrant for the mixed model structure $\text{Top}_m$. Let $U$ be a topological space. Let $U^{\mathcal{P}_0} \to U$ be a cofibrant replacement of $U$ in $\text{Top}_Q$. We obtain a map $U^{\mathcal{P}_0} \to (\Delta_{\mathcal{P}} U)_{\ell}$ for some $\ell \in \text{Obj}(\mathcal{P})$. By Proposition 5.5 we obtain a map $\mathbb{F}^{\mathcal{P}_0} U^{\mathcal{P}_0} \to \Delta_{\mathcal{P}} U$ of $[\mathcal{P}, \text{Top}]_{0}$. Since all topological spaces $\mathcal{P}(\ell, \ell')$ are contractible by hypothesis, the map $\mathbb{F}^{\mathcal{P}_0} U^{\mathcal{P}_0} \to \Delta_{\mathcal{P}} U$ is a weak equivalence of $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$. By Proposition 6.8 the map $\mathbb{F}^{\mathcal{P}_0} \varnothing \to \mathbb{F}^{\mathcal{P}_0} U^{\mathcal{P}_0}$ is a cofibration of $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$. Thus $\mathbb{F}^{\mathcal{P}_0} U^{\mathcal{P}_0}$ is a cofibrant replacement of $\Delta_{\mathcal{P}} U$ in $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$. Using Proposition 5.8 we obtain that the canonical map $\lim_{\to} \mathbb{F}^{\mathcal{P}_0} U^{\mathcal{P}_0} \to U$ is a weak equivalence of $\text{Top}_Q$. We deduce that the functor $\lim_{\to} : [\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}} \to \text{Top}_Q$ is homotopically surjective.

Let $Y$ be a cofibrant diagram of $[\mathcal{P}, \text{Top}_Q]_{0}^{\text{proj}}$. We want to prove that the unit of the adjunction $Y \to \Delta_{\mathcal{P}}(\lim_{\to} Y)$ is a weak equivalence to complete the proof. Every

\[
\begin{array}{ccc}
\mathbb{F}^{\mathcal{P}_0} S^{n-1} & \to & X \\
\downarrow & & \downarrow \\
\mathbb{F}^{\mathcal{P}_0} D^n & \to & \Delta_{\mathcal{P}} \lim_{\to} \mathbb{F}^{\mathcal{P}_0} S^{n-1} \to \Delta_{\mathcal{P}} \lim_{\to} X \to \Delta_{\mathcal{P}} \lim_{\to} \mathbb{F}^{\mathcal{P}_0} D^n
\end{array}
\]

Figure 4. Preparation for applying the cube lemma
cofibrant diagram of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\) is a retract of a cellular object of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\), i.e. of a transfinite composition of pushouts of generating cofibrations. As a first step, consider the commutative diagram in \([\mathcal{P}, \text{Top}]_0\) of Figure 4 obtained using the unit of the adjunction \(\text{Id} \Rightarrow \Delta_{\mathcal{P}} \lim_{\to}\). Suppose that the map \(X \to \Delta_{\mathcal{P}} \lim_{\to} X\) is a pointwise weak equivalence (of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\) or equivalently of \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\)), that \(X\) is projective cofibrant and that \(\Delta_{\mathcal{P}} \lim_{\to} X\) is pointwise mixed cofibrant. The map \(F^\ell_Sn^{-1} \to F^\ell_Dn\) is a pointwise mixed cofibration between pointwise mixed cofibrant diagrams by Proposition 7.1. By Proposition 5.8, the map \(\Delta_{\mathcal{P}} \lim_{\to} F^\ell_Sn^{-1} \cong \Delta_{\mathcal{P}} S^{n-1} \to \Delta_{\mathcal{P}} D^n \cong \Delta_{\mathcal{P}} \lim_{\to} F^\ell_Dn\) is a pointwise mixed cofibration between pointwise mixed cofibrant diagrams as well because all Quillen cofibrations are mixed cofibrations. Since \(X\) is projective cofibrant by hypothesis, it is also pointwise mixed cofibrant by Proposition 7.1. For all topological spaces \(U\), the maps \(F^\ell_SU \to \Delta_{\mathcal{P}} U\) are pointwise weak equivalences since all spaces \(\mathcal{P}(\ell, \ell')\) are contractible. We are ready to apply the cube lemma [Hov99, Lemma 5.2.6] in \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\) by passing to the colimit. Since the functor \(\Delta_{\mathcal{P}} \lim_{\to} : [\mathcal{P}, \text{Top}]_0 \to [\mathcal{P}, \text{Top}]_0\) is colimit-preserving as a composite of two colimit-preserving functors, we obtain the commutative diagram of Figure 5. Using the cube lemma in \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\), we deduce that the map \(Y \to \Delta_{\mathcal{P}} \lim_{\to} Y\) is a pointwise weak equivalence from a projective cofibrant object of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\) to a pointwise mixed cofibrant object of \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\). Moreover, the map \(X \to Y\) is a cofibration both of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\) and of \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\), and the map \(\Delta_{\mathcal{P}} \lim_{\to} X \to \Delta_{\mathcal{P}} \lim_{\to} Y\) is a pointwise mixed cofibration, i.e. a cofibration of \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\) as well. By starting from \(X = \emptyset\) and by iterating the process transfinitely, we obtain two transfinite towers of cofibrations of \([\mathcal{P}, \text{Top}_{m_0}]^\text{inj}\) between pointwise mixed cofibrant diagrams. In this case, the colimit is a homotopy colimit. Thus, for all cellular objects \(Y\) of \([\mathcal{P}, \text{Top}_{Q_0}]^\text{proj}\), the unit of the adjunction \(Y \to \Delta_{\mathcal{P}} \lim_{\to} Y\) is a pointwise weak homotopy equivalence. □

\[\text{Figure 5. The next step in the transfinite sequence}\]
Let us go back to the geometric example of the introduction. The space of paths of a multipointed $d$-space is closed only under the action of $\mathcal{G}$ because we need a specific property for the cofibrant objects to construct the model structure as it is carried out in [Gau09] (cf the second statement of [Gau09, Proposition 4.7]). On the contrary, the space of dipaths of a Grandis $d$-space is closed under the action of the monoid $\mathcal{M} \supset\mathcal{G}$ of nondecreasing continuous maps from $[0,1]$ to itself preserving the extremities. By seeing $\mathcal{M}$ as a one-object category, we obtain a new contravariant non-enriched diagram of topological spaces $\mathcal{D}^\mathcal{M}(X,dX)$. The interpretations of $\varprojlim \mathcal{D}^\mathcal{M}(X,dX)$ and $\varprojlim \mathcal{D}^\mathcal{M}(X,dX)$ are the same. The behavior of the homotopy colimit $\varinjlim \mathcal{D}^\mathcal{M}(X,dX)$ in the enriched case remains the same because Theorem 7.4 can still be applied. However, the homotopy colimit in the non-enriched setting behaves well in this case:

8.1. Proposition. We have a weak homotopy equivalence $\varinjlim \mathcal{D}^\mathcal{M}(X,dX) \simeq dX$ where the homotopy colimit is calculated in the ordinary (i.e. non-enriched) projective model structure.

Proof. Every map $\phi^* : dX \to dX$ is homotopic to the identity by the homotopy $H : [0,1] \times dX \to dX$ taking $(t, \gamma)$ to $\gamma.(t.\phi + (1-t).\text{Id}_{[0,1]})$. We deduce that every map of $\mathcal{D}^\mathcal{M}(X,dX)$ is a weak homotopy equivalence. Since $BM$ is contractible by [Law18], the ordinary (i.e. non-enriched) homotopy colimit $\varinjlim \mathcal{D}^\mathcal{M}(X,dX)$ is in this case weakly homotopy equivalent to $dX$ by [CS02, Corollary 29.2].

References


