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# Investigating The Algebraic Structure of Dihomotopy Types

Philippe Gaucher

*Institut de Recherche Mathématique Avancée*
*ULP et CNRS*
*7 rue René Descartes*
*67084 Strasbourg*
*France*
*gaucher@math.u-strasbg.fr*

**Abstract**

This presentation is the sequel of a paper published in the GETCO'00 proceedings where a research program to construct an appropriate algebraic setting for the study of deformations of higher dimensional automata was sketched. This paper focuses precisely on detailing some of its aspects. The main idea is that the category of homotopy types can be embedded in a new category of dihomotopy types, the embedding being realized by the globe functor. In this latter category, isomorphism classes of objects are exactly higher dimensional automata up to deformations leaving invariant their computer scientific properties as presence or not of deadlocks (or everything similar or related). Some hints to study the algebraic structure of dihomotopy types are given, in particular a rule to decide whether a statement/notion concerning dihomotopy types is or not the lifting of another statement/notion concerning homotopy types. This rule does not enable to guess what is the lifting of a given notion/statement, it only enables to make the verification, once the lifting has been found.

## Contents

## 1 Introduction

This paper is an expository paper which is the sequel of [Gau01b]. We will come back only very succinctly on the explanations given in this latter. A technical appendix explains some of the notions used in the core of the paper and fixes some notations. A reader who would need more information about algebraic topology or homological algebra could refer to [May67,Wei94,Rot88,Hat]. A reader who would need more information about the geometric point of view of concurrency theory could refer to [Gou95,FGR99b].

The purpose is indeed to explain with much more details [1] the speculations of the last paragraph of [Gau01b]. More precisely, we are going to describe a research program whose goal is to construct an appropriate algebraic theory of the deformations of higher dimensional automata (HDA) leaving invariant their computer-scientific properties. Most of the paper is as informal as the preceding one. The term *dihomotopy* (contraction of *directed homotopy*) will be used as an analogue in our context of the usual notion of *homotopy*.

There are two known ways of modeling higher dimensional automata for us to be able to study their deformations. 1) The $\omega$-categorical approach, where strict globular $\omega$-categories are supposed to encode the algebraic structure of the possible compositions of execution paths and homotopies between them, initiated by [Pra91] and continued in [Gau00] where connections with homological ideas of [Gou95] were made. 2) The topological approach which consists, loosely speaking, to locally endow a topological space with a closed partial ordering which is supposed to represent the time : this is the notion of local po-space developed for example in [FGR99b]. The description of these models is sketched in Section 2.

Section 3 is an exposition of the homological constructions which will play a role in the future algebraic investigations. Once again, the $\omega$-categorical case and the topological case are described in parallel.

[1] Even if the limited required number of pages for this paper too entails to make some shortcuts.

In Section 4, the notion of deformation of higher dimensional automata is succinctly recalled. For further details, see [Gau01b].

Afterwards Section 5 exposes the main ideas about the relation between homotopy types and dihomotopy types. And some hints to explore the algebraic structure of *dihomotopy types* are explained (this question is widely open).

Everything is presented in parallel because, as in usual algebraic topology, the $\omega$-categorical approach and the topological approach present a lot of similarities. In a first version, the paper was organized with respect to the main result of [KV91], that is the category equivalence between CW-complexes up to weak homotopy equivalence and weak $\omega$-groupoids up to weak homotopy equivalence. By [Sim98], it seems that this latter result cannot be true, at least with the functors used in Kapranov-Voevodsky's paper. Therefore in this new version, the presentation of some ideas is slightly changed. I thank Sjoerd Crans for letting me know this fact.

## 2 The formalization

### *2.1 The $\omega$-categorical approach*

Several authors have noticed that a higher dimensional automaton can be encoded in a structure of precubical set (Definition C.1). This idea is implemented in [Cri96] where a CaML program translating programs written in Concurrent Pascal into (huge !) text files is presented.

But such object does not contain any information about the way of composing $n$-transitions, hence the idea of adding composition laws. In an $\omega$-category (Definition C.2), the 1-morphisms represent the execution paths, the 2-morphisms the concurrent execution of the 1-source and the 1-target of the 2-morphisms we are considering, etc... The link between this way of modeling higher dimensional automata and the formalization by precubical sets is the realization functor $K \mapsto \Pi(K)$ described in Appendix D.

There exist two equivalent notions of (strict) $\omega$-categories, the globular one and the cubical one [AABS00] : the globular version will be used, although all notions could be adapted to the cubical version. In fact, for some technical reasons (Proposition 3.1), even a more restrictive notion will be necessary :

**Definition 2.1** *[Gau02] An $\omega$-category $\mathcal{C}$ is non-contracting if $s_1 x$ and $t_1 x$ are* 1*-dimensional as soon as $x$ is not* 0*-dimensional. Let $f$ be an $\omega$-functor from $\mathcal{C}$ to $\mathcal{D}$. The morphism $f$ is non-contracting if for any* 1*-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a* 1*-dimensional morphism of $\mathcal{D}$. The category of non-contracting $\omega$-categories with the non-contracting $\omega$-functors is denoted by $\omega Cat_1$.*

The following proposition ensures that this technical restriction is not too small and that it does contain all precubical sets.

**Proposition 2.2** *[Gau02] For any precubical set $K$, $\Pi(K)$ is a non-contracting $\omega$-category. The functor $\Pi$ from the category of cubical sets $Sets^{\square^{preop}}$ to that of $\omega$-categories $\omega Cat$ yields a functor from $Sets^{\square^{preop}}$ to the category of non-contracting $\omega$-categories $\omega Cat_1$.*

### *2.2 The topological approach*

Another way of modeling higher dimensional automata is to use the notion of local po-space. A local po-space is a gluing of the following local situation : 1) a topological space, 2) a partial ordering, 3) as compatibility axiom between both structures, the graph of the partial ordering is supposed to be closed [FGR99b] (cf. Appendix A).

However the category of local po-spaces is too wide, and as in usual algebraic topology, a more restrictive notion is necessary to avoid too pathological situations (for instance think of the Cantor set). A new notion which would play in this context the role played by the CW-complexes in usual algebraic topology is necessary. This is precisely the subject of [GG01] (joined work with Eric Goubault).

Let $n \geq 1$. Let $D^n$ be the closed $n$-dimensional disk defined by the set of points $(x_1, \dots, x_n)$ of $\mathbb{R}^n$ such that $x_1^2 + \cdots + x_n^2 \leq 1$ endowed with the topology induced by that of $\mathbb{R}^n$. Let $S^{n-1} = \partial D^n$ be the boundary of $D^n$ for $n \geq 1$, that is the set of $(x_1, \dots, x_n) \in D^n$ such that $x_1^2 + \cdots + x_n^2 = 1$. Notice that $S^0$ is the discrete two-point topological space $\{-1, +1\}$. Let $I = [0, 1]$. Let $D^0$ be the one-point topological space. And let $e^n := D^n - S^n$. Loosely speaking, globular CW-complexes are gluing of po-spaces $\boldsymbol{D}^{n+1} := Glob(D^n)$ along $\boldsymbol{S}^n := Glob(\partial D^n) = Glob(S^{n-1})$ where $Glob$ is the globe functor (cf. Appendix A).

Notice that there is a canonical inclusion of po-spaces $\boldsymbol{S}^n \subset \boldsymbol{D}^{n+1}$ for $n \geq 1$. By convention, let $\boldsymbol{S}^0 := \{-1, 1\}$ with the trivial ordering (0 and 1 are not comparable). There is a canonical inclusion $\boldsymbol{S}^0 \subset \boldsymbol{D}^1$ which is a morphism of po-spaces.

**Proposition and Definition 2.3** *[GG01] For any $n \geq 1$, $\boldsymbol{D}^n - \boldsymbol{S}^{n-1}$ with the induced partial ordering is a po-space. It is called the $n$-dimensional globular cell. More generally, every local po-space isomorphic to $\boldsymbol{D}^n - \boldsymbol{S}^{n-1}$ for some $n$ will be called a $n$-dimensional globular cell.*

Now we are going to describe the process of attaching globular cells.

(i) Start with a discrete set of points $X^0$.

(ii) Inductively, form the $n$-skeleton $X^n$ from $X^{n-1}$ by attaching globular $n$-cells $\boldsymbol{e}^n_\alpha$ via maps $\phi_\alpha : \boldsymbol{S}^{n-1} \longrightarrow X^{n-1}$ with $\phi_\alpha(\underline{\iota}), \phi_\alpha(\underline{\sigma}) \in X^0$ such that [2] : for every non-decreasing map $\phi$ from $\boldsymbol{I}$ to $\boldsymbol{S}^{n-1}$ such that $\phi(0) = \underline{\iota}$ and $\phi(1) = \underline{\sigma}$, there exists $0 = t_0 < \cdots < t_k = 1$ such that $\phi_\alpha \circ \phi(t_i) \in X^0$ for

[2] This condition will appear to be necessary in the sequel.

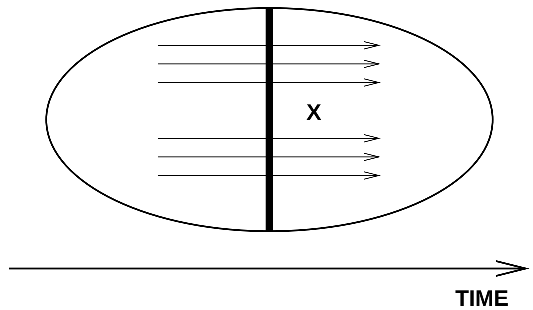



Fig. 1. Symbolic representation of $Glob(X)$ for some topological space $X$

any $0 \leq i \leq k$ which must satisfy

(a) for any $0 \leq i \leq k-1$, there exists a globular cell of dimension $d_i$ with $d_i \leq n-1$ $\psi_i : \boldsymbol{D}^{d_i} \to X^{n-1}$ such that for any $t \in [t_i, t_{i+1}]$, $\phi_\alpha \circ \phi(t) \in \psi_i(\boldsymbol{D}^{d_i})$ ;

(b) for $0 \leq i \leq k-1$, the restriction of $\phi_\alpha \circ \phi$ to $[t_i, t_{i+1}]$ is non-decreasing ;

(c) the map $\phi_\alpha \circ \phi$ is non-constant ;

Then $X^n$ is the quotient space of the disjoint union $X^{n-1} \bigsqcup_\alpha \boldsymbol{D}^n_\alpha$ of $X^{n-1}$ with a collection of $\boldsymbol{D}^n_\alpha$ under the identification $x \sim \phi_\alpha(x)$ for $x \in \boldsymbol{S}^{n-1}_\alpha \subset \partial \boldsymbol{D}^n_\alpha$. Thus as set, $X^n = X^{n-1} \bigsqcup_\alpha \boldsymbol{e}^n_\alpha$ where each $\boldsymbol{e}^n_\alpha$ is a $n$-dimensional globular cell.

(iii) One can either stop this inductive process at a finite stage, setting $X = X^n$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case, $X$ is given the weak topology : A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in $X^n$ for some $n$ (this topology is nothing else but the direct limit of the topology of the $X^n$, $n \in \mathbb{N}$). Such a $X$ is called a globular CW-complex and $X_0$ and the collection of $\boldsymbol{e}^n_\alpha$ and its attaching maps $\phi_\alpha : \boldsymbol{S}^{n-1} \longrightarrow X^{n-1}$ is called the cellular decomposition of $X$.

As trivial examples of globular CW-complexes, there are $\boldsymbol{D}^{n+1}$ and $\boldsymbol{S}^n$ themselves where the 0-skeleton is, by convention, $\{\underline{\iota}, \underline{\sigma}\}$.

We will consider without further mentioning that the segment $\boldsymbol{I}$ is a globular CW-complex, with $\{0, 1\}$ as its 0-skeleton.

**Proposition and Definition 2.4** *[GG01] Let $X$ be a globular CW-complex with characteristic maps $(\phi_\alpha)$. Let $\gamma$ be a continuous map from $\boldsymbol{I}$ to $X$. Then $\gamma([0,1]) \cap X^0$ is finite. Suppose that there exists $0 \leq t_0 < \cdots < t_n \leq 1$ with $n \geq 1$ such that $t_0 = 0$, $t_n = 1$, such that for any $0 \leq i \leq n$, $\gamma(t_i) \in X^0$, and at last such that for any $0 \leq i \leq n-1$, there exists an $\alpha_i$ (necessarily unique) such that for $t \in [t_i, t_{i+1}]$, $\gamma(t) \in \phi_{\alpha_i}(\boldsymbol{D}^{n_\alpha})$. Then such a $\gamma$ is called an execution path if the restriction $\gamma \restriction_{[t_i, t_{i+1}]}$ is non-decreasing.*

By constant execution paths, one means an execution paths $\gamma$ such that $\gamma([0,1]) = \{\gamma(0)\}$. The points (i.e. elements of the 0-skeleton) of a given globular CW-complexes $X$ are also called *states*. Some of them are fairly special:

**Definition 2.5** *Let $X$ be a globular CW-complex. A point $\alpha$ of $X^0$ is initial (resp. final) if for any execution path $\phi$ such that $\phi(1) = \alpha$ (resp. $\phi(0) = \alpha$), then $\phi$ is the constant path $\alpha$.*

Let us now describe the category of *globular CW-complexes.*

**Definition 2.6** *[GG01] The category* **glCW** *of globular CW-complexes is the category having as objects the globular CW-complexes and as morphisms the continuous maps $f : X \longrightarrow Y$ satisfying the two following properties :*

- $f(X^0) \subset Y^0$
- *for every non-constant execution path $\phi$ of $X$, $f \circ \phi$ must not only be an execution path ($f$ must preserve partial order), but also $f \circ \phi$ must be non-constant as well : we say that $f$ must be non-contracting.*

The condition of non-contractibility is very analogous to the notion of non-contracting $\omega$-functors appearing in [Gau00], and is necessary for similar reasons. In particular, if the constant paths are not removed from $\mathbb{P}^{\pm}X$ (see Section 3.2 for the definition), then this latter spaces are homotopy equivalent to the discrete set $X^0$ (the 0-skeleton of $X$ !). And the removing of the constant paths from $\mathbb{P}^{\pm}X$ entails to remove also the constant paths from $\mathbb{P}X$ in order to keep the existence of both natural transformations $\mathbb{P} \to \mathbb{P}^{\pm}$. Then the mappings $\mathbb{P}$ and $\mathbb{P}^{\pm}$ can be made functorial only if we work with non-contracting maps as above [GG01].

One can also notice that by construction, the attaching maps are morphisms of globular CW-complexes. Of course one has

**Theorem 2.7** *[GG01] Every globular CW-complex is a local po-space and this mapping induces a functor from the category of globular CW-complexes to the category of local po-spaces.*

## 3 The homological constructions

The three principal constructions are all based upon the idea of capturing the algebraic structure of the set of *achronal cuts* (cf. [Gau01b] for some explanations of this idea) included in the higher dimensional automaton $M$ we are considering in three simplicial sets which seem to be the basement of an algebraic theory which remains to build. For that, one has to construct in both cases (the categorical and the topological approaches), three *spaces* :

(i) the *space of non-constant execution paths* (this idea will become more precise below) : let us call it the *path space* $\mathbb{P}M$

(ii) the *space of equivalence classes of non-constant execution paths beginning in the same way* : let us call it the *negative semi-path space* $\mathbb{P}^-M$

(iii) the *space of equivalence classes of non-constant execution paths ending in the same way* : let us call it the *positive semi-path space* $\mathbb{P}^+M$.

$$\begin{tikzcd} & \mathbb{P}M \arrow[dl, "h^-"'] \arrow[dr, "h^+"] & \\ \mathbb{P}^- M & & \mathbb{P}^+ M \end{tikzcd}$$

Fig. 2. The fundamental diagram

and one will consider the simplicial nerve of each one.

[Gau01b] Figure 11 will become Figure 2 in both topological and $\omega$-categorical situations. The construction of $h^-$ and $h^+$ is straightforward in both situations.

### 3.1 The $\omega$-categorical approach

**Proposition 3.1** *[Gau02] Let $\mathcal{C}$ be an $\omega$-category. Consider the set $\mathbb{P}\mathcal{C} = \bigcup_{n\geq 1} \mathcal{C}_n$. Then the operators $s_n$, $t_n$ and $*_n$ for $n \geq 1$ are internal to $\mathbb{P}\mathcal{C}$ if and only if $\mathcal{C}$ is non-contracting. In that case, $\mathbb{P}\mathcal{C}$ can be endowed with a structure of $\omega$-category whose $n$-source (resp. $n$-target, $n$-dimensional composition law) is the $(n+1)$-source (resp. $(n+1)$-target, $(n+1)$-dimensional composition law) of $\mathcal{C}$. The $\omega$-category $\mathbb{P}\mathcal{C}$ is called the path $\omega$-category of $\mathcal{C}$, and the mapping $\mathcal{C}$ induces a well-defined functor from $\omega Cat_1$ to $\omega Cat$.*

**Definition 3.2** *Let $\mathcal{C}$ be a non-contracting $\omega$-category. Denote by $\mathcal{R}^-$ (resp. $\mathcal{R}^+$) the reflexive symmetric and transitive closure of $\{(x, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$ (resp. $\{(y, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$) in $\mathbb{P}\mathcal{C} \times \mathbb{P}\mathcal{C}$.*

**Proposition 3.3** *[Gau01c] Let $\alpha \in \{-,+\}$ and let $\mathcal{C}$ be a non-contracting $\omega$-category. Then the universal problem*

> *"There exists a pair $(\mathcal{D}, \mu)$ such that $\mathcal{D}$ is an $\omega$-category and $\mu$ an $\omega$-functor from $\mathbb{P}\mathcal{C}$ to $\mathcal{D}$ such that for any $x, y \in \mathbb{P}\mathcal{C}$, $x\mathcal{R}^\alpha y$ implies $\mu(x) = \mu(y)$."*

*has a solution $(\mathbb{P}^\alpha\mathcal{C}, (-)^\alpha)$. Moreover $\mathbb{P}^\alpha\mathcal{C}$ is generated by the elements of the form $(x)^\alpha$ for $x$ running over $\mathbb{P}\mathcal{C}$. The mappings $\mathbb{P}^-$ and $\mathbb{P}^+$ induce two well-defined functors from $\omega Cat_1$ to $\omega Cat$.*

**Definition 3.4** *The $\omega$-category $\mathbb{P}^-\mathcal{C}$ (resp. $\mathbb{P}^+\mathcal{C}$) is called the negative (resp. positive) semi-path $\omega$-category of $\mathcal{C}$.*

In the sequel, $\mathbb{P}\mathcal{C}$ will be supposed to be a strict globular $\omega$-groupoid in the sense of Brown-Higgins, which implies that $\mathbb{P}^-\mathcal{C}$ and $\mathbb{P}^+\mathcal{C}$ satisfy also the same property : this means concretely that if there exists an homotopy from a given execution path $\gamma$ to another one $\gamma'$, then there exists also an homotopy in the opposite direction [Gau01c].

**Definition 3.5** *[Gau02] The globular simplicial nerve $\mathcal{N}^{gl}$ is the functor from $\omega Cat_1$ to $Sets_+^{\Delta^{op}}$ defined by*

$$\mathcal{N}^{gl}_n(\mathcal{C}) := \omega Cat(\Delta^n, \mathbb{P}\mathcal{C})$$

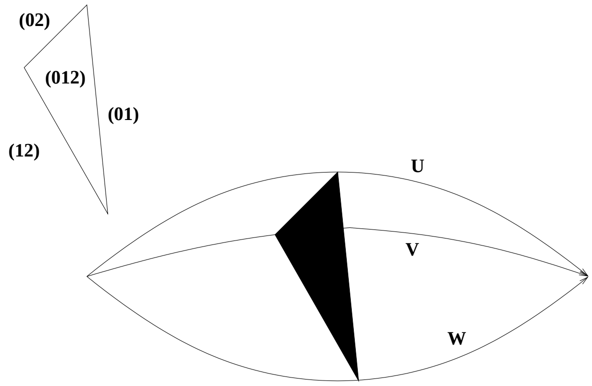



Fig. 3. Globular 2-simplex


Fig. 4. Negative and positive semi-globular 2-simplexes

*for* $n \geq 0$ *and with* $\mathcal{N}^{gl}_{-1}(\mathcal{C}) := \mathcal{C}_0 \times \mathcal{C}_0$, *and endowed with the augmentation map* $\partial_{-1}$ *from* $\mathcal{N}^{gl}_0(\mathcal{C}) = \mathcal{C}_1$ *to* $\mathcal{N}^{gl}_{-1}(\mathcal{C}) = \mathcal{C}_0 \times \mathcal{C}_0$ *defined by* $\partial_{-1}x := (s_0x, t_0x)$.

Geometrically, a simplex of this simplicial nerve looks as in Figure 3.

**Definition 3.6** *Let* $\mathcal{C}$ *be a non-contracting* $\omega$*-category. Then set*

$$\mathcal{N}^{gl^-}_n(\mathcal{C}) := \omega Cat(\Delta^n, \mathbb{P}^-\mathcal{C})$$

*and* $\mathcal{N}^{gl^-}_{-1}(\mathcal{C}) := \mathcal{C}_0$ *with* $\partial_{-1}(x) := s_0x$. *Then* $\mathcal{N}^{gl^-}$ *induces a functor from* $\omega Cat_1$ *to* $Sets_+^{\Delta^{op}}$ *which is called the negative semi-globular nerve or ( branching semi-globular homology) of* $\mathcal{C}$ .

The *positive semi-globular nerve* is defined in a similar way by replacing $-$ by $+$ everywhere in the above definition and by setting $\partial_{-1}(x) = t_0x$. Intuitively, the simplexes in the semi-globular nerves look as in Figure 4 : they correspond to the left or right half part of Figure 3.

### *3.2 The topological approach*

Let $X$ be a globular CW-complex. Let $\alpha, \beta \in X^0$. Denote by $(X, \alpha, \beta)^\perp$ the topological space of non-decreasing non-constant continuous maps $\gamma$ from $[0,1]$ endowed with the usual order to $X$ such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$ and endowed with the compact-open topology. Then

**Definition 3.7** *Let* $X$ *be a globular CW-complex. Then the path space of* $X$

*is the disjoint union*

$$\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} (X,\alpha,\beta)^{\perp}$$

*endowed with the disjoint union topology.*

Now denote by $(X,\alpha)^{\perp^-}$ (resp. $(X,\beta)^{\perp^+}$) the topological space of non-decreasing non-constant continuous maps from $[0,1]$ with the usual order to $X$ such that $\gamma(0) = \alpha$ (resp. $\gamma(1) = \beta$), endowed with the compact-open topology. Then

**Definition 3.8** *Let $X$ be a globular CW-complex. Then the negative semi-path space $\mathbb{P}^-X$ (resp. positive semi-path space $\mathbb{P}^+X$) of $X$ are defined by*

$$\mathbb{P}^-X = \bigsqcup_{\alpha\in X^0} (X,\alpha)^{\perp^-}$$
$$\mathbb{P}^+X = \bigsqcup_{\beta\in X^0} (X,\beta)^{\perp^+}$$

*endowed with the disjoint union topology.*

The reader can notice that in the topological context, we do not need anymore to consider something like the equivalence relations $\mathcal{R}^-$ and $\mathcal{R}^+$. The reason is that, ideologically ("moralement" in french !), a 1-morphism is of length 1. On the contrary, a non-constant execution path is homotopic to any shorter execution path [3]

**Definition 3.9** *[Gau02] The globular simplicial nerve $\mathcal{N}^{gl}$ is the functor from* **glCW** *to $Sets_+^{\Delta^{op}}$ defined by*

$$\mathcal{N}^{gl}_n(X) := S_n(\mathbb{P}X)$$

*for $n \geq 0$ where $S_*$ is the singular simplicial nerve (cf. Appendix B) with $\mathcal{N}^{gl}_{-1}(X) := X^0 \times X^0$, and endowed with the augmentation map $\partial_{-1}$ from $\mathcal{N}^{gl}_0(X) = \mathbb{P}X$ to $\mathcal{N}^{gl}_{-1}(X) = X^0 \times X^0$ defined by $\partial_{-1}\gamma := (\gamma(0), \gamma(1))$.*

**Definition 3.10** *Let $X$ be a globular CW-complex. Then set*

$$\mathcal{N}^{gl^-}_n(X) := S_n(\mathbb{P}^-X)$$

*for $n \geq 0$ and $\mathcal{N}^{gl^-}_{-1}(X) := X^0$ with $\partial_{-1}(\gamma) := \gamma(0)$. Then $\mathcal{N}^{gl^-}$ induces a functor from* **glCW** *to $Sets_+^{\Delta^{op}}$ which is called the branching semi-globular nerve of $X$ .*

The *merging semi-globular nerve* is defined in a similar way by replacing $-$ by $+$ everywhere in the above definition and by setting $\partial_{-1}(\gamma) := \gamma(1)$.

[3] in a "natural way" by considering $H(\gamma(t), u) = \gamma(tu)$. It is the reason why $\mathbb{P}^-X$ and $\mathbb{P}^+X$ are homotopy equivalent to $X^0$ if one does not remove the constant paths from their definition.

$$\begin{tikzcd} \alpha \arrow[r, "u"] & \beta \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] & \gamma \end{tikzcd}$$

(a) $\mathcal{C}$

$$\begin{tikzcd} \alpha_1 \arrow[r, "u_1"] & \alpha_2 \arrow[r, "u_2"] & \beta \arrow[r, bend left, "v"] \arrow[r, bend right, "w"'] & \gamma \end{tikzcd}$$

(b) Subdivision of $u$ in $\mathcal{C}$

Fig. 5. Example of $T$-deformation

# 4 Deforming higher dimensional automata

As already seen in [Gau01b] in the $\omega$-categorical context, there are two types of deformation leaving invariant the computer scientific properties of higher dimensional automata : the *temporal deformations* (or *T-deformations*) and the *spatial deformations* (*S-deformations*). The first type (temporal) is closely related to the notion of homeomorphism because a non-trivial execution path cannot be contracted in the same dihomotopy class [4], and the second one (spatial) to the classical notion of homotopy equivalence.

The $\omega$-categorical case will be only briefly recalled. A temporal deformation corresponds informally to the reflexive symmetric and transitive closure of subdividing in an $\omega$-category a 1-morphism in two parts as in Figure 5. A spatial deformation consists of deforming in the considered $\omega$-category $p$-morphisms with $p \geq 2$, which is equivalent to deforming faces in one the three nerves in the usual sense of homotopy equivalence.

The topological approach is completely similar. A temporal deformation of a globular CW-complex $X$ consists of dividing in two parts a globular 1-dimensional cell of the cellular decomposition of $X$, as in Figure 5. A spatial deformation consists of crushing globular cells of higher dimension.

Now what can we do with the previous homological constructions ? First of all consider the corresponding simplicial homology theories of all these augmented simplicial sets, with the following convention on indices : for $n \geq -1$ and $u \in \{gl, gl^-, gl^+\}$, set $H^u_{n+1}(M) = H_n\left(\mathcal{N}^u(M)\right)$ for $M$ either an $\omega$-category or a globular CW-complex. We obtain this way three homology theories called as the corresponding nerve. One knows that the globular homology sees the globes included in the HDA [Gau00,Gau02] and that the branching (resp. merging) semi-globular homology sees the branching areas (resp. merging areas) in the HDA [Gau00,Gau01a,Gau01c]. Since the three nerves are Kan [5], one can also consider the homotopy groups of these nerves, with the same convention for indices : for $n \geq 1$ and $u \in \{gl, gl^-, gl^+\}$, set $\pi^u_{n+1}(X) = \pi_n\left(\mathcal{N}^u(X), \phi\right)$ for $X$ either an $\omega$-category or a globular CW-

[4] In fact, the T-dihomotopy equivalences in [GG01] are precisely the morphisms of globular CW-complexes inducing an homeomorphim between both underlying topological spaces.

[5] The $\omega$-categorical versions are Kan as soon as $\mathbb{P}\mathcal{C}$ is an $\omega$-groupoid [Gau01c] and the singular simplicial nerve is known to be Kan [May67].

complex. In this latter case, the base-point $\phi$ is in fact a 0-morphism of $\mathbb{P}\mathcal{C}$, that is a 1-morphism of $\mathcal{C}$ if $u = gl$, and an equivalence class of 1-morphism of $\mathcal{C}$ with respect to $\mathcal{R}^-$ (resp. $\mathcal{R}^+$) if $u = gl^-$ (resp. $u = gl^+$). Intuitively, elements of $\pi^{gl}_{n+1}$ are $(n+1)$-dimensional cylinders with achronal basis.

The four first lines of Table 6 are explained in [Gau01b]. The branching and merging (semi-cubical) nerves $\mathcal{N}^{\pm}$ defined in [Gau00,Gau01a] are almost never Kan : in fact as soon as there exists in the $\omega$-category $\mathcal{C}$ we are considering two 1-morphisms $x$ and $y$ such that $x *_0 y$ exists (see Proposition 5.7), both semi-cubical nerves are not Kan.

If the branching and merging (semi-cubical) nerves are replaced by the branching and merging semi-globular nerve, then the "almost" (in fact a "no") becomes a "yes" here because we are not disturbed anymore by the non-simplicial part of the elements of the branching and merging nerves (which is removed by construction).

The lines concerning the (globular, negative and positive semi-globular) homotopy groups need to be explained. The S-invariance of a given nerve implies of course the S-invariance of the corresponding homotopy groups. As for the T-invariance, it is due to the fact that in these homotopy groups, the "base-point" is an execution path (or eventually an equivalence class of). So these homotopy groups contain information only related to achronal cuts crossing the "base-point". Dividing this base-point or any other 1-morphism or 1-dimensional globular cell changes nothing.

The last lines are concerned with the bisimplicial set what we call *biglobular nerve* (for the contraction of bisimplicial globular nerve) described in [Gau01b,Gau02] (cf. Appendix F) and constructed by considering the structure of augmented simplicial object of the category of small categories of the globular nerve. The biglobular nerve inherits the S-invariance of the globular nerve. And its T-invariance is due to the T-invariance of the simplicial nerve functor of small categories. The answer "yes ?" means that it is expected to find "yes" in some sense... It is worth noticing that in a true higher dimensional automaton, 1-morphisms are never invertible because the time is not reversible. So one cannot expect to find a Kan bisimplicial set in the usual sense of the notion.

The last column is not directly concerned with the different types of deformations of HDA, but rather by the question of knowing if the functors contain information from $t = -\infty$ to $t = \infty$. The answer is yes everywhere except for the three homotopy groups functors : the latter contain indeed information only related to achronal cuts crossing the "base-point". One can by the way notice that, in the $\omega$-categorical case :

**Proposition 4.1** *Let $\phi$ and $\psi$ be two 1-morphisms of a non-contracting $\omega$-category $\mathcal{C}$. Suppose that $\phi *_0 \psi$ exists. Then*

(i) *If $\phi *_0 \psi$ is 1-dimensional, then the mapping $(x, y) \mapsto x *_0 y$ partially defined on $\mathcal{C}_n \times \mathcal{C}_n$ induces a morphism of groups $\pi^{gl}_{n+1}(\mathcal{C}, \phi) \times \pi^{gl}_{n+1}(\mathcal{C}, \psi) \rightarrow$*

| Functors | S-invariant | T-invariant | Kan | $-\infty \to +\infty$ |
|---|---|---|---|---|
| $\mathcal{N}^{gl}$ | yes | no | yes | yes |
| $\mathcal{N}^{\pm}$ | almost | no | almost never | yes |
| $H^{gl}$ | yes | no | no meaning | yes |
| $H^{\pm}$ | yes | yes | no meaning | yes |
| $\mathcal{N}^{gl^{\pm}}$ | yes | no | yes | yes |
| $H^{gl^{\pm}}$ | yes | yes | no meaning | yes |
| $\pi^{gl}$ | yes | yes | no meaning | no |
| $\pi^{gl^{\pm}}$ | yes | yes | no meaning | no |
| $\mathcal{N}^{bigl}$ | yes | yes | yes ? | yes |
| $H^{bigl}$ | yes | yes | no meaning | yes |
| $(\pi^{bigl})$ | (yes) | (yes) | (no meaning) | (yes) |

Fig. 6. Behavior w.r.t the two types of deformations

$\pi^{gl}_{n+1}(\mathcal{C}, \phi *_0 \psi)$.

(ii) *If $\phi *_0 \psi$ is 0-dimensional, then the mapping $(x, y) \mapsto x *_0 y$ partially defined on $\mathcal{C}_n \times \mathcal{C}_n$ induces the constant map $\phi *_0 \psi$.*

**Proof.** It is due to the fact that $\pi^{gl}_{n+1}(\mathcal{C}, \phi)$ is the quotient of $\{x \in \mathcal{C}_{n+1}, s_1 x = s_2 x = \cdots = s_n x = t_1 x = t_2 x = \cdots = t_n x = \phi\}$ by the equivalence relation generated by the identifications $X = Y$ where $X$ and $Y$ are $(n+1)$-morphisms and such that there exists a $(n+2)$-morphism $Z$ with $s_{n+1} Z = X$ and $t_{n+1} Z = Y$. □

The above proposition is a hint to correct the drawbacks of the globular and semi-globular homotopy groups.

The last line $\pi^{bigl}$ is explained with Philosophy 5.9.

# 5 The category of dihomotopy types

## *5.1 Towards a construction*

Here both approaches slightly diverge because of a lack of knowledge about the $\omega$-categorical ways of constructing homotopy types. However one can certainly define in both contexts a notion of *weak dihomotopy equivalence* : see [GG01] for the topological context. Then let

- $\omega Grp$ be the category of strict globular $\omega$-groupoids with the $\omega$-functors as morphisms, and $\mathbf{Ho}(\omega Grp)$ its localization by the weak homotopy equivalences

- $\omega Cat_1^{Kan}$ be the category of non-contracting $\omega$-categories $\mathcal{C}$ such that $\mathbb{P}\mathcal{C}$ is an $\omega$-groupoid with the non-contracting $\omega$-functors as morphisms, and $\mathbf{Ho}(\omega Cat_1^{Kan})$ its localization by the weak dihomotopy equivalences
- **CW** the category of CW-complexes with the continuous maps as morphisms, and $\mathbf{Ho}(\mathbf{CW})$ its localization by the weak homotopy equivalences
- **glCW** the category of globular CW-complexes with the morphisms of globular CW-complexes as morphisms, and $\mathbf{Ho}(\mathbf{glCW})$ its localization by the weak dihomotopy equivalences.

**Philosophy 5.1** *Both localizations* $\mathbf{Ho}(\omega Cat_1^{Kan})$ *and* $\mathbf{Ho}(\mathbf{glCW})$ *contain the precubical sets modulo spatial and temporal deformations. However, due to the fact that strict globular* $\omega$*-groupoids do not represent all homotopy types [BH81a], but only those having a trivial Whitehead product,* $\mathbf{Ho}(\omega Cat_1^{Kan})$ *could be not big enough to construct an appropriate algebraic setting.*

After [Sim98], it is clear that the $\omega$-categorical realization functor described in Section D loses some homotopical information and that keeping the complete information requires to work with $\omega$-categories where the associativity of $*_n$ is weakened for any $n \geq 1$. However, this lost homotopical information is only related to the geometric situation in achronal cuts. In particular, this realization functor does not contract 1-morphisms. Therefore the $\mathbf{Ho}(\omega Cat_1^{Kan})$ framework could be sufficient to study questions concerning deadlocks or other similar 1-dimensional phenomena.

**Definition 5.2** *The category* $\mathbf{Ho}(\mathbf{glCW})$ *is called the category of dihomotopy types.*

To describe the relation between the usual situation and the directed situation, we need two last propositions and definitions :

**Proposition 5.3** *[GG01] Let* $X$ *be a CW-complex. Let* $Glob(X)^0 = \{\underline{\iota}, \underline{\sigma}\}$ *where* $\underline{\iota}$ *(resp.* $\underline{\sigma}$*) is the equivalence class of* $(x, 0)$ *(resp.* $(x, 1)$*). Then the cellular decomposition of* $X$ *yields a cellular decomposition of* $Glob(X)$ *and this way,* $Glob(-)$ *induces a functor from* **CW** *to* **glCW***.*

**Proposition 5.4** *Let* $G$ *be an object of* $\omega Grp$*. Then there exists a unique object* $Glob(G)$ *of* $\omega Cat_1^{Kan}$ *such that* $\mathbb{P}Glob(G) = G$*,* $Glob(G)_0 = \{\alpha, \beta\}$ *is a two-element set, and such that* $s_0(Glob(G)\backslash\{\beta\}) = \{\alpha\}$ *and* $t_0(Glob(G)\backslash \{\alpha\}) = \{\beta\}$*. Moreover the mapping* $Glob$ *induces a functor from* $\omega Grp$ *to* $\omega Cat_1^{Kan}$*.*

Both *Glob* functors (called *globe functors*) yield two functors

$$\mathbf{Ho}(\mathbf{CW}) \to \mathbf{Ho}(\mathbf{glCW})$$

and

$$\mathbf{Ho}(\omega Grp) \to \mathbf{Ho}(\omega Cat_1^{Kan})$$

In the topological context, one has :

**Proposition 5.5** *[GG01] Let $X$ and $Y$ be two CW-complexes. Then $X$ and $Y$ are homotopy equivalent if and only if $Glob(X)$ and $Glob(Y)$ are dihomotopy equivalent. Therefore the functor* $\mathbf{Ho}(\mathbf{CW}) \to \mathbf{Ho}(\mathbf{glCW})$ *is an embedding.*

**Question 5.6** *Is it possible to find an $\omega$-categorical construction of* $\mathbf{Ho}(\mathbf{glCW}$ *) ?*

### *5.2 Investigating the algebraic structure of the category of dihomotopy types*

One can check that in both topological and $\omega$-categorical situations, the following fact holds

**Proposition 5.7** *(partially in [Gau01a]) Let $\alpha \in \{-,+\}$. The morphism $h^\alpha$ induces an isomorphism of simplicial sets (not of augmented simplicial sets for trivial reason !) $\mathcal{N}^{gl}(Glob(M)) \simeq \mathcal{N}^{gl^\alpha}(Glob(M))$. Moreover in the $\omega$-categorical case, $\mathcal{N}^{gl}(Glob(M)) \simeq \mathcal{N}^{\alpha}(Glob(M))$ where $\mathcal{N}^\alpha$ are the branching or the merging nerves (depending on the value of $\alpha$) of an $\omega$-category as defined and studied in [Gau00,Gau01a]. Moreover, this common simplicial set is homotopy equivalent to the simplicial nerve of $M$.*

This important proposition together with Proposition 5.5 suggests us a way of investigating the algebraic structure of the category of dihomotopy types.

**Philosophy 5.8** *Let* $\mathbf{Th}$ *be a theorem (or a notion) in usual algebraic topology, i.e. concerning the category of homotopy types. Let* $\mathbf{Th}^{di}$ *be its lifting (i.e. its analogue) on the category of dihomotopy types. Then the statement* $\mathbf{Th}^{di}$ *must specialize into* $\mathbf{Th}$ *on the image of the globe functor.*

Following Baues's philosophy [Bau99], a first goal would be then to lift from the usual situation to the directed situation the Whitehead theorem and the Hurewicz theorems. Concerning the last one, it would be first necessary to understand what is the analogue [6] of the Hurewicz morphism for the category of dihomotopy types.

**Philosophy 5.9** *The target of the Hurewicz morphism in the directed situation is likely to be the biglobular homology $H^{bigl}$. This new Hurewicz morphism must contain is some way all usual Hurewicz morphisms of all achronal cuts.*

[6] The solution given in [Gau00] is naturally wrong : the morphisms $h^-$ and $h^+$ are not the analogues of the Hurewicz morphism. When [Gau00] was being written, It was not known that the correct definition of the globular homology would come from the simplicial homology of a simplicial nerve. Moreover the role of achronal cuts was also not yet understood. The globular homology was introduced as an answer of Goubault's suggestion of finding the analogue of the Hurewicz morphism in "directed homotopy" theory. Then starting from the principle that the branching and merging homology theories could be an analogue of the singular homology, I wondered whether it was possible to construct a morphism abutting to both corner homologies. The globular homology was then designed to be the source of this morphism.

*At last, the source (let us denote it by $\pi^{bigl}$) of the Hurewicz morphism must be S-invariant, T-invariant and must contain information concerning the geometry of the HDA from $t = -\infty$ to $t = +\infty$.*

Suppose $n \geq 2$. After Proposition 4.1, a possible idea in the $\omega$-categorical case would be then to build a chain complex of abelian groups by considering elements

$$(x_1, \ldots, x_p) \in \pi_{n+1}^{gl}(\mathcal{C}, \phi_1) \times \ldots \times \pi_{n+1}^{gl}(\mathcal{C}, \phi_p)$$

for all $p$ and all $p$-uples $(\phi_1, \ldots, \phi_p)$ such that $\phi_1 *_0 \cdots *_0 \phi_p$ exists and by considering the simplicial differential map induced by $*_0$. Let us call the corresponding homology theory the *toroidal homology* $H_*^{tor}(\mathcal{C})$. Of course this construction makes sense only for $n \geq 2$ because the $\pi_2^{gl}$ are not necessarily abelian. Then the classical Hurewicz morphism induces a natural transformation from $H_*^{tor}$ to the $E^2_{*,n+1}$-term of one of the canonical spectral sequences converging to $H^{bigl}$.

As explained in the introduction, the goal would be to reach an homological understanding of the geometry of flows modulo deformations. In particular, we would like to find exact sequences. It is then reasonable to think that

**Philosophy 5.10** *An exact sequence $F_1(M) \to F_2(M) \to F_3(M)$ telling us something about flows $M$ of execution paths modulo spatial and temporal deformations must use functors $F_1$, $F_2$ and $F_3$ invariant by spatial and temporal deformations.*

The weakness of the internal structure of the globular nerve (it is a disjoint union of simplicial sets), its non-invariance with respect to temporal deformations, and its natural correction by considering the *biglobular nerve* suggests that the *biglobular homology* (the total homology of this bisimplicial set) has more interesting homological properties than the globular homology.

Concerning the biglobular nerve, it is worth noticing that this object contains the whole information about the position of achronal simplexes and about the temporal structure of the underlying higher dimensional automaton. So in some sense, the biglobular nerve contains everything related to the geometry of HDA. Since the biglobular nerve is expected to be S-invariant and T-invariant, then it is natural to ask the following question :

**Question 5.11** *Is it possible to recover all other S-invariant and T-invariant functors from the biglobular nerve ? For example, is it possible to recover the semi-globular homology theories ?*

Another natural question would be to relate a given dihomotopy type to the underlying homotopy type (when the flow of execution paths is removed). If the biglobular nerve really contains the complete information, then it should be possible to recover from it the underlying homotopy type.

As last remark, let us have a look at PV diagrams as in Figure 7. They are always constructed by considering a $n$-cube and by digging cubical holes inside. Such examples produce examples of $\omega$-categories or globular CW-

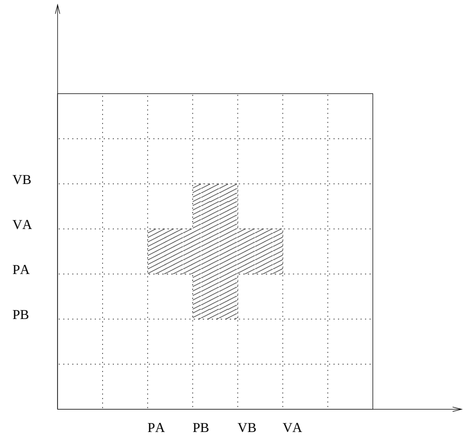



Fig. 7. PV diagram

complexes whose all types of globular homologies do not have any torsion. To classify this kind of examples, the study of *rational dihomotopy types* could be sufficient.

## 6 Conclusion

We have described in this paper a way of constructing the category of dihomotopy types and we have given some hints to investigate its internal algebraic structure. Intuitively, the isomorphism classes of objects in this category represent exactly the higher dimensional automata modulo the deformations which leave invariant their computer-scientific properties. So a good knowledge of the algebraic structure of this category will enable us to classify higher dimensional automata up to dihomotopy and therefore, hopefully, to write new algorithms manipulating directly the equivalence classes of HDA.

## References

[AABS00] Fahd A. A. Al-Agl, R. Brown, and R. Steiner. Multiple categories: the equivalence of a globular and a cubical approach, 2000. arxiv:math.CT/0007009.

[Ait86] I. R. Aitchison. The geometry of oriented cubes. Macquarie Mathematics Reports 86-0082, 1986.

[Bau99] H-J. Baues. *Combinatorial foundation of homology and homotopy.* Springer-Verlag, Berlin, 1999.

[BH81a] R. Brown and P. J. Higgins. The equivalence of $\infty$-groupoids and crossed complexes. *Cahiers Topologie Géom. Différentielle*, 22(4):371–386, 1981.

[BH81b] R. Brown and P. J. Higgins. On the algebra of cubes. *J. Pure Appl. Algebra*, 21(3):233–260, 1981.

[Cri96] R. Cridlig. Implementing a static analyzer of concurrent programs: Problems and perspectives. In *Logical and Operational Methods in the Analysis of Programs and Systems*, pages 244–259, 1996.

[Dus75] J. Duskin. Simplicial methods and the interpretation of "triple" cohomology. *Mem. Amer. Math. Soc.*, 3(issue 2, 163):v+135, 1975.

[FGR99a] L. Fajstrup, E. Goubault, and M. Raussen. Algebraic topology and concurrency. *submitted to Theoretical Computer Science, also technical report, Aalborg University*, 1999.

[FGR99b] L. Fajstrup, E. Goubault, and M. Raußen. Algebraic topology and concurrency. preprint R-99-2008, Aalborg University, June 1999.

[Gau00] P. Gaucher. Homotopy invariants of higher dimensional categories and concurrency in computer science. *Math. Structures Comput. Sci.*, 10:481–524, 2000.

[Gau01a] P. Gaucher. Combinatorics of branchings in higher dimensional automata. *Theory and Applications of Categories*, 8(12):pp.324–376, 2001.

[Gau01b] P. Gaucher. From concurrency to algebraic topology. In *Electronic Notes in Theoretical Computer Science*, volume 39, page 19pp, 2001.

[Gau01c] P. Gaucher. The branching nerve of HDA and the Kan condition, 2001. arxiv:math.AT/0103011.

[Gau02] P. Gaucher. About the globular homology of higher dimensional automata. *to appear in Cahiers de Topologie et Géométrie Différentielle Catégorique*, 2002.

[GG01] P. Gaucher and E. Goubault. Topological deformation of higher dimensional automata. preprint to appear in Homology, Homotopy and Applications, proceedings of ATMCS'01, 2001.

[Gou95] E. Goubault. *The Geometry of Concurrency.* PhD thesis, École Normale Supérieure, 1995.

[Hat] A. Hatcher. Algebraic topology. http://www.math.cornell.edu/˜hatcher/.

[Joh82] P. T. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[KP97] K. H. Kamps and T. Porter. *Abstract homotopy and simple homotopy theory.* World Scientific Publishing Co. Inc., River Edge, NJ, 1997.

[KV91] M. M. Kapranov and V. A. Voevodsky. $\infty$-groupoids and homotopy types. *Cahiers Topologie Géom. Différentielle Catégoriques*, 32(1):29–46, 1991. International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990).

[Mac71] S. MacLane. *Categories for the working mathematician.* Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[May67] J. P. May. *Simplicial objects in algebraic topology.* D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967. Van Nostrand Mathematical Studies, No. 11.

[Nac65] L. Nachbin. *Topology and Order.* Van Nostrand, Princeton, 1965.

[Pra91] V. Pratt. Modeling concurrency with geometry. In ACM Press, editor, *Proc. of the 18th ACM Symposium on Principles of Programming Languages*, 1991.

[Qui73] D. Quillen. Higher algebraic $K$-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Rot88] J. J. Rotman. *An introduction to algebraic topology.* Springer-Verlag, New York, 1988.

[Sim98] C. Simpson. Homotopy types of strict 3-groupoids, 1998. arxiv:math.CT/9810059.

[Ste91] R. Steiner. Tensor products of infinity-categories. University of Glasgow, 1991.

[Str87] R. Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[Wei94] C. A. Weibel. *An introduction to homological algebra.* Cambridge University Press, Cambridge, 1994.

# Technical Appendix

## A Local po-space : definition and examples

If $X$ is a topological space, a binary relation $R$ on $X$ is closed if the graph of $R$ is a closed subset of the cartesian product $X \times X$. If $R$ is a closed partial order $\leq$, then $(X, \leq)$ is called a *po-space* (see for instance [Nac65], [Joh82] and [FGR99a]). Notice that a po-space is necessarily Hausdorff. We say that $(U, \leq)$ is a sub-po-space of $(X, R)$ if and only if it is a po-space such that $U$ is a sub topological space of $X$ and such that $\leq$ is the restriction of $R$ to $U$.

A collection $\mathcal{U}(X)$ of po-spaces $(U, \leq_U)$ covering $X$ is called a *local partial order* if for every $x \in X$, there exists a po-space $(W(x), \leq_{W(x)})$ such that:

- $W(x)$ is an open neighborhood containing $x$,
- the restrictions of $\leq_U$ and $\leq_{W(x)}$ to $W(x) \cap U$ coincide for all $U \in \mathcal{U}(X)$ such that $x \in U$. This can be stated as: $y \leq_U z$ iff $y \leq_{W(x)} z$ for all $U \in \mathcal{U}(X)$ such that $x \in U$ and for all $y, z \in W(x) \cap U$. Sometimes, $W(x)$ will be denoted by $W_X(x)$ to avoid ambiguities. Such a $W_X(x)$ is called a po-neighborhood.

Two local partial orders are equivalent if their union is a local partial order. This defines an equivalence relation on the set of local partial orders of $X$. A topological space together with an equivalence class of local partial order is called a *local po-space*.

A morphism $f$ of local po-spaces (or *dimap*) from $(X,\mathcal{U})$ to $(Y, \mathcal{V})$ is a continuous map from $X$ to $Y$ such that for every $x \in X$,

- there is a po-neighborhood $W(f(x))$ of $f(x)$ in $Y$,
- there exists a po-neighborhood $W(x)$ of $x$ in $X$ with $W(x) \subset f^{-1}(W(f(x))$,
- for $y, z \in W(x)$, $y \leq z$ implies $f(y) \leq f(z)$.

In particular, a dimap $f$ from a po-space $X$ to a po-space $Y$ is a continuous map from $X$ to $Y$ such that for any $y, z \in X$, $y \leq z$ implies $f(y) \leq f(z)$. A morphism $f$ of local po-spaces from $[0, 1]$ endowed with the usual ordering (denoted by $\boldsymbol{I}$) to a local po-space $X$ is called *dipath* or sometime *execution path*.

The category of Hausdorff topological spaces with the continuous maps as morphisms will be denoted by **Haus**. The category of local po-spaces with the dimaps as morphisms will be denoted by **LPoHaus**. The category of general topological spaces without further assumption will be denoted by **Top** and the category of general topological spaces endowed with a partial ordering not necessary closed will be denoted by **PoTop**.

We end this section by an example of po-spaces which matters for this paper. Let us construct the *Globe* $Glob(X)$ associated to a topological space $X$. It is defined as follows. As topological space, $Glob(X)$ is the quotient of the product space $X \times I$ by the relations $(x,0) = (x',0)$ and $(x,1) = (x',1)$ for any $x, x' \in X$. It is equipped with the closed partial order $(x,t) \leq (x',t')$ if and only if $x = x'$ and $t \leq t'$. The equivalence class of $(x,0)$ (resp. $(x,1)$) in $Glob(X)$ is denoted by $\underline{\iota}$ (resp. $\underline{\sigma}$).

# B Simplicial set

For further details, cf. [May67,Wei94].

**Definition B.1** *A simplicial set $A_*$ is a family $(A_n)_{n\geq 0}$ together with face maps $\partial_i : A_n \to A_{n-1}$ and $\epsilon_i : A_n \to A_{n+1}$ for $i = 0,\dots,n$ which satisfy the following identities :*

$$\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i && \textit{if } i<j\\
\epsilon_i\epsilon_j &= \epsilon_{j+1}\epsilon_i && \textit{if } i\leq j\\
\partial_i\epsilon_j &= \begin{cases} \epsilon_{j-1}\partial_i & \textit{if } i<j\\ \textit{Identity} & \textit{if } i=j \textit{ or } i=j+1\\ \epsilon_j\partial_{i-1} & \textit{if } i>j+1\end{cases}
\end{aligned}$$

*A morphism of simplicial sets from $A_*$ to $B_*$ consists of a set map from $A_n$ to $B_n$ for each $n \geq 0$ commuting with all operators defined on both sides. The category of simplicial sets is denoted by $Sets^{\Delta^{op}}$.*

Consider the **topological $n$-simplex** $\Delta^n$ defined by

$$\Delta^n = \{(t_0,\dots,t_n), t_0 \geq 0,\dots,t_n \geq 0 \text{ and } t_0 + \dots t_n = 1\}$$

Here is now the most classical example of simplicial sets :

**Definition B.2** *Let $Y$ be a topological space. The singular simplicial nerve of $Y$ is the simplicial set $S_*(XY)$ defined as follows : $S_n(Y) := \mathbf{Top}(\Delta^n, Y)$ with $\partial_i(f)(t_0,\dots,t_{n-1}) = f(t_0,\dots,t_{i-1},\dots,t_{n-1})$ and $\epsilon_i(f)(t_0,\dots,t_{n+1}) = f(t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_{n+1})$.*

**Definition B.3** *[Dus75] An augmented simplicial set is a simplicial set*

$$((X_n)_{n\geq 0}, (\partial_i : X_{n+1} \longrightarrow X_n)_{0\leq i\leq n+1}, (\epsilon_i : X_n \longrightarrow X_{n+1})_{0\leq i\leq n})$$

*together with an additional set $X_{-1}$ and an additional map $\partial_{-1}$ from $X_0$ to $X_{-1}$ such that $\partial_{-1}\partial_0 = \partial_{-1}\partial_1$. A morphism of augmented simplicial set is a map of $\mathbb{N}$-graded sets which commutes with all face and degeneracy maps. We denote by $Sets_+^{\Delta^{op}}$ the category of augmented simplicial sets.*

If $X_*$ is an augmented simplicial set, one obtains a chain complex of abelian groups ($\mathbb{Z}S$ being the free abelian group generated by the set $S$)

$$\cdots \longrightarrow \mathbb{Z}X_2 \xrightarrow{\partial_0-\partial_1+\partial_2} \mathbb{Z}X_1 \xrightarrow{\partial_0-\partial_1} \mathbb{Z}X_0 \xrightarrow{\partial_{-1}} \mathbb{Z}X_{-1} \longrightarrow 0$$

We will denote $H_{n+1}(X)$ for $n \geq -1$ the $n$-th simplicial homology group of $X_*$. This means for example that $H_1(X)$ will be the quotient of $\partial_{-1} : \mathcal{N}_0^{gl}(\mathcal{C}) \to \mathcal{N}_{-1}^{gl}(\mathcal{C})$ by the image of $\partial_0 - \partial_1 : \mathcal{N}_1^{gl}(\mathcal{C}) \to \mathcal{N}_0^{gl}(\mathcal{C})$.

## C Precubical set, globular $\omega$-category and globular set

**Definition C.1** *[BH81b] [KP97] A precubical set consists of a family of sets $(K_n)_{n\geqslant 0}$ and of a family of face maps $K_n \xrightarrow{\partial_i^\alpha} K_{n-1}$ for $\alpha \in \{-,+\}$ which satisfies the following axiom (called sometime the cube axiom) :*

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \text{ for all } i < j \leqslant n \text{ and } \alpha, \beta \in \{-,+\}.$$

If $K$ is a precubical set, the elements of $K_n$ are called the $n$-cubes. An element of $K_n$ is of dimension $n$. The elements of $K_0$ (resp. $K_1$) can be called the *vertices* (resp. the *arrows*) of $K$.

**Definition C.2** *[BH81a,Str87,Ste91] An $\omega$-category is a set $A$ endowed with two families of maps $(s_n = d_n^-)_{n\geqslant 0}$ and $(t_n = d_n^+)_{n\geqslant 0}$ from $A$ to $A$ and with a family of partially defined 2-ary operations $(*_n)_{n\geqslant 0}$ where for any $n \geqslant 0$, $*_n$ is a map from $\{(a,b) \in A \times A, t_n(a) = s_n(b)\}$ to $A$ ($(a,b)$ being carried over $a *_n b$) which satisfies the following axioms for all $\alpha$ and $\beta$ in $\{-,+\}$ :*

(i) $d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x \text{ if } m < n \\ d_n^\alpha x \text{ if } m \geqslant n \end{cases}$

(ii) $s_n x *_n x = x *_n t_n x = x$

(iii) *if $x *_n y$ is well-defined, then $s_n(x *_n y) = s_n x$, $t_n(x *_n y) = t_n y$ and for $m \neq n$, $d_m^\alpha(x *_n y) = d_m^\alpha x *_n d_m^\alpha y$*

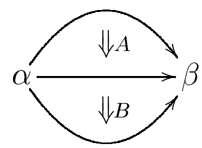
(iv) *as soon as the two members of the following equality exist, then $(x *_n y) *_n z = x *_n (y *_n z)$*

(v) *if $m \neq n$ and if the two members of the equality make sense, then $(x *_n y) *_m (z *_n w) = (x *_m z) *_n (y *_m w)$*

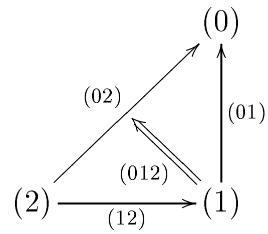
(vi) *for any $x$ in $A$, there exists a natural number $n$ such that $s_n x = t_n x = x$ (the smallest of these numbers is called the dimension of $x$ and is denoted by $dim(x)$).*

A $n$-dimensional element of $\mathcal{C}$ is called a $n$-morphism. A 0-morphism is also called a state of $\mathcal{C}$, and a 1-morphism an arrow. If $x$ is a morphism of an $\omega$-category $\mathcal{C}$, we call $s_n(x)$ the $n$-source of $x$ and $t_n(x)$ the $n$-target of $x$. The category of all $\omega$-categories (with the obvious morphisms) is denoted by $\omega Cat$. The corresponding morphisms are called $\omega$-functors. The set of $n$-dimensional morphisms of $\mathcal{C}$ is denoted by $\mathcal{C}_n$.

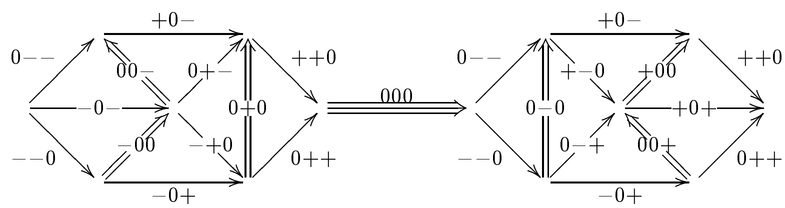
As fundamental examples of $\omega$-categories, there is the $\omega$-category $\Delta^n$ freely generated by the faces of the $n$-simplex [Str87]. To characterize this $\omega$-category, the first step consists of labeling all faces of the $n$-simplex. Its faces are indeed in bijection with strictly increasing sequences of elements of




(a) Composition of two 2-morphims




(b) The $\omega$-category $\Delta^2$




(c) The $\omega$-category $I^3$

Fig. C.1. Some $\omega$-categories (a $k$-fold arrow symbolizes a k-morphism)

$\{0, 1, \ldots, n\}$. A sequence of length $p+1$ will be of dimension $p$. If $x$ is a face, let $R(x)$ be the set of faces of $x$ seen as a sub-simplex. If $X$ is a set of faces, then let $R(X) = \bigcup_{x \in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$ and that $R(\{x\}) = R(x)$. Then $\Delta^n$ is the free $\omega$-categories generated by the $R(x)$ with the rules

(i) For $x$ $p$-dimensional with $p \geq 1$, $s_{p-1}(R(x)) = R(s_x)$ and $t_{p-1}(R(x)) = R(t_x)$ where $s_x$ and $t_x$ are the sets of faces defined below.
(ii) If $X$ and $Y$ are two elements of $\Delta^n$ such that $t_p(X) = s_p(Y)$ for some $p$, then $X \cup Y$ belongs to $\Delta^n$ and $X \cup Y = X *_p Y$.

Let us give the definition of $s_x$ and $t_x$ on some example :

$$s_{(04589)} = \{(4589), (0489), (0458)\}$$

The elements in odd position are removed ;

$$t_{(04589)} = \{(0589), (0459)\}$$

The elements in even position are removed.

Let $\underline{\Delta}$ be the unique small category such that a pre-sheaf over $\underline{\Delta}$ is exactly a simplicial set [May67,Wei94]. The category $\underline{\Delta}$ has for objects the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$ and has for morphisms the non-decreasing monotone functions. One is used to distinguishing in this category the morphisms $\epsilon_i : [n-1] \to [n]$ and $\eta_i : [n+1] \to [n]$ defined as follows for each $n$ and $i = 0, \dots, n$ :

$$\epsilon_i(j) = \left\{ \begin{array}{ll} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{array} \right\}, \quad \eta_i(j) = \left\{ \begin{array}{ll} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{array} \right\}$$

The mapping $n \mapsto \Delta^n$ yields a functor from $\underline{\Delta}$ to $\omega Cat$ by setting $\epsilon_i \mapsto \Delta^{\epsilon_i}$ and $\eta_i \mapsto \Delta^{\eta_i}$ where

- for any face $(\sigma_0 < \dots < \sigma_s)$ of $\Delta^{n-1}$, $\Delta^{\epsilon_i}(\sigma_0 < \dots < \sigma_s)$ is the only face of $\Delta^n$ having $\epsilon_i\{\sigma_0, \dots, \sigma_s\}$ as set of vertices ;
- for any face $(\sigma_0 < \dots < \sigma_r)$ of $\Delta^{n+1}$, $\Delta^{\eta_i}(\sigma_0 < \dots < \sigma_r)$ is the only face of $\Delta^n$ having $\eta_i\{\sigma_0, \dots, \sigma_r\}$ as set of vertices.

Therefore

**Definition C.3** *Let $\mathcal{C}$ be an $\omega$-category. Then the graded set $\omega Cat(\Delta^*, \mathcal{C})$ is naturally endowed with a structure of simplicial sets. It is called the simplicial nerve of $\mathcal{C}$.*

## D $\omega$-categorical realization of a precubical set

Intuitively the $\omega$-categorical realization $\Pi(K)$ of a precubical set $K$ (also called the free $\omega$-category generated by $K$) as defined below contains as $n$-morphisms all composites (or all concatenations) of cubes of $K$ which are $n$-dimensional (this means that somewhere in the composite, a $n$-dimensional cube appears). In particular the 1-morphisms of $\Pi(K)$ will be exactly all arrows of $K$ and all possible compositions of these arrows.

The free $\omega$-category $\Pi(K)$ is constructed as follows. The main ingredient is the free $\omega$-category $I^n$ generated by the faces of the $n$-cube. Its characterization is very similar to that of the $\omega$-category $\Delta^n$ generated by the faces of the $n$-simplex. The faces of the $n$-cube are labeled by the word of length $n$ in the alphabet $\{-, 0, +\}$, the number of zero corresponding to the dimension of the face. Everything is similar, except the definition of $s_x$ and $t_x$. The set $s_x$ is the set of sub-faces of the faces obtained by replacing the $i$-th zero of $x$ by $(-)^i$, and the set $t_x$ is the set of sub-faces of the faces obtained by replacing the $i$-th zero of $x$ by $(-)^{i+1}$. For example, $s_{0+00} = \{\text{-+00}, \text{0++0}, \text{0+0-}\}$ and $t_{0+00} = \{\text{++00}, \text{0+-0}, \text{0+0+}\}$. Figure 1(c) represents the free $\omega$-category generated by the 3-cube (cf. [Gau00] for some examples of calculations). The first construction of $I^n$ is due to Aitchison in [Ait86].

Then to each $x \in K_n$, we associate a copy of $I^n$ denoted by $\{x\} \times I^n$ whose corresponding faces will be denoted by $(x, k_1 \dots k_n)$. We then take the quotient of the direct sum of these $\{x\} \times I^{dim(x)}$ in $\omega Cat$ (which corresponds for the underlying sets to the disjoint union) by the relations

$$(\partial_i^\alpha(x), k_1 \dots k_{n-1}) \sim (x, k_1 \dots k_{i-1}[\alpha]_i k_i \dots k_{n-1})$$

for any $n \geq 1$ and any $x \in K_n$ where the notation $[\alpha]_i$ means that $\alpha$ is put in $i$-th position. This expression means that in the copy of $I^{n-1}$ corresponding to $\partial_i^\alpha(x)$, the face $k_1 \dots k_{n-1}$ must be identified to the face $k_1 \dots k_{i-1}[\alpha]_i k_i \dots k_{n-1}$ in the copy of $I^n$ corresponding to $x$. And one has

**Proposition D.1** *One obtains a well-defined $\omega$-category $\Pi(K)$ and $\Pi$ induces a well-defined functor from the category of precubical sets to that of $\omega$-categories.*

The proof uses the coend construction (cf. [Mac71]).

# E Localization of a category with respect to a collection of morphisms

**Definition E.1** *Let $\mathcal{C}$ be a category (not necessarily small). Let $S$ be a collection of morphisms of $\mathcal{C}$. Consider the following universal problem :*

> *"There exists a pair $(\mathcal{D}, \mu)$ such that $\mu$ is a functor from $\mathcal{C}$ to $\mathcal{D}$ and such that for any $s \in S$, $\mu(s)$ is an invertible morphism of $\mathcal{D}$."*

*Then the solution $(\mathcal{C}[S^{-1}], Q)$, if there exists, is called the localization of $\mathcal{C}$ with respect to $S$.*

# F The biglobular nerve

**Theorem F.1** *[Gau02] Let $\mathcal{C}$ be a non-contracting $\omega$-category.*

(i) *Let $x$ be an $\omega$-functor from $\Delta^n$ to $\mathbb{P}\mathcal{C}$ for some $n \geq 0$. Then the set maps $(\sigma_0 \dots \sigma_r) \mapsto s_0 x((\sigma_0 \dots \sigma_r))$ and $(\sigma_0 \dots \sigma_r) \mapsto t_0 x((\sigma_0 \dots \sigma_r))$ from the underlying set of faces of $\Delta^n$ to $\mathcal{C}_0$ are constant. The unique value of $s_0 \circ x$ is denoted by $S(x)$ and the unique value of $t_0 \circ x$ is denoted by $T(x)$.*

(ii) *For any pair $(\alpha, \beta)$ of 0-morphisms of $\mathcal{C}$, for any $n \geq 1$, and for any $0 \leq i \leq n$, then $\partial_i \left(\mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])\right) \subset \mathcal{N}^{gl}_{n-1}(\mathcal{C}[\alpha,\beta])$.*

(iii) *For any pair $(\alpha, \beta)$ of 0-morphisms of $\mathcal{C}$, for any $n \geq 0$, and for any $0 \leq i \leq n$, then $\epsilon_i \left(\mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])\right) \subset \mathcal{N}^{gl}_{n+1}(\mathcal{C}[\alpha,\beta])$.*

(iv) *By setting, $G^{\alpha,\beta}\mathcal{N}^{gl}_n(\mathcal{C}) := \mathcal{N}^{gl}_n(\mathcal{C}[\alpha,\beta])$ for $n \geq 0$ and $G^{\alpha,\beta}\mathcal{N}^{gl}_{-1}(\mathcal{C}) := \{(\alpha,\beta),(\beta,\alpha)\}$, one obtains a $(\mathcal{C}_0 \times \mathcal{C}_0)$-graduation on the globular nerve ; in particular, one has the direct sum of augmented simplicial sets*

$$\mathcal{N}^{gl}_*(\mathcal{C}) = \bigsqcup_{(\alpha,\beta)\in\mathcal{C}_0\times\mathcal{C}_0} G^{\alpha,\beta}\mathcal{N}^{gl}_*(\mathcal{C})$$

*and $G^{\alpha,\beta}\mathcal{N}^{gl}_*(\mathcal{C}) = \mathcal{N}^{gl}_*(\mathcal{C}[\alpha,\beta])$.*

Let $\mathcal{C}$ be a non-contracting $\omega$-category. Using Theorem F.1, recall that for some $\omega$-functor $x$ from $\Delta^n$ to $\mathbb{P}\mathcal{C}$, one calls $S(x)$ the unique element of the image of $s_0 \circ x$ and $T(x)$ the unique element of the image of $t_0 \circ x$. If $(\alpha, \beta)$ is a pair of $\mathcal{N}^{gl}_{-1}(\mathcal{C})$, set $S(\alpha, \beta) = \alpha$ and $T(\alpha, \beta) = \beta$.

**Proposition F.2** *[Gau02] Let $\mathcal{C}$ be a non-contracting $\omega$-category. Let $x$ and $y$ be two $\omega$-functors from $\Delta^n$ to $\mathbb{P}\mathcal{C}$ with $n \geq 0$. Suppose that $T(x) = S(y)$. Let $x * y$ be the map from the faces of $\Delta^n$ to $\mathcal{C}$ defined by*

$$(x * y)((\sigma_0 \dots \sigma_r)) := x((\sigma_0 \dots \sigma_r)) *_0 y((\sigma_0 \dots \sigma_r)).$$

*Then the following conditions are equivalent :*

(i) *The image of $x * y$ is a subset of $\mathbb{P}\mathcal{C}$.*

(ii) *The set map $x * y$ yields an $\omega$-functor from $\Delta^n$ to $\mathbb{P}\mathcal{C}$ and $\partial_i(x * y) = \partial_i(x) * \partial_i(y)$ for any $0 \leq i \leq n$.*

*On the contrary, if for some $(\sigma_0 \dots \sigma_r) \in \Delta^n$, $(x * y)((\sigma_0 \dots \sigma_r))$ is 0-dimensional, then $x * y$ is the constant map $S(x) = T(y)$.*

In the sequel, we set $(\alpha, \beta) * (\beta, \gamma) = (\alpha, \gamma)$, $S(\alpha, \beta) = \alpha$ and $T(\alpha, \beta) = \beta$. If $x$ is an $\omega$-functor from $\Delta^n$ to $\mathbb{P}\mathcal{C}$, and if $y$ is the constant map $T(x)$ (resp. $S(x)$) from $\Delta^n$ to $\mathcal{C}_0$, then set $x * y := x$ (resp. $y * x := x$).

**Theorem F.3** *Suppose that $\mathcal{C}$ is an object of $\omega Cat_1$. Then for $n \geq 0$, the operations $S$, $T$ and $*$ allow to define a small category $\underline{\mathcal{N}^{gl}_n(\mathcal{C})}$ whose morphisms are the elements of $\mathcal{N}^{gl}_n(\mathcal{C}) \cup \{$constant maps $\Delta^n \to \mathcal{C}_0\}$ and whose objects are the 0-morphisms of $\mathcal{C}$. If $\underline{\mathcal{N}^{gl}_{-1}(\mathcal{C})}$ is the small category whose morphisms are the elements of $\mathcal{C}_0 \times \mathcal{C}_0$ and whose objects are the elements of $\mathcal{C}_0$ with the operations $S$, $T$ and $*$ above defined, then one obtains (by defining the face maps $\partial_i$ and degeneracy maps $\epsilon_i$ in an obvious way on $\{$constant maps $\Delta^n \to \mathcal{C}_0\}$) this way an augmented simplicial object $\underline{\mathcal{N}^{gl}_*}$ in the category of small categories.*

By composing by the classifying space functor of small categories (cf. for example [Qui73] for further details), one obtains a bisimplicial set which is called the *biglobular nerve.*