ENRICHED DIAGRAMS OF TOPOLOGICAL SPACES OVER LOCALLY CONTRACTIBLE ENRICHED CATEGORIES

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Abstract. It is proved that the projective model structure of the category of topologically enriched diagrams of topological spaces over a topologically enriched locally contractible small category is Quillen equivalent to the standard Quillen model structure of topological spaces. We give a geometric interpretation of this fact in directed homotopy.

Contents

1. Introduction

Presentation. The model categories of *flows* [Gau03] (with some updated proofs in [Gau18]) and of *multipointed d-spaces* [Gau09] were introduced for studying some properties of concurrent systems from a topological point of view. The two model categories look very similar in their construction. Moreover, there exists a functor from multipointed d-spaces to flows [Gau09, Theorem 7.2] such that the total left derived functor in the sense of [DHKS04] induces an equivalence of categories between the homotopy categories [Gau09, Theorem 7.5]. However, this functor is not a left adjoint [1] by [Gau09, Proposition 7.3]. Therefore it cannot give rise to a left Quillen equivalence.

The geometric reason of this problem is that the composition of paths is only associative up to homotopy in multipointed d-spaces and, on the contrary, that it is strictly associative in flows. This situation is not specific to directed homotopy of course. The

1991 *Mathematics Subject Classification.* 18C35,55U35,18G55,68Q85.
Key words and phrases. d-space,flow,topological model of concurrency,combinatorial model category, accessible model category, Quillen equivalence, locally presentable category, topologically enriched category, projective model structure, injective model structure.

[1] It can be proved that it is not a right adjoint either.

composition of paths in topological spaces is also associative only up to homotopy in non-directed algebraic topology whereas the composition of morphisms is strictly associative in small categories.

This paper belongs to a series of papers which must lead to a proof that these two model categories are related by a zig-zag of Quillen equivalences. The intermediate model category, the model category of Moore flows, will be introduced in a subsequent work. We want to focus here on a Quillen equivalence which has a geometric interpretation in directed homotopy. This theorem and the material expounded here will be used in a different way in the next papers.

The starting point is the following geometric observation. Let $\mathcal{G}$ be the topological group of nondecreasing homeomorphisms of $[0,1]$. It can be viewed as a one-object topologically enriched category such that $\mathcal{G}$ is the unique space of maps. Consider a topological space X and a set of paths dX closed under the action of $\mathcal{G}$. Typical examples are d-spaces in Grandis' sense [Gra03] or multipointed d-spaces in the sense of [Gau09]. In each case, a part of the structure is forgotten. We suppose dX equipped with its natural topology making the evaluation maps continuous. Such a set of data gives rise to a contravariant diagram of topological spaces over $\mathcal{G}$ denoted by $\mathcal{D}^{\mathcal{G}}(X, dX)$ with the only vertex dX and taking $\phi \in \mathcal{G}$ to the mapping $\phi^* : \gamma \mapsto \gamma.\phi$. The limit $\varprojlim \mathcal{D}^{\mathcal{G}}(X, dX)$ is the space of paths of dX invariant by the action of $\mathcal{G}$. It is equal to the subspace of constant paths of dX. The colimit $\varinjlim \mathcal{D}^{\mathcal{G}}(X, dX)$ is nothing else but the quotient of the space of paths dX by the action of $\mathcal{G}$. So far, the homotopy type of $B\mathcal{G}$ does not interfere. It turns out that it is not the case for the homotopy colimit $\operatorname{holim}\limits_{\longrightarrow} \mathcal{D}^{\mathcal{G}}(X, dX)$. For example, if dX is a singleton (i.e. a constant path), then $\operatorname{holim}\limits_{\longrightarrow} \mathcal{D}^{\mathcal{G}}(X, dX)$ has the homotopy type of $B\mathcal{G}$ by [Hir03, Proposition 14.1.6 and Proposition 18.1.6] which is not contractible, although both $\mathcal{G}$ and dX (which is supposed to be here a singleton) are contractible. The main result of this paper is that one possible way to overcome this problem is to work in the enriched setting. The main theorem of this paper is stated now:

Theorem. *(Theorem 7.6) Let $\mathcal{P}$ be a topologically enriched small category. Suppose that $\mathcal{P}$ is locally contractible (i.e all spaces of maps $\mathcal{P}(\ell, \ell')$ are contractible). Let* **Top** *be the category of Δ-generated spaces. Then the colimit functor from the category $[\mathcal{P}, \mathbf{Top}]_0$ of topologically enriched functors and natural transformations to* **Top** *induces a left Quillen equivalence between the projective model structure and the Quillen model structure.*

Note that the particular case where $\mathcal{P}$ has exactly one map between each pair of objects (i.e. each space $\mathcal{P}(\ell, \ell')$ is a singleton) is trivial. In this case, $[\mathcal{P}, \mathbf{Top}]_0$ is equivalent to **Top** as a category indeed.

Using this theorem, our example can now be reinterpreted in the enriched setting. The diagram $\mathcal{D}^{\mathcal{G}}(X, dX)$ belongs to $[\mathcal{G}^{op}, \mathbf{Top}]_0$ because the mapping $\phi \mapsto \phi^*$ from $\mathcal{G}$ to $\mathbf{TOP}(dX, dX)$, where $\mathbf{TOP}(dX, dX)$ is the space of continuous maps from dX to itself, is continuous. Since the inclusion functor $[\mathcal{G}^{op}, \mathbf{Top}]_0 \subset \mathbf{Top}^{\mathcal{G}^{op}}$ into the category of all contravariant functors from $\mathcal{G}$ to **Top** is colimit-preserving and limit-preserving, nothing changes concerning the interpretations of $\varprojlim \mathcal{D}^{\mathcal{G}}(X, dX)$ and $\varinjlim \mathcal{D}^{\mathcal{G}}(X, dX)$. On the contrary, the behavior of the homotopy colimit is completely different. There exists a cofibrant replacement $\mathcal{D}^{\mathcal{G}}(X, dX)^{cof}$ of $\mathcal{D}^{\mathcal{G}}(X, dX)$ in $[\mathcal{P}, \mathbf{Top}]_0$ together with a pointwise weak homotopy equivalence $\mathcal{D}^{\mathcal{G}}(X, dX)^{cof} \to \Delta_{\mathcal{G}^{op}}(dX)$ (the constant diagram $\Delta_{\mathcal{G}^{op}}(dX)$ belongs to $[\mathcal{G}^{op}, \mathbf{Top}]_0$). Since dX is fibrant, we deduce that the canonical map $\varinjlim \mathcal{D}^{\mathcal{G}}(X, dX)^{cof} \to dX$ is a weak homotopy equivalence because the colimit functor is a left Quillen equivalence. It means that in the enriched setting, $\underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{G}}(X, dX)$ always has the homotopy type of dX. Everything behaves as if we were in the trivial case above.

Outline of the paper. Section 2 collects the notations and some useful facts about locally presentable categories and model categories which are used in this paper. Section 3 establishes the fact that a locally presentable category which is cartesian closed is a locally presentable base in the sense of [BQR98, Definition 1.1] for the closed monoidal structure induced by the binary product (Theorem 3.13). To the best of our knowledge, the proof of this fact was not yet known. The proof is due to Tim Campion with a minor correction which forces us to introduce and to study succinctly the category of all small diagrams over all small categories. Section 4 recalls some facts about the category of Δ-generated spaces, the *Quillen model structure*, the *Cole-Strøm model structure* and the so-called *mixed model structure*. It culminates in the proof that the mixed model structure is accessible. Section 5 introduces the material of enriched diagrams of topological spaces. Some elementary facts which are used in the next sections are proved or recalled. Section 6 introduces two model structures, the projective one, which is combinatorial, on the category of enriched diagrams using the Quillen model structure, and the injective one, which is only accessible, on the category of enriched diagrams using the mixed model structure. This section discusses the interactions between the two model structures. The existence of the projective model structure is a straightforward consequence of Moser's work [Mos18]. We are also able to prove that this model structure is left proper in Theorem 6.7 (it is right proper because all objects are fibrant). The latter result is not a consequence of Moser's work. It is a consequence of the way the Quillen model structure and the Cole-Strøm model structure on Δ-generated spaces interact with each other. Section 7 proves the main theorem of the paper. Section 8 adds a comment based on a surprising answer made by Tyler Lawson in MathOverflow about the monoid of non-decreasing continuous maps from $[0, 1]$ to itself preserving the extremities [Law18] and another one mentioning Shulman's work [Shu09] about enriched homotopical categories. Finally, an appendix proves two particular cases of Theorem 7.6 using [Shu09].

Acknowledgments. I thank Tim Campion, Asaf Karagila and Tyler Lawson for their answers in MathOverflow. I also thank Tim Porter for valuable email discussions.

2. Notations, conventions and prerequisites

We refer to [AR94] for locally presentable categories, to [Ros09] for combinatorial model categories. We refer to [Hov99] and to [Hir03] for more general model categories. We refer to [Kel05] and to [Bor94b, Chapter 6] for enriched categories. *All enriched categories are topologically enriched categories: the word topologically is therefore omitted.*

- All categories are locally small except the category **CAT** of all locally small categorie.
- $\mathcal{K}$ always denotes a locally presentable category.
- **Set** is the category of sets.

- $\mathcal{K}(X, Y)$ is the set of maps in a category $\mathcal{K}$.
- $\mathcal{K}_0$ denotes sometimes the underlying category of an enriched category $\mathcal{K}$. Since we will be working in a cartesian closed category of topological spaces, $\mathcal{K}_0$ is nothing else but the category $\mathcal{K}$ with the topology of the space of maps forgotten.
- **Cat** is the category of all small categories and functors between them.
- $\mathcal{P}$ denotes a nonempty enriched small category.
- $\mathcal{K}^{op}$ denotes the opposite category of $\mathcal{K}$.
- $\mathrm{Obj}(K)$ is the class of objects of $\mathcal{K}$.
- $\mathrm{Mor}(\mathcal{K})$ is the category of morphisms of $\mathcal{K}$ with the commutative squares for the morphisms.
- $\mathcal{K}^I$ is the category of functors and natural transformations from a small category I to $\mathcal{K}$.
- $[I, \mathcal{K}]$ is the enriched category of enriched functors from an enriched small category to an enriched category $\mathcal{K}$, and $[I, \mathcal{K}]_0$ is the underlying category.
- $\Delta_I : \mathcal{K} \to \mathcal{K}^I$ is the constant diagram functor.
- $\varnothing$ is the initial object.
- $\mathbf{1}$ is the final object.
- Id_X is the identity of X.
- $g.f$ is the composite of two maps $f : A \to B$ and $g : B \to C$; the composite of two functors is denoted in the same way.
- If $f : \underline{I} \to \underline{J}$ is a functor between small categories and if $F : \underline{I} \to \mathcal{K}$ is a functor, then $\mathrm{Lan}_f F$ denotes the left Kan extension of F along f.
- $F \Rightarrow G$ denotes a natural transformation from a functor F to a functor G.
- The composite of two natural transformations $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow H$ is denoted by $\nu \odot \mu$ to make the distinction with the composition of maps.
- A subcategory is always isomorphism-closed (replete).
- $f \square g$ means that f satisfies the *left lifting property* (LLP) with respect to g, or equivalently that g satisfies the *right lifting property* (RLP) with respect to f.
- $\mathbf{inj}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \square g\}$.
- $\mathbf{cof}(\mathcal{C}) = \{f \mid \forall g \in \mathbf{inj}(\mathcal{C}), f \square g\}$.
- $\mathbf{cell}(\mathcal{C})$ is the class of transfinite compositions of pushouts of elements of $\mathcal{C}$.
- A *cellular* object X of a combinatorial model category is an object such that the canonical map $\varnothing \to X$ belongs to $\mathbf{cell}(I)$ where I is the set of generating cofibrations.
- A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ means that the class of cofibrations is $\mathcal{C}$, that the class of weak equivalences is $\mathcal{W}$ and that the class of fibrations is $\mathcal{F}$ in this order.
- $(-)^{cof}$ denotes a cofibrant replacement, $(-)^{fib}$ denotes a fibrant replacement.
- $F \dashv G$ denotes an adjunction where F is the left adjoint and G the right adjoint.

We will use the following known facts:

- A functor $F : \mathcal{K} \to \mathcal{L}$ between locally presentable categories is a left adjoint if and only if it is colimit-preserving; indeed, any left adjoint is colimit-preserving; conversely, if F is colimit-preserving, F^{op} is limit-preserving; since every locally presentable category is well-copowered by [AR94, Theorem 1.58] and has a generator, the opposite category $\mathcal{K}^{op}$ is well-powered and has a cogenerator; Hence the Special Adjoint Functor theorem [Bor94a, Theorem 3.3.4] states that F^{op} is a right adjoint.

- A functor $F : \mathcal{K} \to \mathcal{L}$ between locally presentable categories is a right adjoint if and only if it is limit-preserving and accessible by [AR94, Theorem 1.66].
- A transfinite tower (of length λ) of $\mathcal{K}$ consists of an ordinal λ and a colimit-preserving functor D from λ to $\mathcal{K}$; it means that for every limit ordinal $\mu \leqslant \lambda$, the canonical map $\varinjlim_{\nu<\mu} D_\nu \to D_\mu$ is an isomorphism. In any model category $\mathcal{M}$, a colimit of a transfinite tower of cofibrations between cofibrant objects is a homotopy colimit. It is due to the fact that the transfinite tower is a diagram over a direct Reedy category and that, in this case, the tower is Reedy cofibrant for the Reedy model structure which coincides with the projective model structure [Hov99, Theorem 5.2.5].

A weak factorization system $(\mathcal{L}, \mathcal{R})$ of a locally presentable category $\mathcal{K}$ is *accessible* if there is a functorial factorization

$$(A \xrightarrow{f} B) \longmapsto (A \xrightarrow{Lf} Ef \xrightarrow{Rf} B)$$

with $Lf \in \mathcal{L}$, $Rf \in \mathcal{R}$ such that the functor $E : \mathrm{Mor}(\mathcal{K}) \to \mathcal{K}$ is accessible [GKR18, Definition 2.4]. Since colimits are calculated pointwise in $\mathrm{Mor}(\mathcal{K})$, a weak factorization system is accessible if and only if the functors $L : \mathrm{Mor}(\mathcal{K}) \to \mathrm{Mor}(\mathcal{K})$ and $R : \mathrm{Mor}(\mathcal{K}) \to \mathrm{Mor}(\mathcal{K})$ are accessible. By [Ros17, Theorem 4.3], a weak factorization system is accessible if and only if it is small in Garner's sense. In particular, every *small* weak factorization system (i.e. of the form $(\mathbf{cof}(I), \mathbf{inj}(I))$ for a set I) is accessible. A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a locally presentable category is *accessible* if the two weak factorization systems $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are accessible. Every combinatorial model category is an accessible model category.

2.1. **Theorem.** *(Rezk) Let $(\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ be two model structures on the same underlying category such that $\mathcal{W}_1 = \mathcal{W}_2$. Then the model structure $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ is left proper (right proper resp.) if and only if the model structure $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ is left proper (right proper resp.).*

Proof. This amazing result is a consequence of [Rez02, Proposition 2.5]. □

We will be using the following characterization of a Quillen equivalence. A Quillen adjunction $F \dashv G : \mathcal{C} \leftrightarrows \mathcal{D}$ is a Quillen equivalence if and only if for all fibrant objects X of $\mathcal{D}$, the natural map $F(G(X)^{cof}) \to X$ is a weak equivalence of $\mathcal{D}$ (the functor F is then said *homotopically surjective*) and if for all cofibrant objects Y of $\mathcal{C}$, the natural map $Y \to G(F(Y)^{fib})$ is a weak equivalence of $\mathcal{C}$ [Hov99, Proposition 1.3.13]. If all objects of $\mathcal{D}$ are fibrant, the latter assertion is equivalent to saying that for all cofibrant objects Y of $\mathcal{C}$, the unit of the adjunction $Y \to G(F(Y))$ is a weak equivalence of $\mathcal{C}$.

3. On locally presentable bases

We recall the terminology of [Roi94]. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor. Let X be an object of $\mathcal{B}$. The *fibre of p over X* consists of the subcategory of $\mathcal{E}$ generated by the *vertical maps* f, i.e. the maps f such that $F(f) = \mathrm{Id}_X$. A map $\alpha : a \to b$ of $\mathcal{E}$ is *cartesian* if every map $\omega : a' \to b$ of $\mathcal{E}$ with $p(a) = p(a')$ factors uniquely as a composite $\omega = \alpha.\phi$ with ϕ vertical

$$
\begin{tikzcd}
a' \arrow[d, dashed, "\exists!\phi"'] \arrow[dr, "\omega"] & \\
a \arrow[r, "\alpha"] & b
\end{tikzcd}
$$

FIGURE 1. a is cartesian

$$
\begin{tikzcd}
 & b' \\
a \arrow[ur, "\omega"] \arrow[r, "\alpha"] & b \arrow[u, dashed, "\exists!\psi"']
\end{tikzcd}
$$

FIGURE 2. a is cocartesian

(see Figure 1). A map $\alpha : a \to b$ of $\mathcal{E}$ is *cocartesian* if every map $\omega : a \to b'$ of $\mathcal{E}$ with $p(b) = p(b')$ factors uniquely as a composite $\omega = \psi.\alpha$ with ψ vertical (see Figure 2). If for a given morphism $f : x \to y$ of $\mathcal{B}$ and an object $b \in \mathcal{E}_y$, there exists a cartesian morphism $b^f : a \to b$ with $p(b^f) = f$, then a is determined in $\mathcal{E}_x$ up to a unique isomorphism. It is called the *reciprocal image* of b by f and it is denoted by f^*b. If for a given morphism $f : x \to y$ of $\mathcal{B}$ and an object $a \in \mathcal{E}_x$, there exists a cartesian morphism $a_f : a \to b$ with $p(a_f) = f$, then b is determined in $\mathcal{E}_y$ up to a unique isomorphism. It is called the *direct image* of a by f and it is denoted by f_*a.

3.1. **Definition.** *A functor $p : \mathcal{E} \to \mathcal{B}$ is a* bifibred category (over $\mathcal{B}$) *if for every map f: $x \to y$ of $\mathcal{B}$, there exists a reciprocal image functor $f^* : \mathcal{E}_y \to \mathcal{E}_x$ and a direct image functor $f_* : \mathcal{E}_x \to \mathcal{E}_y$ and if the canonical morphisms of functors $f^* \odot g^* \Rightarrow (g.f)^*$ and $(g.f)_* \Rightarrow g_* \odot f_*$ are isomorphisms.*

3.2. **Remark.** *The usual vocabulary is to call $p : \mathcal{E} \to \mathcal{B}$ a* fibred category (over $\mathcal{B}$) *if there is only a reciprocal image functor and a* cofibred category (over $\mathcal{B}$) *if there is only a direct image functor.*

Let $\mathcal{K}$ be a locally presentable category. We introduce the category $\mathcal{DK}$ of all small diagrams over all small categories defined as follows. An object is a functor $F : \underline{I} \to \mathcal{K}$ from a small category $\underline{I}$ to $\mathcal{K}$. A morphism from $F : \underline{I}_1 \to \mathcal{K}$ to $G : \underline{I}_2 \to \mathcal{K}$ is a pair $(f : \underline{I}_1 \to \underline{I}_2, \mu : F \Rightarrow G.f)$ where f is a functor and μ is a natural transformation. If (g, ν) is a map from $G : \underline{I}_2 \to \mathcal{K}$ to $H : \underline{K} \to \mathcal{K}$, then the composite $(g, \nu).(f, \mu)$ is defined by $(g.f, (\nu.f) \odot \mu)$. The identity of $F : \underline{I}_1 \to \mathcal{K}$ is the pair $(\mathrm{Id}_{\underline{I}_1}, \mathrm{Id}_F)$. If $(h, \xi) : (H : \underline{K} \to \mathcal{K}) \to (I : \underline{L} \to \mathcal{K})$ is another map of $\mathcal{DK}$, then we have

$$
\begin{aligned}
((h,\xi).(g,\nu)).(f,\mu) &= (h.g, \xi.g \odot \nu).(f,\mu) \\
&= (h.g.f, \xi.g.f \odot \nu.f \odot \mu) \\
&= (h,\xi).(g.f, \nu.f \odot \mu) \\
&= (h,\xi).((g,\nu).(f,\mu)).
\end{aligned}
$$

Thus the composition law is associative and the category $\mathcal{DK}$ is well-defined.

3.3. **Proposition.** *The forgetful functor* $p : \mathcal{DK} \to \mathbf{Cat}$ *is a bifibred category over* $\mathbf{Cat}$.

Proof. Let $f : \underline{I} \to \underline{J}$ be a functor between small categories. For a functor $G : \underline{J} \to \mathcal{K}$, let $f^*(G) = G.f$. There is a canonical map $f^*(G) \to G$ in $\mathcal{DK}$ defined by the pair $(f, \mathrm{Id}_{G.f} : G.f \Rightarrow G.f)$. For a functor $F : \underline{I} \to \mathcal{K}$, let $f_*(F) = \operatorname{Lan}_f F$. Since $\mathcal{K}$ is locally presentable and hence bicomplete, there is an adjunction $f_* \dashv f^*$ between $\mathcal{K}^{\underline{I}}$ and $\mathcal{K}^{\underline{J}}$. There is a canonical map $F \to f_*(F)$ in $\mathcal{DK}$ defined by the pair $(f, \eta_f : F \Rightarrow f_*(F).f)$ where $\eta_f : \mathrm{Id} \Rightarrow f^*.f_*$ is the unit of the adjunction. Let $F : \underline{I}_1 \to \mathcal{K}$ and $G : \underline{I}_2 \to \mathcal{K}$ be two objects of $\mathcal{DK}$. Let $\omega = (f, \mu) : F \to G$ be a map of $\mathcal{DK}$. A factorization of ω as a composite in $\mathcal{DK}$ (with the left-hand map vertical)

$$F \xrightarrow{\omega^f = (\mathrm{Id}_{\underline{I}_1}, \overline{\omega})} f^*(G) \longrightarrow G$$

implies $\mathrm{Id}_{G.f} . \overline{\omega} = \mu$. We obtain $\overline{\omega} = \mu$ as the unique possible choice. A factorization of ω as a composite in $\mathcal{DK}$ (with the right-hand map vertical)

$$F \longrightarrow f_*(F) \xrightarrow{\omega_f = (\mathrm{Id}_{\underline{I}_2}, \overline{\omega})} G$$

is equivalent to finding a commutative diagram of natural transformations

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta_f"] \arrow[rr, Rightarrow, bend right, "\mu"'] & f^* f_*(F) \arrow[r, Rightarrow, "\overline{\omega}.f"] & f^*(G), \end{tikzcd}$$

which can be rewritten as the commutative square

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta_f"] \arrow[d, equal] & f^* f_*(F) \arrow[d, Rightarrow, "f^*(\overline{\omega})"] \\ F \arrow[r, Rightarrow, "\mu"] & f^*(G). \end{tikzcd}$$

By adjunction, it is equivalent to finding a commutative square as follows

$$\begin{tikzcd} f_* F \arrow[r, equal] \arrow[d, equal] & f_*(F) \arrow[d, Rightarrow, "\overline{\omega}"] \\ f_* F \arrow[r, Rightarrow, "\epsilon_f \odot f_*(\mu)"] & G \end{tikzcd}$$

where ϵ_f is the counit of the adjunction. We obtain $\overline{\omega} = \epsilon_f \odot f_*(\mu)$ as the unique possible choice. It means that f^* and f_* satisfies the properties of the inverse and the direct image respectively. Let $g : \underline{I}_2 \to \underline{I}_3$ be another map of $\mathbf{Cat}$. One has for all functors $H : \underline{I}_3 \to \mathcal{K}$ the isomorphisms of functors $f^*(g^*(H)) \cong H.g.f \cong (g.f)^*(H)$. One also has for all functors $K : \underline{I}_1 \to \mathcal{K}$ the isomorphisms of functors $(g.f)_*(K) \cong \operatorname{Lan}_{g.f} K \cong \operatorname{Lan}_g(\operatorname{Lan}_f K) = (g_*.f_*)(K)$. $\square$

3.4. **Proposition.** *The category* $\mathcal{DK}$ *is cocomplete. The colimit functor induces a well-defined functor* $\varinjlim : \mathcal{DK} \to \mathcal{K}$ *which is a left adjoint.*

It is actually possible to prove that $\mathcal{DK}$ is locally presentable. The principle of the proof is that the fibred category $\mathcal{DK} \to \mathbf{Cat}$ is the one associated to the pseudofunctor

$$\begin{tikzcd} \mathrm{Lan}_{\iota_k}(F.\iota_k) \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\mu_u"'] & F \arrow[d, equal] \\ \mathrm{Lan}_{\iota_{k'}}(F.\iota_{k'}) \arrow[r, Rightarrow] & F \end{tikzcd}$$

FIGURE 3. Diagram of Proposition 3.5

$\mathbf{Cat}^{op} \to \mathbf{CAT}$ associating to a small category $\underline{I}$ the functor category $\mathcal{K}^{\underline{I}}$. One can check that it is accessible in the sense of [MP89, Definition 5.3.1]. Thus, by [MP89, Theorem 5.3.4], $\mathcal{DK}$ is accessible. The proof is complete thanks to Proposition 3.4.

Proof. Every fibre over a small category is a category of diagrams over a fixed small category: therefore all fibres of the bifibred category $p : \mathcal{DK} \to \mathbf{Cat}$ are cocomplete. Moreover, the category of small categories is locally presentable, and therefore cocomplete as well. Using [Roi94, Proposition 3.3°], we deduce that $\mathcal{DK}$ is cocomplete. Consider the functor $\mathcal{I} : \mathcal{K} \to \mathcal{DK}$ which takes an object X of $\mathcal{K}$ to the constant diagram $\Delta_{\mathbf{1}}X$ over the terminal small category $\mathbf{1}$. Then we have the sequence of natural isomorphisms (where $F : \underline{I} \to \mathcal{K}$ is an object of $\mathcal{DK}$)

$$\begin{aligned} \mathcal{DK}(F, \mathcal{I}(X)) &\cong \mathcal{DK}(F, \Delta_{\mathbf{1}}X) && \text{by definition of } \mathcal{I} \\ &\cong \mathcal{K}^{\underline{I}}(F, f^*(\Delta_{\mathbf{1}}X)) && \text{where } f : \underline{I} \to \mathbf{1} \text{ is the canonical functor} \\ &\cong \mathcal{K}^{\underline{I}}(F, \Delta_{\underline{I}}X) && \text{by definition of } f \\ &\cong \mathcal{K}(\varinjlim F, X) && \text{by adjunction.} \end{aligned}$$

This sequence of natural isomorphisms implies that the mapping $\varinjlim : \mathcal{DK} \mapsto \mathcal{K}$ yields a well-defined functor and that it is a left adjoint. $\square$

Let $F : \underline{J} \to \mathcal{K}$ and $G : \underline{J} \to \mathcal{K}$ be two objects of $\mathcal{DK}$. Then a functor $f : \underline{I} \to \underline{J}$ induces a restriction set map

$$f^! : \mathcal{K}^{\underline{J}}(F, G) \to \mathcal{K}^{\underline{I}}(F.f, G.f)$$

by taking a natural transformation $\mu : F \Rightarrow G$ to the natural transformation $\mu.f : F.f \Rightarrow G.f$. We have $(f.g)^! = g^!.f^!$.

Let $(\underline{I}_k)_{k\in\mathbb{K}}$ be a small diagram of small categories. Let $\underline{I} = \varinjlim \underline{I}_k$. Let $\iota_k : \underline{I}_k \to \underline{I}$ be the canonical functor. Let $u : k \to k'$ be a map of $\mathbb{K}$. If $\phi_u : \underline{I}_k \to \underline{I}_{k'}$ is the corresponding functor in the diagram of small categories, then $\iota_{k'}.\phi_u = \iota_k$ and there is a restriction set map $(\phi_u)^! : \mathcal{K}^{\underline{I}_{k'}}(F.\iota_{k'}, G.\iota_{k'}) \to \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)$. We obtain a diagram of sets $\mathbb{K}^{op} \to \mathbf{Set}$ taking k to $\mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)$. The set map $(\phi_u)^!$ yields a set map

$$\mathcal{K}^{\underline{I}}(\mathrm{Lan}_{\iota_{k'}}(F.\iota_{k'}), G) \cong \mathcal{K}^{\underline{I}_{k'}}(F.\iota_{k'}, G.\iota_{k'}) \to \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k) \cong \mathcal{K}^{\underline{I}}(\mathrm{Lan}_{\iota_k}(F.\iota_k), G).$$

Let $\mu_u = (\phi_u)^!(\mathrm{Id}_{\mathrm{Lan}_{\iota_{k'}}(F.\iota_{k'})}) : \mathrm{Lan}_{\iota_k}(F.\iota_k) \Rightarrow \mathrm{Lan}_{\iota_{k'}}(F.\iota_{k'})$. Then the map above is the precomposition by μ_u.

3.5. **Proposition.** *The natural transformation μ_u is the unique natural transformation from $\mathrm{Lan}_{\iota_k}(F.\iota_k)$ to $\mathrm{Lan}_{\iota_{k'}}(F.\iota_{k'})$ making the following diagram of Figure 3 commute where*

$$\begin{tikzcd}
\mathrm{Lan}_{\phi_u}(F.\iota_k) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F.\iota_{k'} \arrow[d, equal] \\
F.\iota_{k'} \arrow[r, Rightarrow] & F.\iota_{k'}
\end{tikzcd}$$

FIGURE 4. Equivalent form for the diagram of Proposition 3.5

the horizontal natural transformations are the counits of the adjunction characterizing left Kan extensions.

Proof. We have $\iota_{k'}.\phi_u = \iota_k$. Therefore, by adjunction, there is a bijection between the set of commutative squares as in the statement of the proposition and the set of commutative squares of the form of Figure 4 since $\mathrm{Lan}_{\iota_{k'}}.\mathrm{Lan}_{\phi_u} = \mathrm{Lan}_{\iota_k}$. $\square$

3.6. **Corollary.** *Let $u : k \to k'$ and $v : k' \to k''$ be two maps of $\mathbb{K}$. Then $\mu_{v.u} = \mu_v \odot \mu_u$.*

Corollary 3.6 yields a well-defined diagram $\mathbb{K} \to \mathcal{K}^{\underline{I}}$ defined on objects by the mapping $k \mapsto \mathrm{Lan}_{\iota_k}(F.\iota_k)$.

3.7. **Proposition.** *With the notations above. Suppose that the diagram of small categories $(\underline{I}_k)_{k\in\mathbb{K}}$ is λ-directed. Let $F, G : \underline{I} \to \mathcal{K}$ be two objects of $\mathcal{DK}$. Then there is the natural bijection of sets $\mathcal{K}^{\underline{I}}(F,G) \cong \varprojlim \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)$.*

Proof. The family of maps $(\iota_k)^! : \mathcal{K}^{\underline{I}}(F,G) \to \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)_{k\in\mathbb{K}}$ yields a well-defined cone because of the equality $(h.k)^! = k^!.h^!$. Let $(f_k)_{k\in\mathbb{K}} : S \xrightarrow{\bullet} \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)_{k\in\mathbb{K}}$ be another cone of maps. We have to prove that it factors uniquely as a composite

$$S \xrightarrow{f} \mathcal{K}^{\underline{I}}(F,G) \xrightarrow{\bullet} \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k)_{k\in\mathbb{K}}.$$

Every object $c \in \underline{I}$ can be written $c = \iota_k(d)$ for some k and some d. Let $f(s)_c$ be defined by $f_k(s)_d : F(c) \to G(c)$. If $c = \iota_{k'}(d')$ is another possible choice, then there exists a cospan of maps $k \to k'' \leftarrow k'$ of $\mathbb{K}$ by hypothesis and $f_k(s)_d = f_{k''}(s)_{d''} = f_{k'}(s)_{d'}$ since $(f_k)_{k\in\mathbb{K}}$ is a cone of maps. Thus the definition $f(s)_c = f_k(s)_d$ is independant of the choice of k and d in the equality $c = \iota_k(d)$. We want to prove now that the map $f(s)_c : F(c) \to G(c)$ is natural with respect to c. Let $c \to d$ be a map of $\underline{I}$. Then it is equal to a finite composite $\iota_{k_1}(u_1)\dots\iota_{k_n}(u_n)$ where u_i is a map of $\underline{I}_{k_i}$. The source of $\iota_{k_n}(u_n)$ is c, which implies that $c = \iota_{k_n}(c')$ where c' is the source of u_n. The target of $\iota_{k_1}(u_1)$ is d, which implies that $d = \iota_{k_1}(d')$ where d' is the target of u_1. Since $\mathbb{K}$ is λ-directed, there exists an object k of $\mathbb{K}$ and maps $\phi_1 : k_1 \to k, \dots, \phi_n : k_n \to k$ of $\mathbb{K}$. We write $\iota_{k_1}(u_1)\dots\iota_{k_n}(u_n) = \iota_k(\phi_1(u_1)\dots\phi_n(u_n))$. And we have $c = \iota_k(\phi_n(c'))$ and $d = \iota_k(\phi_1(d'))$. The map $\phi_1(u_1)\dots\phi_n(u_n) : \phi_n(c') \to \phi_1(d')$ gives rise to the commutative square of $\mathcal{K}$

$$\begin{tikzcd}
F(c) = F(\iota_k(\phi_n(c'))) \arrow[r, "f(s)_c"] \arrow[d, "F(\iota_k(\phi_1(u_1)\dots\phi_n(u_n)))"'] & G(c) = G(\iota_k(\phi_n(c'))) \arrow[d, "G(\iota_k(\phi_1(u_1)\dots\phi_n(u_n)))"] \\
F(d) = F(\iota_k(\phi_1(d'))) \arrow[r, "f(s)_d"] & G(d) = G(\iota_k(\phi_1(d'))).
\end{tikzcd}$$

We deduce that f induces a well-defined set map $f : S \to \mathcal{K}^{\underline{I}}(F, G)$ and it is clearly the unique choice. $\square$

3.8. **Proposition.** *With the notations above. Suppose that the diagram of small categories $(\underline{I}_k)_{k\in\mathbb{K}}$ is λ-directed. Then $F = \varinjlim \mathrm{Lan}_{\iota_k}(F.\iota_k)$, the colimit being calculated in the functor category $\mathcal{K}^{\underline{I}}$.*

Proof. Let $F, G : \underline{I} \to \mathcal{K}$ be two objects of $\mathcal{DK}$. Then

$$\begin{aligned} \mathcal{K}^{\underline{I}}(F, G) &\cong \varprojlim \mathcal{K}^{\underline{I}_k}(F.\iota_k, G.\iota_k) && \text{by Proposition 3.7} \\ &\cong \varprojlim \mathcal{K}^{\underline{I}}(\mathrm{Lan}_{\iota_k}(F.\iota_k), G) && \text{by adjunction} \\ &\cong \mathcal{K}^{\underline{I}}(\varinjlim \mathrm{Lan}_{\iota_k}(F.\iota_k), G) && \text{by definition of the (co)limit.} \end{aligned}$$

The proof is complete by the Yoneda lemma. $\square$

3.9. **Corollary.** *With the notations above. Suppose that the diagram of small categories $(\underline{I}_k)_{k\in\mathbb{K}}$ is λ-directed. Then $F = \varinjlim F.\iota_k$, the colimit being calculated in $\mathcal{DK}$.*

Proof. The formula $F = \varinjlim \mathrm{Lan}_{\iota_k}(F.\iota_k)$ of Proposition 3.8 is exactly the description of the colimit in a bifibred category as given in [Roi94, Proposition 3.3°]. $\square$

3.10. **Definition.** [BQR98, Definition 1.1] *Let λ be a regular cardinal. A locally λ-presentable base is a symmetric monoidal closed category $\mathcal{V}$ which is locally λ-presentable and such that*

- *The unit of the tensor product is λ-presentable*
- *The tensor product of two λ-presentable objects of $\mathcal{V}$ is λ-presentable.*

A locally presentable base is a locally λ-presentable base for some regular cardinal λ.

3.11. **Proposition.** [Kar18] *Let λ be a regular cardinal. Then there exist arbitrarily large regular cardinals μ such that $\mu^\lambda = \mu$.*

Proof. Let $\kappa > \lambda$. We want to find a regular cardinal $\mu > \kappa$ such that $\mu^\lambda = \mu$. Suppose that $2^\kappa = \aleph_\alpha$. Take $\mu = \aleph_{\alpha+1}$. Then μ is regular since it is the successor of $\aleph_\alpha$. We write

$$\mu^\lambda = \aleph_{\alpha+1}^\lambda = \aleph_\alpha^\lambda \aleph_{\alpha+1} = (2^\kappa)^\lambda \aleph_{\alpha+1} = 2^{\kappa\lambda} \aleph_{\alpha+1} = \aleph_\alpha \aleph_{\alpha+1} = \mu,$$

the second equality being due to Hausdorff's formula for exponentiation [HJ99, Chapter 9 Theorem 3.11]. $\square$

3.12. **Proposition.** *Let $\mathcal{K}$ be a locally λ-presentable category. Let μ be a regular cardinal such that $\mu^\lambda = \mu$. Then every μ-presentable object is a μ-small λ-directed colimit of λ-presentable objects.*

Proof. Consider a μ-presentable object A of $\mathcal{K}$. Using [AR94, Remark 1.30(2)], write $A = \varinjlim F$ where $F : \underline{K} \to \mathcal{K}$ is a μ-small diagram of λ-presentable objects. The point is that $\underline{K}$ is not necessarily λ-filtered or λ-directed (as far as we understand [AR94, Remark 1.30(2)]). Consider the set $\mathcal{S}$ of λ-small subcategories of $\underline{K}$ ordered by the inclusion. The poset $(\mathcal{S}, \subset)$ is λ-directed because $\lambda^2 = \lambda$ and because the smallest subcategory generated by a union of subcategories is generated by the finite compositions of morphisms of this union. For $\underline{I} \in \mathcal{S}$, let $F_{\underline{I}}$ be the composite functor

$$F_{\underline{I}} : \underline{I} \subset \underline{K} \xrightarrow{F} \mathcal{K}.$$

Every inclusion $\underline{I} \subset \underline{J}$ gives rise to a map $F_{\underline{I}} \to F_{\underline{J}}$ in $\mathcal{DK}$. By Corollary 3.9, we have

$$\varinjlim_{\underline{I} \in \mathcal{S}} F_{\underline{I}} = F$$

in $\mathcal{DK}$. The latter colimit is λ-directed since the poset $(\mathcal{S}, \subset)$ is λ-directed. The poset $(\mathcal{S}, \subset)$ contains at most $\mu^\lambda = \mu$ elements (we have to choose λ maps among μ maps), and therefore at most $\mu^2 = \mu$ maps. Thus the colimit $\varinjlim_{\underline{I} \in \mathcal{S}} F_{\underline{I}} = F$ is μ-small. We obtain the sequence of isomorphisms

$$\begin{aligned}
\varinjlim_{\underline{I} \in \mathcal{S}} \left(\varinjlim F_{\underline{I}}\right) &\cong \varinjlim \left(\varinjlim_{\underline{I} \in \mathcal{S}} F_{\underline{I}}\right) && \text{by Proposition 3.4} \\
&\cong \varinjlim F && \text{by } F = \varinjlim_{\underline{I} \in \mathcal{S}} F_{\underline{I}} \\
&\cong A.
\end{aligned}$$

Each $\varinjlim F_{\underline{I}}$ is a λ-small colimit of λ-presentable objects. Using [AR94, Proposition 1.16], we deduce that each $\varinjlim F_{\underline{I}}$ is λ-presentable. We have rewritten A as a μ-small λ-directed colimit of λ-presentable objects. $\square$

3.13. **Theorem.** *(T. Campion)* [Cam18] *Let $\mathcal{K}$ be a locally presentable category which is cartesian closed. Then it is a locally presentable base for the closed monoidal structure induced by the binary product.*

Proof. The proof is reproduced here for the convenience of the reader and because it contains a minor correction which forces us to use Proposition 3.11 and Proposition 3.12 above. Since we have the isomorphisms

$$\mathcal{K}(Z, X \times Y) \cong \mathcal{K}(Z, X) \times \mathcal{K}(Z, Y) \cong (\mathcal{K} \times \mathcal{K})((Z, Z), (X, Y)),$$

the functor $(X, Y) \mapsto X \times Y$ is a right adjoint. It is therefore accessible. We choose a big enough regular cardinal λ such that the functor $(X, Y) \mapsto X \times Y$ is λ-accessible, the category $\mathcal{K}$ is locally λ-presentable and the terminal object (i.e. the unit of the binary product) is λ-presentable. We choose a regular cardinal $\mu > \lambda$ such that the binary product of two λ-presentable objects is μ-presentable. Using Proposition 3.11, we can suppose that $\mu^\lambda = \mu$ (it is the correction).

We are going to prove that the class of μ-presentable objects is closed under the binary product to complete the proof. Let A and B be two μ-presentable objects. Using Proposition 3.12, write $A = \varinjlim_{k \in K} A_k$ and $B = \varinjlim_{\ell \in L} B_\ell$ as μ-small λ-directed (and therefore λ-filtered) colimits of λ-presentable objects. Let $I = K \times L$ which is μ-small and λ-filtered. The projections $\pi_1 : K \times L \to K$ and $\pi_2 : K \times L \to L$ are right cofinal. Indeed, for $k \in K$, $k{\downarrow}\pi_1 = (k{\downarrow}K) \times L$ is a product of filtered categories, and so filtered itself and therefore nonempty and connected. This implies that π_1 (and also π_2) is right cofinal. We obtain the isomorphisms $A \cong \varinjlim_{i \in I} A_{\pi_1(i)}$ and $B \cong \varinjlim_{i \in I} B_{\pi_2(i)}$ by [Hir03, Theorem 14.2.5(1)]. We deduce the isomorphism of $\mathcal{K} \times \mathcal{K}$

$$(A, B) \cong \varinjlim_{i \in I} \left(A_{\pi_1(i)}, B_{\pi_2(i)}\right).$$

Since I is λ-filtered and since the functor $(X,Y) \mapsto X \times Y$ is supposed to be λ-accessible, we obtain the isomorphism

$$A \times B \cong \varinjlim_{i \in I} \left(A_{\pi_1(i)} \times B_{\pi_2(i)}\right).$$

Let $C = \varinjlim_{j \in J} C_j$ be a μ-filtered colimit. Then we have

$$\begin{aligned}
\mathcal{K}(A \times B, C) &\cong \varprojlim_{i \in I} \mathcal{K}\left(A_{\pi_1(i)} \times B_{\pi_2(i)}, \varinjlim_{j \in J} C_j\right) && \text{by } A \times B = \varinjlim_{i \in I} \left(A_{\pi_1(i)} \times B_{\pi_2(i)}\right) \\
&\cong \varprojlim_{i \in I} \varinjlim_{j \in J} \mathcal{K}(A_{\pi_1(i)} \times B_{\pi_2(i)}, C_j) && \text{since } A_{\pi_1(i)} \times B_{\pi_2(i)} \text{ is } \mu\text{-presentable} \\
&\cong \varinjlim_{j \in J} \varprojlim_{i \in I} \mathcal{K}(A_{\pi_1(i)} \times B_{\pi_2(i)}, C_j) && \text{since } I \text{ is } \mu\text{-small and } J \text{ is } \mu\text{-filtered} \\
&\cong \varinjlim_{j \in J} \mathcal{K}(A \times B, C_j) && \text{by } A \times B = \varinjlim_{i \in I} \left(A_{\pi_1(i)} \times B_{\pi_2(i)}\right).
\end{aligned}$$

□

4. Quillen and mixed model structures of topological spaces

The category **Top** denotes the category of Δ-*generated spaces*, i.e. the colimits of simplices. For a tutorial about these topological spaces, see for example [Gau09, Section 2]. The category **Top** is locally presentable (see [FR08, Corollary 3.7]), cartesian closed and it contains all CW-complexes. The internal hom functor is denoted by $\mathbf{TOP}(-,-)$. The forgetful functor from **Top** to **Set** is fibre-small and topological. The category **Top** is a full coreflective subcategory of the category $\mathcal{TOP}$ of general topological spaces.

The category **Top** can be viewed as a category enriched over itself. It is also locally presentable in the enriched sense by [Mos18, Proposition 2.4] (see also [Kel82, Corollary 7.3]). It is tensored and cotensored over itself because **Top** is cartesian closed: the tensor product is the binary product and the unit is the singleton. A category enriched over **Top** is called an *enriched category*. As already said in Section 2, the adjective "topologically" is omitted because all enrichments in this paper are over **Top**.

We recall Cole's theorem enabling to mix model structures.

4.1. **Theorem.** [Col06, Theorem 2.1] *Let $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ and $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be two model structures on the same underlying category with $\mathcal{W}_1 \subset \mathcal{W}_2$ and with $\mathcal{F}_1 \subset \mathcal{F}_2$. Then there exists a unique model structure $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$ such that $\mathcal{W}_m = \mathcal{W}_2$ and $\mathcal{F}_m = \mathcal{F}_1$. Moreover, we have $\mathcal{C}_1 \cap \mathcal{W}_1 = \mathcal{C}_m \cap \mathcal{W}_m$ and $\mathcal{C}_2 \subset \mathcal{C}_m$.*

4.2. **Proposition.** *With the notations of Theorem 4.1. Suppose that the underlying category $\mathcal{K}$ is locally presentable. Suppose that the weak factorization system $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1)$ is accessible and that the model structure $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ is combinatorial (i.e. cofibrantly generated). Then the model structure $(\mathcal{C}_m, \mathcal{W}_m, \mathcal{F}_m)$ is accessible.*

Proof. There is the equality of weak factorization systems $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1) = (\mathcal{C}_m \cap \mathcal{W}_m, \mathcal{F}_m)$ by Theorem 4.1. Thus the right-hand weak factorization system is accessible because the left-hand one is accessible by hypothesis. The other factorization is obtained as follows: first f factors as a composite $f = R_2(f).L_2(f)$ with $L_2(f) \in \mathcal{C}_2$ and $R_2(f) \in \mathcal{W}_2 \cap \mathcal{F}_2$. Since $\mathcal{C}_2 \subset \mathcal{C}_m$ by Theorem 4.1, $L_2(f) \in \mathcal{C}_m$. Then $R_2(f)$ factors as a composite

$R_2(f) = \ell.k$ with $k \in \mathcal{C}_1 \cap \mathcal{W}_1 = \mathcal{C}_m \cap \mathcal{W}_m$ and $\ell \in \mathcal{F}_1 = \mathcal{F}_m$. By the 2-out-of-3 property, $\ell \in W_2 = \mathcal{W}_m$. Thus the second factorization is $f = \ell.(k.L_2(f))$. The functor $R_2 : \mathrm{Mor}(\mathcal{K}) \to \mathrm{Mor}(\mathcal{K})$ is accessible by [Dug01, Proposition 7.1] since the model structure $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ is combinatorial by hypothesis. Since $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1)$ is accessible by hypothesis, we deduce that the weak factorization system $(\mathcal{C}_m, \mathcal{W}_m \cap \mathcal{F}_m)$ is accessible. □

The model category **Top** can be equipped with the standard Quillen model structure $(\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ in which the weak equivalences are the weak homotopy equivalences [Hov99, Section 2.4]. There is another well-known model structure $(\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ on **Top** called the Cole-Strøm model structure. The weak equivalences are the homotopy equivalences; the fibrations are the Hurewicz fibrations; the cofibrations are the strong Hurewicz cofibrations. A general proof of its existence can be found in [BR13, Corollary 5.23]; The monomorphism hypothesis is automatically satisfied because **Top** is locally presentable [BR13, Remark 5.20]. All topological spaces are fibrant and cofibrant for the Cole-Strøm model structure. By using Proposition 4.2, we obtain the mixed model structure: the weak equivalences are the weak homotopy equivalences and the fibrations are the Hurewicz fibrations. All topological spaces are fibrant for this model structure. The cofibrations (the cofibrant objects resp.) of the mixed model structure are called the *mixed cofibrations* (the *mixed cofibrant objects* resp.). All Quillen cofibrations are mixed cofibrations because $\mathcal{C}_2 \subset \mathcal{C}_m$. By [Col06, Proposition 3.6], a map $f : A \to X$ is a mixed cofibration if and only if it is a closed Hurewicz cofibration and f factors as a composite $f : A \to X' \to X$ such that the left-hand map is a Quillen cofibration and the right-hand map is a homotopy equivalence. In particular, the cofibrant objects of the mixed model structure are the topological spaces homotopy equivalent to a cofibrant object of the Quillen model structure [Col06, Corollary 3.7].

4.3. **Notation.** *By convention,* $\mathbf{Top}_Q$ *denotes the category of* Δ*-generated spaces equipped with the Quillen model structure and* $\mathbf{Top}_m$ *denotes the category of* Δ*-generated spaces equipped with the mixed model structure.*

Convention. *The words cofibration and cofibrant without further precision mean cofibration and cofibrant in* $\mathbf{Top}_Q$*. The words mixed cofibration and mixed cofibrant mean cofibration and cofibrant in* $\mathbf{Top}_m$*.*

4.4. **Corollary.** *The model category* $\mathbf{Top}_m$ *is accessible.*

With the notations of Proposition 4.2 applied to **Top**, it is known that the weak factorization system $(\mathcal{C}_1, \mathcal{W}_1 \cap \mathcal{F}_1)$ is not small by [Rap10, Remark 4.7]. It is unlikely that the weak factorization system $(\mathcal{C}_1 \cap \mathcal{W}_1, \mathcal{F}_1)$ be small but we are not aware of a proof of this fact. Thus it is unlikely that the mixed model category $\mathbf{Top}_m$ be combinatorial.

Sketch of proof. It suffices to check that the factorization of a map by a strong cofibration which is a homotopy equivalence followed by a Hurewicz fibration is accessible. We can use the construction of [BR13, Definition 3.2]. The middle space is given by an accessible functor as soon as the underlying category is locally presentable. □

Note that $\mathbf{Top}_m$ is proper. Indeed, it has the same class of weak equivalences as the model category $\mathbf{Top}_Q$. And the latter is known to be proper by [Hir03, Theorem 13.1.11].

$$\begin{tikzcd}
\mathcal{P}(\ell_1,\ell_2) \arrow[r, "(G_{\ell_1,\ell_2},\eta_{\ell_1})"] \arrow[d, "(\eta_{\ell_2},F_{\ell_1,\ell_2})"'] & \mathbf{TOP}(G(\ell_1),G(\ell_2)) \times \mathbf{TOP}(F(\ell_1),G(\ell_1)) \arrow[d] \\
\mathbf{TOP}(F(\ell_2),G(\ell_2)) \times \mathbf{TOP}(F(\ell_1),F(\ell_2)) \arrow[r] & \mathbf{TOP}(F(\ell_1),G(\ell_2))
\end{tikzcd}$$

FIGURE 5. Enriched natural transformation

Thus the former is proper as well by Theorem 2.1. The mixed model structure $\mathbf{Top}_m$ is also monoidal closed for the binary product by [Col06, Proposition 6.6].

5. Enriched diagrams over a small enriched category

Let $\mathcal{P}$ be a nonempty enriched small category. Denote by $\mathcal{P}(\ell,\ell')$ the space of maps from ℓ to ℓ'. The underlying category is denoted by $\mathcal{P}_0$ and we have

$$\mathcal{P}_0(\ell,\ell') = \mathbf{Top}(\{0\},\mathcal{P}(\ell,\ell'))$$

for all objects ℓ and ℓ' of $\mathcal{P}$.

An enriched functor from $\mathcal{P}$ to $\mathbf{Top}$ is a functor F of $\mathbf{Top}^{\mathcal{P}_0}$ such that the set map

$$\mathcal{P}_0(\ell_1,\ell_2) \longrightarrow \mathbf{Top}(F(\ell_1),F(\ell_2))$$

induces a continuous map

$$F_{\ell_1,\ell_2} : \mathcal{P}(\ell_1,\ell_2) \longrightarrow \mathbf{TOP}(F(\ell_1),F(\ell_2)).$$

An enriched natural transformation $\eta : F \Rightarrow G$ from an enriched functor F to an enriched functor G is, by definition [Bor94b, Diagram 6.13], a family of continuous maps

$$\eta_\ell : \{0\} \to \mathbf{TOP}(F(\ell),G(\ell))$$

such that the diagram of $\mathbf{Top}$ depicted in Figure 5 commutes for all $\ell_1,\ell_2 \in \mathrm{Obj}(\mathcal{P})$. Since $\mathbf{Top}$ is cartesian closed, we have

$$\mathbf{Top}(F(\ell),G(\ell))) = \mathbf{Top}(\{0\},\mathbf{TOP}(F(\ell),G(\ell))).$$

Therefore η is just an ordinary natural transformation from F to G in $\mathbf{Top}^{\mathcal{P}_0}$. The underlying category $[\mathcal{P},\mathbf{Top}]_0$ of the enriched category of enriched functors $[\mathcal{P},\mathbf{Top}]$ can then be identified with a full subcategory of the category $\mathbf{Top}^{\mathcal{P}_0}$ of functors $F : \mathcal{P} \to \mathbf{Top}$ such that the set map $\mathcal{P}_0(\ell_1,\ell_2) \longrightarrow \mathbf{Top}(F(\ell_1),F(\ell_2))$ induces a continuous map $\mathcal{P}(\ell_1,\ell_2) \longrightarrow \mathbf{TOP}(F(\ell_1),F(\ell_2))$ for all $\ell_1,\ell_2 \in \mathrm{Obj}(\mathcal{P})$.

It is well-known that the enriched category $[\mathcal{P},\mathbf{Top}]$ is tensored and cotensored (e.g. see [Mos18, Lemma 5.2]). For an enriched diagram $F : \mathcal{P} \to \mathbf{Top}$, and a topological space U, the enriched diagram $F \otimes U : \mathcal{P} \to \mathbf{Top}$ is defined by $F \otimes U = F(-) \times U$ and $F^U : \mathcal{P} \to \mathbf{Top}$ is defined by $F^U = \mathbf{TOP}(U, F(-))$.

5.1. **Proposition.** *The category $[\mathcal{P},\mathbf{Top}]_0$ is locally presentable.*

Proof. By Theorem 3.13, the category $\mathbf{Top}$ together with the binary product is a locally presentable base. Using [BQR98, Example 6.2], we deduce that the enriched category $[\mathcal{P},\mathbf{Top}]$ is enriched locally presentable. The proof is complete using [BQR98, Proposition 6.6]. $\square$

5.2. **Proposition.** *A functor F of $\mathbf{Top}^{\mathcal{P}_0}$ belongs to the full subcategory $[\mathcal{P}, \mathbf{Top}]_0$ if and only if for all $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$, the set map $\mathcal{P}(\ell, \ell') \times F(\ell) \to F(\ell')$ defined by the mapping $(\phi, \gamma) \mapsto F(\phi)(\gamma)$ is continuous.*

Note that $\Delta_{\mathcal{P}}\varnothing$ belongs to $[\mathcal{P}, \mathbf{Top}]_0$ just because $\mathrm{Id}_\varnothing$ is continuous.

Proof. This comes from the bijection of sets

$$\mathbf{Top}(\mathcal{P}(\ell, \ell'), \mathbf{TOP}(F(\ell'), F(\ell))) \cong \mathbf{Top}(\mathcal{P}(\ell, \ell') \times F(\ell'), F(\ell)).$$

□

5.3. **Proposition.** *The inclusion functor $[\mathcal{P}, \mathbf{Top}]_0 \subset \mathbf{Top}^{\mathcal{P}_0}$ is colimit-preserving and limit-preserving.*

Proof. Since the category $[\mathcal{P}, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{\mathcal{P}_0}$, it suffices to prove that $[\mathcal{P}, \mathbf{Top}]_0$ is closed under the colimits and the limits of $\mathbf{Top}^{\mathcal{P}_0}$. Let $(F_i)_{i \in I}$ be a small diagram of functors of $[\mathcal{P}, \mathbf{Top}]_0$. The case of colimits comes from the fact that the colimit of the maps

$$\mathcal{P}(\ell, \ell') \times F_i(\ell) \to F_i(\ell')$$

in the category of diagrams $\mathrm{Mor}(\mathbf{Top})$ is

$$\mathcal{P}(\ell, \ell') \times (\varinjlim F_i(\ell)) \to \varinjlim F_i(\ell')$$

because $\mathbf{Top}$ is cartesian closed and because colimits in $\mathrm{Mor}(\mathbf{Top})$ are calculated pointwise. The case of limits comes from the fact that the limit of the maps

$$\mathcal{P}(\ell, \ell') \times F_i(\ell) \to F_i(\ell')$$

in the category of diagrams $\mathrm{Mor}(\mathbf{Top})$ is

$$\mathcal{P}(\ell, \ell') \times \varprojlim F_i(\ell) \to \varprojlim F_i(\ell')$$

because the functor $\varprojlim$ commutes with binary products as any right adjoint and because limits in $\mathrm{Mor}(\mathbf{Top})$ are calculated pointwise. □

5.4. **Notation.** *Let $\mathbb{F}^{\mathcal{P}}_\ell X = \mathcal{P}(\ell, -) \times X \in [\mathcal{P}, \mathbf{Top}]_0$ where X is a topological space and where ℓ is an object of $\mathcal{P}$.*

5.5. **Proposition.** *For every enriched functor $F : \mathcal{P} \to \mathbf{Top}$, every $\ell \in \mathrm{Obj}(\mathcal{P})$ and every topological space X, we have the natural bijection of sets*

$$[\mathcal{P}, \mathbf{Top}]_0(\mathbb{F}^{\mathcal{P}}_\ell X, F) \cong \mathbf{Top}(X, F(\ell)).$$

In particular, the functor $\mathbb{F}^{\mathcal{P}}_\ell : \mathbf{Top} \to [\mathcal{P}, \mathbf{Top}]_0$ is colimit-preserving for all $\ell \in \mathrm{Obj}(\mathcal{P})$.

Proof. We have the sequence of natural homeomorphisms

$$[\mathcal{P}, \mathbf{Top}](\mathbb{F}^{\mathcal{P}}_\ell X, F) \cong [\mathcal{P}, \mathbf{Top}](\mathcal{P}(\ell, -), \mathbf{TOP}(X, F(-)) \tag{1}$$
$$\cong \mathbf{TOP}(X, F(\ell)), \tag{2}$$

(1) because the enriched category $[\mathcal{P}, \mathbf{Top}]$ is tensored and cotensored, (2) by the enriched Yoneda lemma. By applying the functor $\mathbf{Top}(\{0\}, -)$, we obtain the desired bijection. □

5.6. **Corollary.** *Let $f : X \to Y$ be a map of $\mathbf{Top}$. The map of enriched diagrams $\mathbb{F}^{\mathcal{P}}_\ell X \to \mathbb{F}^{\mathcal{P}}_\ell Y$ induced by f satisfies the LLP with respect to a map of diagrams $D \to E$ of $[\mathcal{P}, \mathbf{Top}]_0$ if and only if f satisfies the LLP with respect to the continuous map $D_\ell \to E_\ell$.*

5.7. **Theorem.** *The inclusion functor $i^{\mathcal{P}} : [\mathcal{P}, \mathbf{Top}]_0 \subset \mathbf{Top}^{\mathcal{P}_0}$ has both a left adjoint and a right adjoint. In other terms, the category $[\mathcal{P}, \mathbf{Top}]_0$ is both a reflective and a coreflective subcategory of $\mathbf{Top}^{\mathcal{P}_0}$. If $i^{\mathcal{P}}_! : \mathbf{Top}^{\mathcal{P}_0} \to [\mathcal{P}, \mathbf{Top}]_0$ is the left adjoint, then for all $\ell \in \mathrm{Obj}(\mathcal{P})$ and all topological spaces U, $i^{\mathcal{P}}_!(\mathcal{P}_0(\ell,-) \times U) = \mathcal{P}(\ell,-) \times U$.*

Proof. Since the inclusion functor is colimit-preserving, it is in particular accessible and it is also a left adjoint because both the categories $[\mathcal{P}, \mathbf{Top}]_0$ and $\mathbf{Top}^{\mathcal{P}_0}$ are locally presentable. Since it is moreover limit-preserving, it is a right adjoint. We have the sequence of bijections

$$[\mathcal{P}, \mathbf{Top}]_0(i^{\mathcal{P}}_!(\mathcal{P}_0(\ell,-) \times U), Y) \cong \mathbf{Top}^{\mathcal{P}_0}(\mathcal{P}_0(\ell,-) \times U, Y) \tag{3}$$

$$\cong \int_{\ell'} \mathbf{Top}(\mathcal{P}_0(\ell,\ell') \times U, Y(\ell')) \tag{4}$$

$$\cong \int_{\ell'} \mathbf{Set}(\mathcal{P}_0(\ell,\ell'), \mathbf{Top}(U, Y(\ell'))) \tag{5}$$

$$\cong \mathbf{Set}^{\mathcal{P}_0}(\mathcal{P}_0(\ell,-), \mathbf{Top}(U, Y(-))) \tag{6}$$

$$\cong \mathbf{Top}(U, Y(\ell)) \tag{7}$$

$$\cong [\mathcal{P}, \mathbf{Top}]_0(\mathcal{P}(\ell,-) \times U, Y), \tag{8}$$

(3) because $[\mathcal{P}, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{\mathcal{P}_0}$ by Proposition 5.2 and by adjunction, (4) by [ML98, page 219 (2)], (5) because $\mathcal{P}_0(\ell,\ell')$ is a set, (6) by [ML98, page 219 (2)], (7) by Yoneda, and finally (8) by Proposition 5.5. The proof is complete thanks to Yoneda. $\square$

5.8. **Proposition.** *Let $\ell \in \mathrm{Obj}(\mathcal{P})$. Let U be a topological space. Then there is the natural homeomorphism $\varinjlim \mathbb{F}^{\mathcal{P}}_\ell U \cong U$.*

Proof. There is the sequence of bijections (V being another topological space):

$$\mathbf{Top}(\varinjlim \mathbb{F}^{\mathcal{P}}_\ell U, V) \cong \mathbf{Top}^{\mathcal{P}_0}(\mathbb{F}^{\mathcal{P}}_\ell U, \Delta_{\mathcal{P}} V) \tag{9}$$

$$\cong [\mathcal{P}, \mathbf{Top}]_0(\mathbb{F}^{\mathcal{P}}_\ell U, \Delta_{\mathcal{P}} V) \tag{10}$$

$$\cong \mathbf{Top}\left(\{0\}, [\mathcal{P}, \mathbf{Top}](\mathcal{P}(\ell,-) \times U, \Delta_{\mathcal{P}} V)\right) \tag{11}$$

$$\cong \mathbf{Top}\left(\{0\}, [\mathcal{P}, \mathbf{Top}](\mathcal{P}(\ell,-), \mathbf{TOP}(U, \Delta_{\mathcal{P}} V(-)))\right) \tag{12}$$

$$\cong \mathbf{Top}\left(\{0\}, \mathbf{TOP}(U, V)\right) \tag{13}$$

$$\cong \mathbf{Top}(U, V), \tag{14}$$

(9) by definition of the colimit, (10) because $[\mathcal{P}, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{\mathcal{P}_0}$ and because the constant diagram functor belongs to $[\mathcal{P}, \mathbf{Top}]_0$, (11) by definition of the enriched category $[\mathcal{P}, \mathbf{Top}]$, (12) since the enriched category $[\mathcal{P}, \mathbf{Top}]$ is tensored and cotensored, (13) by the enriched Yoneda lemma, and finally (14) by definition of the enrichment of $\mathbf{Top}$. The proof is complete thanks to the (ordinary) Yoneda lemma. $\square$

6. The homotopy theory of enriched diagrams of topological spaces

6.1. **Notation.** *Let $n \geqslant 1$. Denote by $\mathbf{D}^n = \{b \in \mathbb{R}^n, |b| \leqslant 1\}$ the n-dimensional disk, and by $\mathbf{S}^{n-1} = \{b \in \mathbb{R}^n, |b| = 1\}$ the $(n-1)$-dimensional sphere. By convention, let $\mathbf{D}^0 = \{0\}$ and $\mathbf{S}^{-1} = \varnothing$.*

6.2. **Theorem.** *The category $[\mathcal{P}, \mathbf{Top}]_0$ can be endowed with a structure of combinatorial model category as follows:*

- *The set of generating cofibrations is the set of maps*
$$\{\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \to \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \mid n \geqslant 0, \ell \in \mathrm{Obj}(\mathcal{P})\}.$$
- *The set of generating trivial cofibrations is the set of maps*
$$\{\mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \to \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^{n+1} \mid n \geqslant 0, \ell \in \mathrm{Obj}(\mathcal{P})\}.$$
- *A map $F \to G$ is a weak equivalence if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a weak equivalence of $\mathbf{Top}_Q$, i.e. the weak equivalences are the pointwise weak homotopy equivalences*
- *A map $F \to G$ is a fibration if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a fibration of $\mathbf{Top}_Q$, i.e. the fibrations are the pointwise Serre fibrations.*

This model structure, denoted by $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$, is called the projective model structure. *The cofibrations are called the projective cofibrations.*

Proof. The existence of an accessible model structure is a consequence of Theorem 3.13, [Mos18, Theorem 6.5(ii)] and of the fact that all objects of $\mathbf{Top}_Q$ are fibrant (it suffices to use the adjunction $[\mathcal{P}, \mathbf{Top}]_0 \leftrightarrows \mathbf{Top}^{\mathrm{Obj}(\mathcal{P})}$ and the Quillen Path Object Argument, which implies the acyclicity condition). It is cofibrantly generated thanks to Corollary 5.6 and because the set of inclusions $\{\mathbf{S}^{n-1} \subset \mathbf{D}^n \mid n \geqslant 0\}$ ($\{\mathbf{D}^n \subset \mathbf{D}^{n+1} \mid n \geqslant 0\}$ resp.) is a set of generating (trivial resp.) cofibrations of $\mathbf{Top}_Q$. □

In [Pia91], Piacenza proves a similar result by working in the category of Hausdorff k-spaces in the sense of [Vog71]. We did not read his proof in detail (which is much longer). We do not know if Piacenza's proof can be adapted to Δ-generated spaces, especially because Piacenza works with Hausdorff spaces. It is known that Hausdorff k-spaces do not behave very well for algebraic topology problems and that weak Hausdorff k-spaces are much better (see the end of the introduction of [Gau09] for some bibliographical research about this problem).

[Shu09, Theorem 24.4] and [Mos18, Theorem 4.4] (the latter is a generalization of the former to the framework of accessible model categories) give sufficient conditions for the projective model structure to exist in an enriched situation. They could be applied to prove Theorem 6.2 if e.g. we assumed that all topological spaces $\mathcal{P}(\ell, \ell')$ were cofibrant in $\mathbf{Top}_Q$.

6.3. **Corollary.** *The adjunction $i_!^{\mathcal{P}} \dashv i^{\mathcal{P}}$ of Theorem 5.7 is a Quillen adjunction between the projective model structures of $\mathbf{Top}^{\mathcal{P}_0}$ and $[\mathcal{P}, \mathbf{Top}]_0$.*

6.4. **Theorem.** *Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. The category $[\mathcal{P}, \mathbf{Top}_m]_0$ can be endowed with a structure of accessible model category characterized as follows:*

- *A map $F \to G$ is a cofibration if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a cofibration of $\mathbf{Top}_m$, i.e. the cofibrations are the pointwise mixed cofibrations.*

- *A map $F \to G$ is a weak equivalence if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a weak equivalence of $\mathbf{Top}_m$, i.e. the weak equivalences are the pointwise weak homotopy equivalences*

This model structure, denoted by $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$, is called the injective mixed model structure*. The fibrations are called the injective mixed fibrations.*

Proof. By [Col06, Proposition 6.4], the cartesian closed category **Top** equipped with the mixed model structure is a monoidal model category. The proof is complete using Theorem 3.13, Corollary 4.4 and [Mos18, Theorem 6.5(i)] □

6.5. **Corollary.** *Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. Then the projective model structure $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is proper.*

Proof. By hypothesis, all topological spaces $\mathcal{P}(\ell, \ell')$ are mixed cofibrant. We deduce that the functors $- \times \mathcal{P}(\ell, \ell') : \mathbf{Top}_m \to \mathbf{Top}_m$ preserve mixed cofibrations for all $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$ because $(\mathbf{Top}_m, \times)$ is a closed monoidal model category. Using [Mos18, Proposition 8.1(i)], we obtain that the injective model structure $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$ is left proper. By Theorem 2.1, we deduce that $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is left proper[2]. The model category $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is right proper because the fibrations are the pointwise fibrations and because all topological spaces are fibrant. □

We can actually remove the hypothesis of Corollary 6.5 but the proof is a little bit more involved.

6.6. **Theorem.** *Let λ be a ordinal. Let $M : \lambda \to$ **Top** and $N : \lambda \to$ **Top** be two λ-sequences of topological spaces. Let $f : M \to N$ be a map of λ-sequences. Suppose that for each ordinal α with $\alpha + 1 \leqslant \lambda$, the map $M_\alpha \to M_{\alpha+1}$ and $N_\alpha \to N_{\alpha+1}$ are pushouts of a map of the form $\mathrm{Id} \times \ldots \times \mathrm{Id} \times g \times \mathrm{Id} \times \ldots \times \mathrm{Id}$ where g is a cofibration of* **Top**. *Suppose moreover that the map $M_0 \to N_0$ is a weak homotopy equivalence. Then the map $\varinjlim f : \varinjlim M \to \varinjlim N$ is a weak homotopy equivalence as well.*

Proof. All maps of the form $\mathrm{Id} \times \ldots \times \mathrm{Id} \times g \times \mathrm{Id} \times \ldots \times \mathrm{Id}$ where g is a cofibration of **Top** are cofibrations of the Cole-Strøm model structure of **Top**. We deduce that the colimit is a homotopy colimit in **Top** equipped with the Cole-Strøm model structure. By [DI04, Theorem A.7], this colimit is therefore a homotopy colimit in **Top** equipped with the Quillen model structure ([DI04, Theorem A.7] is written in the category of general topological spaces $\mathcal{TOP}$ but the argument remains valid in **Top** because **Top** is a coreflective subcategory of $\mathcal{TOP}$). We can also prove this theorem by using [Gau07, Proposition 7.2] and [Gau07, Proposition 7.3]. The proofs of these two propositions are written down in the category of weakly Hausdorff k-spaces but they remain valid in **Top**. The key point is that the cofibrations of the Cole-Strøm model structure of **Top** are closed T_1-inclusions and that compact spaces are $\aleph_0$-small relative to closed T_1-inclusions by [Hov99, Proposition 2.4.2]. In both arguments, the Cole-Strøm model structure plays a key role. □

6.7. **Theorem.** *The projective model structure $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is proper.*

[2]We cannot apply [Mos18, Proposition 8.1(i)] directly to $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ because the spaces of maps of $\mathcal{P}$ are not supposed to be cofibrant in the Quillen model structure of **Top**.

Proof. It suffices to prove that it is left proper because all objects are fibrant. In a model category, weak equivalences are closed under retract. Therefore it suffices to prove that the pushout of a weak equivalence along a transfinite composition of pushouts of maps of the form $\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \to \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n$ is still a weak equivalence. Consider first the following situation:

$$\begin{tikzcd}
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \arrow[r] \arrow[d] & F \arrow[r, "f"] \arrow[d] & G \arrow[d] \\
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \arrow[r] & H \arrow[r, "\widetilde{f}"] & K.
\end{tikzcd}$$

For all $\ell' > 0$, we obtain the diagram of **Top**

$$\begin{tikzcd}
\mathcal{P}(\ell,\ell') \times \mathbf{S}^{n-1} \arrow[r] \arrow[d] & F(\ell') \arrow[r, "f_{\ell'}"] \arrow[d] & G(\ell') \arrow[d] \\
\mathcal{P}(\ell,\ell') \times \mathbf{D}^n \arrow[r] & H(\ell') \arrow[r, "\widetilde{f}_{\ell'}"] & K(\ell').
\end{tikzcd}$$

If $f_{\ell'}$ is a weak homotopy equivalence, then $\widetilde{f}_{\ell'}$ is weak homotopy equivalence by Theorem 6.6. Thus if f is a pointwise weak equivalence, then $\widetilde{f}$ is a pointwise weak equivalence. Again by Theorem 6.6, this process can be iterated transfinitely since colimits in $[\mathcal{P}, \mathbf{Top}]_0$ are calculated pointwise by Proposition 5.3. □

6.8. **Proposition.** *For all (trivial resp.) cofibrations $f : U \to V$ of $\mathbf{Top}_Q$ and all $\ell \in$ Obj$(\mathcal{P})$, the map of diagrams $\mathbb{F}_\ell^{\mathcal{P}}U \to \mathbb{F}_\ell^{\mathcal{P}}V$ is a (trivial resp.) projective cofibration.*

Proof. The map of diagrams $\mathbb{F}_\ell^{\mathcal{P}}U \to \mathbb{F}_\ell^{\mathcal{P}}V$ satisfies the LLP with respect to a (trivial) fibration $D \to E$ if and only if f satisfies the LLP with respect to the continuous map $D_\ell \to E_\ell$ by Corollary 5.6. The proof is complete because the fibrations and the trivial fibrations are the pointwise ones. □

6.9. **Corollary.** *The combinatorial model category $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is tractable.*

Proof. It is a consequence of the fact that the maps $\varnothing \subset \mathbf{S}^{n-1}$ and $\varnothing \subset \mathbf{D}^n$ are cofibrations for all $n \geqslant 0$ and that $\mathbb{F}_\ell^{\mathcal{P}}\varnothing = \Delta_{\mathcal{P}}\varnothing$ is the initial object of $[\mathcal{P}, \mathbf{Top}]_0$ for all $\ell \in \mathrm{Obj}(\mathcal{P})$. The proof is complete thanks to Proposition 6.8. □

7. The main theorem

7.1. **Proposition.** *Suppose that all topological spaces $\mathcal{P}(\ell,\ell')$ are homotopy equivalent to a cofibrant space. Then any cofibration of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is a cofibration of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$. In other terms, the identity functor induces a left Quillen adjoint* $\mathrm{Id} : [\mathcal{P}, \mathbf{Top}_Q]_0^{proj} \to [\mathcal{P}, \mathbf{Top}_m]_0^{inj}$.

Another way to formulate this proposition is that any projective cofibration of the model category $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is a pointwise mixed cofibration of $\mathbf{Top}^{\mathcal{P}_0}$. Note that a projective cofibration of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is not necessarily a pointwise cofibration of $\mathbf{Top}^{\mathcal{P}_0}$. Indeed the diagram $\mathbb{F}_\ell^{\mathcal{P}}U$ is projective cofibrant for any cofibrant space U by Proposition 6.8. But the vertices of this diagram are only homotopy equivalent to a cofibrant space.

Proof. Every cofibration of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ is a retract of a transfinite composition of pushouts of maps of $\{\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \subset \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \mid n \geqslant 0, \ell \in \mathrm{Obj}(\mathcal{P})\}$. Therefore it suffices to prove that the maps $\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \subset \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n$ are pointwise mixed cofibrations of $\mathbf{Top}^{\mathcal{P}_0}$ for all $n \geqslant 0, \ell \in \mathrm{Obj}(\mathcal{P})$. It suffices to prove that for all $\ell, \ell' \in \mathrm{Obj}(\mathcal{P})$ and all $n \geqslant 0$, the map $\mathcal{P}(\ell, \ell') \times \mathbf{S}^{n-1} \subset \mathcal{P}(\ell, \ell') \times \mathbf{D}^n$ is a mixed cofibration. The latter fact comes from the facts that $\mathcal{P}(\ell, \ell')$ is cofibrant in $\mathbf{Top}_m$, that any cofibration is a mixed cofibration and that $(\mathbf{Top}_m, \times)$ is a monoidal model structure. □

7.2. **Corollary.** *Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. Then the identity functor induces a left Quillen equivalence*

$$\mathrm{Id} : [\mathcal{P}, \mathbf{Top}_Q]_0^{proj} \to [\mathcal{P}, \mathbf{Top}_m]_0^{inj}.$$

7.3. **Proposition.** *There is a Quillen adjunction $\varinjlim \dashv \Delta_{\mathcal{P}}$ between the model categories $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ and $\mathbf{Top}_Q$.*

Proof. There is the sequence of bijections (X being an object of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ and U being a topological space)

$$\mathbf{Top}(\varinjlim X, U) \cong \mathbf{Top}^{\mathcal{P}_0}(X, \Delta_{\mathcal{P}}U) \cong [\mathcal{P}, \mathbf{Top}]_0(X, \Delta_{\mathcal{P}}U),$$

the left-hand bijection by definition of the colimit and by Proposition 5.3, the right-hand bijection because the category $[\mathcal{P}, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{\mathcal{P}_0}$ and because the constant diagram functor belongs to $[\mathcal{P}, \mathbf{Top}]_0$. The right adjoint $\Delta_{\mathcal{P}} : \mathbf{Top}_Q \to [\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ takes (trivial resp.) fibrations to pointwise (trivial resp.) fibrations. We deduce that it is a right Quillen adjoint. □

7.4. **Proposition.** *There is a categorical adjunction $\Delta_{\mathcal{P}} \dashv \varprojlim$ between the categories $[\mathcal{P}, \mathbf{Top}]_0$ and $\mathbf{Top}$.*

Proof. There is the sequence of bijections (X being an object of $[\mathcal{P}, \mathbf{Top}]_0$ and U being a topological space)

$$\mathbf{Top}(U, \varprojlim X) \cong \mathbf{Top}^{\mathcal{P}_0}(\Delta_{\mathcal{P}}U, X) \cong [\mathcal{P}, \mathbf{Top}]_0(\Delta_{\mathcal{P}}U, X),$$

the left-hand bijection by definition of the limit and by Proposition 5.3, the right-hand bijection because the category $[\mathcal{P}, \mathbf{Top}]_0$ is a full subcategory of $\mathbf{Top}^{\mathcal{P}_0}$ and because the constant diagram functor belongs to $[\mathcal{P}, \mathbf{Top}]_0$. □

The following corollary is useless for the sequel. But it can be noted anyway:

7.5. **Corollary.** *Suppose that all topological spaces $\mathcal{P}(\ell, \ell')$ are homotopy equivalent to a cofibrant space. There is a Quillen adjunction $\Delta_{\mathcal{P}} \dashv \varprojlim$ between the model categories $\mathbf{Top}_m$ and $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$.*

Proof. The left adjoint $\Delta_{\mathcal{P}} : \mathbf{Top}_m \to [\mathcal{P}, \mathbf{Top}_m]_0^{inj}$ takes (trivial resp.) mixed cofibrations to pointwise (trivial resp.) mixed cofibrations. We deduce that it is a left Quillen adjoint. □

We can now prove the main theorem of the paper.

7.6. **Theorem.** *Suppose that all spaces $\mathcal{P}(\ell, \ell')$ are contractible. The Quillen adjunction $\varinjlim \dashv \Delta_{\mathcal{P}}$ is a Quillen equivalence between the model categories $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ and $\mathbf{Top}_Q$.*

$$
\begin{tikzcd}
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \arrow[rr] \arrow[d] \arrow[ddr] & & X \arrow[ddr] & \\
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \arrow[ddr] & & & \\
& \Delta_{\mathcal{P}} \varinjlim \mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \arrow[rr] \arrow[d] & & \Delta_{\mathcal{P}} \varinjlim X \\
& \Delta_{\mathcal{P}} \varinjlim \mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n & &
\end{tikzcd}
$$

Figure 6. Preparation for applying the cube lemma

$$
\begin{tikzcd}
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{S}^{n-1} \arrow[r] \arrow[d] \arrow[ddr] & X \arrow[d] \arrow[ddr] & \\
\mathbb{F}_\ell^{\mathcal{P}}\mathbf{D}^n \arrow[r] \arrow[ddr] & Y \arrow[ul, phantom, "\ulcorner", very near start] \arrow[ddr, dashed] & \\
& \Delta_{\mathcal{P}}\mathbf{S}^{n-1} \arrow[r] \arrow[d] & \Delta_{\mathcal{P}} \varinjlim X \arrow[d] \\
& \Delta_{\mathcal{P}}\mathbf{D}^n \arrow[r] & \Delta_{\mathcal{P}} \varinjlim Y. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Figure 7. The next step in the transfinite sequence

Proof. Since all spaces $\mathcal{P}(\ell,\ell')$ are contractible by hypothesis, i.e. homotopy equivalent to a point, they are cofibrant for the mixed model structure $\mathbf{Top}_m$. Let U be a topological space. Let $U^{cof} \to U$ be a cofibrant replacement of U in $\mathbf{Top}_Q$. We obtain a map $U^{cof} \to (\Delta_{\mathcal{P}}U)(\ell)$ for some $\ell \in \mathrm{Obj}(\mathcal{P})$. By Proposition 5.5, we obtain a map $\mathbb{F}_\ell^{\mathcal{P}}U^{cof} \to \Delta_{\mathcal{P}}U$ of $[\mathcal{P},\mathbf{Top}]_0$. Since all topological spaces $\mathcal{P}(\ell,\ell')$ are contractible by hypothesis, the map $\mathbb{F}_\ell^{\mathcal{P}}U^{cof} \to \Delta_{\mathcal{P}}U$ is a weak equivalence of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$. By Proposition 6.8, the map $\mathbb{F}_\ell^{\mathcal{P}}\varnothing \to \mathbb{F}_\ell^{\mathcal{P}}U^{cof}$ is a cofibration of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$. Thus $\mathbb{F}_\ell^{\mathcal{P}}U^{cof}$ is a cofibrant replacement of $\Delta_{\mathcal{P}}U$ in $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$. Using Proposition 5.8, we obtain that the canonical map $\varinjlim \mathbb{F}_\ell^{\mathcal{P}}U^{cof} \cong U^{cof} \to U$ is a weak equivalence of $\mathbf{Top}_Q$. We deduce that the functor $\varinjlim : [\mathcal{P},\mathbf{Top}_Q]_0^{proj} \to \mathbf{Top}_Q$ is homotopically surjective.

Let Y be a cofibrant diagram of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$. Since all objects of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$ are fibrant, we need to prove that the unit of the adjunction $Y \to \Delta_{\mathcal{P}}(\varinjlim Y)$ is a weak equivalence to complete the proof. Every cofibrant diagram of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$ is a retract of a cellular object of $[\mathcal{P},\mathbf{Top}_Q]_0^{proj}$, i.e. of a transfinite composition of pushouts of generating cofibrations. And in a model category, the retract of a weak equivalence is a weak equivalence. We can therefore assume without lack of generality that Y is cellular. As a first step, consider the commutative diagram in $[\mathcal{P},\mathbf{Top}]_0$ of Figure 6 obtained using the unit of the adjunction $\mathrm{Id} \Rightarrow \Delta_{\mathcal{P}} \varinjlim$. Suppose that the map $X \to \Delta_{\mathcal{P}} \varinjlim X$

is a pointwise weak equivalence (of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ or equivalently of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$), that X is projective cofibrant and that $\Delta_{\mathcal{P}} \varinjlim X$ is pointwise mixed cofibrant. The map $\mathbb{F}_\ell^{\mathcal{P}} \mathbf{S}^{n-1} \to \mathbb{F}_\ell^{\mathcal{P}} \mathbf{D}^n$ is a pointwise mixed cofibration between pointwise mixed cofibrant diagrams by Proposition 7.1. By Proposition 5.8, the map

$$\Delta_{\mathcal{P}} \varinjlim \mathbb{F}_\ell^{\mathcal{P}} \mathbf{S}^{n-1} \cong \Delta_{\mathcal{P}} \mathbf{S}^{n-1} \to \Delta_{\mathcal{P}} \mathbf{D}^n \cong \Delta_{\mathcal{P}} \varinjlim \mathbb{F}_\ell^{\mathcal{P}} \mathbf{D}^n$$

is a pointwise mixed cofibration between pointwise mixed cofibrant diagrams as well because all Quillen cofibrations are mixed cofibrations. Since X is projective cofibrant by hypothesis, it is also pointwise mixed cofibrant by Proposition 7.1. For all topological spaces U, the maps $\mathbb{F}_\ell^{\mathcal{P}} U \to \Delta_{\mathcal{P}} U$ are pointwise weak equivalences since all spaces $\mathcal{P}(\ell, \ell')$ are contractible. We are ready to apply the cube lemma [Hov99, Lemma 5.2.6] in $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$ by passing to the colimit. Since the functor

$$\Delta_{\mathcal{P}}. \varinjlim : [\mathcal{P}, \mathbf{Top}]_0 \to [\mathcal{P}, \mathbf{Top}]_0$$

is colimit-preserving by Proposition 7.3 and Proposition 7.4 as a composite of two colimit-preserving functors, we obtain the commutative diagram of Figure 7. Using the cube lemma in $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$, we deduce that the map $Y \to \Delta_{\mathcal{P}} \varinjlim Y$ is a pointwise weak equivalence from a projective cofibrant object of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ to a pointwise mixed cofibrant object of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$. Moreover, the map $X \to Y$ is a cofibration both of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$ and of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$, and the map $\Delta_{\mathcal{P}} \varinjlim X \to \Delta_{\mathcal{P}} \varinjlim Y$ is a pointwise mixed cofibration, i.e. a cofibration of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$ as well. By starting from $X = \varnothing$ and by iterating the process transfinitely, we obtain two transfinite towers of cofibrations of $[\mathcal{P}, \mathbf{Top}_m]_0^{inj}$ between pointwise mixed cofibrant diagrams. In this case, the colimit is a homotopy colimit. Thus, for all cellular objects Y of $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$, the unit of the adjunction $Y \to \Delta_{\mathcal{P}} \varinjlim Y$ is a pointwise weak homotopy equivalence. □

8. Concluding remarks

We conclude this paper with two remarks.

About the monoid of nondecreasing continuous maps from $[0,1]$ to itself. Let us go back to the geometric example of the introduction. The space of paths of a multipointed d-space is closed only under the action of $\mathcal{G}$ because we need a specific property for the cofibrant objects to construct the model structure as it is carried out in [Gau09] (cf the second statement of [Gau09, Proposition 4.7]). On the contrary, the space of dipaths of a Grandis d-space is closed under the action of the monoid $\mathcal{M} \supset \mathcal{G}$ of nondecreasing continuous maps from $[0,1]$ to itself preserving the extremities. By seeing $\mathcal{M}$ as a one-object category, we obtain a new contravariant non-enriched diagram of topological spaces $\mathcal{D}^{\mathcal{M}}(X, dX)$. The interpretations of $\varprojlim \mathcal{D}^{\mathcal{M}}(X, dX)$ and $\varinjlim \mathcal{D}^{\mathcal{M}}(X, dX)$ are the same. The behavior of the homotopy colimit $\underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{M}}(X, dX)$ in the enriched case remains the same because Theorem 7.6 can still be applied. However, the homotopy colimit *in the non-enriched setting* behaves well in this case:

8.1. **Proposition.** *We have a weak homotopy equivalence $\underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{M}}(X, dX) \simeq dX$ where the homotopy colimit is calculated in the ordinary (i.e. non-enriched) projective model structure.*

Proof. Every map $\phi^* : dX \to dX$ is homotopic to the identity by the homotopy $H : [0,1] \times dX \to dX$ taking (t,γ) to $\gamma.(t.\phi + (1-t).\mathrm{Id}_{[0,1]})$. We deduce that every map of $\mathcal{D}^{\mathcal{M}}(X, dX)$ is a weak homotopy equivalence. Since $B\mathcal{M}$ is contractible by [Law18], the ordinary (i.e. non-enriched) homotopy colimit $\varinjlim \mathcal{D}^{\mathcal{M}}(X, dX)$ is in this case weakly homotopy equivalent to dX by [CS02, Corollary 29.2]. $\square$

Toward a generalization of the main theorem. Every enriched functor $f : \mathcal{P}_1 \to \mathcal{P}_2$ gives rise to a Quillen adjunction $f_* \dashv f^* : [\mathcal{P}_1, \mathbf{Top}_Q]_0^{proj} \leftrightarrows [\mathcal{P}_2, \mathbf{Top}_Q]_0^{proj}$ where $f^*(G) = G.f$ is the precomposition by f and where $f_*F(-) = \int^{\ell'} \mathcal{P}_2(f(\ell'), -) \times F(\ell')$ is the enriched left Kan extension. The main theorem can be reformulated as follows:

8.2. **Theorem.** *Suppose that $\mathcal{P}$ is locally contractible. Let $f : \mathcal{P} \to \mathrm{Thin}(\mathcal{P})$ be the unique functor where $\mathrm{Thin}(\mathcal{P})$ is the enriched small category which has the same objects as $\mathcal{P}$ and exactly one map between each object. Then the functor*

$$f^* : [\mathrm{Thin}(\mathcal{P}), \mathbf{Top}_Q]_0^{proj} \to [\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$$

is a right Quillen equivalence.

Sketch of proof. There is the categorical equivalence $\mathbf{Top} \simeq [\mathrm{Thin}(\mathcal{P}), \mathbf{Top}]_0$. The composite functor $\mathbf{Top} \simeq [\mathrm{Thin}(\mathcal{P}), \mathbf{Top}]_0 \xrightarrow{f^*} [\mathcal{P}, \mathbf{Top}]_0$ is the constant diagram functor $\Delta_{\mathcal{P}}$. By the 2-out-of-3 property, this composite is a right Quillen equivalence if and only if f^* is a right Quillen equivalence. The proof is complete thanks to Theorem 7.6. $\square$

This raises the question of generalizing Theorem 8.2 by considering an enriched functor $f : \mathcal{P}_1 \to \mathcal{P}_2$ between enriched small categories which is essentially surjective and locally a weak homotopy equivalence. Similar questions are studied in [Shu09, Proposition 22.5 and Proposition 22.9] in the context of enriched homotopical categories. Unfortunately, Shulman's work cannot be used here, at least without adaptation, because the goodness condition is not satisfied. The main obstacle is that the injection of a point need not be a cofibration (cf. [Shu09, Definition 23.11]).

Appendix A. Variant for the particular cases of $\mathcal{G}$ and $\mathcal{M}$

The obstacle mentioned in Section 8 to apply [Shu09] does not exist in the particular case of $\mathcal{G}$ and $\mathcal{M}$. Indeed we have the following proposition:

A.1. **Proposition.** *For any map ϕ of $\mathcal{G}$, the injection $\{\phi\} \subset \mathcal{G}$ is a mixed cofibration. For any map ϕ of $\mathcal{M}$, the injection $\{\phi\} \subset \mathcal{M}$ is a mixed cofibration.*

Proof. We write the proof for $\mathcal{G}$. It is similar for $\mathcal{M}$. The homotopy $H : \mathcal{G} \times [0,1] \to \mathcal{G}$ defined by $(\psi, t) \mapsto t\psi + (1-t)\phi$ between ϕ and $\mathrm{Id}_{\mathcal{G}}$ satisfies $H(\phi, t) = \phi$ for all $t \in [0,1]$. Thus the inclusion $\{\phi\} \subset \mathcal{G}$ is an inclusion of a strong deformation retract. Let $k : \mathcal{TOP} \to \mathbf{Top}$ be the right adjoint of the inclusion functor $\mathbf{Top} \subset \mathcal{TOP}$. The space $\mathcal{G}$ is equipped with the final topology making the inclusion $\mathcal{G} \subset \mathbf{TOP}([0,1],[0,1])$ continuous. The space $\mathbf{TOP}([0,1],[0,1])$ is equal to $k(\mathcal{TOP}([0,1],[0,1]))$ where $\mathcal{TOP}([0,1],[0,1])$ is the set of continuous maps from $[0,1]$ to $[0,1]$ equipped with the compact-open topology. We have a composite of continuous maps in $\mathcal{TOP}$

$$\mathcal{G} \longrightarrow \mathbf{TOP}([0,1],[0,1]) \longrightarrow \mathcal{TOP}([0,1],[0,1])$$

since the underlying set of $\mathbf{TOP}([0,1],[0,1])$ and $\mathcal{TOP}([0,1],[0,1])$ are the same and $\mathbf{TOP}([0,1],[0,1])$ have more open sets than the compact-open topology. Since $[0,1]$ is compact metrisable with the metric defined by $d_1(x,y) = |x-y|$, the topological space $\mathcal{TOP}([0,1],[0,1])$ is metrisable, and the compact-open topology is induced by the metric

$$d_1(f,g) = \max_{x \in [0,1]} d_1(f(x), g(x))$$

by [Hat02, Proposition A.13]. We obtain a composite of continuous maps

$$q : \mathcal{G} \longrightarrow \mathbf{TOP}([0,1],[0,1]) \longrightarrow \mathcal{TOP}([0,1],[0,1]) \xrightarrow{f \mapsto d_1(f,\phi)} [0,1]$$

such that $q^{-1}(0) = \{\phi\}$. By [Str66, Theorem 3] (see also [BR13, Proposition 2.3]), we deduce that the inclusion $\{\phi\} \subset \mathcal{G}$ is a cofibration of the Cole-Strøm model structure. Since this map is homotopic to $\mathrm{Id}_{\{\phi\}}$ which is a cofibration of $\mathbf{Top}_Q$, we deduce by [Col06, Proposition 3.6] that the map $\{\phi\} \subset \mathcal{G}$ is a mixed cofibration. □

A.2. **Theorem.** *Let $\mathcal{P}$ be an enriched small category. The category $[\mathcal{P}, \mathbf{Top}]_0$ can be endowed with a structure of accessible model category characterized as follows:*

- *A map $F \to G$ is a weak equivalence if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a weak equivalence of $\mathbf{Top}_m$, i.e. the weak equivalences are the pointwise weak homotopy equivalences*
- *A map $F \to G$ is a fibration if and only if for all $\ell \in \mathrm{Obj}(\mathcal{P})$, the continuous map $F(\ell) \to G(\ell)$ is a fibration of $\mathbf{Top}_m$, i.e. the fibrations are the pointwise Hurewicz fibrations.*

This model structure, denoted by $[\mathcal{P}, \mathbf{Top}_m]_0^{proj}$, is called the projective mixed model structure. *The cofibrations are called the projective mixed cofibrations. This model structure is proper. It is Quillen equivalent to the projective model structure $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$*

Proof. The existence of an accessible model structure is a consequence of Theorem 3.13, [Mos18, Theorem 6.5(ii)] and of the fact that all objects of $\mathbf{Top}_m$ are fibrant. It is proper by Theorem 2.1 and Theorem 6.7. The identity functor induces a left Quillen equivalence $[\mathcal{P}, \mathbf{Top}_Q]_0^{proj} \to [\mathcal{P}, \mathbf{Top}_m]_0^{proj}$ for obvious reasons. □

A.3. **Theorem.** *Suppose that $\mathcal{P}$ is either $\mathcal{G}$ or $\mathcal{M}$ viewed as one-object enriched categories. Let $f : \mathcal{P} \to \mathrm{Thin}(\mathcal{P})$ be the unique functor where $\mathrm{Thin}(\mathcal{P})$ is the enriched small category which has the same objects as $\mathcal{P}$ and exactly one map between each object. Then the functor*

$$f^* : [\mathrm{Thin}(\mathcal{P}), \mathbf{Top}_Q]_0^{proj} \to [\mathcal{P}, \mathbf{Top}_Q]_0^{proj}$$

is a right Quillen equivalence.

Proof. The first step is to replace the projective model structures by the projective mixed model structures. Indeed, it suffices by Theorem A.2 to prove that the Quillen adjunction

$$f_* \dashv f^* : [\mathrm{Thin}(\mathcal{P}), \mathbf{Top}_m]_0^{proj} \leftrightarrows [\mathcal{P}, \mathbf{Top}_m]_0^{proj}$$

is a Quillen equivalence to complete the proof. It then suffices to prove that the total derived functors induce an equivalence of categories between the homotopy categories by [Hov99, Proposition 1.3.13]. To prove this fact, we can work in the more general setting of enriched homotopical categories in the sense of [Shu09]. We want to apply [Shu09, Proposition 22.5]. In the language of [Shu09], we have to prove that the two enriched small categories Thin($\mathcal{P}$) and $\mathcal{P}$ are very good for the tensor structure generated by the binary product of **Top**. To prove the latter fact, we have to jump to [Shu09, Theorem 23.12]. It is easy to check that the monoidal model category ($\mathbf{Top}_m, \times$) is simplicial and that the binary product gives rise to a Quillen two-variable enriched adjunctions. All spaces of maps of Thin($\mathcal{P}$) and $\mathcal{P}$ are cofibrant in $\mathbf{Top}_m$. To check the hypotheses of [Shu09, Theorem 23.12], it remains to check that any injection of a singleton in one of the space of maps of $\mathcal{G}$ or $\mathcal{M}$ is actually a mixed cofibration (cf. [Shu09, Definition 23.11]). It is precisely what is proved in Proposition A.1. □

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