

Homotopy invariants of higher dimensional categories and concurrency in computer science

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The strict globular ω -categories formalize the execution paths of a parallel automaton and the homotopies between them. One associates to such (and any) ω -category \mathcal{C} three homology theories. The first one is called the globular homology. It contains the oriented loops of \mathcal{C} . The two other ones are called the negative (respectively, positive) corner homology. They contain in a certain manner the branching areas of execution paths or negative corners (respectively, the merging areas of execution paths or positive corners) of \mathcal{C} . Two natural linear maps called the negative (respectively, the positive) Hurewicz morphism from the globular homology to the negative (respectively, positive) corner homology are constructed. We explain the reason why these constructions allow the reinterpretation of some geometric problems coming from computer science.

1. Introduction

The use of geometric notions to describe the behaviour of concurrent machines is certainly not new: in particular, they have been used for *progress graphs* (Dijkstra 1968), HDA (Pratt 1991; van Glabbeek 1991) and the simplicial models of Herlihy and Shavit (1994) and Herlihy and Rajsbaum (1994). The purpose of this article is to provide a new setting for the homotopy of execution paths in concurrent automata in order to improve the homological approach of Goubault (1995). We can point out that some other approaches to this question already exist, such as partially ordered topological spaces in Goubault (1995) and partial posets in Sokolowski (1999). Pratt has already noted that a good structure to deal with execution paths and homotopies between them is the structure of globular ω -category (Pratt 1991). The aim of this paper is threefold. First, the use of the concept of a globular ω -category to describe concurrent machines is justified on some well-known examples and in a very informal way. Second, we associate to every globular ω -category three homology theories and two natural maps between them, and explain why their content is interesting for some geometric problems coming from computer science. Third, as in algebraic topology, a notion of homotopic ω -categories is proposed and we prove that the preceding homology theories are invariant with respect to it (only in a particular case for the corner homologies).

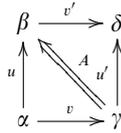


Fig. 1. A 2-transition

This paper is organised as follows. In Section 2, we make precise the notion of paths and homotopies between paths, homotopies between homotopies, *etc.*. The notion of globular ω -category is recalled, and the link between the usual formalization of concurrent automata using cubical sets and the new formalization by globular ω -categories is explained. We give in a very informal way the geometric and computer science meaning of the homology theories that will be constructed in this paper. In Section 3, the globular homology of a globular ω -category is defined. Some examples of globular cycles are given and the globular complex is related to a derived functor. Section 4 is devoted to the construction of the negative and positive corner homologies of an ω -category. Then we introduce in Section 5 a technical tool to fill shells in the cubical singular nerve of an ω -category, and in Section 6, this notion of the filling of a shell is used to construct two families of connections on the cubical singular nerve of an ω -category. This allows us to prove that the corner homologies are the homologies associated to two new simplicial nerves. In Section 7, we construct the two natural maps from the globular homology to the negative and positive corner homologies. In Section 8, a notion of homotopy equivalence of ω -categories is proposed. We will prove the following property: *Let f and g be two non 1-contracting ω -functors from \mathcal{C} to \mathcal{D} . If f and g are homotopic, then for any natural number n , $H_n^{gl}(f) = H_n^{gl}(g)$ (Theorem 8.6).* In a very particular case, it is also possible to relate the homotopy of paths in ω -categories to the corner homology theories H_n^- and H_n^+ (Theorem 8.7). In Section 9, some conjectures and perspectives both in mathematics and computer science are exposed. In Section 10, we prove that the cubical singular nerve of the free ω -category generated by a cubical set K is nothing else but the free cubical ω -category generated by K . This allows us to propose a direct construction of the globular homology and of the corner homologies of a cubical set without using any globular ω -category.

2. Presentation of the geometric ideas of this work

2.1. Informal introduction to execution paths and the homotopies between them

A sequential machine (that is, without concurrency) is a set of states, also called 0-transitions, and a set of 1-transitions from a given state to another one. A concurrent machine, like the previous one, consists of a set of states and a set of 1-transitions, but also has the capability of carrying out several 1-transitions at the same time.

In Figure 1, if we work in cartesian coordinates in such a way that $A = [0, 1] \times [0, 1]$ with $\alpha = (0, 0)$ and $\delta = (1, 1)$, the set of continuous maps (c_1, c_2) from $[0, 1]$ to A such that $c_1(0) = c_2(0) = 0$, $c_1(1) = c_2(1) = 1$ and $t \leq t'$ implies $c_1(t) \leq c_1(t')$ and $c_2(t) \leq c_2(t')$

represents all the simultaneous possible executions of u and v . Coordinates represent the evolution of u and v , which means the local time taken to execute u or v . If $(c_1, c_2)(]0, 1[)$ is entirely included in the interior of A , it is a ‘true parallelism’. If $(c_1, c_2)(]0, 1[)$ is entirely included in the edge of the square, u and v are sequentially carried out by the automaton. More generally, the concurrent execution of n 1-transitions can be represented by an n -cube. This has already been noted, for example, in Pratt (1991) and Goubault (1995).

Definition 2.1. A cubical set consists of a family of sets $(K_n)_{n \geq 0}$, of a family of face maps $K_n \xrightarrow{\partial_i^\alpha} K_{n-1}$ for $\alpha \in \{-, +\}$ and of a family of degeneracy maps $K_{n-1} \xrightarrow{\epsilon_i} K_n$ with $1 \leq i \leq n$ that satisfy the following relations:

- 1 $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ for all $i < j \leq n$ and $\alpha, \beta \in \{-, +\}$
- 2 $\epsilon_i \epsilon_j = \epsilon_{j+1} \epsilon_i$ for all $i \leq j \leq n$
- 3 $\partial_i^\alpha \epsilon_j = \epsilon_{j-1} \partial_i^\alpha$ for $i < j \leq n$ and $\alpha \in \{-, +\}$
- 4 $\partial_i^\alpha \epsilon_j = \epsilon_j \partial_{i-1}^\alpha$ for $i > j \leq n$ and $\alpha \in \{-, +\}$
- 5 $\partial_i^\alpha \epsilon_i = Id.$

The corresponding category of cubical sets, with an obvious definition of its morphisms, is isomorphic to the category of presheaves $Sets^{\square^{op}}$ over a small category \square . This latter can be described in a nice way as follows (Crans 1995). The objects of \square are the sets $\underline{n} = \{1, \dots, n\}$ where n is a natural number greater than or equal to 1 and an arrow f from \underline{n} to \underline{m} is a function f^* from \underline{m} to $\underline{n} \cup \{-, +\}$ such that $f^*(k) \leq f^*(k') \in \underline{n}$ implies $k \leq k'$ and $f^*(k) = f^*(k') \in \underline{n}$ implies $k = k'$.

Let X be a topological space. Let $[0, 1]$ denote the interval between 0 and 1. Set $K_n = C^0([0, 1]^n, X)$ the set of continuous maps from the n -box $[0, 1]^n$ to X . Set

$$\begin{aligned} \partial_i^-(f)(x_1, \dots, x_p) &= f(x_1, \dots, [0]_i, \dots, x_p) \\ \partial_i^+(f)(x_1, \dots, x_p) &= f(x_1, \dots, [1]_i, \dots, x_p) \\ \epsilon_i(f)(x_1, \dots, x_p) &= f(x_1, \dots, \hat{x}_i, \dots, x_p). \end{aligned}$$

Then we obtain a cubical set K , which is called the cubical singular nerve of the topological space X .

In Figure 1, we call A a homotopy between the two 1-paths uw' and vu' . We call a 2-path a homotopy between two 1-paths and by induction on $n \geq 2$, we call an n -path a homotopy between two $(n - 1)$ -paths. This notion of homotopy is different from the classical one in the sense that only the 1-paths uw' and vu' are homotopic and because the 1-paths are oriented. For example, still in Figure 1, u is not homotopic with u', v or v' .

In Figure 2, there are an initial state α , a final state β , two 1-transitions u and v , and two 2-transitions or homotopies A and B between u and v . Choosing an orientation for A and B , for example $s_1A = s_1B = u$ and $t_1A = t_1B = v$ (s for source and t for target), we see that $s_0s_1A = s_0t_1A = \alpha$ and $t_0s_1A = t_0t_1A = \beta$. These are precisely the globular equations that appear in the axioms of globular ω -categories.

The concatenation yields an associative composition law on the set of 1-transitions of an ω -category. It turns out that there is also a natural composition law on the set of

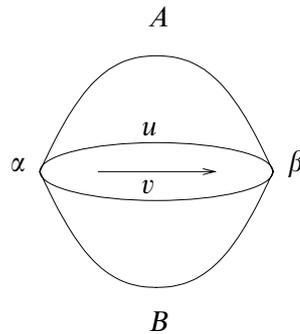


Fig. 2. A 3-dimensional hole

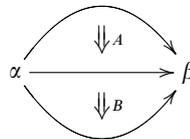


Fig. 3. Composition of two 2-morphisms

2-transitions. In Figure 3, with the convention of orientation $t_1A = s_1B$, we can compose A and B . We denote this composition using $A *_1 B$. We can see that $s_1(A *_1 B) = s_1A$ and $t_1(A *_1 B) = t_1B$. The composition of higher dimensional morphisms must be associative because it corresponds to the concatenation of the real execution paths contained in A and B .

Thus the geometric properties of transitions of concurrent machines can be encoded in a structure of cubical set. And their associated set of execution paths and the homotopies between them have the natural structure of a globular ω -category. All these ideas have already appeared in Pratt (1991). Pratt uses the term n -complex, which is in fact nothing other than a small n -category. We use the notations of Steiner (1991) and Street (1987) for the following definition, which has already appeared in Brown and Higgins (1981a).

Definition 2.2. An ω -category is a set A endowed with two families of maps $(s_n = d_n^-)_{n \geq 0}$ and $(t_n = d_n^+)_{n \geq 0}$ from A to A and with a family of partially defined 2-ary operations $(*_n)_{n \geq 0}$ where for any $n \geq 0$, $*_n$ is a map from $\{(a, b) \in A \times A, t_n(a) = s_n(b)\}$ to A ((a, b) being carried over $a *_n b$) that satisfies the following axioms for all α and β in $\{-, +\}$:

- 1 $d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x & \text{if } m < n \\ d_n^\alpha x & \text{if } m \geq n. \end{cases}$
- 2 $s_n x *_n x = x *_n t_n x = x$.
- 3 If $x *_n y$ is well defined, then $s_n(x *_n y) = s_n x$, $t_n(x *_n y) = t_n y$ and for $m \neq n$, $d_m^\alpha(x *_n y) = d_m^\alpha x *_n d_m^\alpha y$.
- 4 As soon as the two members of the following equality exist, then $(x *_n y) *_n z = x *_n (y *_n z)$.

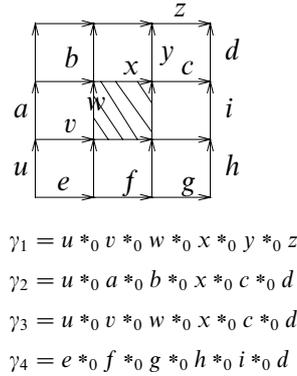


Fig. 4. Example of (an idealized) distributed database

- 5 If $m \neq n$ and if the two members of the equality make sense, then $(x *_n y) *_m (z *_n w) = (x *_m z) *_n (y *_m w)$.
- 6 For any x in A , there exists a natural number n such that $s_n x = t_n x = x$ (the smallest of these numbers is called the dimension of x and is denoted by $\dim(x)$).

We will sometimes use the notation $d_n^- := s_n$ and $d_n^+ = t_n$. If x is a morphism of an ω -category \mathcal{C} , we call $s_n(x)$ the n -source of x and $t_n(x)$ the n -target of x . The category of all ω -categories (with the obvious morphisms) is denoted by ωCat . The corresponding morphisms are called ω -functors.

If S is a set, the free abelian group generated by S is denoted by $\mathbb{Z}S$. By definition, an element of $\mathbb{Z}S$ is a formal linear combination of elements of S .

Definition 2.3. Let \mathcal{C} be an ω -category. Let \mathcal{C}_n be the set of n -dimensional morphisms of \mathcal{C} . Two n -morphisms x and y are *homotopic* if there exists $z \in \mathbb{Z}\mathcal{C}_{n+1}$ such that $s_n z - t_n z = x - y$. This property is denoted by $x \sim y$.

If $x \sim y$, the pair (x, y) belongs to the reflexive, symmetric and transitive closure of the binary relation generated by all pairs $(s_n(u), t_n(u))$, where u runs over \mathcal{C}_{n+1} .

Figure 4 is a very simple example of a distributed database. The hole in the middle corresponds to a mutual exclusion. See Fajstrup *et al.* (1998b) for a complete treatment. The two 1-paths γ_1 and γ_2 are homotopic because there exists a 2-morphism between γ_1 and γ_3 and another one between γ_2 and γ_3 . On the other hand, none of the previous three 1-paths is homotopic to γ_4 because of the oriented hole in the middle.

2.2. The free ω -category generated by a cubical set

How may we mathematically associate to every cubical set K its corresponding set of execution paths and higher dimensional homotopies? The link between the two formalizations is as follows. We need to describe precisely the free ω -category associated to each n -cube of K . In a very informal way, it consists of seeing the faces of an n -cube as words of length n in the alphabet $\{-, 0, +\}$. The term 0_n means 0 n times, that is, the interior of the n -cube. And we say that the face $k_1 \dots k_n$ is at the source of x if $k_i \neq 0$ implies

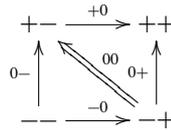
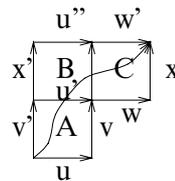


Fig. 5. The ω -category I^2



$$(A *_0 w *_0 x) *_1 (v' *_0 ((u' *_0 C) *_1 (B *_0 w')))$$

Fig. 6. Composition of three squares

$k_i = (-1)^i$, and we say that $k_1 \dots k_n$ is at the target of x if $k_i \neq 0$ implies $k_i = (-1)^{i+1}$. We will make precise the construction of I^n in Section 4.1. Once this is done, it suffices to paste the ω -categories associated to every n -cube of K in the same way that they are pasted in K . More concretely, every cubical set K is in a canonical way the direct limit of the elementary n -cubes included in it. This is due to the fact that any functor from a small category to the category *Sets* of sets is a canonical direct limit of representable functors, the set of those functors being dense in the set of all set-valued functors. More precisely, we have $K = \int^{n \in \square} K_n \cdot \square(-, \underline{n})$ where the integral sign is the coend construction (Mac Lane 1971) and $K_n \cdot \square(-, \underline{n})$ means the sum of ‘cardinal of K_n ’ copies of $\square(-, \underline{n})$. So $F(K) = \int^{n \in \square} K_n \cdot I^n$ is an ω -category containing as 1-morphisms all arrows of K and all possible compositions of these arrows, as 2-morphisms all homotopies between the execution paths, etc..

Consider, for example, the 2-cube of Figure 5. The 2-face 00 is oriented from the side $\{-0, 0+\}$ to the side $\{0-, +0\}$. The corresponding ω -category I^2 contains all possible compositions of faces of the 2-cube. Therefore, as set, we have

$$I^2 = \{--, -+, +-, ++, -0, 0-, +0, 0+, -0 *_0 0+, 0- *_0 +0, 00\}.$$

In Figure 6, the three 2-morphisms A , B and C are not drawn, and are supposed to be oriented to the north west. The composition of the three squares A , B and C is equal to

$$(A *_0 w *_0 x) *_1 (v' *_0 ((u' *_0 C) *_1 (B *_0 w')))$$

in the free ω -category generated by this cubical set.

The map F induces a functor from the category of cubical sets to the category of ω -categories. Indeed, this is the left Kan extension of the functor Q from \square to ωCat defined as follows (Mac Lane 1971). It maps \underline{n} to I^n . Let ϵ_i be the surjective morphism from \underline{n} to $\underline{n-1}$ for $1 \leq i \leq n$ defined by $(\epsilon_i)^*(l) = l$ if $l < i$ and $(\epsilon_i)^*(l) = l + 1$. Let ∂_i^α be the injective morphism from $\underline{n-1}$ to \underline{n} for $1 \leq i \leq n$ and for $\alpha = \pm$ defined by $(\partial_i^\alpha)^*(l) = l$

execution paths on the left with the small number (four) of execution paths on the right, which are essentially the same! This means that an algorithm that could take into account this notion of ω -category up to homotopy would be more efficient than any other algorithm.

Instead of dealing directly with ω -category up to homotopy, a more fruitful approach consists of building some functors from ω -categories to, for example, abelian groups, invariant up to homotopy. These functors contain, at least theoretically, the relevant geometric information because of their invariance up to homotopy in the above sense. A common way to construct such invariants consists of constructing functors from the category of ω -categories to the category $Comp(Ab)$ of chain complexes of groups and to consider the associated homology groups.

Definition 2.4. A *chain complex of groups* is a family of abelian groups $(C_n)_{n \geq 0}$ together with a family of linear maps $\partial_n : C_{n+1} \rightarrow C_n$ such that $\partial_n \circ \partial_{n+1} = 0$ for any $n \geq 0$.

Since the image of ∂_{n+1} is included in the kernel of ∂_n , the quotient group

$$H_n(C_*, \partial_*) = Ker(\partial_n) / Im(\partial_{n+1})$$

is well defined. It is called the n -th homology group of the group complex $(C_n)_{n \geq 0}$. The map H_n yields a functor from $Comp(Ab)$ to the category Ab of abelian groups. See for example Rotman (1979) or Weibel (1994) for an introduction to the theory of these mathematical objects.

The first homology theory will be the globular homology $H_*^{gl}(\mathcal{C})$ (Definition 3.1). An example of a globular cycle of dimension 1 is $\gamma_1 - \gamma_4$ in Figure 4. We call this an oriented 1-dimensional loop. An example of a globular cycle of dimension 2 is $A - B$ in Figure 2.

The two other homology theories will be called the negative and positive corner homologies $H_*^\pm(\mathcal{C})$ (Definition 4.3). The cycles of the negative one correspond to the branching areas of execution paths (or negative corner) and those of the positive one correspond to the merging areas of execution paths (or positive corner). In the case of $\gamma_1 - \gamma_4$ in Figure 4, there is one branching area on the left and one merging area on the right, and similarly for Figure 2.

The idea then is to associate to any oriented loop of any dimension its corresponding negative or positive corners. This is the underlying geometric meaning of the morphisms h_*^\pm from $H_*^{gl}(\mathcal{C})$ to $H_*^\pm(\mathcal{C})$ (Proposition 7.5 and 7.7). We can immediately see an application of these maps. Looking back to the Swiss Flag example of Figure 8, it is clear that the cokernel of h_1^- is not empty, because of the deadlock and the forbidden area. A negative corner that yields a non-trivial element in this cokernel is drawn in Figure 9. In the same way, the cokernel of h_1^+ in the Swiss Flag example is still not empty, because of the unreachable state and the unreachable area. A positive corner that yields a non-zero element of this cokernel is represented in Figure 9.

These geometric remarks ensure that the homology groups that we are going to construct contain relevant information about the geometry of concurrency.

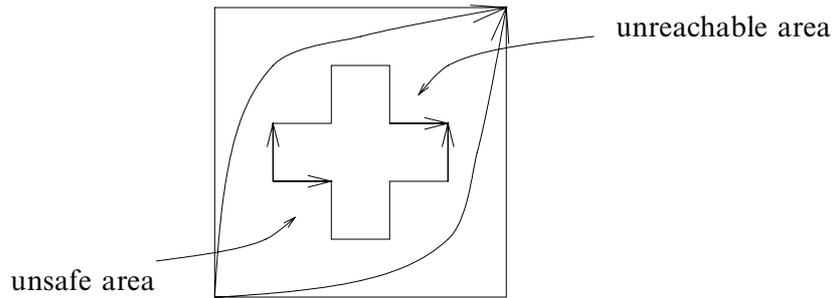


Fig. 9. Unsafe area and unreachable area in a concurrent machine with semaphores

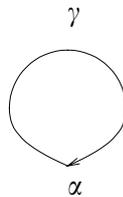


Fig. 10. A loop that does not give rise to a globular cycle

3. Globular homology of ω -category

3.1. Definition

The starting point is an ω -category \mathcal{C} .

Definition 3.1. Let $(C_*^{gl}(\mathcal{C}), \partial^{gl})$ be the chain complex defined as follows: $C_0^{gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_1$, and for $n \geq 1$, $C_n^{gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n$, $\partial^{gl}(x) = (s_0x, t_0x)$ if $x \in \mathbb{Z}\mathcal{C}_1$ and for $n \geq 1$, $x \in \mathbb{Z}\mathcal{C}_{n+1}$ implies $\partial^{gl}(x) = s_nx - t_nx$. This complex is called the *globular complex* of \mathcal{C} , and its corresponding homology the *globular homology*.

There is a difference between the 1-dimensional case and the other cases. A loop as in Figure 10 where γ is a 1-morphism such that $s_0\gamma = t_0\gamma = \alpha$ does not yield a globular 1-cycle. However, a loop as in Figure 11 where A is a n -morphism with $n \geq 2$ and such that $s_{n-1}A = t_{n-1}A = \gamma$ yields a globular n -cycle.

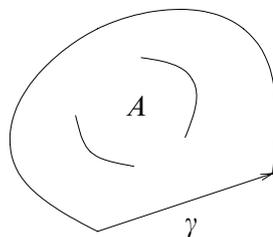


Fig. 11. Example of a globular cycle in a higher dimension

3.2. Functorial property of the globular homology

Now we give an important technical definition for the sequel.

Definition 3.2. Let f be an ω -functor from \mathcal{C} to \mathcal{D} . The morphism f is *non 1-contracting* if for any 1-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a 1-dimensional morphism of \mathcal{D} .

The category of ω -categories with the non 1-contracting ω -functors is denoted by ωCat_1 . The category of cubical sets equipped with the non 1-contracting morphisms is denoted by $Sets_1^{\square op}$.

If f is a non 1-contracting ω -functor from \mathcal{C} to \mathcal{D} , then for any morphism $x \in \mathcal{C}$ of dimension greater than 1, $f(x)$ is of dimension greater than one as well. This is due to the equality $f(s_1x) = s_1f(x)$.

Let f be an ω -functor from \mathcal{C} to \mathcal{D} . Then f induces for all $n \geq 0$ a linear morphism f_* from $\mathbb{Z}\mathcal{C}_n$ to $\mathbb{Z}\mathcal{D}_n$ by setting $f_*(x) = f(x) \bmod \mathcal{D}_{\leq n-1}$: we use this notation to mean that $f_*(x) = f(x)$ if $f(x)$ is n -dimensional and $f_*(x) = 0$ otherwise. For $n \geq 2$, $f_*(s_{n-1} - t_{n-1})(x) = (s_{n-1} - t_{n-1})f_*(x)$, therefore $f_*\partial^{gl}(x) = \partial^{gl}f_*(x)$. The latter equality is no longer true if x is 1-dimensional because an ω -functor can contract 1-morphisms and because $\partial^{gl}(x) = (s_0(x), t_0(x))$. So the globular homology does not yield a functor from ωCat to Ab but only a functor from ωCat_1 to Ab .

3.3. Homological property

Now we give a homological property of the globular complex to justify this construction. The starting point is the small category $Glob$ defined as follows: the objects are all natural numbers and the arrows are generated by s and t in $Glob(m, m - 1)$ for any $m > 0$ and quotiented by the relations $ss = st, ts = tt$. We can depict $Glob$ as follows:

$$\begin{array}{ccccccc} & \xrightarrow{t} & & \xrightarrow{t} & & \xrightarrow{t} & & \xrightarrow{t} \\ & \searrow & & \searrow & & \searrow & & \searrow \\ & s & & s & & s & & s \end{array} \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array}$$

Definition 3.3. A *globular set* is a covariant functor from $Glob$ to $Sets$. The corresponding category is denoted $[Glob, Sets]$. A *globular group* is a covariant functor from $Glob$ to the category Ab of abelian groups. The corresponding category is denoted by $[Glob, Ab]$.

The notion of a globular set has already appeared in many works and is certainly not new (Street 1998; Penon 1999; Batanin 1998).

If \mathcal{C} is an ω -category, we use \mathcal{C}_n to denote the set of n -dimensional morphisms of \mathcal{C} with $n \geq 0$.

Definition 3.4. Let Gl be the map from ω -categories to globular groups defined as follows. If $n \geq 0$, set $Gl(\mathcal{C})_n = \mathbb{Z}\mathcal{C}_n$. For any $s, t \in Glob(n + 1, n)$, set $Gl(s)(x) = s_n(x)$ and $Gl(t)(x) = t_n(x)$.

Unfortunately, $Gl(-)$ is not a functor because an ω -functor might be n -contracting for $n \geq 2$. That is, suppose that f is an ω -functor from \mathcal{C} to \mathcal{D} such that for a 2-morphism x of \mathcal{C} , $f(x)$ is 1-dimensional. Then $Gl(f)(x) = 0 \in Gl(\mathcal{D})_2$ and $s_1Gl(f)(x) = 0$, but $Gl(f)(s_1(x)) = f(s_1(x)) \in Gl(\mathcal{D})_1$ and $f(s_1(x)) \neq 0$.

If M is a globular group, let $H(M)$ be the cokernel of the additive map from M_1 to $M_0 \oplus M_0$ that maps x to $(s(x), t(x))$. The map H induces a right exact additive functor from $[Glob, Ab]$ to Ab . Since $[Glob, Ab]$ has enough projectives, we can deal with the left derived functors $L_n(H)$ of H (see Weibel (1994) or Rotman (1979) for the definition of projective object and right exact functor).

Proposition 3.1. For any ω -category \mathcal{C} , we have $H_*^{gl}(\mathcal{C}) \cong L_*(H)(Gl(\mathcal{C}))$.

Before proving this theorem, we need to recall some standard facts about category of diagrams. We are going to solve Exercice (2.3.13) of Weibel (1994) because later we will need a precise description of a family of projective globular groups that allows us to resolve any globular group.

Let ev_k be the functor from $[Glob, Ab]$ to Ab such that $ev_k(M) = M(k)$, k being a natural number. This functor is exact and by the special adjoint functor theorem has a left adjoint denoted by $k_!$. We need to make $k_!$ explicit for later use.

Proposition 3.2. If M is an abelian group, set

$$k_!(M)(l) = \bigoplus_{h \in Glob(k,l)} M_h$$

where M_h is a copy of M . If $x \in M$, let $h.x$ be the corresponding element of $k_!(M)(l)$. If $f : l \rightarrow l'$ is an arrow of I , we set $k_!(M)(f)(h.x) = (fh).x$. Then $k_!$ is a globular group and this is the left adjoint of ev_k .

Proof. Let N be a globular group. We introduce the map

$$F : [Glob, Ab](k_!(M), N) \longrightarrow Ab(M, N(k))$$

defined by $F(u)(x) = u(Id_k.x)$ and the map

$$G : Ab(M, N(k)) \longrightarrow [Glob, Ab](k_!(M), N)$$

defined by $G(v)(f.x) = N(f)(v(x))$. The arrow $G(v)$ is certainly a morphism of globular groups from $k_!(M)$ to N . In fact, if $l \xrightarrow{f} l'$ is an arrow of $Glob$, and if $h.x$ is an element of $k_!(M)(l)$, then $G(v)(fh.x) = N(fh)(v(x)) = N(f)(G(v)(h.x))$. Therefore the diagram

$$\begin{array}{ccc} k_!(M)(l) & \xrightarrow{G(v)} & N(l) \\ \downarrow k_!(M)(f) & & \downarrow N(f) \\ k_!(M)(l') & \xrightarrow{G(v)} & N(l') \end{array}$$

commutes. Now we have to verify that F and G are inverses of each other. Indeed,

$$F(G(v))(x) = G(v)(Id_k.x) = N(Id)(v(x)),$$

and therefore $F(G(v)) = v$. Conversely,

$$G(F(u))(h.x) = N(h)(F(u)(x)) = N(h)(u(Id_k.x)) = u(k_!(M)(h)(Id_k.x)) = u(h.x),$$

and therefore $G(F(u)) = u$. □

Now we set

$$\mathcal{F} = \left\{ \bigoplus_{k \in \mathbb{N}} k_!(L)/L \text{ free module} \right\}$$

and can state the following proposition.

Proposition 3.3. All elements of \mathcal{F} are projective globular groups. Any globular group can be resolved by elements of \mathcal{F} .

Proof. Let X be a globular group. For any $k \in \mathbb{N}$, let L_k be a free abelian group and $L_k \rightarrow X(k)$ an epimorphism of abelian groups. Then the epimorphisms $L_k \rightarrow X(k)$ for all k induce a natural transformation $\bigoplus_{k \in \mathbb{N}} k_!(L_k) \rightarrow X$, which is certainly itself an epimorphism. Left adjoint functors and coproducts preserve projective objects (Borceux 1994), and hence we have the conclusion. \square

We are now in a position to prove the following proposition.

Proposition 3.4. For any ω -category \mathcal{C} , the equality $H_*^{gl}(\mathcal{C}) \cong L_*(H)(Gl(\mathcal{C}))$ holds.

Proof. If M is a globular group, we introduce the complex of abelian groups $(C_*^{gl}(M), \partial^{gl})$ defined as follows: $C_0^{gl}(M) = M_0 \oplus M_0$ and for $n \geq 1$, $C_n^{gl}(M) = M_n$, with the differential map $\partial^{gl}(x) = (s(x), t(x))$ if $x \in M_1$, and $\partial^{gl}(x) = s(x) - t(x)$ if $x \in M_n$ with $n \geq 2$. We have $H_0(C_*^{gl}(M), \partial) = H(M)$. Let k be a natural number and let L be a free abelian group. If $p > 0$, let us prove that $H_p(C_*^{gl}(k_!(L))) = 0$. Let $X = x_p$ be a cycle of $C_p(k_!(L))$. By construction, for all $p > k$, one has $k_!(L)(p) = 0$. Therefore, if $p > k$, then $X = 0$, and hence $H_p(C_*^{gl}(k_!(L))) = 0$ whenever $p > k$. Now let us look at the case $p \leq k$. We have $0 = \partial^{gl}(X) = s_{p-1}(x_p) - t_{p-1}(x_p)$. Then there exists x_p^s and x_p^t such that $x_p = s^{k-p} \cdot x_p^s + t^{k-p} \cdot x_p^t$. The equality $s_{p-1}(x_p) = t_{p-1}(x_p)$ implies $s^{k-p+1} \cdot x_p^s + s^{k-p+1} \cdot x_p^t = t^{k-p+1} \cdot x_p^s + t^{k-p+1} \cdot x_p^t$. Therefore $x_p = 0$.

Now we can deduce from Proposition 3.3 that for any projective globular group P ,

$$H_p(C_*^{gl}(P), \partial) = 0$$

for all $p > 0$. It is thus easy to check that for any natural number p , the equality $H_p(C_*^{gl}(M), \partial) = L_p(H)(M)$ holds. Indeed, the case $p = 0$ is trivial and the commutative diagram

$$\begin{array}{ccccccc} H_{p+1}(C_*^{gl}(P), \partial) & \longrightarrow & H_{p+1}(C_*^{gl}(M), \partial) & \longrightarrow & H_p(C_*^{gl}(K), \partial) & \longrightarrow & H_p(C_*^{gl}(P), \partial) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_{p+1}(H)(P) & \longrightarrow & L_{p+1}(H)(M) & \longrightarrow & L_p(H)(K) & \longrightarrow & L_p(H)(P) \end{array}$$

with P projective allows to make the induction on p . \square

4. Positive and negative corner homology of an ω -category

4.1. The pasting scheme Λ^n and the ω -category I^n

We need to describe precisely the ω -category associated to the n -cube.

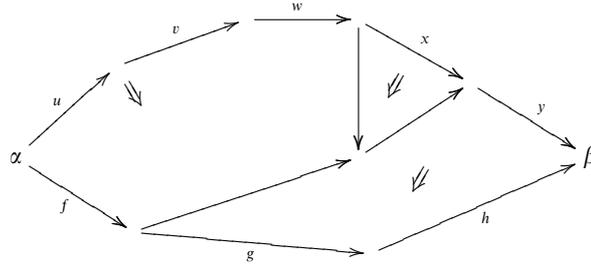


Fig. 12. A pasting scheme

Definition 4.1. A *pasting scheme* is a triple (A, E, B) where A is a \mathbb{N} -graded set, and E_j^i and B_j^i are two binary relations over $A_i \times A_j$ with $j \leq i$ satisfying:

- (1) The set E_i^i is the diagonal of A_i .
- (2) For $k > 0$, and for any $x \in A_k$, there exists $y \in A_{k-1}$ with $x E_{k-1}^k y$.
- (3) For $k < n$, $w E_k^n x$ if and only if there exists u and v such that $w E_{n-1}^n u E_k^{n-1} x$ and $w E_{n-1}^n v B_k^{n-1} x$.
- (4) If $w E_{n-1}^n z E_k^{n-1} x$, then either $w E_k^n x$ or there exists v such that $w B_{n-1}^n v E_k^{n-1} x$.

and such that (A, B, E) satisfies the same properties. If $x \in A_i$, we set $\dim(x) = i$.

If x is an element of the pasting scheme (A, E, B) , we use $R(x)$ to denote the smallest pasting scheme of (A, E, B) containing x .

Intuitively, a pasting scheme is a pasting of faces of several dimensions together (Johnson 1989). Kapranov and Voedvosky have their own formalization using some particular chain complexes of abelian groups (Kapranov and Voedvodsky 1991). We can see Figure 3 as a pasting scheme (S, E, B) . It suffices to set $S = \{\alpha, \beta, s_1 A, t_1 A, t_1 B, A, B\}$ endowed with the binary relations $B_1^2 = \{(A, s_1 A), (B, s_1 B)\}$, $E_1^2 = \{(A, t_1 A), (B, t_1 B)\}$, $B_0^2 = E_0^2 = \{\}$, $B_0^1 = \{(s_1 A, \alpha), (t_1 A, \alpha), (t_1 B, \alpha)\}$, $E_0^1 = \{(s_1 A, \beta), (t_1 A, \beta), (t_1 B, \beta)\}$. Figure 12 shows another more complicated example of a pasting scheme.

Here we only want to recall the construction of the free ω -category I^n generated by the faces of the n -cube. For more details see Crans (1995), for an analogous construction for simplices see Street (1987), and for some explicit calculations on I^n see Aitchison (1986).

Set $\underline{n} = \{1, \dots, n\}$ and let Λ^n be the set of maps from \underline{n} to $\{-, 0, +\}$. We say that an element x of Λ^n is of dimension p if $x^{-1}(0)$ is a set of p elements. We can identify the elements of Λ^n with the words of length n in the alphabet $\{-, 0, +\}$. The set Λ^n is supposed to be graded by the dimension of its elements. The set Λ^0 is the set of maps from the empty set to $\{-, 0, +\}$, and therefore it is a singleton.

Let $y \in \Lambda^i$. Let r_y be the map from $(\Lambda^n)_i$ to $(\Lambda^n)_{\dim(y)}$ defined as follows (with $x \in (\Lambda^n)_i$): for $k \in \underline{n}$, $x(k) \neq 0$ implies $r_y(x)(k) = x(k)$ and if $x(k)$ is the l -th zero of the sequence $x(1), \dots, x(n)$, then $r_y(x)(k) = y(\ell)$. If for any ℓ between 1 and i , $y(\ell) \neq 0$ implies $y(\ell) = (-)^\ell$, we set $b_y(x) := r_y(x)$. If for any ℓ between 1 and i , $y(\ell) \neq 0$ implies $y(\ell) = (-)^{\ell+1}$, we set $e_y(x) := r_y(x)$. We thus introduce the following binary relations: the set B_j^i of pairs (x, z) in $(\Lambda^n)_i \times (\Lambda^n)_j$ such that there exists y such that $z = b_y(x)$ and the set E_j^i of pairs (x, z)

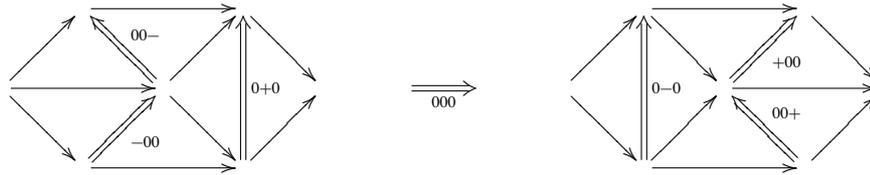


Fig. 13. The ω -category I^3

in $(\Lambda^n)_i \times (\Lambda^n)_j$ such that there exists y such that $z = e_y(x)$. Then Λ^n is a pasting scheme, and we have the following theorem.

Theorem 4.1. If $X \subset \Lambda^n$, let $R(X)$ be the sub-pasting scheme of $(\Lambda^n, B_j^i, E_j^i)$ generated by X . There is one and only one ω -category I^n such that:

- 1 The underlying set of I^n is included in the set of sub-pasting schemes of $(\Lambda^n, B_j^i, E_j^i)$ and it contains all pasting schemes like $R(\{x\})$ where x runs over Λ^n .
- 2 All elements of I^n are a composition of $R(\{x\})$ where x runs over Λ^n .
- 3 For x p -dimensional with $p \geq 1$, one has

$$s_{p-1}(R(\{x\})) = R(\{b_y(x), \dim(y) = p - 1\})$$

$$t_{p-1}(R(\{x\})) = R(\{e_y(x), \dim(y) = p - 1\}).$$

- 4 If $R(X)$ and $R(Y)$ are two elements of I^n such that $t_p(R(X)) = s_p(R(Y))$ for some p , then $R(X \cup Y) \in I^n$ and $R(X \cup Y) = R(X) *_p R(Y)$.

The oriented 2-cube is drawn in Figure 5. With the rules exposed in the above theorem, we can calculate $s_2R(00)$. We actually have $s_2R(00) = R(\{-0, 0+\})$. But $t_0R(-0) = R(-+)$ $= s_0R(0+)$. Then $s_2R(00) = R(\{-0\} \cup \{0+\}) = R(-0) *_0 R(0+)$.

The ω -category generated by a 3-cube is drawn in Figure 13. Let us give the example of the calculation of $s_2R(000)$. We have

$$s_2R(000) = R(\{-00, 0+0, 00-\}) = R(\{-00, 0++\} \cup \{-0-, 0+0\} \cup \{00-, 0++\})$$

since $0++, -0-, 0++ \in R(\{-00, 0+0, 00-\})$. It is easy to verify that $t_1R(\{-00, 0++\}) = s_1R(\{-0-, 0+0\})$ and $t_1R(\{-0-, 0+0\}) = s_1R(\{00-, 0++\})$. Therefore

$$s_2R(000) = R(\{-00, 0++\}) *_1 R(\{-0-, 0+0\}) *_1 R(\{00-, 0++\}).$$

It is then easy to verify that $R(\{-00, 0++\}) = R(-00) *_0 R(0++)$, $R(\{-0-, 0+0\}) = R(-0-) *_0 R(0+0)$ and $R(\{00-, 0++\}) = R(00-) *_0 R(0++)$.

The oriented 4-cube is represented in Figure 14.

4.2. The cubical singular nerve

The map that sends every ω -category \mathcal{C} to $\mathcal{N}^\square(\mathcal{C})_* = \omega Cat(I^*, \mathcal{C})$ induces a functor from ωCat to the category of cubical sets. If x is an element of $\omega Cat(I^n, \mathcal{C})$, then $\epsilon_i(x)$ is the ω -functor from I^{n+1} to \mathcal{C} defined by $\epsilon_i(x)(k_1 \dots k_{n+1}) = x(k_1 \dots \widehat{k}_i \dots k_{n+1})$ for all i between

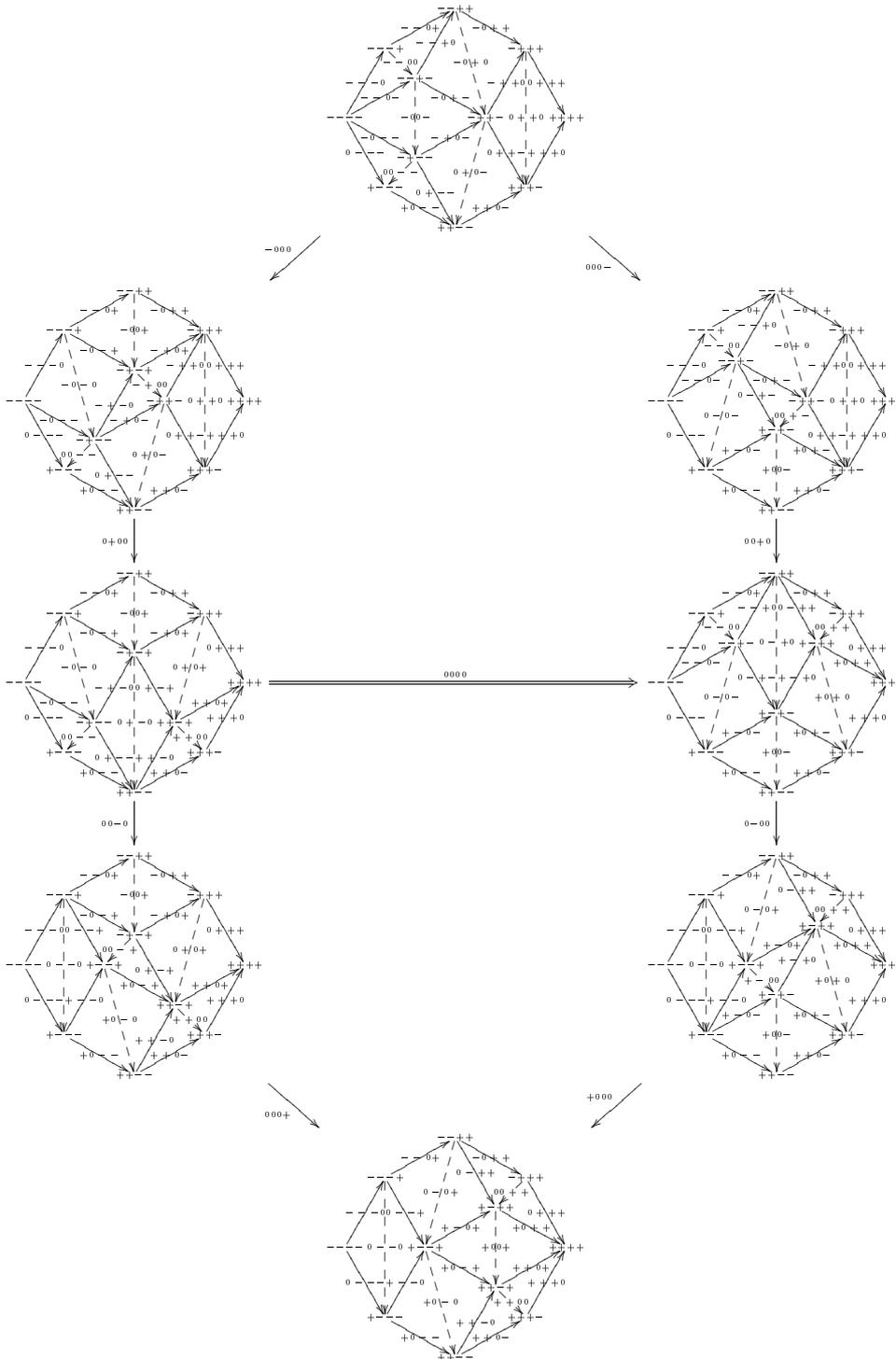


Fig. 14. The ω -category I^4

1 and $n + 1$ and $\partial_i^\alpha(x)$ is the ω -functor from I^{n-1} to \mathcal{C} defined by $\partial_i^\alpha(x)(k_1 \dots k_{n-1}) = x(k_1 \dots k_{i-1} \alpha k_i \dots k_{n-1})$ for all i between 1 and n .

The arrow ∂_i^α for a given i such that $1 \leq i \leq n$ induces a natural transformation from $\omega\text{Cat}(I^n, -)$ to $\omega\text{Cat}(I^{n-1}, -)$, and therefore, by Yoneda, corresponds to an ω -functor δ_i^α from I^{n-1} to I^n . This functor is defined on the faces of I^{n-1} by $\delta_i^\alpha(k_1 \dots k_{n-1}) = R(k_1 \dots [\alpha]_i \dots k_{n-1})$. The notation $[\dots]_i$ means that the term inside the brackets are in the i -th place.

Definition 4.2. The cubical set $\omega\text{Cat}(I^*, \mathcal{C})$ is called the *cubical singular nerve* of the ω -category \mathcal{C} .

This functor is a right adjoint. Its left adjoint is the functor F .

4.3. Construction of the corner homologies

The starting point is the cubical singular nerve $\omega\text{Cat}(I^*, \mathcal{C})$ of \mathcal{C} , which contains all n -cubes included in \mathcal{C} . The main idea in building the positive and negative corner homology of an ω -category \mathcal{C} is to separate the two differential maps $\partial^- = \sum_i (-1)^{i+1} \partial_i^-$ and $\partial^+ = \sum_i (-1)^{i+1} \partial_i^+$ and to separately consider the chain complexes of groups $(\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C}), \partial^\pm)$ (a bit like in Goubault (1995), where the author separates the horizontal and vertical differential maps of a bicomplex). However, the following proposition holds.

Proposition 4.2. Both chain complexes of groups $(\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C}), \partial^-)$ and $(\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C}), \partial^+)$ are acyclic.

Proof. It turns out that ϵ_1 is a chain retraction of $(\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C}), \partial^-)$. If $x \in \mathbb{Z}\omega\text{Cat}(I^0, \mathcal{C})$, then $\partial^\alpha \epsilon_1 x = \partial_1^\alpha \epsilon_1 x = x$. And for $x \in \mathbb{Z}\omega\text{Cat}(I^n, \mathcal{C})$, with $n \geq 1$, we actually have:

$$\begin{aligned} \partial^\alpha \circ \epsilon_1(x) + \epsilon_1 \circ \partial^\alpha(x) &= \sum_{i=1}^{i=n+1} (-1)^{i+1} \partial_i^\alpha \epsilon_1(x) + \sum_{i=1}^{i=n} (-1)^{i+1} \epsilon_1 \partial_i^\alpha(x) \\ &= x + \sum_{i=2}^{i=n+1} (-1)^{i+1} \epsilon_1 \partial_{i-1}^\alpha(x) + \sum_{i=1}^{i=n} (-1)^{i+1} \epsilon_1 \partial_i^\alpha(x) \\ &= x + \sum_{i=1}^{i=n} (-1)^i \epsilon_1 \partial_i^\alpha(x) + \sum_{i=1}^{i=n} (-1)^{i+1} \epsilon_1 \partial_i^\alpha(x) \\ &= x \end{aligned} \quad \square$$

The previous proposition entails the following definition.

Definition 4.3. Let \mathcal{C} be an ω -category and $\alpha \in \{-, +\}$. We denote by $\omega\text{Cat}(I^n, \mathcal{C})^\alpha$ the subset of elements x of $\omega\text{Cat}(I^n, \mathcal{C})$ satisfying the following conditions:

- The element x is a non-degenerate element of the cubical nerve.
- Any element of the form $\partial_{i_1}^\alpha \dots \partial_{i_p}^\alpha(x)$ is non-degenerate in the cubical nerve.

Then $\partial^\alpha(\mathbb{Z}\omega\text{Cat}(I^{*+1}, \mathcal{C})^\alpha) \subset \mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^\alpha$ by construction. We thus set

$$H_*^\alpha(\mathcal{C}, \mathbb{Z}) = H_*(\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^\alpha, \partial^\alpha),$$

and we call these homology theories the *negative (or positive according to α) corner homology* of \mathcal{C} . The cycles are called the *negative (or positive) corners* of \mathcal{C} . The maps H_*^\pm induce functors from ωCat_1 to Ab .

The second part of the definition is essential. Indeed, if a is a 1-dimensional morphism of \mathcal{C} , then the following element of $\omega\text{Cat}(I^2, \mathcal{C})$ is non-degenerate although its image by ∂_1^- and ∂_2^- are degenerate elements of $\text{Cat}(I^1, \mathcal{C})$:

$$\begin{array}{ccc}
 s_0(a) & \xrightarrow{a} & t_0(a) \\
 s_0(a) \uparrow & \searrow a & \uparrow a \\
 s_0(a) & \xrightarrow{s_0(a)} & s_0(a)
 \end{array}$$

The following proposition characterizes the elements of $\omega\text{Cat}(I^*, \mathcal{C})^\alpha$.

Proposition 4.3. Assume that x is an element of $\omega\text{Cat}(I^*, \mathcal{C})$. Then x is in $\omega\text{Cat}(I^*, \mathcal{C})^\alpha$ if and only if all $x(\alpha\dots 0\dots\alpha)$ (the notation $\alpha\dots 0\dots\alpha$ meaning that 0 appears only once) are 1-dimensional morphisms of \mathcal{C} .

Proof. If x is in $\omega\text{Cat}(I^n, \mathcal{C})^\alpha$, then all $\partial_{i_1}^\alpha \dots \partial_{i_p}^\alpha(x)$ are non-degenerate in the cubical nerve. But $y \in \omega\text{Cat}(I^1, \mathcal{C})$ is non-degenerate if and only if $y(R(0))$ is 1-dimensional. This gives the necessity of the condition. Conversely, assume that $x \notin \omega\text{Cat}(I^n, \mathcal{C})^\alpha$. Then there exists i between 1 and n such that $\partial_{i_1}^\alpha \dots \partial_{i_p}^\alpha x = \epsilon_i(z)$ with $p < n$ and some i_1, \dots, i_p and with $z \in \omega\text{Cat}(I^{n-p-1}, \mathcal{C})$. Then

$$\{x(\alpha\dots[-]_{i\dots}\alpha), x(\alpha\dots[0]_{i\dots}\alpha), x(\alpha\dots[+]_{i\dots}\alpha)\}$$

is a singleton for some i , and therefore $x(\alpha\dots[0]_{i\dots}\alpha)$ is 0-dimensional. Hence we have the sufficiency of the condition. □

As with the globular homology, the corner homologies do not yield functors from ωCat to Ab .

4.4. Examples of corners

Proposition 4.4. Let \mathcal{C} be an ω -category. The group $H_0^-(\mathcal{C})$ is the free abelian group generated by the final states of \mathcal{C} . The group $H_0^+(\mathcal{C})$ is the free abelian group generated by the initial states of \mathcal{C} .

Proof. The proof is obvious. □

Figure 15 provides a very simple example of a 1-dimensional negative corner. It consists of two ω -functors x and y from I^1 to \mathcal{C} such that $x(R(-)) = y(R(-))$ and such that $x(R(0))$ and $y(R(0))$ are 1-dimensional. Figure 16 shows an example of a 2-dimensional negative corner. If we suppose A and B to be oriented such that $s_1A = u *_0 x$, $t_1A = v *_0 y$, $s_1B = u *_0 z$ and $t_1B = v *_0 t$, then $A - B$ is a negative corner.

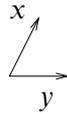


Fig. 15. A 1-dimensional negative corner

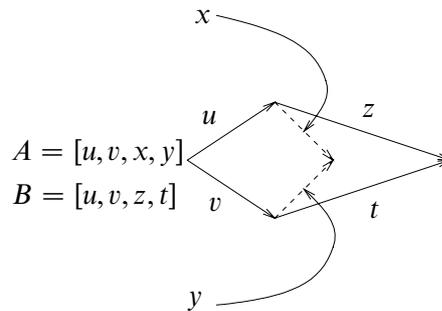


Fig. 16. A 2-dimensional negative corner

5. Filling of shells in the cubical singular nerve

In this section we develop a technical tool that will enable us to construct some operations on the cubical singular nerve of a globular ω -category (Section 6 and Section 10) and to construct the two Hurewicz morphisms (Section 7). The notion of shell and of the filling of (thin or not thin) shells has already appeared in Brown and Higgins (1981b) in the framework of ω -groupoids and in Al-Agl (1989) in the framework of cubical ω -categories (see Definition 10.1).

5.1. *Reminder about the concept of freeness of I^n*

A key property of I^n is its freeness, which means the following. Let p be some natural number. Let us call a realization (Λ^p, f_i) of (Λ^p, E, B) in an ω -category \mathcal{C} a family of maps f_i from $(\Lambda^p)_i$ to $\tau_i \mathcal{C}$, where $\tau_i \mathcal{C}$ is the ω -category obtained by keeping only those cells of \mathcal{C} of dimension lower than or equal to i . The realization (Λ^p, f_i) is called n -extendable if there exists only one functor f from $\tau_n I^p$ to \mathcal{C} such that for any $k \leq n$, the following commutative diagram holds:

$$\begin{array}{ccc}
 \tau_k I^p & \longrightarrow & \tau_n I^p \\
 R() \uparrow & & \downarrow f \\
 (\Lambda^p)_k & \xrightarrow{f_k} & \mathcal{C}
 \end{array}$$

By convention, any realization (Λ^p, f_i) is 0-appropriate and is already 0-extendable since f_0 is already a functor from $(I^p)_0$ to \mathcal{C} . Suppose we set up a notion of n -appropriate

realization. Then we have

$$\begin{array}{ccc}
 (\Lambda^p)_{n+1} & & \\
 \downarrow R() & \searrow f_{n+1} & \\
 \tau_{n+1}I^p & \dashrightarrow & \mathcal{C} \\
 \downarrow s_n & \downarrow t_n & \downarrow s_n \quad \downarrow t_n \\
 \tau_n I^p & \xrightarrow{f} & \mathcal{C}
 \end{array}$$

A realization (Λ^p, f_i) is $(n + 1)$ -appropriate if $s_n f_{n+1} = f s_n R$ and $t_n f_{n+1} = f t_n R$, and in this case, (Λ^p, f_i) is $(n + 1)$ -extendable (cf. Johnson (1989, page 224)). Thus to construct an ω -functor from I^n for some natural number n to an ω -category \mathcal{C} , it suffices to construct a realization that is appropriate in all dimensions.

5.2. Filling of shells using the freeness of I^n

Definition 5.1. An element $x \in \omega Cat(I^n, \mathcal{C})$ is *thin* if $x(R(0_n))$ is of dimension strictly lower than n . An element that is not thin is called *thick*.

Definition 5.2. An n -shell in the cubical singular nerve is a family of $2(n + 1)$ elements x_i^\pm of $\omega Cat(I^n, \mathcal{C})$ such that $\partial_i^\alpha x_j^\beta = \partial_{j-1}^\beta x_i^\alpha$ for $1 \leq i < j \leq n + 1$ and $\alpha, \beta \in \{-, +\}$.

Definition 5.3. The n -shell (x_i^\pm) is *fillable* if:

- 1 The sets $\{x_i^{(-)j}, 1 \leq i \leq n + 1\}$ and $\{x_i^{(-)i+1}, 1 \leq i \leq n + 1\}$ each have exactly one thick element and the others are thin.
- 2 If $x_{i_0}^{(-)i_0}$ and $x_{i_1}^{(-)i_1+1}$ are these two thick elements, then there exists $u \in \mathcal{C}$ such that $s_n(u) = x_{i_0}^{(-)i_0}(0_n)$ and $t_n(u) = x_{i_1}^{(-)i_1+1}(0_n)$.

The main proposition of this section is the following, which is an analogue of Al-Agl (1989 Proposition 2.7.3).

Proposition 5.1. Let (x_i^\pm) be a fillable n -shell with u as above. Then there exists one and only one element x of $\omega Cat(I^{n+1}, \mathcal{C})$ such that $x(0_{n+1}) = u$, and for $1 \leq i \leq n + 1$, and $\alpha \in \{-, +\}$ such that $\partial_i^\alpha x = x_i^\alpha$.

Proof. The underlying idea of the proof is as follows. If one wants to define an ω -functor from I^{n+1} to an ω -category \mathcal{C} , it suffices to construct $2(n + 1)$ ω -functors from the $2(n + 1)$ n -faces of I^{n+1} to \mathcal{C} that coincide on the intersection of their definition domains, and to fill correctly the interior of I^{n+1} .

We have necessarily $x(k_1 \dots [\pm]_i \dots k_n) = x_i^\pm(k_1 \dots k_n)$ and $x(0_{n+1}) = u$. Therefore there is at most one such realization (Λ^{n+1}, x_i) . It suffices to show that x is $(n + 1)$ -extendable. It is certainly 0-appropriate, and therefore 0-extendable. Suppose we have proved that this realization is p -appropriate and therefore p -extendable for every $p < n + 1$. First suppose that $p < n$. We want to prove that (Λ^{n+1}, x_i) is $(p + 1)$ -appropriate, that is, that $s_p x(k_1 \dots k_{n+1}) = x s_p R(k_1 \dots k_{n+1})$ and $t_p x(k_1 \dots k_{n+1}) = x t_p R(k_1 \dots k_{n+1})$ for $k_1 \dots k_{n+1}$ of dimension $p + 1$.

Let us verify the first equality. We have

$$\begin{aligned} s_p x(k_1 \dots k_{n+1}) &= s_p(\partial_i^{k_i} x)(k_1 \dots \widehat{k}_i \dots k_{n+1}) \text{ since } p + 1 < n + 1 \\ &= s_p x_i^{k_i}(k_1 \dots \widehat{k}_i \dots k_{n+1}) \\ &= x_i^{k_i} s_p R(k_1 \dots \widehat{k}_i \dots k_{n+1}) \text{ since the } x_i^\alpha \text{ are } \omega\text{-functors.} \end{aligned}$$

But $s_p R(k_1 \dots \widehat{k}_i \dots k_{n+1}) = \Psi(R(X_1), \dots, R(X_s))$ where Ψ consists only of compositions of X_h of dimension lower than p . Then

$$\begin{aligned} s_p x(k_1 \dots k_{n+1}) &= \Psi(x_i^{k_i} R(X_1), \dots, x_i^{k_i} R(X_s)) \\ &= \Psi((\partial_i^{k_i} x)R(X_1), \dots, (\partial_i^{k_i} x)R(X_s)) \\ &= x\Psi(\delta_i^{k_i} R(X_1), \dots, \delta_i^{k_i} R(X_s)) \text{ since } x \text{ is } p\text{-extendable} \\ &= x s_p R(k_1 \dots k_{n+1}) \text{ since } \delta_i^{k_i} \text{ is an } \omega\text{-functor.} \end{aligned}$$

It remains to prove that x is $(n + 1)$ -extendable. We have to prove that $s_n x(0_{n+1}) = x s_n R(0_{n+1})$ and $t_n x(0_{n+1}) = x t_n R(0_{n+1})$. Let us verify the first equality. We have $s_n R(0_{n+1}) = \Psi'(\delta_{i_0}^{(-)j_0}(0_n), Y_1, \dots, Y_s)$ where Ψ' is a function containing only composition maps and where Y_1, \dots, Y_s are elements of I^{n+1} of the form $R(\{x\})$ where x is a face of the $(n + 1)$ -cube. Then

$$\begin{aligned} x s_n R(0_{n+1}) &= x \Psi'(\delta_{i_0}^{(-)j_0}(0_n), Y_1, \dots, Y_s) \\ &= \Psi'(x(\delta_{i_0}^{(-)j_0}(0_n)), x(Y_1), \dots, x(Y_s)) \text{ since } x \text{ is } n\text{-extendable} \\ &= x(\delta_{i_0}^{(-)j_0}(0_n)) \text{ since } \dim(x(Y_1)), \dots, \dim(x(Y_s)) < n \\ &= x_{i_0}^{(-)j_0}(0_n) \\ &= s_n(u) \text{ by definition of } u \\ &= s_n x(0_{n+1}) \text{ by definition of } x \quad \square \end{aligned}$$

6. Two new simplicial nerves

As an immediate application of Section 5, we construct two families of connections on the cubical singular nerve of any ω -category. They will be useful later.

6.1. Cubical set with connections

Definition 6.1. (Al-Agl 1989) A cubical set with connections consists of a cubical set

$$((K_n)_{n \geq 0}, \partial_i^\alpha, \epsilon_i)$$

together with two additional families of degeneracy maps

$$\Gamma_i^\alpha : K_n \longrightarrow K_{n+1}$$

with $\alpha \in \{-, +\}$, $n \geq 1$ and $1 \leq i \leq n$ and satisfying the following axioms:

- 1 $\partial_i^\alpha \Gamma_j^\beta = \Gamma_{j-1}^\beta \partial_i^\alpha$ for all $i < j$ and all $\alpha, \beta \in \{-, +\}$
- 2 $\partial_i^\alpha \Gamma_j^\beta = \Gamma_j^\beta \partial_{i-1}^\alpha$ for all $i > j + 1$ and all $\alpha, \beta \in \{-, +\}$

- 3 $\partial_j^\pm \Gamma_j^\pm = \partial_{j+1}^\pm \Gamma_j^\pm = Id$
- 4 $\partial_j^\pm \Gamma_j^\mp = \partial_{j+1}^\pm \Gamma_j^\mp = \epsilon_j \partial_j^\pm$
- 5 $\Gamma_i^\pm \Gamma_j^\pm = \Gamma_{j+1}^\pm \Gamma_i^\pm$ if $i \leq j$
- 6 $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_{j+1}^\mp \Gamma_i^\pm$ if $i < j$
- 7 $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_j^\mp \Gamma_{i-1}^\pm$ if $i > j + 1$
- 8 $\Gamma_i^\pm \epsilon_j = \epsilon_{j+1} \Gamma_i^\pm$ if $i < j$
- 9 $\Gamma_i^\pm \epsilon_j = \epsilon_i \epsilon_j$ if $i = j$
- 10 $\Gamma_i^\pm \epsilon_j = \epsilon_j \Gamma_{i-1}^\pm$ if $i > j$.

There is an obviously defined small category Γ , such that the category of cubical sets with connections is exactly the category of presheaves over Γ . Hence the category of cubical set with connections is denoted by $Sets^{\Gamma^{op}}$.

The category of cubical sets with connections equipped with the non 1-contracting morphisms is denoted by $Sets_1^{\Gamma^{op}}$.

Looking back to the cubical singular nerve of a topological space, we can endow it with connections as follows:

$$\begin{aligned}\Gamma_i^-(f)(x_1, \dots, x_p) &= f(x_1, \dots, \max(x_i, x_{i+1}), \dots, x_p) \\ \Gamma_i^+(f)(x_1, \dots, x_p) &= f(x_1, \dots, \min(x_i, x_{i+1}), \dots, x_p).\end{aligned}$$

By keeping in a cubical set with connections only the morphisms Γ_i^- , or by exchanging the role of the face maps ∂_i^+ and ∂_i^- and by keeping only the morphisms Γ_i^+ , we obtain exactly a cubical set with connections in the sense of Brown and Higgins.

6.2. Construction of connections on the cubical singular nerve

Theorem 6.1. Let \mathcal{C} be an ω -category and let n be a natural number greater than or equal to 1. For any x in $\omega Cat(I^n, \mathcal{C})$ and for any i between 1 and n , we introduce two realizations $\Gamma_i^-(x)$ and $\Gamma_i^+(x)$ from Λ^{n+1} to \mathcal{C} by setting

$$\begin{aligned}\Gamma_i^-(x)(k_1 \dots k_{n+1}) &= x(k_1 \dots \max(k_i, k_{i+1}) \dots k_{n+1}) \\ \Gamma_i^+(x)(k_1 \dots k_{n+1}) &= x(k_1 \dots \min(k_i, k_{i+1}) \dots k_{n+1})\end{aligned}$$

where the set $\{-, 0, +\}$ is ordered by $- < 0 < +$. Then $\Gamma_i^-(x)$ and $\Gamma_i^+(x)$ yield ω -functors from I^{n+1} to \mathcal{C} , meaning two elements of $\omega Cat(I^{n+1}, \mathcal{C})$. Moreover, in this way, the cubical nerve of \mathcal{C} is equipped with a structure of a cubical complex with connections.

Proof. The construction of Γ_i^\pm is exactly the same as the one of connections on the cubical singular nerve of a topological space. Thus there is nothing to verify in the axioms of a cubical complex with connections except the relations mixing the two families of degeneracies Γ_i^+ and Γ_i^- : all other axioms are already verified in Brown and Higgins (1981b). Therefore it remains to verify that $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_{j+1}^\mp \Gamma_i^\pm$ if $i < j$, and $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_j^\mp \Gamma_{i-1}^\pm$ if $i > j + 1$, which can be done quickly. The only remaining point to be verified is that all realizations $\Gamma_i^\pm(x)$ yield ω -functors.

If x is an element of $\omega Cat(I^1, \mathcal{C})$, we can depict $\Gamma_1^-(x)$ as in Figure 17 and $\Gamma_1^+(x)$

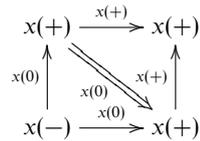


Fig. 17. $\Gamma_1^-(x)$

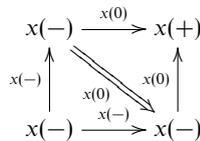


Fig. 18. $\Gamma_1^+(x)$

as in Figure 18. One can see immediately that $\Gamma_1^-(x)$ and $\Gamma_1^+(x)$ yield two elements of $\omega Cat(I^2, \mathcal{C})$.

Suppose we have proved that all realizations $\Gamma_i^\pm(x)$ for $1 \leq i \leq n$ yield ω -functors if $x \in \omega Cat(I^n, \mathcal{C})$. Now we want to prove that all realizations $\Gamma_i^\pm(y)$ for $1 \leq i \leq n + 1$ yield ω -functors if $y \in \omega Cat(I^{n+1}, \mathcal{C})$.

Because of the axioms of a cubical set with connections and because of the induction hypothesis, the $(\partial_j^z \Gamma_i^\pm(y))$ are ω -functors from I^n to \mathcal{C} . The family $(\partial_j^z \Gamma_i^\pm(y))$ is also an n -shell. We can fill it in a canonical way because the top dimensional elements are the same. □

We use $Sets^{\Delta^{op}}$ to denote the category of simplicial sets (May 1967). If A is a simplicial set, the axioms of simplicial sets imply that $C(A) = (\mathbb{Z}A_*, \partial = \sum (-1)^i \partial_i)$, where $\mathbb{Z}A_n$ means the free abelian group generated by the set A_n , is a chain complex. It is called the unnormalized chain complex of A . The normalized chain complex of A is the quotient chain complex $N(A) = C(A)/D(A)$ where $D(A)$ is the sub-complex of $C(A)$ generated by the degenerate elements. It turns out that the canonical morphism of a chain complex from $C(A)$ to $N(A)$ is a quasi-isomorphism (Weibel 1994).

As a consequence of the previous construction, we obtain two new simplicial nerves.

Proposition 6.2. Let \mathcal{C} be an ω -category and $\alpha \in \{-, +\}$. We set

$$\mathcal{N}_n^\alpha(\mathcal{C}) = \omega Cat(I^{n+1}, \mathcal{C})^\alpha,$$

and for all $n \geq 0$ and all $0 \leq i \leq n$,

$$\partial_i : \mathcal{N}_n^\alpha(\mathcal{C}) \longrightarrow \mathcal{N}_{n-1}^\alpha(\mathcal{C})$$

is the arrow ∂_{i+1}^α , and

$$\epsilon_i : \mathcal{N}_n^\alpha(\mathcal{C}) \longrightarrow \mathcal{N}_{n+1}^\alpha(\mathcal{C})$$

is the arrow Γ_{i+1}^α . In this way we obtain a simplicial set

$$(\mathcal{N}_*^\alpha(\mathcal{C}), \partial_i, \epsilon_i)$$

called the *negative (or positive according to α) corner simplicial nerve* of \mathcal{C} . The non-normalized chain complex associated to it gives exactly the corner homology of \mathcal{C} (of degree greater than or equal to 1). The maps \mathcal{N}^α induce a functor from ωCat_1 to $\text{Sets}^{\Delta^{op}}$.

Proof. The axioms of simplicial sets are immediate consequences of the axioms of cubical set with connections. □

Notice that the indices are shifted by one. Intuitively, these simplicial nerves consist of cutting an oriented n -hypercube by a hyperplane close to a corner (the negative one or the positive one): the intersection we get is the oriented $(n - 1)$ -simplex in the sense of Street (1987).

7. The oriented Hurewicz morphisms

In this section, we construct natural morphisms from the globular homology of an ω -category \mathcal{C} to its two corner homology theories. We call these maps the negative and positive oriented Hurewicz morphisms. Intuitively, they map any oriented loop with corners to its corresponding negative or positive corners (except for the 0-dimensional, see below).

7.1. The 0-dimensional case

The projection from $\mathbb{Z}\mathcal{C}_0 \times \mathbb{Z}\mathcal{C}_0$ to $\mathbb{Z}\mathcal{C}_0$ on the first (respectively, the second) component yields a natural group morphism from $H_0^{gl}(\mathcal{C})$ to $H_0^-(\mathcal{C})$ (respectively, $H_0^+(\mathcal{C})$). Indeed, if

$$X = (s_0(x - y), t_0(x - y)),$$

then the first component (respectively, the second) of X induces 0 on the corner homology. Thus we obtain a natural morphism h_0^\pm from $H_0^{gl}(\mathcal{C})$ to $H_0^\pm(\mathcal{C})$.

7.2. The 1-dimensional case

If x is a 1-dimensional morphism of \mathcal{C} , let $\square_1(x)$ be the element of $\omega\text{Cat}(I^1, \mathcal{C})$ defined by

$$\square_1(x)(R(-)) = s_0(x), \square_1(x)(R(0)) = x, \square_1(x)(R(+)) = t_0(x).$$

We extend \square_1 by linearity. If z is a 2-dimensional morphism of \mathcal{C} , let $\square_2^-(z)$ be the element of $\omega\text{Cat}(I^2, \mathcal{C})$ defined by the diagram

$$\begin{array}{ccc} t_0(z) & \xrightarrow{t_0(z)} & t_0(z) \\ \uparrow s_1(z) & \searrow z & \uparrow t_0(z) \\ s_0(z) & \xrightarrow{t_1(z)} & t_0(z) \end{array}$$

and let $\square_2^+(z)$ be the element of $\omega Cat(I^2, \mathcal{C})$ defined by

$$\begin{array}{ccc} s_0(z) & \xrightarrow{s_1(z)} & t_0(z) \\ \uparrow s_0(z) & \searrow z & \uparrow t_1(z) \\ s_0(z) & \xrightarrow{s_0(z)} & s_0(z) \end{array}$$

We extend $\square_2^-()$ and $\square_2^+()$ by linearity. Thus we get the following proposition.

Proposition 7.1. The natural linear map h_1^\pm from $\mathbb{Z}\mathcal{C}_1$ to $\mathbb{Z}\omega Cat(I^1, \mathcal{C})^\pm$, which maps x_1 to $\square_1(x_1)$ induces a natural map (still denoted by h_1^\pm) from $H_1^{gl}(\mathcal{C})$ to $H_1^\pm(\mathcal{C})$. We call it the 1-dimensional oriented Hurewicz morphism.

Proof. The proof is quite simple. If x_1 is a 1-dimensional globular cycle, then

$$\partial^\pm \square_1(x_1) = \square_1(x_1)(\pm) = 0$$

because of the definition of $\square_1(x_1)$. And a 1-dimensional globular boundary $s_1(x_2) - t_1(x_2)$ is mapped to $\partial^\pm(\square_2^-(x_2))$. □

7.3. The higher dimensional case

Proposition 7.2. For any natural number n greater than or equal to 2, there exists a unique natural map \square_n^- from \mathcal{C}_n (the n -dimensional cells of \mathcal{C}) to $\omega Cat(I^n, \mathcal{C})$ such that:

- 1 The equality $\square_n^-(x)(0_n) = x$ holds.
- 2 If $n \geq 3$ and $1 \leq i \leq n - 2$, then $\partial_i^\pm \square_n^- = \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- s_{n-1}$.
- 3 If $n \geq 2$ and $n - 1 \leq i \leq n$, then $\partial_i^- \square_n^- = \square_{n-1}^- d_{n-1}^{(-)}$ and $\partial_i^+ \square_n^- = \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}$.

Moreover, for $1 \leq i \leq n$, we have $\partial_i^\pm \square_n^- s_n u = \partial_i^\pm \square_n^- t_n u$ for any $(n + 1)$ -morphism u .

Proof. Let $k_1 \dots k_{n-1} \in \Lambda^{n-1}$. Then the natural map $ev_{k_1 \dots k_{n-1}} \partial_i^\pm \square_n^-$ from \mathcal{C}_n to \mathcal{C}_{n-1} that sends $x \in \mathcal{C}_n$ to $(\partial_i^\pm \square_n^- x)(k_1 \dots k_{n-1})$ corresponds by Yoneda to an ω -functor $f_{k_1 \dots k_{n-1}}$ from 2_{n-1} to 2_n , where 2_{n-1} (respectively, 2_n) is the free ω -category generated by a $(n - 1)$ -morphism A (respectively, an n -morphism B). Set

$$f_{k_1 \dots k_{n-1}}(A) = d_{n_{k_1 \dots k_{n-1}}}^{2_{k_1 \dots k_{n-1}}}(B),$$

where we still use the convention $d^- = s$ and $d^+ = t$. Then $f_{k_1 \dots k_{n-1}} = d_{n_{k_1 \dots k_{n-1}}}^{2_{k_1 \dots k_{n-1}}}$ because of the freeness of 2_{n-1} . Moreover, the inequality $n_{k_1 \dots k_{n-1}} \leq n - 1$ holds. Therefore

$$\begin{aligned} (\partial_i^\pm \square_n^- s_n u)(k_1 \dots k_{n-1}) &= d_{n_{k_1 \dots k_{n-1}}}^{2_{k_1 \dots k_{n-1}}} s_n u \\ &= d_{n_{k_1 \dots k_{n-1}}}^{2_{k_1 \dots k_{n-1}}} t_n u \\ &= (\partial_i^\pm \square_n^- t_n u)(k_1 \dots k_{n-1}) \end{aligned}$$

Therefore for $1 \leq i \leq n$, we have $\partial_i^\pm \square_n^- s_n u = \partial_i^\pm \square_n^- t_n u$ for any $(n + 1)$ -morphism u .

Suppose the proposition is true for $p < n$ with $n \geq 2$, and take an n -dimensional morphism x . Set $h_i^\pm = \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- s_{n-1}(x)$ for $1 \leq i \leq n - 2$, and set $h_i^- = \square_{n-1}^- d_{n-1}^{(-)}$ and $h_i^+ = \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x)$ for $i \geq n - 1$. We are going to verify that (h_i^\pm) is a fillable

$(n-1)$ -shell. It is sufficient to prove that for any i and any j between 1 and n , and any $\alpha, \beta \in \{-, +\}$, the equality $\partial_i^\alpha h_j^\beta = \partial_{j-1}^\beta h_i^\alpha$ holds as soon as $1 \leq i < j \leq n$.

First treat the case $i < j \leq n-2$. We have

$$\begin{aligned} \partial_i^\alpha h_j^\beta &= \partial_i^\alpha \Gamma_{n-2}^- \partial_j^\beta \square_{n-1}^- s_{n-1}(x), \text{ since } j < n-1 \\ &= \Gamma_{n-3}^- \partial_i^\alpha \partial_j^\beta \square_{n-1}^- s_{n-1}(x), \text{ since } i < n-2 \\ &= \Gamma_{n-3}^- \partial_{j-1}^\beta \partial_i^\alpha \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{j-1}^\beta h_i^\alpha, \text{ since } i < n-1. \end{aligned}$$

Now treat the case $i < j = n-1$. We have

$$\begin{aligned} \partial_i^\pm h_j^- &= \partial_i^\pm \square_{n-1}^- d_{n-1}^{(-)n-1} x \\ &= \partial_{j-1}^- \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- d_{n-1}^{(-)n-1} x \\ &= \partial_{j-1}^- h_i^\pm. \end{aligned}$$

We also have

$$\begin{aligned} \partial_i^\pm h_{n-1}^+ &= \partial_i^\pm \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\ &= \epsilon_{n-2} \partial_i^\pm \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\ &= \epsilon_{n-2} \partial_{n-2}^+ \partial_i^\pm \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{n-2}^+ \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{n-2}^+ h_i^\pm. \end{aligned}$$

Now treat the case $i < j = n$ and $i < n-1$. We have

$$\begin{aligned} \partial_i^\pm h_n^- &= \partial_i^\pm \square_{n-1}^- d_{n-1}^{(-)n-1}(x) \\ &= \partial_{n-1}^- \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- d_{n-1}^{(-)n-1}(x) \\ &= \partial_{n-1}^- h_i^\pm \end{aligned}$$

and

$$\begin{aligned} \partial_i^\pm h_n^+ &= \partial_i^\pm \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\ &= \epsilon_{n-2} \partial_i^\pm \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\ &= \epsilon_{n-2} \partial_{n-2}^+ \partial_i^\pm \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{n-1}^+ \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{n-1}^+ h_i^\pm. \end{aligned}$$

Finally, treat the case $i = n-1$ and $j = n$. We have

$$\begin{aligned} \partial_{n-1}^- h_n^- &= \partial_{n-1}^- \square_{n-1}^- d_{n-1}^{(-)n}(x) \\ &= \partial_{n-1}^- \square_{n-1}^- d_{n-1}^{(-)n-1}(x) \\ &= \partial_{n-1}^- h_{n-1}^- \\ \partial_{n-1}^- h_n^+ &= \partial_{n-1}^- \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\ &= \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \end{aligned}$$

$$\begin{aligned}
 &= \partial_{n-1}^+ \square_{n-1}^- d_{n-1}^{(-)^{n-1}}(x) \\
 &= \partial_{n-1}^+ h_{n-1}^- \\
 \partial_{n-1}^+ h_n^- &= \partial_{n-1}^+ \square_{n-1}^- d_{n-1}^{(-)^n}(x) \\
 &= \partial_{n-1}^- \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\
 &= \partial_{n-1}^- h_{n-1}^+ \\
 \partial_{n-1}^+ h_n^+ &= \partial_{n-1}^+ \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}(x) \\
 &= \partial_{n-1}^+ h_{n-1}^+ \quad \square
 \end{aligned}$$

Corollary 7.3. Let \mathcal{C} be an ω -category and let $n \geq 1$. Set $\mathcal{C}_{\leq n} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$. There exists one and only one natural map \square_n^- from $\mathcal{C}_{\leq n}$ to $\omega\text{Cat}(I^n, \mathcal{C})^-$ such that the following axioms hold:

- 1 If $n \geq 3$ and $1 \leq i \leq n - 2$, then $\partial_i^\pm \square_n^- = \Gamma_{n-2}^- \partial_i^\pm \square_{n-1}^- s_{n-1}$.
- 2 If $n \geq 2$ and $n - 1 \leq i \leq n$, then $\partial_i^- \square_n^- = \square_{n-1}^- d_{n-1}^{(-)^i}$ and $\partial_i^+ \square_n^- = \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^- s_{n-1}$.

Moreover, for $1 \leq i \leq n$, we have $\partial_i^\pm \square_n^- s_n = \partial_i^\pm \square_n^- t_n$.

Let $D_*^-(\mathcal{C})$ be the acyclic group complex generated by the degenerate elements of the negative simplicial nerve of \mathcal{C} with the conventions $D_n^-(\mathcal{C}) \subset \mathbb{Z}\omega\text{Cat}(I^n, \mathcal{C})^-$ and $D_1^-(\mathcal{C}) = D_0^-(\mathcal{C}) = 0$. Hence we get the following proposition.

Proposition 7.4. The natural linear map h_n^- from $\mathbb{Z}\mathcal{C}_n$ to $\mathbb{Z}\omega\text{Cat}(I^n, \mathcal{C})^-$ that sends x_n to $\square_n^- x_n$ for $n \geq 1$ and associates $(x_0, y_0) \in \mathbb{Z}\mathcal{C}_0 \times \mathbb{Z}\mathcal{C}_0$ to $x_0 \in \mathbb{Z}\mathcal{C}_0$ induces a natural complex morphism

$$h^- : C_*^{gl}(\mathcal{C}) \longrightarrow \mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^- / D_*^-(\mathcal{C}).$$

Proof. Let $n \geq 2$ and let $x_n \in \mathbb{Z}\mathcal{C}_n$. We have to compare $\sum_{j=1}^{j=n} (-1)^{j+1} \partial_j^- \square_n^- x_n$ and $\square_{n-1}^- (s_{n-1}(x_n) - t_{n-1}(x_n))$ modulo elements of $D_*^-(\mathcal{C})$. We get

$$\begin{aligned}
 &\sum_{j=1}^{j=n} (-1)^{j+1} \partial_j^- \square_n^- x_n \\
 &= \sum_{j=1}^{j=n-2} (-1)^{j+1} \Gamma_{n-2}^- \partial_j^- \square_{n-1}^- s_{n-1} x_n + (-1)^n \square_{n-1}^- (d_{n-1}^{(-)^{n-1}}(x_n) - d_{n-1}^{(-)^n}(x_n)) \\
 &= \square_{n-1}^- (s_{n-1}(x_n) - t_{n-1}(x_n)) \text{ mod } D_*^-(\mathcal{C}).
 \end{aligned}$$

Now let us treat the case $n = 1$. Let $x_1 \in \mathbb{Z}\mathcal{C}_1$. We immediately see that $h_0^-(s_0(x_1), t_0(x_1))$ and $\partial^-(\square_1^-(x_1))$ are equal. □

Corollary 7.5. The natural linear map h_n^- from $\mathbb{Z}\mathcal{C}_n$ to $\mathbb{Z}\omega\text{Cat}(I^n, \mathcal{C})^-$ that sends x_n to $\square_n^- x_n$ for $n \geq 1$ and associates $(x_0, y_0) \in \mathbb{Z}\mathcal{C}_0 \times \mathbb{Z}\mathcal{C}_0$ to $x_0 \in \mathbb{Z}\mathcal{C}_0$ induces a natural linear map from the globular homology to the negative corner homology of \mathcal{C} .

Proof. The proof follows from the fact that for $n \geq 2$, the n -th homology group of the quotient chain complex $\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^- / D_*^-(\mathcal{C})$ is the $(n - 1)$ -th homology group of the normalized chain complex associated to the corner simplicial nerve of \mathcal{C} . □

Now let us expose the construction of h_n^+ (without proof).

Proposition 7.6. Let \mathcal{C} be an ω -category and let $n \geq 1$. Set $\mathcal{C}_{\leq n} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$. There exists one and only one natural map \square_n^+ from $\mathcal{C}_{\leq n}$ to $\omega\text{Cat}(I^n, \mathcal{C})^+$ such that the following axioms hold:

- 1 If $n \geq 3$ and $1 \leq i \leq n-2$, then $\partial_i^{\pm} \square_n^+ = \Gamma_{n-2}^+ \partial_i^{\pm} \square_{n-1}^+ s_{n-1}$.
- 2 If $n \geq 2$ and $n-1 \leq i \leq n$, then $\partial_i^+ \square_n^+ = \square_{n-1}^+ d_{n-1}^{(-)^{i+1}}$ and $\partial_i^- \square_n^+ = \epsilon_{n-1} \partial_{n-1}^+ \square_{n-1}^+ s_{n-1}$.

Moreover, for $1 \leq i \leq n$, we have $\partial_i^{\pm} \square_n^+ s_n = \partial_i^{\pm} \square_n^+ t_n$.

Proposition 7.7. The natural linear map h_n^+ from $\mathbb{Z}\mathcal{C}_n$ to $\mathbb{Z}\omega\text{Cat}(I^n, \mathcal{C})^+$ that sends x_n to $\square_n^+ x_n$ for $n \geq 1$ and sends $(x_0, y_0) \in \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_0$ to y_0 induces a natural complex morphism

$$h^+ : C_*^{gl}(\mathcal{C}) \longrightarrow \mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^+ / D_*^+(\mathcal{C})$$

where $D_*^+(\mathcal{C})$ is the sub-complex of $\mathbb{Z}\omega\text{Cat}(I^*, \mathcal{C})^+$ generated by the degenerate elements of the positive corner simplicial nerve. Therefore h_n^+ induces a natural linear map from $H_n^{gl}(\mathcal{C})$ to $H_n^+(\mathcal{C})$.

8. Toward an ‘oriented algebraic topology’

8.1. Homotopic ω -categories

Now we want to speculate about the notion of homotopic ω -categories. We proceed as in algebraic topology by defining a notion of homotopy between non 1-contracting ω -functors, and hence we deduce a notion of homotopy equivalence of ω -category. We are obliged to work with non 1-contracting ω -functors because of the globular and corner homologies.

Intuitively, we could say that two non 1-contracting ω -functors f and g from \mathcal{C} to \mathcal{D} are homotopic if $f(\mathcal{C})$ and $g(\mathcal{C})$ have the same ‘oriented topology’. So a first attempt of definition could be: the ω -functors f and g are homotopic if for any $x \in \mathcal{C}$, $f(x) \sim g(x)$. Unfortunately, $f(x)$ and $g(x)$ do not necessarily have the same dimension. So this definition does not make sense, except if x is 0-dimensional: in this case, $f(x) \sim g(x)$ means $f(x) = g(x)$. It is plausible to think that if f and g were homotopic, then $C_*^{gl}(f)$ and $C_*^{gl}(g)$ would be two chain homotopic morphisms from $C_*^{gl}(\mathcal{C})$ to $C_*^{gl}(\mathcal{D})$. So we propose the following definition.

Definition 8.1. Let f and g be two non 1-contracting ω -functors from an ω -category \mathcal{C} to an ω -category \mathcal{D} . The morphisms f and g are *homotopic* if the following conditions hold:

- 1 For any 0-dimensional x of \mathcal{C} , one has $f(x) = g(x)$.
- 2 There exists a linear map h_1 from $\mathbb{Z}\mathcal{C}_1$ to $\mathbb{Z}\mathcal{D}_2$ such that $(s_1 - t_1)h_1(x) = f(x) - g(x)$ for any 1-morphism x of \mathcal{C} .
- 3 For any $n \geq 2$, there exists a linear map h_n from $\mathbb{Z}\mathcal{C}_n$ to $\mathbb{Z}\mathcal{D}_{n+1}$ such that for any n -morphism x of \mathcal{C} , we have

$$h_{n-1}(s_{n-1} - t_{n-1})(x) + (s_n - t_n)h_n(x) = f(x) - g(x) \text{ mod } \mathbb{Z}\mathcal{D}_{n-1}.$$

We denote this property by $f \sim_{(h,*)} g$, or more simply $f \sim g$ whenever it is not necessary to specify the homotopy map.

Proposition 8.1. The binary relation ‘is homotopic to’ is an equivalence relation on the collection of non 1-contracting ω -functors from a given ω -category \mathcal{C} to a given ω -category \mathcal{D} .

Proof. We have $f \sim f$ since $f \sim_{(0)} f$. If $f \sim_{(h,*)} g$, then $g \sim_{(-h,*)} f$. Now suppose that $f \sim_{(h_1^*)} g$ and $g \sim_{(h_2^*)} k$. Then $f \sim_{(h_1^*+h_2^*)} k$. \square

Proposition 8.2. The homotopy equivalence of non 1-contracting ω -functors is compatible with the composition of non 1-contracting ω -functors in the following sense. Take a diagram in ωCat_1

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} \mathcal{E} \xrightarrow{k} \mathcal{F} .$$

If $g \sim h$, then $g \circ f \sim h \circ f$, $k \circ g \sim k \circ h$ and $k \circ g \circ f \sim k \circ h \circ f$.

Proof. Suppose that $g \sim_H h$. Define H^f ($n \geq 1$) by:

- 1 If $x \in \mathcal{C}_n$ and if $\dim(f(x)) < n$, then $H_n^f(x) := 0$.
- 2 If $x \in \mathcal{C}_n$ and if $\dim(f(x)) = n$, then $H_n^f(x) := H_n(f(x))$.

If x is 1-dimensional, then

$$\begin{aligned} g \circ f(x) - h \circ f(x) &= (s_1 - t_1)H_1(f(x)) \text{ since } f \text{ is non 1-contracting} \\ &= (s_1 - t_1)H_1^f(x). \end{aligned}$$

If x is of dimension n greater than 2, then either:
 $\dim(f(x)) < n$, and in this case

$$\begin{aligned} g \circ f(x) - h \circ f(x) \text{ mod } \mathbb{Z}\mathcal{E}_{n-1} &= 0 \\ &= H_{n-1}(s_{n-1} - t_{n-1})(f(x)) + (s_n - t_n)H_n^f(x) \\ &= H_{n-1} \circ f \circ (s_{n-1} - t_{n-1})(x) + (s_n - t_n)H_n^f(x) \text{ since } f \text{ is an } \omega\text{-functor} \\ &= H_{n-1}^f(s_{n-1} - t_{n-1})(x) + (s_n - t_n)H_n^f(x) \end{aligned}$$

since $H_{n-1} \circ f \circ (s_{n-1} - t_{n-1})(x) = 0$.

Or:

$\dim(f(x)) = n$, and in that case

$$\begin{aligned} g \circ f(x) - h \circ f(x) \text{ mod } \mathbb{Z}\mathcal{E}_{n-1} &= H_{n-1}(s_{n-1} - t_{n-1})(f(x)) + (s_n - t_n)H_n^f(x) \\ &= H_{n-1} \circ f \circ (s_{n-1} - t_{n-1})(x) + (s_n - t_n)H_n^f(x) \text{ since } f \text{ is an } \omega\text{-functor} \end{aligned}$$

Since $f(x)$ is n -dimensional, we have $f(s_{n-1}(x)) = s_{n-1} \circ f(x)$ and $f(t_{n-1}(x)) = t_{n-1} \circ f(x)$ are $(n - 1)$ -dimensional. Therefore

$$g \circ f(x) - h \circ f(x) \text{ mod } \mathbb{Z}\mathcal{E}_{n-1} = H_{n-1}^f(s_{n-1} - t_{n-1})(x) + (s_n - t_n)H_n^f(x).$$

Therefore $g \circ f \sim_{H^f} h \circ f$.

Now define kH by ${}^kH_n(x) := k(H_n(x))$ for $x \in \mathcal{C}_n$. Then for x 1-dimensional, we have

$$\begin{aligned} & k \circ g(x) - k \circ h(x) \\ &= k \circ (s_1 - t_1)H_1(x) \\ &= (s_1 - t_1){}^kH_1(x) \text{ since } k \text{ is an } \omega\text{-functor.} \end{aligned}$$

And for x of dimension greater than 2, we have

$$\begin{aligned} & k \circ g(x) - k \circ h(x) \text{ mod } \mathbb{Z}\mathcal{F}_{n-1} \\ &= k(H_{n-1}(s_{n-1} - t_{n-1})(x) + (s_n - t_n)H_n(x)) \\ &= {}^kH_{n-1}(s_{n-1} - t_{n-1})(x) + (s_n - t_n){}^kH_n(x) \text{ since } k \text{ is an } \omega\text{-functor.} \end{aligned}$$

Therefore $k \circ g \sim_{kH_*} k \circ h$. □

Proposition 8.3. Let f and g be two non 1-contracting ω -functors such that for any x of dimension lower than n , we have $f(x)$ and $g(x)$ are two homotopic $\dim(x)$ -dimensional morphisms. Then f and g are homotopic as n -functor from $\tau_n\mathcal{C}$ to $\tau_n\mathcal{D}$ (when we consider only the morphisms of dimension lower than n). In other words, the ‘oriented topology’ is the same in dimension lower than n .

Proof. For any x of dimension $1 \leq d \leq n$, there exists $h_d(x) \in \mathbb{Z}\mathcal{D}_{d+1}$ such that $(s_d - t_d)(h_d(x)) = f(x) - g(x)$. By convention, we take $h_d(x) = 0$ whenever $f(x) = g(x)$. However $s_{d-1}(f(x) - g(x)) = s_{d-1}(s_d - t_d)(h_d(x)) = 0$ and in the same way, we have $t_{d-1}(f(x) - g(x)) = t_{d-1}(s_d - t_d)(h_d(x)) = 0$. Therefore $h_{d-1}(s_{d-1}x) = h_{d-1}(t_{d-1}x) = 0$ and

$$h_{d-1}(s_{d-1} - t_{d-1})(x) + (s_d - t_d)h_d(x) = f(x) - g(x). \quad \square$$

Definition 8.2. Let \mathcal{C} and \mathcal{D} be two ω -categories. They are *homotopic* if and only if there exists a non 1-contracting ω -functor f from \mathcal{C} to \mathcal{D} and a non 1-contracting ω -functor g from \mathcal{D} to \mathcal{C} such that $f \circ g \sim Id_{\mathcal{D}}$ and $g \circ f \sim Id_{\mathcal{C}}$. We say that f and g are *homotopy equivalences* between the two ω -categories \mathcal{C} and \mathcal{D} .

Proposition 8.4. The homotopy equivalence is indeed an equivalence relation on the collection of ω -categories.

Proof. This relation is obviously reflexible and symmetric. It remains to prove the transitivity. Let us consider the following diagram in ωCat_1 :

$$\mathcal{C} \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{h} \\ \xleftarrow{k} \end{array} \mathcal{E}$$

Suppose that $g \circ f \sim Id_{\mathcal{C}}$, $f \circ g \sim Id_{\mathcal{D}}$, $h \circ k \sim Id_{\mathcal{E}}$ and $k \circ h \sim Id_{\mathcal{D}}$. Then $g \circ k \circ h \circ f \sim g \circ f \sim Id_{\mathcal{C}}$ and $h \circ f \circ g \circ k \sim h \circ k \sim Id_{\mathcal{E}}$. □

Now we can give some examples of homotopic ω -categories.

Proposition 8.5. For any natural number $p \geq 1$ and $q \geq 1$, 2_p (the free ω -category generated by one p -morphism) and 2_q are homotopic.

Proof. If $p = q$, it is trivial. Suppose that $p > q$. Let f be the only ω -functor from $\langle A \rangle = 2_p$ to $\langle B \rangle = 2_q$ such that $f(A) = B$ and let g be the unique functor from $\langle B \rangle$ to $\langle A \rangle$ such that $g(B) = s_q(A)$. Then $f \circ g = Id_{2_q}$ so $f \circ g$ and Id_{2_q} are homotopic as ω -functors. Now consider $g \circ f$ and Id_{2_p} . Set

$$\begin{aligned} h_r &= 0 \text{ if } 1 \leq r < q \\ h_r(s_r A) &= 0 \text{ and } h_r(t_r A) = s_{r+1} A \text{ if } q \leq r < p \\ h_r(A) &= 0 \text{ if } r \geq p. \end{aligned}$$

First suppose that $q = 1$. Then we have $(s_1 - t_1)h_1(s_1 A) = 0 = (g \circ f)(s_1 A) - s_1 A$ and $(s_1 - t_1)h_1(t_1 A) = s_1 A - t_1 A = (g \circ f)(t_1 A) - t_1 A$, and for any $1 < r < p$, we have

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(s_r A) + (s_r - t_r)h_r(s_r A) &= -s_r A \\ &= g \circ f(s_r A) - s_r A \text{ mod } (2_p)_{r-1} \end{aligned}$$

and

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(t_r A) + (s_r - t_r)h_r(t_r A) &= -s_r A + s_r A - t_r A \\ &= s_1 A - t_r A \text{ mod } (2_p)_{r-1} \text{ since } r > 1 \\ &= g \circ f(t_r A) - t_r A. \end{aligned}$$

In order to complete the case $q = 1$, now suppose that $r = p$. Then

$$\begin{aligned} h_{p-1}(s_{p-1} - t_{p-1})(A) + (s_p - t_p)h_p(A) &= -A \\ &= s_1 A - A \text{ mod } (2_p)_{p-1} \text{ since } p \geq 2 \\ &= g \circ f(A) - A. \end{aligned}$$

Now suppose that $q > 1$. The different cases may be treated in the same way. First set $r = 1$. Then $(s_1 - t_1)h_1(s_1 A) = 0 = (g \circ f)(s_1 A) - s_1 A$ and $(s_1 - t_1)h_1(t_1 A) = s_1 A - t_1 A = (g \circ f)(t_1 A) - t_1 A$, and for any $1 < r < q$, we have

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(s_r A) + (s_r - t_r)h_r(s_r A) &= 0 \\ &= g \circ f(s_r A) - s_r A \end{aligned}$$

and

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(t_r A) + (s_r - t_r)h_r(t_r A) &= 0 \\ &= g \circ f(t_r A) - t_r A. \end{aligned}$$

For $r = q$, we have

$$\begin{aligned} h_{q-1}(s_{q-1} - t_{q-1})(s_q A) + (s_q - t_q)h_q(s_q A) &= 0 \\ &= g \circ f(s_q A) - s_q A \end{aligned}$$

and

$$\begin{aligned} h_{q-1}(s_{q-1} - t_{q-1})(t_q A) + (s_q - t_q)h_q(t_q A) &= s_q A - t_q A \\ &= g \circ f(t_q A) - t_q A. \end{aligned}$$

For $q < r < p$, we have

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(s_r A) + (s_r - t_r)h_r(s_r A) &= -s_r A \\ &= s_q A - s_r A \bmod (2_p)_{r-1} \text{ since } q \leq r - 1 \\ &= g \circ f(s_r A) - s_r A \end{aligned}$$

and

$$\begin{aligned} h_{r-1}(s_{r-1} - t_{r-1})(t_r A) + (s_r - t_r)h_r(t_r A) &= -s_r A + s_r A - t_r A \\ &= s_q A - t_r A \bmod (2_p)_{r-1} \text{ since } q \leq r - 1 \\ &= g \circ f(t_r A) - t_r A \bmod (2_p)_{r-1}. \end{aligned}$$

The remaining case is $r = p$:

$$\begin{aligned} h_{p-1}(s_{p-1} - t_{p-1})(A) + (s_p - t_p)h_p(A) &= -A \\ &= s_q A - A \bmod (2_p)_{p-1} \text{ since } q \leq p - 1 \\ &= g \circ f(A) - A. \quad \square \end{aligned}$$

Definition 8.3. Let \mathcal{C} be an ω -category. Let I and F be some sets of 0-morphisms of \mathcal{C} . The *bilocalization* of \mathcal{C} with respect to I and F the sub-category of \mathcal{C} consists of the n -morphisms f such that $s_0 f \in I$ and $t_0 f \in F$ with the induced structure of an ω -category. This ω -category is denoted by $\mathcal{C}(I, F)$.

In the following, the set I will always be the set of initial states and the set F the set of final states of the considered ω -category.

Now we prove that the bilocalization of the ω -category \mathcal{C} of Figure 4 with respect to I (the intersection of u and e) and F (the intersection of z and d) is homotopic to the ω -category of Figure 7, denoted by $G_1[A, B]$. Let f be the unique ω -functor from $\mathcal{C}(s_0(u), t_0(z))$ to $G_1[A, B]$ that maps any 1-path homotopic to γ_1 to A and any 1-path homotopic to γ_4 to B . Let g be the unique ω -functor from $G_1[A, B]$ to $\mathcal{C}(s_0(u), t_0(z))$ that maps A to γ_1 and B to γ_4 . Then $f \circ g = Id_{G_1[A, B]}$. It remains to prove that $g \circ f$ and $Id_{\mathcal{C}(s_0(u), t_0(z))}$ are homotopic ω -functors. For any 1-morphism x of $\mathcal{C}(s_0(u), t_0(z))$, $g \circ f(x)$ and x are homotopic 1-morphisms. Let $h_1(x)$ be the element of $\mathbb{Z}\mathcal{C}(s_0(u), t_0(z))_2$ such that $(s_1 - t_1)h_1(x) = g \circ f(x) - x$. Take $h_2 = 0$. We have to verify that for any 2-morphism C , we have

$$h_1(s_1 - t_1)C + (s_2 - t_2)h_2C = g \circ f(C) - C \bmod \mathcal{C}(s_0(u), t_0(z))_1.$$

Suppose for example that $s_1 C$ is homotopic to γ_1 . Then $h_1 s_1 C$ is the unique element of $\mathbb{Z}\mathcal{C}(s_0(u), t_0(z))_2$ such that $(s_1 - t_1)h_1 s_1 C = \gamma_1 - s_1 C$. And $h_1 t_1 C$ is the unique element of $\mathbb{Z}\mathcal{C}(s_0(u), t_0(z))_2$ such that $(s_1 - t_1)h_1 t_1 C = \gamma_1 - t_1 C$. Then $h_1(s_1 - t_1)C$ is the unique element of $\mathbb{Z}\mathcal{C}(s_0(u), t_0(z))_2$ such that $(s_1 - t_1)h_1(s_1 - t_1)C = t_1 C - s_1 C$. Therefore $h_1(s_1 - t_1)C = -C$. Hence we have the result.

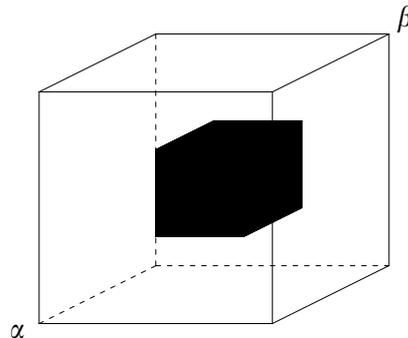


Fig. 19. A 2-semaphore

Now we will give some other examples without proof. In Figure 8, the bilocalization of the depicted ω -categories with respect to their sets of initial and final states are homotopic.

Figure 19 represents a cubical set with a 3-dimensional cubical hole. Then the bilocalization of this 3-category with respect to its sets of initial and final states is homotopic to $G_2[A, B]$, the 2-category of Figure 2 generated by two non homotopic 2-morphisms A and B having the same 1-source and the same 1-target.

8.2. Invariance of the globular and corner homologies

Theorem 8.6. Let f and g be two non 1-contracting ω -functors from \mathcal{C} to \mathcal{D} . Suppose that f and g are homotopic. Then for all natural numbers n , we have f and g induce linear maps from $H_n^{gl}(\mathcal{C})$ to $H_n^{gl}(\mathcal{D})$ and, moreover, $H_n^{gl}(f) = H_n^{gl}(g)$.

Proof. Take two homotopic ω -functors f and g . Let x_1 be a globular 1-cycle. Then $(f - g)(x_1) = (s_1 - t_1)h_1(x)$. Therefore $(f - g)(x_1) = \partial(h_1(x))$. Now take a globular n -cycle x_n with $n \geq 2$. Then $s_{n-1}x_n = t_{n-1}x_n$. Therefore $f(x_n) - g(x_n) = (s_n - t_n)h_n x_n \text{ mod } \mathcal{D}_{n-1}$. \square

The analogous statement for the corner homologies is still a conjecture and only proved in the following particular case (see Proposition 8.3).

Theorem 8.7. Let f and g be two non 1-contracting ω -functor from \mathcal{C} to \mathcal{D} . Suppose that for any x of dimension strictly lower than n , $f(x) = g(x)$, and that for x n -dimensional, $f(x)$ and $g(x)$ are two homotopic n -morphisms of \mathcal{D} . Then for any $p \leq n$, f_p (respectively, g_p) yield linear maps from $H_p^\alpha(\mathcal{C})$ to $H_p^\alpha(\mathcal{D})$ for $\alpha \in \{-, +\}$, and, moreover, $H_p^\alpha(f) = H_p^\alpha(g)$.

Proof. We will give the proof for $\alpha = -$. The homology of the non-normalized complex associated to a simplicial group is equal to its homotopy (Weibel 1994, Theorem 8.3.8). Therefore it suffices to find a homotopy between $f(x)$ and $g(x)$ in $\mathcal{N}^-(\mathcal{C})$ for any $x \in \omega Cat(I^n, \mathcal{C})$. We can suppose without loss of generality that there exists an $(n + 1)$ -dimensional morphism u of \mathcal{C} such that $s_n(u) = f(x)(0_n)$ and $t_n(u) = g(x)(0_n)$. Then consider the following realizations h_i^\pm of I^n for $1 \leq i \leq n + 1$: for i between 1 and $n - 1$, $h_i^\pm = \Gamma_{n-1}^- \partial_i^\pm x = \Gamma_{n-1}^- \partial_i^\pm y$, $h_n^- = d_n^\epsilon(u)$, $h_{n+1}^- = d_n^{\epsilon+1}(u)$ with ϵ equal to 0 or 1 depending on the parity of n , and finally $h_n^+ = h_{n+1}^+ = \epsilon_n \partial_n^+ x = \epsilon_n \partial_n^+ y$. We obtain the fillable n -shell

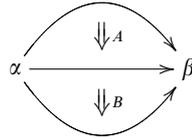


Fig. 20. Composition of two 2-morphisms

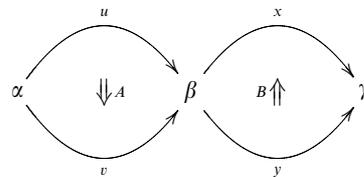


Fig. 21. Two 0-composable 2-morphisms

that has already appeared in Proposition 7.2. The corresponding element of $\omega Cat(I^{n+1}, \mathcal{C})$ yields a homotopy in $\mathcal{N}^-(\mathcal{C})$ between $f(x)$ and $g(x)$. \square

Conjecture 8.8. Let f and g be two non 1-contracting ω -functors from \mathcal{C} to \mathcal{D} . Suppose that f and g are homotopic. Then for all natural numbers n , we have f and g induce linear maps from $H_n^\pm(\mathcal{C})$ to $H_n^\pm(\mathcal{D})$, and, moreover, $H_n^\pm(f) = H_n^\pm(g)$.

9. Some open questions and perspectives

9.1. Some interesting problems in mathematics

First, we return to the globular homology of an ω -category. We propose here a small modification of its definition. Consider the ω -category \mathcal{C} of Figure 20 where A and B are two 2-morphisms that are supposed to be composable. Then $H_2^{gl}(\mathcal{C}) \neq 0$ since $C_3^{gl}(\mathcal{C}) = 0$ and since $(s_1 - t_1)(A *_1 B - A - B) = 0$. However, this globular 2-cycle corresponds to nothing real in \mathcal{C} . As consequence of this small calculation, we obtain that $H_2^{gl}(I^3) \neq 0$ (cf. Figure 13). It suffices to consider, for example, the globular 2-cycle $C *_1 D - C - D$ with $C = R(-00) *_0 R(0++)$ and $D = R(-0-) *_0 R(0+0)$.

Consider the ω -category of Figure 21 where α, β and γ are 0-morphisms, u, v, x and y are 1-morphisms and A and B two 2-morphisms. There are at least two elements of $\mathbb{Z}\mathcal{C}_2$ between $u *_0 x$ and $v *_0 y : A *_0 y - u *_0 B, A *_0 x - v *_0 B$. Therefore $A *_0 y - u *_0 B - A *_0 x + v *_0 B$ is a globular 2-cycle, which means nothing geometrically.

So we propose to modify the definition of the globular homology as follows. Let $\widehat{\omega Cat}_1$ be the category whose objects are globular ω -categories such that any $(n+1)$ -morphism X is invertible with respect to $*_n$ as soon as $n \geq 1$ (that is, there exists an $(n+1)$ -morphism X^{-1} such that $s_n X^{-1} = t_n X, t_n X^{-1} = s_n X, X *_n X^{-1} = s_n X, X^{-1} *_n X = t_n X$) and whose morphisms are non 1-contracting ω -functors. Let us use $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ to denote the left adjoint

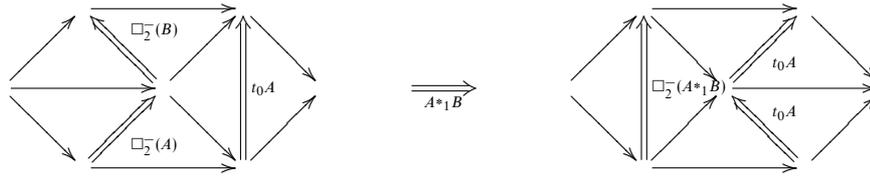


Fig. 22. $h_2^-(A *_1 B - A - B)$ is a boundary

functor to the forgetful functor from $\widehat{\omega Cat}_1$ to ωCat_1 . We set $C_0^{gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_0 \oplus \mathbb{Z}\mathcal{C}_1$, $C_1^{gl}(\mathcal{C}) = \mathbb{Z}\mathcal{C}_1$, and $C_n^{gl}(\mathcal{C})$ for $n \geq 2$ is the free abelian group generated by $\widehat{\mathcal{C}}_n$ quotiented by the relations $A + B = A *_1 B$ mod $\mathbb{Z}\mathcal{C}_{n-1}$ if A and B are two n -morphisms such that $t_{n-1}A = s_{n-1}B$. With the same differential map, we obtain a new globular homology theory. Let us denote it by H_*^{new-gl} .

With this new homology theory, the above problems disappear. It is obvious for the first, so we will just show the reasoning for the second one. In $C_2^{new-gl}(\mathcal{C})$ one has

$$\begin{aligned}
 A *_0 x - v *_0 B &= A *_0 x + (v *_0 B)^{-1} \\
 &= A *_0 x + v *_0 B^{-1} \\
 &= (A *_0 x) *_1 (v *_0 B^{-1}) \\
 &= (A *_1 v) *_0 (x *_1 B^{-1}) \\
 &= A *_0 B^{-1} \\
 &= (u *_1 A) *_0 (B^{-1} *_1 y) \\
 &= (u *_0 B^{-1}) *_1 (A *_0 y) \\
 &= u *_0 B^{-1} + A *_0 y \\
 &= (u *_0 B)^{-1} + A *_0 y \\
 &= A *_0 y - u *_0 B.
 \end{aligned}$$

It suffices to consider the labelled 3-cube of Figure 22 to see that $h_2^-(A *_1 B - A - B)$ is a boundary in the negative corner homology. In the same way, we can prove that $h_2^+(A *_1 B - A - B)$ is also a boundary, this time in the positive corner homology. There is a canonical linear map $H_*^{gl}(\mathcal{C}) \rightarrow H_*^{new-gl}(\mathcal{C})$. Therefore there exists at least for $n = 0, 1, 2$ a natural linear map \widehat{h}_n^\pm such that the following diagram commutes:

$$\begin{array}{ccc}
 H_n^{gl}(\mathcal{C}) & \longrightarrow & H_*^{new-gl}(\mathcal{C}) \\
 \downarrow h_n^\pm & & \downarrow \widehat{h}_n^\pm \\
 H_n^\pm(\mathcal{C}) & \longrightarrow & H_n^\pm(\widehat{\mathcal{C}})
 \end{array}$$

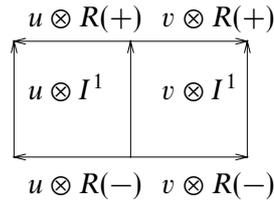


Fig. 23. Filling of corners

We are led to the following conjectures.

Conjecture 9.1.

- 1 The group $H_p^{new-gl}(I^n)$ vanishes for every natural number n and any $p > 0$.
- 2 For every $n \geq 2$, if A and B are two n -morphisms such that $t_{n-1}A = s_{n-1}B$, then $\square_n^\pm(A *_1 B - A - B)$ is a boundary in the corresponding corner homology of $\widehat{\mathcal{C}}$.
- 3 The canonical map $H_n^\pm(\mathcal{C}) \rightarrow H_n^\pm(\widehat{\mathcal{C}})$ is an isomorphism for every $n \geq 0$.
- 4 As consequence of the above two conjectures, h_n^\pm factorizes through the new globular homology theory.

This new definition of the globular homology gives rise to a new definition of homotopic non 1-contracting ω -functors and gives rise to the conjecture that the corner homology theories are still invariant with respect to this new equivalence relation.

Elements like $A *_1 B - A - B$ could be called thin globular cycles. On the corner homologies, the analogous elements are the linear combinations of $x \in \omega Cat(I^n, \mathcal{C})$ for some given natural number n such that $x(0_n)$ is of dimension strictly lower than n . We have therefore the following conjecture.

Conjecture 9.2. (About the thin elements of the corner complexes of a free globular ω -category \mathcal{C}) Let x_i be elements of $\omega Cat(I^n, \mathcal{C})^\pm$ and let λ_i be natural numbers, where i runs over some set I . Suppose for any i that $x_i(0_n)$ is of dimension strictly lower than n . Then $\sum_i \lambda_i x_i$ is a boundary if and only if it is a cycle.

With these new facts about the thin elements, Conjecture 8.8 becomes as follows.

Conjecture 9.3. Let f and g be two homotopic non 1-contracting ω -functors from \mathcal{C} to \mathcal{D} where \mathcal{C} and \mathcal{D} are two free ω -categories. Let x be an element of $\omega Cat(I^n, \mathcal{C})^\pm$. Then $f(x) - g(x) = B + T$ where B is a boundary of $\mathbb{Z}\omega Cat(I^n, \mathcal{D})^\pm$ and T a linear combination of thin elements of $\mathbb{Z}\omega Cat(I^n, \mathcal{D})^\pm$.

There exists a unique biclosed monoidal structure \otimes on ωCat such that $I^m \otimes I^n = I^{m+n}$. See, for example, Crans (1995) for an explicit construction using globular pasting scheme theory. The effect of the functor $- \otimes I^1$ is to fill corners as shown in Figure 23. We are led to the following conjectures.

Conjecture 9.4. If \mathcal{C} is an ω -category, the ω -functor from \mathcal{C} to $I^1 \otimes \mathcal{C}$ that maps u to $R(\mp) \otimes u$ yields an isomorphism from $H_*^\pm(\mathcal{C})$ to $H_*^\pm(I^1 \otimes \mathcal{C})$ and the zero map from $H_*^\mp(\mathcal{C})$ to $H_*^\mp(I^1 \otimes \mathcal{C})$ for $* > 0$.

We can easily check that $u \mapsto R(\overline{\top}) \otimes u$ induces 0 from $H_1^{\overline{\top}}(\mathcal{C})$ to $H_1^{\overline{\top}}(I^1 \otimes \mathcal{C})$. We conclude this section with three other problems. The conjectures are easy to verify in lower dimension.

Conjecture 9.5.

- 1 The corner homology groups of I^n vanish in dimension strictly greater than 0. In other words, if $p > 0$, then $H_p^{\pm}(I^n) = 0$.
- 2 Let 2_n be the free ω -category generated by an n -morphism. Then $p > 0$ implies $H_p^{\pm}(2_n) = 0$.
- 3 Let G_n be the oriented n -globe with $n > 0$, that is, the free ω -category generated by two non homotopic n -morphisms having the same $(n-1)$ -source and the same $(n-1)$ -target. Then $H_p^{\pm}(G_n) = 0$ if $p \neq n$ and $p > 0$. Moreover, the equality $H_n^{\pm}(G_n) = \mathbb{Z}$ holds.

All the conjectures of this section will be the subject of future papers.

9.2. *Perspectives in computer science*

The study of the cokernel of the negative Hurewicz morphism would allow us to detect the deadlock in concurrent machines. In an analogous way the cokernel of the positive Hurewicz morphism would allow us to detect the unreachable states in a concurrent machine. This is useful for detecting the dead code in concurrent machines and for analyzing the safety properties of a machine (Godefroid and Wolper 1991): proving that a property is false is equivalent to proving that some states are unreachable.

We exhibited in Figure 8 a 1-category homotopic to an ω -category. The 1-category we obtain suggests some relations with the graph of oriented connected components introduced in Fajstrup *et al.* (1998a).

We think also that some problems of confidentiality in computer science involve the construction of a relative corner homology. The problem stands as follows: take a concurrent machine with a flow of inputs and a flow of outputs, every input and output having a confidentiality level; such a machine is confidential if the flow of inputs of confidentiality level lower than l determines the flow of outputs of confidentiality level l (otherwise an observer could deduce from observations of outputs of confidentiality levels l some information about inputs of confidentiality level greater than l). The geometric problem that arises from this situation stands as follows: if some n -transitions are the inputs and some others are the outputs, the problem is to know whether inputs determine outputs over the set of all possible execution paths of the machine. In the 1-dimensional case, using bicomplexes, we have already found a relation between this problem and the vertical and horizontal H_1 , and we suspect that in higher dimension this problem is related in some way to the relative oriented Hurewicz morphisms (Gaucher 1997a; 1997b; 1997c).

10. Direct construction of the globular and corner homologies of a cubical set

In this final section we explain how to obtain the globular and corner homologies of a cubical set by using the free cubical ω -category generated by it, instead of considering the free globular one. This approach could be useful from an algorithmic viewpoint.

10.1. *Cubical ω -category*

The notion of a cubical ω -category appears in the (already cited) works of Brown, Higgins and Al-Agl.

Definition 10.1. A cubical ω -category consists of a cubical set with connections

$$((K_n)_{n \geq 0}, \partial_i^\alpha, \epsilon_i, \Gamma_i^\alpha)$$

together with a family of associative operations $+_j$ defined on $\{(x, y) \in K_n \times K_n, \partial_i^+ x = \partial_i^- y\}$ for $1 \leq j \leq n$ such that:

- 1 $(x +_j y) +_j z = x +_j (y +_j z).$
- 2 $\partial_j^-(x +_j y) = \partial_j^-(x).$
- 3 $\partial_j^+(x +_j y) = \partial_j^+(y).$
- 4 $\partial_i^\alpha(x +_j y) = \begin{cases} \partial_i^\alpha(x) +_{j-1} \partial_i^\alpha(y) & \text{if } i < j \\ \partial_i^\alpha(x) +_j \partial_i^\alpha(y) & \text{if } i > j. \end{cases}$
- 5 $(x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t).$ We will denote the two members of this equality by

$$\left[\begin{array}{cc} x & z \\ y & t \end{array} \right] \begin{array}{c} \uparrow i \\ \downarrow j \end{array}$$

- 6 $\epsilon_i(x +_j y) = \begin{cases} \epsilon_i(x) +_{j+1} \epsilon_i(y) & \text{if } i \leq j \\ \epsilon_i(x) +_j \epsilon_i(y) & \text{if } i > j. \end{cases}$
- 7 $\Gamma_i^\pm(x +_j y) = \begin{cases} \Gamma_i^\pm(x) +_{j+1} \Gamma_i^\pm(y) & \text{if } i < j \\ \Gamma_i^\pm(x) +_j \Gamma_i^\pm(y) & \text{if } i > j. \end{cases}$
- 8 If $i = j, \Gamma_i^-(x +_j y) = \left[\begin{array}{cc} \epsilon_{j+1}(y) & \Gamma_j^-(y) \\ \Gamma_j^-(x) & \epsilon_j(y) \end{array} \right] \begin{array}{c} \uparrow j \\ \downarrow j+1 \end{array}$
- 9 If $i = j, \Gamma_i^+(x +_j y) = \left[\begin{array}{cc} \epsilon_j(x) & \Gamma_j^+(y) \\ \Gamma_j^+(x) & \epsilon_{j+1}(x) \end{array} \right] \begin{array}{c} \uparrow j \\ \downarrow j+1 \end{array}$
- 10 $\Gamma_j^+ x +_{j+1} \Gamma_j^- x = \epsilon_j x$ and $\Gamma_j^+ x +_j \Gamma_j^- x = \epsilon_{j+1} x.$
- 11 $\epsilon_i \partial_i^- x +_i x = x +_i \epsilon_i \partial_i^+ x = x.$

The corresponding category with the obvious morphisms is denoted by $\infty Cat.$

Look back again to the cubical singular nerve of a topological space $X.$ We can equip it with operations $+_j$ as follows:

$$(f +_j g)(x_1, \dots, x_p) = \begin{cases} f(x_1, \dots, 2x_i, \dots, x_p) & \text{if } x_i \leq 1/2 \\ g(x_1, \dots, 2x_i - 1, \dots, x_p) & \text{if } x_i \geq 1/2. \end{cases}$$

All of the axioms of cubical ω -categories are satisfied except for the associativity axiom.

It turns out that ωCat and ∞Cat are equivalent. The 2-dimensional case is solved in Spencer (1977) (which was followed by Spencer and Wong (1983)) and the 3-dimensional case is solved in Al-Agl (1989). Recently, Richard Steiner has developed the methods of Al-Agl to prove the result in all dimensions, as conjectured in Al-Agl. The corresponding result for groupoids was already known from earlier results of Brown and Higgins (Brown and Higgins 1981a; 1981b). The category equivalence is realized by the functor $\gamma : \infty Cat \rightarrow \omega Cat$ defined as follows ($G \in \infty Cat$):

$$(\gamma G)_n = \{x \in G_n, \partial_j^\alpha x \in \epsilon_1^{j-1} G_{n-j} \text{ for } 1 \leq j \leq n, \alpha = 0, 1\}.$$

Using general category theory arguments, one can prove that the forgetful functor U from ∞Cat to $Sets^{\square^{op}}$ has a left adjoint functor ρ , which defines therefore the free cubical ω -category generated by a cubical set. We will see in Section 10 that it can be constructed explicitly by considering the cubical singular complex of the free globular ω -category generated by K .

10.2. The globular and corner homologies of a cubical set

First, we will look at the cubical singular nerve.

Proposition 10.1. Let \mathcal{C} be a globular ω -category. For any strictly positive natural number n and any j between 1 and n , there exists one and only one natural map $+_j$ from the set of pairs (x, y) of $\mathcal{N}^{\square}(\mathcal{C})_n \times \mathcal{N}^{\square}(\mathcal{C})_n$ such that $\partial_j^+(x) = \partial_j^-(y)$ to the set $\mathcal{N}^{\square}(\mathcal{C})_n$ that satisfies the following properties:

$$\partial_j^-(x +_j y) = \partial_j^-(x) \tag{1}$$

$$\partial_j^+(x +_j y) = \partial_j^+(y) \tag{2}$$

$$\partial_i^\alpha(x +_j y) = \begin{cases} \partial_i^\alpha(x) +_{j-1} \partial_i^\alpha(y) & \text{if } i < j \\ \partial_i^\alpha(x) +_j \partial_i^\alpha(y) & \text{if } i > j. \end{cases} \tag{3}$$

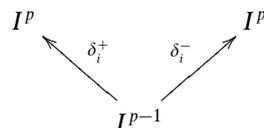
Moreover, these operations induce the structure of a cubical ω -category on $\mathcal{N}^{\square}(\mathcal{C})$.

Proof. We will only give a sketch of the proof.

Step 1. First, we observe that the functor from ωCat to the category $Sets$ of sets

$$\mathcal{C} \mapsto \omega Cat(I^p, \mathcal{C}) \times_j \omega Cat(I^p, \mathcal{C}) = \{(x, y) \in \omega Cat(I^p, \mathcal{C}) \times \omega Cat(I^p, \mathcal{C}), \partial_i^+ x = \partial_i^- y\}$$

is representable. We use $I^p +_j I^p$ to denote the representing ω -category. It is equal to the direct limit of the diagram



We use ϕ_- and ϕ_+ to denote the two canonical embeddings of I^p in $I^p +_j I^p$, respectively, in the first and the second term.

Step 2. Using Yoneda, constructing a natural map

$$+_j : \omega\text{Cat}(I^n, \mathcal{C}) \times_j \omega\text{Cat}(I^n, \mathcal{C}) \longrightarrow \omega\text{Cat}(I^n, \mathcal{C})$$

is equivalent to constructing an ω -functor $\eta_{n,j}$ from I^n to $I^n +_j I^n$ satisfying the dual properties. If $i < j$, the natural transformation of functors $(\partial_i^z, \partial_i^z)$ yields an ω -functor (δ_i^z, δ_i^z) from $I^{n-1} +_{j-1} I^{n-1}$ to $I^n +_j I^n$. It is easy to see that this ω -functor comes from the morphism of the pasting scheme that associates to $(\beta, k_1 \dots k_{n-1}) \in I^{n-1} +_{j-1} I^{n-1}$ $(\beta, k_1 \dots [x]_i \dots k_{n-1}) \in I^n +_j I^n$ with $\beta \in \{-, +\}$. If $i > j$, then the natural transformation of functors $(\partial_i^z, \partial_i^z)$ yields an ω -functor (δ_i^z, δ_i^z) from $I^{n-1} +_j I^{n-1}$ to $I^n +_j I^n$. It is easy to see that this ω -functor comes from the morphism of the pasting scheme that associates to $(\beta, k_1 \dots k_{n-1}) \in I^{n-1} +_j I^{n-1}$ $(\beta, k_1 \dots [x]_i \dots k_{n-1}) \in I^n +_j I^n$ with $\beta \in \{-, +\}$. The properties that are required for the operations $+_j$ entail the following relations for the $\eta_{n,j}$:

$$\eta_{n,j} \circ \delta_j^\pm = \phi_\pm \circ \delta_j^\pm \tag{4}$$

$$\eta_{n,j} \circ \delta_i^z = (\delta_i^z, \delta_i^z) \circ \eta_{n-1,j-1} \text{ if } i < j \tag{5}$$

$$\eta_{n,j} \circ \delta_i^z = (\delta_i^z, \delta_i^z) \circ \eta_{n-1,j} \text{ if } i > j. \tag{6}$$

Step 3. The point is that it is difficult to find a formula for the composition of all cells of $I^p +_j I^p$ (except in lower dimension). It is simpler to find this formula in a free ω -category generated by a composable pasting scheme because composition means union in such a context (Johnson 1989). It turns out that $I^p +_j I^p$ is exactly the free globular ω -category generated by the composable pasting scheme defined as follows. Set

$$(I^p +_j I^p)_q = (\{-\} \times (I^p)_q) \cup (\{+\} \times (I^p)_q) / \equiv$$

where \equiv is the equivalence relation induced by the binary relation

$$(-, k_1 \dots [x]_j \dots k_{p-1}) \equiv (+, k_1 \dots [-]_j \dots k_{p-1})$$

for every k_1, \dots, k_{p-1} in $\{-, 0, +\}$ together with the binary relations E and B defined by (with $x \in (I^p)_i$ and $y \in (I^p)_j$)

$$E_j^i = \{((a, x), (a, y)) \in (I^p +_j I^p) \times (I^p +_j I^p) / x E_j^i y \text{ and } a \in \{-, +\}\}$$

$$B_j^i = \{((a, x), (a, y)) \in (I^p +_j I^p) \times (I^p +_j I^p) / x B_j^i y \text{ and } a \in \{-, +\}\}.$$

Now we are in a position to prove the following property $P(n)$ by induction on n : ‘for any j between 1 and n , there exists one and only one ω -functor $\eta_{n,j}$ from I^n to $I^n +_j I^n$ satisfying Conditions 4, 5 and 6; moreover, $\eta_{n,j}(R(0_n)) = R(\{(-, 0_n), (+, 0_n)\})$ ’. This latter equality illustrates the interest of globular pasting schemes.

Step 4. If $n = 1$, we have to construct an ω -functor from I^1 to $I^1 +_1 I^1$. The hypotheses lead us to set $\eta_{1,1}(R(-)) = R((-,-))$ and $\eta_{1,1}(R(+)) = R((+,+))$. Thus there exists one and only one suitable ω -functor $\eta_{1,1}$, and this is the unique one that satisfies

$$\eta_{1,1}(R(0)) = R((-,-)) *_0 R((+,+)) = R((-,-), (+,+)).$$

So $P(1)$ is true. Suppose we have proved $P(k)$ for $k < n$ where n is a natural number greater

than 2. We have to construct an ω -functor $\eta_{n,j}$ for any j between 1 and n from I^n to $I^n +_j I^n$. The induction hypothesis and Conditions 4, 5 and 6 entail the value of $\eta_{n,j}$ on all faces of I^n of dimension at most $n - 1$. It remains to prove that $\eta_{n,j}(R(0_n)) = R(\{(-, 0_n), (+, 0_n)\})$ is the one and only solution. It suffices to verify that $s_{n-1}R(\{(-, 0_n), (+, 0_n)\}) = \eta_{n,j}(s_{n-1}R(0_n))$ and that $t_{n-1}R(\{(-, 0_n), (+, 0_n)\}) = \eta_{n,j}(t_{n-1}R(0_n))$. Let us verify the first equality. One has

$$s_{n-1}R(0_n) = R\left(\delta_1^{(-)1}(0_{n-1}), \dots, \delta_n^{(-)n}(0_{n-1})\right)$$

by the construction of I^n . By the induction hypothesis, $\eta_{n,j}$ is $(n - 1)$ -extendable. Since composition means union, and because of Conditions 4, 5, and 6, one has

$$\begin{aligned} & \left[\bigcup_{h=1}^{h=j-1} (\delta_h^{(-)h}, \delta_h^{(-)h}) \circ \eta_{n-1,j-1}(0_{n-1}) \right] \cup \left(\phi_- \circ \delta_j^{(-)j}(0_{n-1}) \right) \\ & \cup \left[\bigcup_{h=j+1}^{h=n} (\delta_h^{(-)h}, \delta_h^{(-)h}) \circ \eta_{n-1,j}(0_{n-1}) \right] \\ & = \left[\bigcup_{h=1}^{h=j-1} R\left((-, \delta_h^{(-)h}(0_{n-1})), (+, \delta_h^{(-)h}(0_{n-1}))\right) \right] \cup R\left((-)^j, \delta_j^{(-)j}(0_{n-1})\right) \\ & \cup \left[\bigcup_{h=j+1}^{h=n} R\left((-, \delta_h^{(-)h}(0_{n-1})), (+, \delta_h^{(-)h}(0_{n-1}))\right) \right] \\ & \subset \eta_{n,j}(s_{n-1}R(0_n)). \end{aligned}$$

It suffices to verify that

$$s_{n-1}R((-, 0_n), (+, 0_n)) = R\left(\left\{((-)^j, \delta_j^{(-)j}(0_{n-1})), (\pm, \delta_h^{(-)h}(0_{n-1}))/h \neq j\right\}\right)$$

in the pasting scheme $I^n +_j I^n$ to complete the proof. □

Let us define a natural map \square_n from $\tau_n \mathcal{C}$ (the set of morphisms of \mathcal{C} of dimension lower than or equal to n) to $\omega \text{Cat}(I^n, \mathcal{C})$ by induction on n as follows. One sets $\square_0 = \square_0^-$ and $\square_1 = \square_1^-$.

Proposition 10.2. For any natural number n greater than or equal to 2, there exists a unique natural map \square_n from \mathcal{C} to $\omega \text{Cat}(I^n, \mathcal{C})$ such that:

- 1 The equality $\square_n(x)(0_n) = x$ holds.
- 2 One has $\partial_1^\alpha \square_n = \square_{n-1} d_{n-1}^{(-)\alpha}$ for $\alpha = \pm$.
- 3 For $1 < i \leq n$, one has $\partial_i^\alpha \square_n = \epsilon_1 \partial_{i-1}^\alpha \square_{n-1} s_{n-1}$.

Moreover, for $1 \leq i \leq n$, we have $\partial_i^\pm \square_n s_n u = \partial_i^\pm \square_n t_n u$ for any $(n + 1)$ -morphism u and for all $u \in \tau_n \mathcal{C}$, $\square_n(u) \in \gamma \mathcal{N}^{\square}(\mathcal{C})_n$.

Proof. The induction equations define a fillable $(n-1)$ -shell as defined in Proposition 5.1. \square

Proposition 10.3. For all $n \geq 0$, the evaluation map $ev_{0_n} : x \mapsto x(0_n)$ from $\omega Cat(I^n, \mathcal{C})$ to \mathcal{C} induces a bijection from $\gamma \mathcal{N}^\square(\mathcal{C})_n$ to $\tau_n \mathcal{C}$.

Proof. It is obvious for $n = 0$ and $n = 1$. Let us suppose that $n \geq 2$ and let us proceed by induction on n . Since $ev_{0_n} \square_n(u) = u$ by the previous proposition, the evaluation map ev from $\gamma \mathcal{N}^\square(\mathcal{C})_n$ to $\tau_n \mathcal{C}$ is surjective. Now let us prove that $x \in \gamma \mathcal{N}^\square(\mathcal{C})_n$ and $y \in \gamma \mathcal{N}^\square(\mathcal{C})_n$, and $x(0_n) = y(0_n) = u$ imply $x = y$. Since x and y are in $\gamma \mathcal{N}^\square(\mathcal{C})_n$, one sees immediately that the four elements $\partial_1^\pm x$ and $\partial_1^\pm y$ are in $\gamma \mathcal{N}^\square(\mathcal{C})_{n-1}$. Since all other $\partial_i^\pm x$ and $\partial_i^\pm y$ are thin, $\partial_1^- x(0_{n-1}) = \partial_1^- y(0_{n-1}) = s_{n-1}u$ and $\partial_1^+ x(0_{n-1}) = \partial_1^+ y(0_{n-1}) = t_{n-1}u$. By the induction hypothesis, $\partial_1^- x = \partial_1^- y = \square_{n-1}(s_{n-1}u)$ and $\partial_1^+ x = \partial_1^+ y = \square_{n-1}(t_{n-1}u)$. By hypothesis, one can set $\partial_j^\alpha x = \epsilon_1^{j-1} x_j^\alpha$ and $\partial_j^\alpha y = \epsilon_1^{j-1} y_j^\alpha$ for $2 \leq j \leq n$. And one gets $x_j^\alpha = (\partial_1^\alpha)^{j-1} \partial_j^\alpha x = (\partial_1^\alpha)^j x = (\partial_1^\alpha)^j y = y_j^\alpha$. Therefore $\partial_j^\alpha x = \partial_j^\alpha y$ for all $\alpha \in \{-, +\}$ and all $j \in [1, \dots, n]$. By Proposition 5.1, one gets $x = y$. \square

The above proof also shows that the map that associates to any cube x of the cubical singular nerve of \mathcal{C} the cube $\square_{dim(x)}(x(0_{dim(x)}))$ is exactly the usual folding operator as exposed in Al-Agl (1989).

Now let us remark that the free globular ω -category generated by a cubical set K can be also obtained by considering the image by the functor γ of $\rho(K)$. Beware of the fact that in Al-Agl's Ph.D. thesis, globular ω -categories contain identity operators (his ω -categories are \mathbb{N} -graded). So the correct statement is $(\gamma\rho(K))_n = \tau_n F(K)$ where $\tau_n F(K)$ is the n -category obtained by keeping only the p -morphisms with $p \leq n$. It suffices to prove the previous equality for $K = I^n$ since γ is a left adjoint functor (Al-Agl 1989), therefore it commutes with all direct limits.

Corollary 10.4. Let K be a cubical set. Then $\mathcal{N}^\square(F(K))$ is the free cubical ω -category $\rho(K)$ generated by K .

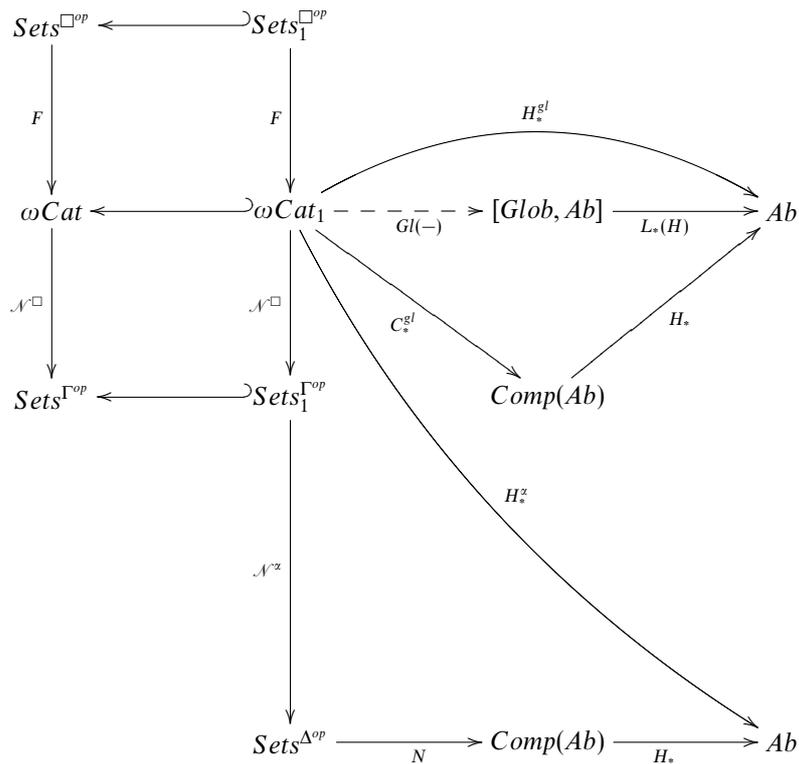
Proof. By Al-Agl (1989, Proposition 2.7.3), any n -cube x of $\rho(K)$ (respectively, of $\mathcal{N}^\square(F(K))$) is determined by its $(n-1)$ -shell of $(n-1)$ -faces $(\partial_j^\pm x)_{1 \leq j \leq n+1}$ and by its image in $\gamma\rho(K)$ (respectively, $\gamma \mathcal{N}^\square(F(K))$). \square

Now take a cubical set K . As a consequence of the above remarks, it is possible to construct $H_*^{gl}(K)$ and $H_*^\pm(K)$ and the two morphisms h_*^\pm by using the free cubical ω -category generated by K instead of using the globular one. Let us still use $\gamma\rho(K)$ to denote the globular ω -category obtained by removing all identity elements. It is exactly the free globular ω -category generated by K . We set $H_*^{gl}(K) := H_*^{gl}(\gamma\rho(K))$, and since $\mathcal{N}^\square(F(K))$ is the free cubical ω -category generated by K , we set $H_*^\pm(K) = H_*(\mathbb{Z}\rho(K)_*^\pm, \partial^\pm)$ where

$$\rho(K)_n^\pm = \{x \in \rho(K)_n, \forall i_1, \dots, i_{n-1}, \partial_{i_1}^\pm \dots \partial_{i_{n-1}}^\pm x \in \rho(K)_1\}.$$

The two morphisms h_*^\pm from $H_*^{gl}(K)$ to $H_*^\pm(K)$ are constructed as in Proposition 7.2: the only tool to be used is again Al-Agl (1989, Proposition 2.7.3).

Appendix A. The categories and functors used in this paper



The above diagram is commutative in the sense that two different ways between the same pair of points give the same transformation. All these transformations are functors except $Gl(-)$. This diagram summarizes all transformations or functors constructed in this paper.

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