

FLOWS REVISITED: THE MODEL CATEGORY STRUCTURE AND ITS LEFT DETERMINEDNESS

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ABSTRACT. Flows are a topological model of concurrency which enables to encode the notion of refinement of observation and to understand the homological properties of branchings and mergings of execution paths. Roughly speaking, they are Grandis' d -spaces without an underlying topological space. They just have an underlying homotopy type. This note is twofold. First, we give a new construction of the model category structure of flows which is more conceptual thanks to Isaev's results. It avoids the use of difficult topological arguments. Secondly, we prove that this model category is left determined by adapting an argument due to Olschok. The introduction contains some speculations about what we expect to find out by localizing this minimal model category structure.

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1. INTRODUCTION

1.1. Topological models of concurrency. There is a multitude of topological models of concurrency: flows which are the subject of this paper and which are introduced in [Gau03], but also d -spaces [Gra03], streams [Kri09], equilogical spaces [Gra04a], spaces with distinguished cubes [Gra05] [Gra04b], multipointed d -spaces [Gau09a] etc... All these mathematical devices contain the same basic examples coming from concurrency (e.g. the geometric realizations of precubical sets), a local ordering modeling the direction of time and its irreversibility, execution paths, a set or a topological space of states and a notion of homotopy between execution paths to model concurrency. Grandis' d -spaces give rise to a vast literature studying directed fundamental groups and directed components of directed spaces whatever the definition we give to this notion of *directed space*.

This paper belongs to the sequence of papers [Gau03] [Gau06a] [Gau09b] [Gau07] [Gau06b] [Gau06c] [Gau05a] [Gau05b]. The main feature of the model category of flows is to enable the formalization and the study of the notion of refinement of observation (the

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cofibrant replacement functor plays a crucial role in the formalization indeed). The model category of flows also enables the study of the homological properties of the branching areas and merging areas of execution paths in concurrent systems, in particular a long exact sequence, and their interaction with the refinement of observation, actually their invariance with respect to them. Since flows have also a labeled version (see [Gau08, Section 6]), they can be used for modeling the path spaces of process algebras for any synchronization algebra [Gau08] [Gau10].

1.2. Speculative digression. One of the challenges of this line of research is to understand the homotopical localization of the model category of (labeled) flows with respect to the T-homotopy equivalences (i.e. the refinement of observation) as defined in [Gau06b]. The paper [Gau05a] proves not only that the class of weak equivalences contains more equivalences than the dihomotopy equivalences as defined in [Gau06a], but also that there is no hope to obtain a model category structure on flows such that the weak equivalences are exactly these dihomotopy equivalences, even if a notion of fibrant object (the homotopy continuous flows [Gau06a, Definition 4.3]) with the associated Whitehead theorem exists for dihomotopy equivalences [Gau06a, Theorem 4.6]. By now, we only know by studying examples that the weak equivalences of this Bousfield localization seem to be dihomotopy equivalences in the sense of [Gau06a] up to a kind of bisimulation. In particular, it means that the weak equivalences of this Bousfield localization likely preserve causality (but not the underlying homotopy type or the branching and merging homologies). It is actually possible to Bousfield localize the model category of flows by the whole class of bisimulations in Joyal-Nielsen-Winskel's sense [JNW96] since this class of maps is accessible. However, we would then have to deal with weak equivalences breaking the causal structure. The latter phenomenon is explained in [Gau17, Theorem 12.4] within the combinatorial framework of Cattani-Sassone higher dimensional transition systems but it can be easily adapted and generalized to many other frameworks of directed homotopy, including the one of flows.

1.3. Purpose of this note. The construction of the model category of flows as carried out in [Gau03] is quite long and tricky. It makes use of rather complicated topological lemmas, in particular because colimits of flows are difficult to understand. Indeed, flows are roughly speaking *small categories without identities (precategories ? pseudocategories ?) enriched over topological spaces*. Therefore new paths are created as soon as states are identified in a colimit, which may generate complicated modifications of the topology of the path space (e.g. see the proof of [Gau03, Theorem 15.2]). The first purpose of this note is to simplify this construction using [Isa18]. We then explain in a second part why the model category of flows is left determined in the sense of [RT03] by adapting an argument due to Marc Olschok for the model category of topological spaces. The latter fact is a new result (it was mentioned without proof in [Gau17, Section 12]).

1.4. Organization. Section 2 recalls what we need to use from Isaev's paper. Section 3 explains the new construction of the model category structure of flows (Theorem 3.11). Section 4 recalls the notion of left determined model category and proves that the category of flows is left determined (Theorem 4.3). Section 5 makes some final comments.

1.5. **Notations.** All categories are locally small. The category of sets is denoted by **Set**. The set of maps in a category \mathcal{K} from X to Y is denoted by $\mathcal{K}(X, Y)$. The initial (final resp.) object, if it exists, is always denoted by \emptyset (**1** resp.). The identity of an object X is denoted by Id_X . The composite of two maps $f : A \rightarrow B$ and $g : B \rightarrow C$ is denoted by $g.f$. A subcategory is always isomorphism-closed (replete). Let f and g be two maps of a category \mathcal{K} . Denote by $f \square g$ when f satisfies the *left lifting property* (LLP) with respect to g , or equivalently g satisfies the *right lifting property* (RLP) with respect to f . Let \mathcal{C} be a class of maps. Let us introduce the notations $\mathbf{inj}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f \square g\}$ and $\mathbf{cof}(\mathcal{C}) = \{f \mid \forall g \in \mathbf{inj}(\mathcal{C}), f \square g\}$. The class of morphisms of \mathcal{K} that are transfinite compositions of pushouts of elements of \mathcal{C} is denoted by $\mathbf{cell}(\mathcal{C})$. We refer to [AR94] for locally presentable categories, to [Ros09] for combinatorial model categories. We refer to [Hov99] and to [Hir03] for more general model categories.

2. ISAEV APPROACH FOR CONSTRUCTING MODEL CATEGORIES

Let \mathcal{K} be a locally presentable category. A combinatorial model category structure is characterized by its set of generating cofibrations and by its class of fibrant objects by [Joy, Proposition E.1.10]. Therefore, for a given set of maps I , there exists at most one combinatorial model category structure on \mathcal{K} such that the set of generating cofibrations is I and such that all objects are fibrant. In [Isa18], it is expounded several methods to obtain model category structures such that all objects are fibrant. We summarize in the next theorem what we are going to need in this note.

2.1. **Theorem.** [Isa18, Theorem 4.3, Proposition 4.4, Proposition 4.5 and Corollary 4.6] *Let \mathcal{K} be a locally presentable category. Let I be a set of maps of \mathcal{K} such that the domains of the maps of I are cofibrant. Suppose that for every map $i : U \rightarrow V \in I$, the relative codiagonal map $V \sqcup_U V \rightarrow V$ factors as a composite $V \sqcup_U V \rightarrow C_U(V) \rightarrow V$ such that the left-hand map belongs to $\mathbf{cof}(I)$. Let $J_I = \{V \rightarrow C_U(V) \mid U \rightarrow V \in I\}$. Suppose that there exists a path functor $\text{Path} : \mathcal{K} \rightarrow \mathcal{K}$, i.e. an endofunctor of \mathcal{K} equipped with two natural transformations $\tau : \text{Id} \Rightarrow \text{Path}$ and $\pi : \text{Path} \Rightarrow \text{Id} \times \text{Id}$ such that the composite $\pi.\tau$ is the diagonal. Moreover we suppose that the path functor satisfies the following hypotheses:*

- (1) *With $\pi = (\pi_0, \pi_1)$, $\pi_0 : \text{Path}(X) \rightarrow X$ and $\pi_1 : \text{Path}(X) \rightarrow X$ have the RLP with respect to I .*
- (2) *The map $\pi : \text{Path}(X) \rightarrow X \times X$ has the RLP with respect to the maps of J_I .*

Then there exists a unique model category structure on \mathcal{K} such that the set of generating cofibrations is I and such that the set of generating trivial cofibrations is J_I . Moreover, all objects are fibrant.

Unlike in [Isa18], we can drop the hypothesis about the smallness of the domains and the codomains of the maps of I with respect to I by [Bek00, Proposition 1.3] because the ambient category is supposed to be locally presentable. Note that every map of J_I is a split monomorphism since the composite $V \sqcup_U V \rightarrow C_U(V) \rightarrow V$ is the relative codiagonal. Therefore every object is fibrant indeed.

3. THE MODEL CATEGORY OF FLOWS

3.1. Notation. *The category \mathbf{Top} denotes a bicomplete locally presentable cartesian closed full subcategory of the category of general topological spaces containing all CW-complexes.*

The category of Δ -generated spaces, i.e. the colimits of simplices, or equivalently the colimits of the segment $[0, 1]$ by [Faj10, Proposition 3.17], satisfies these hypotheses [FR08]. It is also possible to add weak separability hypotheses like this one: for every continuous map $g : \Delta^n \rightarrow X$ where Δ^n is the topological n -simplex with $n \geq 0$, $g(\Delta^n)$ is closed in X . For a tutorial about these topological spaces, see for example [Gau09a, Section 2]. We suppose \mathbf{Top} equipped with the standard Quillen model category structure.

3.2. Notation. *The internal hom functor is denoted by $\mathbf{TOP}(-, -)$.*

3.3. Definition. [Gau03] *A flow X consists of a topological space $\mathbb{P}X$ of execution paths, a discrete space X^0 of states, two continuous maps s and t from $\mathbb{P}X$ to X^0 called the source and target map respectively, and a continuous and associative map*

$$* : \{(x, y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$$

*such that $s(x * y) = s(x)$ and $t(x * y) = t(y)$. A morphism of flows $f : X \longrightarrow Y$ consists of a set map $f^0 : X^0 \longrightarrow Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ such that $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ and $f(x * y) = f(x) * f(y)$. The corresponding category is denoted by \mathbf{Flow} .*

3.4. Notation. *For a topological space X , let $\mathbf{Glob}(X)$ be the flow defined by $\mathbf{Glob}(X)^0 = \{0, 1\}$ and $\mathbb{P}\mathbf{Glob}(X) = X$ with $s = 0$ and $t = 1$. The \mathbf{Glob} mapping induces a functor from the category \mathbf{Top} of topological spaces to the category \mathbf{Flow} of flows.*

We need to recall the two following easy propositions:

3.5. Proposition. [Gau03, Proposition 13.2] *A morphism of flows $f : X \longrightarrow Y$ satisfies the RLP with respect to $\mathbf{Glob}(U) \longrightarrow \mathbf{Glob}(V)$ if and only if for any $\alpha, \beta \in X^0$, $\mathbb{P}_{\alpha, \beta}X \longrightarrow \mathbb{P}_{f(\alpha), f(\beta)}Y$ satisfies the RLP with respect to $U \longrightarrow V$.*

3.6. Proposition. [Gau03, Proposition 16.2] *Let f be a morphism of flows. Then the following conditions are equivalent:*

- (1) *f is bijective on states*
- (2) *f satisfies the RLP with respect to $R : \{0, 1\} \longrightarrow \{0\}$ and $C : \emptyset \subset \{0\}$.*

We will also need this new proposition which does not seem to be proved in one of our previous papers about flows:

3.7. Proposition. *The globe functor $\mathbf{Glob} : \mathbf{Top} \rightarrow \mathbf{Flow}$ preserves connected colimits.*

Note that the connectedness hypothesis is necessary. Indeed, V and W being two topological spaces, the flow $\mathbf{Glob}(V \sqcup W)$ has two states whereas the flow $\mathbf{Glob}(V) \sqcup \mathbf{Glob}(W)$ has four states.

Proof. Let V be a topological space. Giving a map from the flow $\mathbf{Glob}(V)$ to a flow X is equivalent to choosing two states α and β of X (the image of the states 0 and 1 of

$\text{Glob}(V)$) and a continuous map from V to $\mathbb{P}_{\alpha,\beta}X$. Thus the following natural bijection of sets holds

$$(1) \quad \mathbf{Flow}(\text{Glob}(V), X) \cong \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \mathbf{Top}(V, \mathbb{P}_{\alpha,\beta}X).$$

We obtain the sequence of natural bijections ($\varinjlim V_i$ being a connected colimit of topological spaces)

$$\begin{aligned} \mathbf{Flow}(\text{Glob}(\varinjlim V_i), X) &\cong \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \mathbf{Top}(\varinjlim V_i, \mathbb{P}_{\alpha,\beta}X) && \text{by (1)} \\ &\cong \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \varprojlim \mathbf{Top}(V_i, \mathbb{P}_{\alpha,\beta}X) && \text{by definition of a (co)limit} \\ &\cong \varprojlim \bigsqcup_{(\alpha,\beta) \in X^0 \times X^0} \mathbf{Top}(V_i, \mathbb{P}_{\alpha,\beta}X) && \text{by connectedness of the limit} \\ &\cong \varprojlim \mathbf{Flow}(\text{Glob}(V_i), X) && \text{by (1)} \\ &\cong \mathbf{Flow}(\varinjlim \text{Glob}(V_i), X) && \text{by definition of a (co)limit.} \end{aligned}$$

The proof is complete using the Yoneda lemma. \square

3.8. Notation. [Gau03, Notation 7.6] *Let U be a topological space. Let X be a flow. The flow $\{U, X\}_S$ is defined as follows:*

- (1) *The set of states of $\{U, X\}_S$ is X^0 .*
- (2) *For $\alpha, \beta \in X^0$, the topological space $\mathbb{P}_{\alpha,\beta}\{U, X\}_S$ is $\mathbf{TOP}(U, \mathbb{P}_{\alpha,\beta}X)$.*
- (3) *For $\alpha, \beta, \gamma \in X^0$, the composition law*

$$* : \mathbb{P}_{\alpha,\beta}\{U, X\}_S \times \mathbb{P}_{\beta,\gamma}\{U, X\}_S \longrightarrow \mathbb{P}_{\alpha,\gamma}\{U, X\}_S$$

is the composite

$$\mathbb{P}_{\alpha,\beta}\{U, X\}_S \times \mathbb{P}_{\beta,\gamma}\{U, X\}_S \cong \mathbf{TOP}(U, \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X) \longrightarrow \mathbf{TOP}(U, \mathbb{P}_{\alpha,\gamma}X)$$

induced by the composition law of X .

The flow $\{U, X\}_S$ is functorial with respect to U and X (contravariant with respect to U and covariant with respect to X). The flow $\{\emptyset, Y\}_S$ is the flow having the same set of states as Y and exactly one non-constant execution path between two points of Y^0 . The flow $\{\{0\}, X\}_S$ is canonically isomorphic to X for all flows X . Maybe the latter assertion deserves a little explanation because it is precisely why cartesian closedness matters. For all topological spaces U , we have the natural bijection

$$\mathbf{Top}(U, \mathbb{P}_{\alpha,\beta}X) \cong \mathbf{Top}(U \times \{0\}, \mathbb{P}_{\alpha,\beta}X) \cong \mathbf{Top}(U, \mathbf{TOP}(\{0\}, \mathbb{P}_{\alpha,\beta}X)).$$

Thus by Yoneda, we obtain the homeomorphism $\mathbb{P}_{\alpha,\beta}X \cong \mathbf{TOP}(\{0\}, \mathbb{P}_{\alpha,\beta}X)$.

3.9. Notation. *Let $n \geq 1$. Denote by $\mathbf{D}^n = \{b \in \mathbb{R}^n, |b| \leq 1\}$ the n -dimensional disk, and by \mathbf{S}^{n-1} the $(n-1)$ -dimensional sphere. By convention, let $\mathbf{D}^0 = \{0\}$ and $\mathbf{S}^{-1} = \emptyset$. Let $I_+^{gl} = \{\text{Glob}(\mathbf{S}^{n-1}) \subset \text{Glob}(\mathbf{D}^n) \mid n \geq 0\} \cup \{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\}$.*

We recall an elementary lemma about the model category **Top**:

3.10. Lemma. *Let $i : U \rightarrow V$ be a cofibration of \mathbf{Top} . Then for all topological spaces X , the map $i^* : \mathbf{TOP}(V, X) \rightarrow \mathbf{TOP}(U, X)$ is a fibration. If moreover i is a weak equivalence, then the map $i^* : \mathbf{TOP}(V, X) \rightarrow \mathbf{TOP}(U, X)$ is a trivial fibration.*

We write the proof to draw the reader's attention one more time on the cartesian closedness hypothesis.

Proof. By adjunction, the map i^* satisfies the RLP with respect to a map j if and only if X satisfies the RLP with respect to the pushout product of i and j . Since X is fibrant, X satisfies the RLP with respect to any trivial cofibration. Hence the result holds because the pushout product of two cofibrations i and j is trivial as soon as one of them is trivial. \square

We can now easily carry out the construction of the model category structure.

3.11. Theorem. *There exists a unique model category structure such that I_+^{gl} is the set of generating cofibrations and such that all objects are fibrant.*

Proof. We have to check the hypotheses of Theorem 2.1. The category \mathbf{Flow} is locally presentable since \mathbf{Top} is locally presentable (see for example the proof of [Gau05a, Proposition 6.11]). That all $\mathbf{Glob}(\mathbf{S}^{n-1})$ for all $n \geq 0$ are cofibrant comes from the fact that the $(n-1)$ -sphere is cofibrant in \mathbf{Top} . We can factor the relative codiagonal map $\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \rightarrow \mathbf{D}^n$ as a composite $\mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \subset \mathbf{D}^{n+1} \rightarrow \mathbf{D}^n$ for all $n \geq 0$. Thus for $U \rightarrow V$ being one of the maps $\mathbf{Glob}(\mathbf{S}^{n-1}) \subset \mathbf{Glob}(\mathbf{D}^n)$ for $n \geq 0$, we set $C_U(V) = \mathbf{Glob}(\mathbf{D}^{n+1})$. We have a pushout diagram of topological spaces

$$\begin{array}{ccc} \mathbf{S}^n & \longrightarrow & \mathbf{D}^n \sqcup_{\mathbf{S}^{n-1}} \mathbf{D}^n \\ \downarrow & & \downarrow \\ \mathbf{D}^{n+1} & \longrightarrow & \mathbf{D}^{n+1} \end{array}$$

which gives rise to the pushout diagram of flows

$$\begin{array}{ccc} \mathbf{Glob}(\mathbf{S}^n) & \longrightarrow & \mathbf{Glob}(\mathbf{D}^n) \sqcup_{\mathbf{Glob}(\mathbf{S}^{n-1})} \mathbf{Glob}(\mathbf{D}^n) \\ \downarrow & & \downarrow \\ \mathbf{Glob}(\mathbf{D}^{n+1}) & \longrightarrow & \mathbf{Glob}(\mathbf{D}^{n+1}) \end{array}$$

for all $n \geq 0$ by Proposition 3.7. This implies that for $U \rightarrow V$ being one of the maps $\mathbf{Glob}(\mathbf{S}^{n-1}) \subset \mathbf{Glob}(\mathbf{D}^n)$ for $n \geq 0$, the map $V \sqcup_U V \rightarrow C_U(V)$ belongs to $\mathbf{cell}(I_+^{gl})$. The map $C : \emptyset \rightarrow \{0\}$ gives rise to the relative codiagonal map $\{0\} \sqcup \{0\} \rightarrow \{0\}$. Thus we set $C_\emptyset(\{0\}) = \{0\}$. In this case, the map $V \sqcup_U V \rightarrow C_U(V)$ is $\text{Id}_{\{0\}}$ which belongs to $\mathbf{cell}(I_+^{gl})$. The map $R : \{0, 1\} \rightarrow \{0\}$ gives rise to the relative codiagonal map $\text{Id}_{\{0\}}$. Thus we set $C_{\{0,1\}}(\{0\}) = \{0\}$. In this case, the map $V \sqcup_U V \rightarrow C_U(V)$ is also $\text{Id}_{\{0\}}$ which belongs to $\mathbf{cell}(I_+^{gl})$. The set of generating trivial cofibrations will be therefore the set of maps $\mathbf{Glob}(\mathbf{D}^n) \subset \mathbf{Glob}(\mathbf{D}^{n+1})$ for $n \geq 0$. Let $\text{Path}(X) = \{[0, 1], X\}_S$ for all flows X . The composite map $\{0, 1\} \subset [0, 1] \rightarrow \{0\}$ yields a natural composite map of flows $X \cong \{\{0\}, X\}_S \rightarrow \text{Path}(X) \rightarrow \{\{0, 1\}, X\}_S$ which is constant on states and which gives rise to the composite continuous map $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbf{TOP}([0, 1], \mathbb{P}_{\alpha, \beta} X) \rightarrow \mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\alpha, \beta} X$ on the spaces of paths for all $(\alpha, \beta) \in X^0 \times X^0$. We obtain a natural composite map

of flows $X \xrightarrow{\tau} \text{Path}(X) \xrightarrow{\pi} X \times X$ since the set of states of $X \times X$ is $X^0 \times X^0$ and the space of paths from (α, α') to (β, β') is $\mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\alpha', \beta'} X$ by [Gau03, Theorem 4.17]. We have obtained a path object in the sense of Theorem 2.1. Since the maps π_0 and π_1 are bijective on states, they satisfy the RLP with respect to $\{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\}$ by Proposition 3.6. By Proposition 3.5, the maps π_0 and π_1 satisfy the RLP with respect to $\text{Glob}(\mathbf{S}^{n-1}) \subset \text{Glob}(\mathbf{D}^n)$ for $n \geq 0$ if and only if the evaluation maps $\mathbf{TOP}([0, 1], \mathbb{P}_{\alpha, \beta} X) \rightrightarrows \mathbb{P}_{\alpha, \beta} X$ on 0 and 1 satisfy the RLP with respect to the inclusion $\mathbf{S}^{n-1} \subset \mathbf{D}^n$ for $n \geq 0$ and for all $(\alpha, \beta) \in X^0 \times X^0$, i.e. if and only if the evaluation maps $\mathbf{TOP}([0, 1], \mathbb{P}_{\alpha, \beta} X) \rightrightarrows \mathbb{P}_{\alpha, \beta} X$ are trivial fibrations for all $(\alpha, \beta) \in X^0 \times X^0$. The latter fact is a consequence of Lemma 3.10 and from the fact that the inclusions $\{0\} \subset [0, 1]$ and $\{1\} \subset [0, 1]$ are trivial cofibrations of **Top**. Finally we have to check that the map $\pi : \text{Path}(X) \rightarrow X \times X$ satisfies the RLP with respect to the maps $\text{Glob}(\mathbf{D}^n) \subset \text{Glob}(\mathbf{D}^{n+1})$ for $n \geq 0$. By Proposition 3.5 again, it suffices to prove that the map $\mathbf{TOP}([0, 1], \mathbb{P}_{\alpha, \beta} X) \rightarrow \mathbf{TOP}(\{0, 1\}, \mathbb{P}_{\alpha, \beta} X) = \mathbb{P}_{\alpha, \beta} X \times \mathbb{P}_{\alpha, \beta} X$ is a fibration of topological spaces for all $(\alpha, \beta) \in X^0 \times X^0$. By Lemma 3.10 again, this comes from the fact that the inclusion $\{0, 1\} \subset [0, 1]$ is a cofibration of **Top**. \square

3.12. Corollary. *This model category structure coincides with the one of [Gau03].*

4. LEFT DETERMINEDNESS OF THE MODEL CATEGORY OF FLOWS

Let us now recall the definition of a left determined model category:

4.1. Definition. *Let I be a set of maps of a locally presentable category \mathcal{K} . A class of maps \mathcal{W} is a localizer (with respect to I) or a I -localizer if \mathcal{W} satisfies:*

- *Every map satisfying the RLP with respect to the maps of I belongs to \mathcal{W} .*
- *\mathcal{W} is closed under retract and satisfies the 2-out-of-3 property.*
- *The class of maps $\mathbf{cof}(I) \cap \mathcal{W}$ is closed under pushout and transfinite composition.*

The class of all maps is a I -localizer. The class of I -localizers is closed under arbitrarily large intersection. Therefore there exists a smallest I -localizer for the inclusion denoted by \mathcal{W}_I .

4.2. Definition. [RT03] *A combinatorial model category \mathcal{K} with the set of generating cofibrations I is left determined if the class of weak equivalences is \mathcal{W}_I .*

Consider a combinatorial model category \mathcal{K} such that all objects are fibrant with a class of weak equivalences \mathcal{W} and a set of generating cofibrations I . The localizer \mathcal{W}_I could be strictly smaller than \mathcal{W} . If \mathcal{W}_I is the class of weak equivalences of a model category structure on \mathcal{K} , then all objects of this model category structure are fibrant, and therefore $\mathcal{W} = \mathcal{W}_I$. To the best of our knowledge, we can only say, using [RT03, Theorem 2.2], that every combinatorial model category such that all objects are fibrant is left determined if we assume Vopěnka's principle. Since in any model category, two fibrant objects are weakly equivalent if and only if they are related by a span of trivial fibrations, and since all trivial fibrations belong to the smallest localizer, it is also true that there is an equivalence of categories $\mathcal{K}[\mathcal{W}_I^{-1}] \simeq \mathcal{K}[\mathcal{W}^{-1}]$ between the categorical localizations of \mathcal{K} with respect to \mathcal{W}_I and \mathcal{W} if all objects of the combinatorial model category \mathcal{K} are fibrant. Note that in [Isa18], a localizer is just the class of weak equivalences of a model

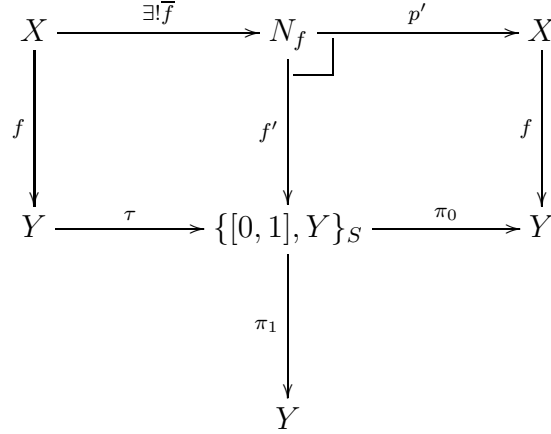


FIGURE 1.

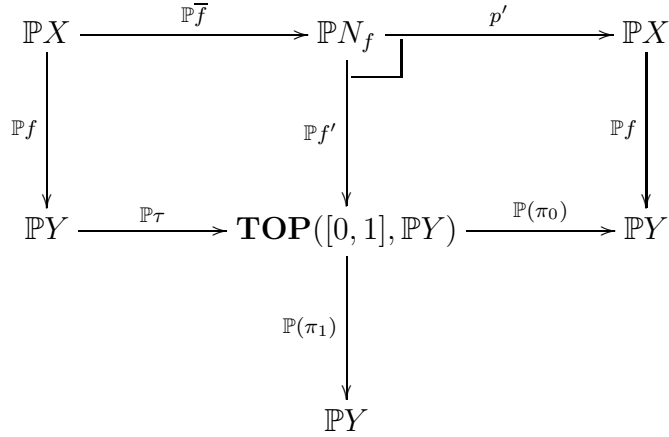


FIGURE 2.

category structure. In the latter sense, a model category of fibrant objects has a minimal localizer indeed.

In our case, it is possible to conclude that the model category is left determined without assuming Vopěnka's principle by adapting a technique we learned from Marc Olschok for the model category of topological spaces.

4.3. Theorem. *The model category of flows is left determined.*

Proof. Let $f : X \rightarrow Y$ be a weak equivalence of flows. Then f factors as a composite $f = f_2 \cdot f_1$ where f_1 is a trivial cofibration, i.e. $f_1 \in \mathbf{cof}(\{\mathbf{Glob}(\mathbf{D}^n) \subset \mathbf{Glob}(\mathbf{D}^{n+1}) \mid n \geq 0\})$ and where f_2 is a trivial fibration. In particular f_2 satisfies the RLP with respect to $C : \emptyset \rightarrow \{0\}$ and $R : \{0, 1\} \rightarrow \{0\}$. Thus f_2 is bijective on states by Proposition 3.6. The functor $X \mapsto X^0$ from \mathbf{Flow} to \mathbf{Set} is colimit-preserving since it has a right adjoint (the functor taking a set S to the flow with the set of states S and exactly one path between each pair of states). Therefore f_1 is bijective on states since the maps $\mathbf{Glob}(\mathbf{D}^n) \subset \mathbf{Glob}(\mathbf{D}^{n+1})$ for all $n \geq 0$ are bijective on states. We deduce that f is bijective on states. Consider the commutative diagram of flows of Figure 1 where the existence of \bar{f} comes from the universal property of the pullback. All arrows are

bijjective on states. Using the fact that the functor $\mathbb{P} : \mathbf{Flow} \rightarrow \mathbf{Top}$ is limit-preserving by [Gau03, Theorem 4.17], one obtains the commutative diagram of topological spaces of Figure 2. By [Hat02, Proposition 4.64], the map $\mathbb{P}(\pi_1).\mathbb{P}(f')$ is a Hurewicz fibration, and therefore a fibration of the model category of \mathbf{Top} . By Proposition 3.5, $\pi_1.f'$ satisfies the RLP with respect to all trivial cofibrations of flows, i.e. $\pi_1.f'$ is a fibration of flows. The maps $\mathbb{P}(\pi_0)$ and $\mathbb{P}(\pi_1)$ are trivial fibrations of the model category of \mathbf{Top} by Lemma 3.10. By Proposition 3.5 and Proposition 3.6, π_0 and π_1 satisfy the RLP with respect to all cofibrations of \mathbf{Flow} , i.e. π_0 and π_1 are trivial fibrations of flows. Thus p' is a trivial fibration of \mathbf{Flow} since it is a pullback of a trivial fibration. Since f is a weak equivalence of flows by hypothesis, we deduce by the 2-out-of-3 property that f' is a weak equivalence of \mathbf{Flow} . Thus $\pi_1.f'$ is a trivial fibration of flows as well. We have $p'.\bar{f} = \text{Id}_X$. Since p' is a trivial fibration, it belongs to the smallest localizer. Therefore by the 2-out-of-3 property, \bar{f} belongs to the smallest localizer. Since $\pi_1.f'$ is a trivial fibration, it belongs to the smallest localizer as well. Since $\pi_1.f'.\bar{f} = \pi_1.\tau.f = f$, we deduce that f belongs to the smallest localizer. \square

5. CONCLUDING REMARKS

The hypothesis that \mathbf{Top} is locally presentable can be removed. Theorem 3.11 and Theorem 4.3 hold by working in any bicomplete cartesian closed full subcategory of the general category of topological spaces containing all CW-complexes. But then, we have to check that all domains and all codomains of the maps of I_{gl}^+ are small relative to $\mathbf{cell}(I_{gl}^+)$. This is done in [Gau03, Section 11] and there is no known way to avoid the use of some difficult topological arguments. However, the model category of flows is left proper by [Gau07, Theorem 7.4] and [Gau09a, Section A] but not cellular because of the presence of $R : \{0, 1\} \rightarrow \{0\}$ in the generating cofibrations. So outside the framework of locally presentable categories, we have no tools to prove the existence of any homotopical localization and to study the homotopical localization of \mathbf{Flow} with respect to the refinement of observation.

REFERENCES

- [AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*. Cambridge University Press, Cambridge, 1994. 3
- [Bek00] T. Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000. 3
- [Faj10] L. Fajstrup. Classification of dicoverings. *Topology Appl.*, 157(15):2402–2412, 2010. 4
- [FR08] L. Fajstrup and J. Rosický. A convenient category for directed homotopy. *Theory and Applications of Categories*, 21(1):pp 7–20, 2008. 4
- [Gau03] P. Gaucher. A model category for the homotopy theory of concurrency. *Homology, Homotopy and Applications*, 5(1):p.549–599, 2003. 1, 2, 4, 5, 7, 9
- [Gau05a] P. Gaucher. Flow does not model flows up to weak dihomotopy. *Applied Categorical Structures*, 13:371–388, 2005. 1, 2, 6
- [Gau05b] P. Gaucher. Homological properties of non-deterministic branchings and mergings in higher dimensional automata. *Homology, Homotopy and Applications*, 7(1):p.51–76, 2005. 1
- [Gau06a] P. Gaucher. Inverting weak dihomotopy equivalence using homotopy continuous flow. *Theory and Applications of Categories*, 16(3):pp 59–83, 2006. 1, 2
- [Gau06b] P. Gaucher. T-homotopy and refinement of observation (III) : Invariance of the branching and merging homologies. *New York J. Math.*, 12:319–348 (electronic), 2006. 1, 2

- [Gau06c] P. Gaucher. T-homotopy and refinement of observation (IV) : Invariance of the underlying homotopy type. *New York J. Math.*, 12:63–95 (electronic), 2006. 1
- [Gau07] P. Gaucher. T-homotopy and refinement of observation, part II: Adding new T-homotopy equivalences. *Internat. J. Math. Math. Sci.*, 2007:Article ID 87404, 20 pages, 2007. doi:10.1155/2007/87404. 1, 9
- [Gau08] P. Gaucher. Towards a homotopy theory of process algebra. *Homology Homotopy Appl.*, 10(1):353–388 (electronic), 2008. 2
- [Gau09a] P. Gaucher. Homotopical interpretation of globular complex by multipointed d-space. *Theory Appl. Categ.*, 22:No. 22, 588–621 (electronic), 2009. 1, 4, 9
- [Gau09b] P. Gaucher. T-homotopy and refinement of observation (I) : Introduction. *Electronic Notes in Theoretical Computer Science*, 230:103–110, 2009. <http://dx.doi.org/10.1016/j.entcs.2009.02.019>. 1
- [Gau10] P. Gaucher. Combinatorics of labelling in higher dimensional automata. *Theoretical Computer Science*, 411(11-13):1452–1483, 2010. doi:10.1016/j.tcs.2009.11.013. 2
- [Gau17] P. Gaucher. Combinatorics of past-similarity in higher dimensional transition systems. *Theory Appl. Categ.*, 32:No. 33, 1107–1164 (electronic), 2017. 2
- [Gra03] M. Grandis. Directed homotopy theory. I. *Cah. Topol. Géom. Différ. Catég.*, 44(4):281–316, 2003. 1
- [Gra04a] M. Grandis. Inequilogical spaces, directed homology and noncommutative geometry. *Homology Homotopy Appl.*, 6(1):413–437, 2004. 1
- [Gra04b] M. Grandis. Normed combinatorial homology and noncommutative tori. *Theory Appl. Categ.*, 13:No. 7, 114–128, 2004. 1
- [Gra05] M. Grandis. Directed combinatorial homology and noncommutative tori (the breaking of symmetries in algebraic topology). *Math. Proc. Cambridge Philos. Soc.*, 138(2):233–262, 2005. 1
- [Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. 9
- [Hir03] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. 3
- [Hov99] M. Hovey. *Model categories*. American Mathematical Society, Providence, RI, 1999. 3
- [Isa18] V. Isaev. On fibrant objects in model categories. *Theory Appl. Categ.*, 33:No. 3, 43–66 (electronic), 2018. 2, 3, 7
- [JNW96] A. Joyal, M. Nielsen, and G. Winskel. Bisimulation from open maps. *Inf. Comput.*, 127(2):164–185, 1996. 2
- [Joy] A. Joyal. Notes on quasicategories. available at <http://preview.tinyurl.com/o6bkdv8>. 3
- [Kri09] S. Krishnan. A convenient category of locally preordered spaces. *Appl. Categ. Structures*, 17(5):445–466, 2009. 1
- [Ros09] J. Rosický. On combinatorial model categories. *Appl. Categ. Structures*, 17(3):303–316, 2009. 3
- [RT03] J. Rosický and W. Tholen. Left-determined model categories and universal homotopy theories. *Trans. Amer. Math. Soc.*, 355(9):3611–3623 (electronic), 2003. 2, 7

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