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Flow does not Model Flows up to Weak Dihomotopy

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Abstract. We prove that the category of flows cannot be the underlying category of a model category whose corresponding homotopy types are the flows up to weak dihomotopy. Some hints are given to overcome this problem. In particular, a new approach of dihomotopy involving simplicial presheaves over an appropriate small category is proposed. This small category is obtained by taking a full subcategory of a locally presentable version of the category of flows.

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1. Introduction

The category of flows **Flow** is introduced in [11] as a geometric model of *higher-dimensional automata* allowing to study dihomotopy from the point of view of model category theory. Roughly speaking, a *weak dihomotopy equivalence* is a morphism of flows preserving computer scientific properties like the presence or absence of *deadlocks*, of *unreachable states* and of *initial and final states* so that it suffices to work in the categorical localization. The class of weak dihomotopy equivalences is divided in two subclasses, the one of *weak S-homotopy equivalences* [11] and the one of *T-homotopy equivalences* [13]. What is concerned is the construction of a model structure (in the sense of [19] or [16]) on the category of flows whose weak equivalences are exactly the weak dihomotopy equivalences. This way, it becomes possible to study the categorical localization of the category of flows with respect to weak dihomotopy equivalence using the tools of algebraic topology.

This is partially done in [11] where a model structure whose weak equivalences are exactly the weak S-homotopy equivalences is constructed. Unfortunately, this model structure does not contain enough weak equivalences because the T-homotopy equivalences are not inverted in its homotopy category.

The most elementary example of T-homotopy equivalence which is not inverted by the model structure constructed in [11] is the unique morphism ϕ dividing a directed segment in a composition of two directed segments (Figure 1 and Notation 2.5).

It is already known that a weak dihomotopy equivalence, whatever it is, must not be a non-trivial pushout of the morphism of flows $R : \{0, 1\} \longrightarrow \{0\}$ (that is: identifying two distinct states) because such a pushout either does not preserve one initial or final state, or creates a loop or a 1-dimensional branching or a 1-dimensional merging (Figure 2 and Figure 3). The main theorem of this paper is then:

THEOREM (Theorem 5.7). *In any model structure on the category of flows such that ϕ is a weak equivalence, there exists a non-trivial pushout of R which is a weak equivalence. In other terms, there does not exist any model structure on the category of flows whose weak equivalences are exactly the weak dihomotopy equivalences.*

The deep cause of this phenomenon is clearly the presence of $R : \{0, 1\} \longrightarrow \{0\}$ in the class of cofibrations (Lemma 5.3). The end of the paper is then devoted to proving that it is at least possible to get rid of R because:

THEOREM (Theorem 6.2). *Consider the model category of flows whose weak equivalences are exactly the weak S-homotopy equivalences constructed in* [11] *and recalled in Theorem* 6.1. *Then it is Quillen equivalent to another model category whose all cofibrations are monomorphisms. The new model category contains more objects but has the same weak S-homotopy types. In particular, the sets, that is to say the flows with empty path space, are replaced by the simplicial sets and the epimorphism* $R : \{0, 1\} \longrightarrow \{0\}$ *by the effective monomorphism* $\{0, 1\} \subset [0, 1]$.

Here is now an outline of the paper. Section 2 recalls what is necessary to know about flows to understand this work. Section 3 proves that the restriction of any model structure on the category of flows to the category of sets gives rise to a model structure on the category of sets. This leads us to studying in Section 4 the weak factorization systems of the category of sets. Then Section 5 proves the main theorem of this paper. At last, Section 6 gives some new directions or research to solve the problem appearing in this paper.

2. Reminder about Flows

Let **Top** be the category of compactly generated topological spaces, i.e. of weak Hausdorff k-spaces. More details for this kind of topological spaces can be found in [7, 18] and the appendix of [17].

DEFINITION 2.1 ([11]). A *flow X* consists of a topological space $\mathbb{P}X$, a discrete space X^0, two continuous maps s and t called respectively the source map and the target map from $\mathbb{P}X$ to X^0 and a continuous and associative map $* : \{(x, y) \in$

$\mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$ such that $s(x * y) = s(x)$ and $t(x * y) = t(y)$. A morphism of flows $f : X \longrightarrow Y$ consists of a set map $f^0 : X^0 \longrightarrow Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ such that $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ and $f(x * y) = f(x) * f(y)$. The corresponding category is denoted by **Flow**.

The category **Flow** is complete and cocomplete. The topological space X^0 is called the 0-*skeleton* of X. The elements of the 0-skeleton X^0 are called *states* or *constant execution paths*. The elements of $\mathbb{P}X$ are called *non-constant execution paths*. An *initial state* (resp. a *final state*) is a state which is not the target (resp. the source) of any non-constant execution path.

For the sequel, the category of sets **Set** is identified with the full subcategory of **Flow** consisting of the flows X such that $\mathbb{P}X = \varnothing$.

DEFINITION 2.2 ([11]). Let Z be a topological space. Then the *globe* of Z is the flow $\mathrm{Glob}(Z)$ defined as follows: $\mathrm{Glob}(Z)^0 = \{0, 1\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = 0$, $t = 1$ and the composition law is trivial.

NOTATION 2.3 ([11]). If Z and T are two topological spaces, then $\mathrm{Glob}(Z) * \mathrm{Glob}(T)$ is the flow obtained by identifying the final state of $\mathrm{Glob}(Z)$ with the initial state of $\mathrm{Glob}(T)$. In other terms, one has the pushout of flows:

$$\begin{array}{ccc} \{0\} & \xrightarrow{0 \mapsto 1} & \mathrm{Glob}(Z) \\ {\scriptstyle 0 \mapsto 0}\downarrow & & \downarrow \\ \mathrm{Glob}(T) & \longrightarrow & \mathrm{Glob}(Z) * \mathrm{Glob}(T) \end{array}$$

NOTATION 2.4 ([11]). For $\alpha, \beta \in X^0$, let $\mathbb{P}_{\alpha,\beta}X$ be the subspace of $\mathbb{P}X$ equipped the Kelleyfication of the relative topology consisting of the non-constant execution paths γ of X with beginning $s(\gamma) = \alpha$ and with ending $t(\gamma) = \beta$.

The morphism of flows ϕ is going to play an important role in this paper:

NOTATION 2.5. The morphism of flows $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ is the unique morphism $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ such that $\phi([0, 1]) = [0, 1] * [0, 1]$ where the flow $\overrightarrow{I} = \mathrm{Glob}(\{[0, 1]\})$ is the directed segment. It corresponds to Figure 1.

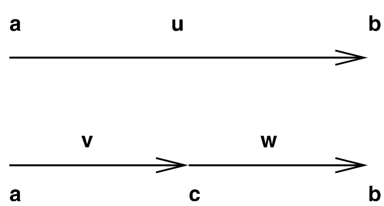


Figure 1. Simplest example of T-homotopy equivalence.

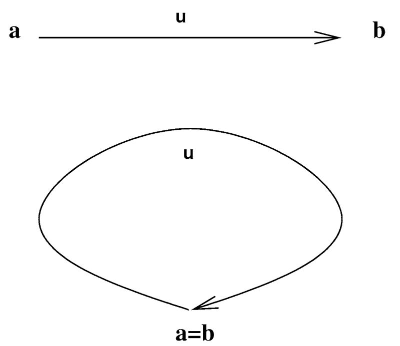


Figure 2. Non-authorized identification.

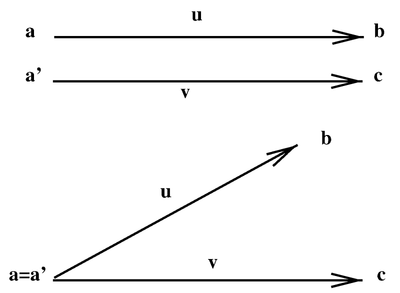


Figure 3. Non-authorized identification.

The morphism of flows $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ is an example of T-homotopy equivalence, as introduced in [13]. We would want the morphism ϕ to be a weak equivalence since it is an example of refinement of observation. On the contrary, the non-trivial pushouts of the morphism $R : \{0, 1\} \longrightarrow \{0\}$ must absolutely be removed from the class of weak equivalences because any such pushout either does not preserve one initial or final state, or creates a loop or a 1-dimensional branching or a 1-dimensional merging (Figure 2 and Figure 3). So such a pushout cannot be a dihomotopy equivalence.

3. Restriction to Set of a Model Structure on Flow

For any category $\mathcal{C}$, $\mathrm{Map}(\mathcal{C})$ denotes the class of morphisms of $\mathcal{C}$. In a category $\mathcal{C}$, an object x is *a retract* of an object y if there exist $f : x \longrightarrow y$ and $g : y \longrightarrow x$ of $\mathcal{C}$ such that $g \circ f = \mathrm{Id}_x$. A *functorial factorization* (α, β) of $\mathcal{C}$ is a pair of functors from $\mathrm{Map}(\mathcal{C})$ to $\mathrm{Map}(\mathcal{C})$ such that for any f object of $\mathrm{Map}(\mathcal{C})$, $f = \beta(f) \circ \alpha(f)$.

DEFINITION 3.1. Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the *left lifting property* (LLP) with respect to p (or p has the *right lifting property* (RLP) with respect to i) if for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & \nearrow^{g} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & Y \end{array}$$

there exists g making both triangles commutative.

DEFINITION 3.2 ([1]). Let $\mathcal{C}$ be a category. A *weak factorization system* is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that the class $\mathcal{L}$ is the class of morphisms having the LLP with respect to $\mathcal{R}$, such that the class $\mathcal{R}$ is the class of morphisms having the RLP with respect to $\mathcal{L}$ and such that any morphism of $\mathcal{C}$ factors as a composite $r \circ \ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. The weak factorization system is *functorial* if the factorization $r \circ \ell$ is a functorial factorization.

In a weak factorization system $(\mathcal{L}, \mathcal{R})$, the class $\mathcal{L}$ (resp. $\mathcal{R}$) is completely determined by $\mathcal{R}$ (resp. $\mathcal{L}$).

DEFINITION 3.3 ([16]). A *model category* is a complete and cocomplete category equipped with three classes of morphisms $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ (resp. called the classes of cofibrations, fibrations and weak equivalences) such that:

(1) the class of morphisms $\mathcal{W}$ is closed under retracts and satisfies the two-out-of-three axiom i.e.: if f and g are morphisms of $\mathcal{C}$ such that $g \circ f$ is defined and two of f, g and $g \circ f$ are weak equivalences, then so is the third,

(2) the pairs $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are both functorial weak factorization systems.

The triple $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ is called a *model structure*. An element of $\mathrm{Cof} \cap \mathcal{W}$ is called a *trivial cofibration*. An element of $\mathrm{Fib} \cap \mathcal{W}$ is called a *trivial fibration*.

LEMMA 3.4. *If $f : X \longrightarrow Y$ is a morphism of flows such that either the space $\mathbb{P}X$ or the space $\mathbb{P}Y$ is non-empty, then f satisfies the LLP with respect to any set map.*

Proof. This is due to the fact that there does not exist any continuous map from a non-empty space to an empty space. □

THEOREM 3.5. *Let* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *be a model structure on* **Flow**. *Then*

$$(\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set}), \mathrm{Fib} \cap \mathrm{Map}(\mathbf{Set}), \mathcal{W} \cap \mathrm{Map}(\mathbf{Set}))$$

is a model structure on **Set**.

Proof. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two morphisms of flows. If f and g belong to $\mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$, then $\mathbb{P}X = \mathbb{P}Y = \mathbb{P}Z = \varnothing$, and so $g \circ f \in \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. If f and $g \circ f$ belong to $\mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$, then $\mathbb{P}X = \mathbb{P}Y = \varnothing$ and $\mathbb{P}Z = \varnothing$. So $g \in \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. If g and $g \circ f$ belong to $\mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$, then $\mathbb{P}Y = \mathbb{P}Z = \varnothing$ and $\mathbb{P}X = \varnothing$. So $f \in \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. The class $\mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$ is closed under retracts since the only retract of the empty set is the empty set. So $\mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$ satisfies the two-out-of-three axiom and is closed under retracts.

Let $g \in \mathrm{Map}(\mathbf{Set})$ satisfying the RLP with respect to any morphism of $\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set})$. By Lemma 3.4, g satisfies the RLP with respect to any morphism of $\mathrm{Map}(\mathbf{Flow}) \setminus \mathrm{Map}(\mathbf{Set})$. But $\mathrm{Cof} \subset (\mathrm{Map}(\mathbf{Flow}) \setminus \mathrm{Map}(\mathbf{Set})) \cup (\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set}))$. So $g \in \mathrm{Fib} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. Thus, the class $\mathrm{Fib} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$ is exactly the class of morphisms of sets satisfying the RLP with respect to $\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set})$. Let $h \in \mathrm{Map}(\mathbf{Set})$. Then $h = r \circ i$ with $i \in \mathrm{Cof}$ and $r \in \mathrm{Fib} \cap \mathcal{W}$. So r induces a continuous map from the path space of its domain to the empty space. So both r and i are set maps. Therefore the pair $(\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set}), \mathrm{Fib} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set}))$ is a functorial weak factorization system.

And for similar reasons, the pair $(\mathrm{Cof} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set}), \mathrm{Fib} \cap \mathrm{Map}(\mathbf{Set}))$ is a functorial weak factorization system as well. □

4. The Weak Factorization Systems of the Category of Sets

Theorem 3.5 leads us to studying the possible weak factorization systems of the category of sets. Let us describe them now.

Let $\underline{\mathrm{All}}$ be the class of all morphisms of sets. Let $\underline{\mathrm{Iso}}$ be the class of bijections. Let $\underline{\mathrm{Mono}}$ be the class of injections. Let $\underline{\mathrm{Epi}}$ be the class of surjections. Let $\underline{\mathrm{SplitMono}}$ be the class of set maps having a left inverse. Let $\underline{\mathrm{Empty}}$ be the class of set maps whose domain is the empty set. Let $\underline{\mathrm{NonEmpty}}$ be the class of set maps whose domain is non-empty.

Let $C : \varnothing \longrightarrow \{0\}$. Let $C^+ : \{0\} \longrightarrow \{0, 1\}$ with $C^+(0) = 0$. Let $R : \{0, 1\} \longrightarrow \{0\}$. One has $C \in \underline{\mathrm{Empty}} \subset \underline{\mathrm{Mono}}$, $C \notin \underline{\mathrm{SplitMono}}$, $C^+ \in \underline{\mathrm{SplitMono}} \subset \underline{\mathrm{Mono}}$ and $R \in \underline{\mathrm{Epi}}$.

NOTATION 4.1. Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the collection of morphisms of $\mathcal{C}$ that satisfy the RLP with respect to any morphism of K is denoted by $\mathbf{inj}(K)$ and the collection of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the collection of morphisms of $\mathcal{C}$ that satisfies the LLP with respect to any morphism that satisfies the RLP with respect to any element of K. This is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$.

LEMMA 4.2. *For any set of morphisms of sets K, the pair $(\mathbf{cof}(K), \mathbf{inj}(K))$ is a functorial weak factorization system.*

Proof. This is due to the fact that any set is small (in the sense of model categories) so the small object argument applies [16]. This is also a consequence of the fact that **Set** is locally presentable [2] and of [3], Proposition 1.3. □

LEMMA 4.3. *For any set of morphisms of sets K, the class of morphisms of sets $\mathbf{cof}(K)$ is exactly the class of retracts of transfinite compositions of pushouts of elements of K.*

Proof. This is due to the fact that **Set** is cocomplete, that any set is small and to [16], Corollary 2.1.15. □

LEMMA 4.4. *One has:*

(1) *if* $K = \varnothing$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{Iso}}, \underline{\mathrm{All}})$,
(2) *if* $K = \{C\}$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}})$,
(3) *if* $K = \{C^+\}$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{SplitMono}}, \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}})$,
(4) *if* $K = \{R\}$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{Epi}}, \underline{\mathrm{Mono}})$,
(5) *if* $K = \{R, C\}$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{All}}, \underline{\mathrm{Iso}})$,
(6) *if* $K = \{R, C^+\}$, *then* $(\mathbf{cof}(K), \mathbf{inj}(K)) = (\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}})$.

Proof. By Lemma 4.3, the class $\mathbf{cof}(K)$ is exactly the class of retracts of transfinite compositions of pushouts of maps of K. Hence $\mathbf{cof}(\varnothing) = \underline{\mathrm{Iso}}$, $\mathbf{cof}(\{C\}) = \underline{\mathrm{Mono}}$, $\mathbf{cof}(\{C^+\}) = \underline{\mathrm{SplitMono}}$, $\mathbf{cof}(\{R\}) = \underline{\mathrm{Epi}}$, $\mathbf{cof}(\{R, C\}) = \underline{\mathrm{All}}$ and $\mathbf{cof}(\{R, C^+\}) = \underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}}$. The equalities $\mathbf{inj}(\varnothing) = \underline{\mathrm{All}}$, $\mathbf{inj}(\{C\}) = \underline{\mathrm{Epi}}$ and $\mathbf{inj}(\{R\}) = \underline{\mathrm{Mono}}$ are clear. The equality $\mathbf{inj}(\{C^+\}) = \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}}$ is a consequence of $\mathbf{inj}(\{C\}) = \underline{\mathrm{Epi}}$ and of the fact that there does not exist any set map from a non-empty set to the empty set. And $\mathbf{inj}(\{R, C\}) = \mathbf{inj}(\{R\}) \cap \mathbf{inj}(\{C\}) = \underline{\mathrm{Mono}} \cap \underline{\mathrm{Epi}} = \underline{\mathrm{Iso}}$. At last: $\mathbf{inj}(\{R, C^+\}) = \mathbf{inj}(\{R\}) \cap \mathbf{inj}(\{C^+\}) = \underline{\mathrm{Mono}} \cap (\underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}}) = \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}}$. □

THEOREM 4.5 (Goodwillie). *The six weak factorization systems* $(\underline{\mathrm{Iso}}, \underline{\mathrm{All}})$, $(\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}})$, $(\underline{\mathrm{SplitMono}}, \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}})$, $(\underline{\mathrm{Epi}}, \underline{\mathrm{Mono}})$, $(\underline{\mathrm{All}}, \underline{\mathrm{Iso}})$ *and* $(\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}})$ *are the only possible weak factorization systems on the category of sets.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system on the category of sets. Then $\underline{\mathrm{Iso}} \subset \mathcal{L} \cap \mathcal{R}$.

First of all, if $\mathcal{L} = \underline{\mathrm{Iso}}$, then $\mathcal{R} = \underline{\mathrm{All}}$ by Lemma 4.4. Let us suppose now that $\mathcal{L} \backslash \underline{\mathrm{Iso}} \neq \varnothing$.

If $C \in \mathcal{L}$, then $\underline{\mathrm{Mono}} \subset \mathcal{L}$ since $\mathcal{L}$ is closed under pushouts and transfinite composition. So if $\mathcal{L} \subset \underline{\mathrm{Mono}}$, then $\mathcal{L} = \underline{\mathrm{Mono}}$ and necessarily $(\mathcal{L}, \mathcal{R}) = (\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}})$ by Lemma 4.4.

If $C \in \mathcal{L}$ and if $\mathcal{L} \not\subset \underline{\mathrm{Mono}}$, let $f \in \mathcal{L} \backslash \underline{\mathrm{Mono}}$. Then R is a retract of f and so $R \in \mathcal{L}$. Therefore in this case, $\mathcal{L} = \underline{\mathrm{All}}$ since $\mathcal{L}$ is closed under pushouts and transfinite compositions and necessarily $\mathcal{R} = \underline{\mathrm{Iso}}$ by Lemma 4.4.

Let us suppose now that $C \notin \mathcal{L}$ and that $C^+ \in \mathcal{L}$. Then $\underline{\mathrm{SplitMono}} \subset \mathcal{L}$ since $\mathcal{L}$ is closed under pushouts and transfinite composition. So if $\mathcal{L} \subset \underline{\mathrm{Mono}}$, then $\mathcal{L} = \underline{\mathrm{SplitMono}}$ and $\mathcal{R} = \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}}$ by Lemma 4.4. And if $\mathcal{L} \not\subset \underline{\mathrm{Mono}}$, then $R \in \mathcal{L}$ like above. So $\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}} \subset \mathcal{L}$ since $\mathcal{L}$ is closed under pushouts and transfinite composition. And therefore since $C \notin \mathcal{L}$, one has $\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}} = \mathcal{L}$ and $\mathcal{R} = \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}}$ by Lemma 4.4.

Let us suppose now that $C \notin \mathcal{L}$ and that $C^+ \notin \mathcal{L}$. Then $R \in \mathcal{L}$ like above. And $\underline{\mathrm{Epi}} \subset \mathcal{L}$ since $\mathcal{L}$ is closed under pushouts and transfinite composition. So in this case, one has $\mathcal{L} = \underline{\mathrm{Epi}}$ and $\mathcal{R} = \underline{\mathrm{Mono}}$ by Lemma 4.4. □

It is even possible to prove the:

THEOREM 4.6 (Goodwillie's exercise). *The nine model structures of the category of sets are:*

1. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{All}}, \underline{\mathrm{All}}, \underline{\mathrm{Iso}})$,
2. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{All}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}})$,
3. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{All}}, \underline{\mathrm{Iso}}, \underline{\mathrm{All}})$,
4. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{Iso}}, \underline{\mathrm{All}}, \underline{\mathrm{All}})$,
5. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{Epi}}, \underline{\mathrm{Mono}}, \underline{\mathrm{All}})$,
6. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}}, \underline{\mathrm{All}})$,
7. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{SplitMono}}, \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}}, \underline{\mathrm{All}})$,
8. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}}, \underline{\mathrm{All}})$,
9. $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}})$.

Proof. Exercise for the idle mathematician proposed by Goodwillie in Don Davis' mailing-list "Algebraic Topology". □

Only Theorem 4.5 is necessary for the proof of the main theorem.

5. Proof of the Main Theorem

LEMMA 5.1. *Let* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of* R. *Then the only possibilities for the weak factorization system* $(\mathrm{Cof} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set}), \mathrm{Fib} \cap \mathrm{Map}(\mathbf{Set}))$ *are* $(\underline{\mathrm{Iso}}, \underline{\mathrm{All}})$, $(\underline{\mathrm{Mono}}, \underline{\mathrm{Epi}})$ *and* $(\underline{\mathrm{SplitMono}}, \underline{\mathrm{Epi}} \cup \underline{\mathrm{Empty}})$.

Proof. The morphism R is not a weak equivalence, so it cannot be a trivial cofibration. So $R \notin \mathrm{Cof} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. The proof is then complete with Theorem 4.5. □

LEMMA 5.2. *Let* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of* R. *Then the only possibilities for the weak factorization system* $(\mathrm{Cof} \cap \mathrm{Map}(\mathbf{Set}), \mathrm{Fib} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set}))$ *are* $(\underline{\mathrm{Epi}}, \underline{\mathrm{Mono}})$, $(\underline{\mathrm{All}}, \underline{\mathrm{Iso}})$ *and* $(\underline{\mathrm{Iso}} \cup \underline{\mathrm{NonEmpty}}, \underline{\mathrm{Iso}} \cup \underline{\mathrm{Empty}})$.

Proof. The morphism R is not a weak equivalence, so it cannot be a trivial fibration. So $R \notin \mathrm{Fib} \cap \mathcal{W} \cap \mathrm{Map}(\mathbf{Set})$. The proof is then complete with Theorem 4.5. □

Table I. The last nine possibilities.

	$(\underline{\text{Epi}}, \underline{\text{Mono}})$	$(\underline{\text{All}}, \underline{\text{Iso}})$	$(\underline{\text{Iso}} \cup \underline{\text{NonEmpty}}, \underline{\text{Iso}} \cup \underline{\text{Empty}})$
$(\underline{\text{Iso}}, \underline{\text{All}})$	$\mathcal{W} = \underline{\text{Mono}}$	possible	$\mathcal{W} = \underline{\text{Iso}} \cup \underline{\text{Empty}}$
$(\underline{\text{Mono}}, \underline{\text{Epi}})$	$\underline{\text{Mono}} \not\subset \underline{\text{Epi}}$	$\mathcal{W} = \underline{\text{Mono}}$	$\underline{\text{Iso}} \cup \underline{\text{Empty}} \not\subset \underline{\text{Epi}}$
$(\underline{\text{SplitMono}}, \underline{\text{Epi}} \cup \underline{\text{Empty}})$	$\underline{\text{SplitMono}} \not\subset \underline{\text{Epi}}$	$\mathcal{W} = \underline{\text{SplitMono}}$	$\underline{\text{Iso}} \cup \underline{\text{Empty}} \subset \mathcal{W}$

LEMMA 5.3. *Let* $(\text{Cof}, \text{Fib}, \mathcal{W})$ *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of* R. *Then both* $C : \varnothing \longrightarrow \{0\}$ *and* $R : \{0, 1\} \longrightarrow \{0\}$ *are cofibrations.*

Proof. By Theorem 3.5, the triple $(\text{Cof} \cap \text{Map}(\mathbf{Set}), \text{Fib} \cap \text{Map}(\mathbf{Set}), \mathcal{W} \cap \text{Map}(\mathbf{Set}))$ yields a model structure on the category of sets. By Lemma 5.1 and Lemma 5.2, we then have $3 \times 3 = 9$ possibilities for this restriction. These nine possibilities are summarized in Table I.

Three situations are impossible because the class of trivial cofibrations (resp. of trivial fibrations) must be a subclass of the one of cofibrations (resp. of fibrations): $\underline{\text{Mono}} \not\subset \underline{\text{Epi}}$, $\underline{\text{Iso}} \cup \underline{\text{Empty}} \not\subset \underline{\text{Epi}}$ and $\underline{\text{SplitMono}} \not\subset \underline{\text{Epi}}$.

Four other situations are impossible since the class $\mathcal{W}$ of weak equivalences cannot satisfy the two-out-of-three axiom: $\mathcal{W} = \underline{\text{Mono}}$ (twice), $\mathcal{W} = \underline{\text{Iso}} \cup \underline{\text{Empty}}$ and $\mathcal{W} = \underline{\text{SplitMono}}$.

The column $(\underline{\text{Iso}} \cup \underline{\text{NonEmpty}}, \underline{\text{Iso}} \cup \underline{\text{Empty}})$ implies that the class $\mathcal{W}$ of weak equivalences satisfies $\underline{\text{Iso}} \cup \underline{\text{Empty}} \subset \mathcal{W}$. Consider the composite $\varnothing \longrightarrow X \longrightarrow Y$ for any set map from X to Y. One deduces that $\mathcal{W} = \underline{\text{All}}$. So $\underline{\text{SplitMono}} = \underline{\text{All}} \cap (\underline{\text{Iso}} \cup \underline{\text{NonEmpty}})$: contradiction.

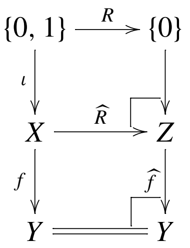
Therefore it remains the case $(\underline{\text{Iso}}, \underline{\text{All}}), (\underline{\text{All}}, \underline{\text{Iso}})$ which does correspond to a possible model structure for the category of sets, that is $(\text{Cof}, \text{Fib}, \mathcal{W}) = (\underline{\text{All}}, \underline{\text{All}}, \underline{\text{Iso}})$. □

LEMMA 5.4. *Let* $(\text{Cof}, \text{Fib}, \mathcal{W})$ *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of* R. *Then any trivial fibration induces a bijection between the* 0*-skeletons.*

Proof. By Lemma 5.3, any trivial fibration r satisfies the RLP with respect to R and C. Thus, the set map r^0 is bijective. □

LEMMA 5.5. *Let* $(\text{Cof}, \text{Fib}, \mathcal{W})$ *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of* R. *Then for any trivial cofibration* f, *the set map* f^0 *is one-to-one.*

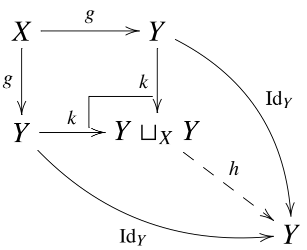
Proof. Let us suppose that there exists a trivial cofibration $f : X \longrightarrow Y$ such that f^0 is not one-to-one, that is there exists $(\alpha, \beta) \in X^0 \times X^0$ such that $\alpha \neq \beta$ and $f^0(\alpha) = f^0(\beta)$. Then consider the diagram of flows



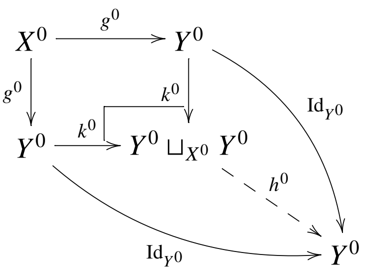
with $\iota(0) = \alpha$, $\iota(1) = \beta$. Since f is a trivial cofibration, $\widehat{f}$ is a trivial cofibration as well. Since $\widehat{f} \circ \widehat{R}$ is a weak equivalence, the morphism $\widehat{R}$ therefore belongs to $\mathcal{W}$. Contradiction. □

THEOREM 5.6. *Let* (Cof, Fib, $\mathcal{W}$) *be a model structure on* **Flow** *whose weak equivalences are never a non-trivial pushout of R. Then for any* $f \in \mathcal{W}$, f^0 *is a bijection.*

Proof. Let $g : X \longrightarrow Y$ of Cof $\cap\, \mathcal{W}$. By Lemma 5.5, g^0 is a one-to-one set map. Let us suppose that g^0 is not bijective. Consider the diagram of flows



Since g is a trivial cofibration of (Cof, Fib, $\mathcal{W}$), the morphism of flows k is a trivial cofibration of (Cof, Fib, $\mathcal{W}$) as well. Since $h \circ k$ is an isomorphism and therefore a weak equivalence, h is therefore a weak equivalence of (Cof, Fib, $\mathcal{W}$). Moreover, h^0 is epi and not bijective since g^0 is not bijective and since one has the diagram of sets



The morphism of flows h factors as $h = r \circ i$ with $i \in$ Cof $\cap\, \mathcal{W}$ and $r \in$ Fib. By Lemma 5.5 again, i^0 is one-to-one. Since $h = r \circ i \in \mathcal{W}$, the morphism of flows r belongs to $\mathcal{W}$ as well. So $r \in$ Fib $\cap\, \mathcal{W}$. By Lemma 5.4, r^0 is bijective. Therefore h^0 is a one-to-one set map. Contradiction. So g^0 is bijective.

Any morphism f of $\mathcal{W}$ factors as a composite $f = q \circ j$ with $q \in \mathrm{Fib} \cap \mathcal{W}$ and $j \in \mathrm{Cof} \cap \mathcal{W}$. Since q^0 and j^0 are both bijective, $f^0 = q^0 \circ j^0$ is bijective as well. □

The weak S-homotopy model structure of flows constructed in [11] and the Cole–Strøm model structure of flows constructed in [12] are two examples of model category structure on **Flow** such that any weak equivalence is never a non-trivial pushout of R.

THEOREM 5.7. *In any model structure on the category of flows containing $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ as weak equivalence, there exists a non-trivial pushout of R which is a weak equivalence. In other terms, there does not exist any model structure on the category of flows whose weak equivalences are exactly the weak dihomotopy equivalences.*

Proof. The morphism of flows $\phi : \overrightarrow{I} \longrightarrow \overrightarrow{I} * \overrightarrow{I}$ does not induce a bijection between $\overrightarrow{I}^0 = \{0, 1\}$ and the 0-skeleton of $\overrightarrow{I} * \overrightarrow{I}$ which is a 3-element set. □

COROLLARY 5.8. *For any model structure on the category of flows such that ϕ is a weak equivalence, there exists a weak equivalence which does not preserve the branching homology or the merging homology.*

6. Towards Other Models for Dihomotopy

Theorem 5.7 shows that the category of flows cannot be the underlying category of a model category whose corresponding homotopy types are the flows up to weak dihomotopy. The cause of the problem seems to be the unavoidable presence of $R : \{0, 1\} \longrightarrow \{0\}$ in the class of cofibrations (Proposition 5.3). In particular, it prevents the weak S-homotopy model structure of **Flow** from being cellular in the sense of [15].

Let $n \geqslant 1$. Let $\mathbf{D}^n$ be the closed n-dimensional disk and let $\mathbf{S}^{n-1}$ be its boundary. Let $\mathbf{D}^0 = \{0\}$. Let $\mathbf{S}^{-1} = \varnothing$ be the empty space. Let us recall the:

THEOREM 6.1 ([11]). *The category of flows* **Flow** *is given a structure of cofibrantly generated model category such that:*

1. *the set of generating cofibrations is the union of $\{R, C\}$ and the set of morphisms* $\mathrm{Glob}(f)$ *for f running over the set of inclusions* $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ *for $n \geqslant 0$,*
2. *the set of generating trivial cofibrations is the set of morphisms* $\mathrm{Glob}(f)$ *for f running over the set of inclusions* $\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n \times [0, 1])$,
3. *a morphism $f : X \longrightarrow Y$ of* **Flow** *is a weak equivalence if and only if $f : X^0 \longrightarrow Y^0$ is a bijection of sets and for any $(\alpha, \beta) \in X^0 \times X^0$, $f : \mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a weak homotopy equivalence of topological spaces, that is a* weak S-homotopy equivalence.

This model structure is called the weak S-homotopy model structure *of* **Flow**. *In this model structure, any object is fibrant.*

We are going to prove in this section the:

THEOREM 6.2. *The model category* **Flow** *is Quillen equivalent to a model category whose all cofibrations are monomorphisms* (*and even effective monomorphisms in the sense of* [15], *or regular monomorphisms in the sense of* [5, 6]).

For this purpose, let us introduce the notion of *flow over a monoidal category*. Let $(\mathcal{C}, \otimes)$ be a monoidal category.

An object X of $\mathbf{Flow}(\mathcal{C}, \otimes)$ consists of a set X^0 called the 0-*skeleton* of X and for any $(\alpha, \beta) \in X^0 \times X^0$ an object $\mathbb{P}_{\alpha,\beta}X$ of $\mathcal{C}$ such that there exists a morphism $* : \mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ of $\mathcal{C}$ for any $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$ satisfying the associativity axiom: for any $(\alpha, \beta, \gamma, \delta) \in X^0 \times X^0 \times X^0 \times X^0$, the following diagram is commutative

$$\begin{array}{ccc} \mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\gamma}X \otimes \mathbb{P}_{\gamma,\delta}X & \xrightarrow{(*,\mathrm{Id})} & \mathbb{P}_{\alpha,\gamma}X \otimes \mathbb{P}_{\gamma,\delta}X \\ \Big\downarrow{\scriptstyle (\mathrm{Id},*)} & & \Big\downarrow{\scriptstyle *} \\ \mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\delta}X & \xrightarrow{\quad * \quad} & \mathbb{P}_{\alpha,\delta}X \end{array}$$

A morphism $f : X \longrightarrow Y$ of $\mathbf{Flow}(\mathcal{C}, \otimes)$ consists of a set map $f : X^0 \longrightarrow Y^0$ together with morphisms $f : \mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ such that the following diagram is commutative for any $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$

$$\begin{array}{ccc} \mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\gamma}X & \xrightarrow{(f,f)} & \mathbb{P}_{f(\alpha),f(\beta)}Y \otimes \mathbb{P}_{f(\beta),f(\gamma)}Y \\ \Big\downarrow{\scriptstyle *} & & \Big\downarrow{\scriptstyle *} \\ \mathbb{P}_{\alpha,\gamma}X & \xrightarrow{\quad f \quad} & \mathbb{P}_{f(\alpha),f(\gamma)}Y \end{array}$$

NOTATION 6.3. Let Z be an object of $\mathcal{C}$. Denote by $\mathrm{Glob}(Z)$ the flow such that $\mathrm{Glob}(Z)^0 = \{0, 1\}$ and $\mathbb{P}_{0,1}\mathrm{Glob}(Z) = Z$. This defines a full and faithful functor $\mathrm{Glob} : \mathcal{C} \longrightarrow \mathbf{Flow}(\mathcal{C}, \otimes)$.

NOTATION 6.4. From now on, the category **Flow** is denoted by $\mathbf{Flow}(\mathbf{Top}, \times)$.

NOTATION 6.5. The pair $(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ denotes the monoidal model category of simplicial sets [14].

NOTATION 6.6. The geometric realization functor is denoted by $|-| : \Delta^{\mathrm{op}}\mathbf{Set} \longrightarrow \mathbf{Top}$. The singular nerve functor is denoted by $\mathrm{Sing} : \mathbf{Top} \longrightarrow \Delta^{\mathrm{op}}\mathbf{Set}$.

For the sequel, the categories $\Delta^{\mathrm{op}}\mathbf{Set}$ and **Top** are supposed to be equipped with their standard cofibrantly generated model structure.

The following two lemmas are necessary for the proof of Theorem 6.9.

LEMMA 6.7. *Let $f : U \longrightarrow V$ be a morphism of simplicial sets. Then*

$$\mathrm{Glob}(f) : \mathrm{Glob}(U) \longrightarrow \mathrm{Glob}(V)$$

satisfies the LLP with respect to the morphism $g : X \longrightarrow Y$ of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *if and only if for any $(\alpha, \beta) \in X^0 \times X^0$, f satisfies the LLP with respect to $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{g(\alpha),g(\beta)}Y$.*

Proof. Obvious. □

The category of sets can be viewed as a full subcategory of the category of flows over simplicial sets by identifying a set X with the flow Y such that $Y = X$ and $\mathbb{P}Y = \varnothing$.

LEMMA 6.8. *Let $f : U \longrightarrow V$ be a set map. Then $f : U \longrightarrow V$ satisfies the LLP with respect to the morphism $g : X \longrightarrow Y$ of* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *if and only if f satisfies the LLP with respect to g^0.*

Proof. Obvious. □

THEOREM 6.9. *The category of flows* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *is given a structure of cofibrantly generated model category such that:*

(1) *the set $\mathcal{I}$ of generating cofibrations is the union of $\{R, C\}$ and the set of morphisms* $\mathrm{Glob}(f)$ *for f running over the set of generating cofibrations of the cofibrantly generated model category* $\Delta^{\mathrm{op}}\mathbf{Set}$ *of simplicial sets,*
(2) *the set $\mathcal{J}$ of generating trivial cofibrations is the set of morphisms* $\mathrm{Glob}(f)$ *for f running over the set of generating trivial cofibrations of the cofibrantly generated model category* $\Delta^{\mathrm{op}}\mathbf{Set}$ *of simplicial sets,*
(3) *a morphism $f : X \longrightarrow Y$ of* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *is a weak equivalence if and only if $f : X^0 \longrightarrow Y^0$ is a bijection of sets and for any $(\alpha, \beta) \in X^0 \times X^0$, $f : \mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a weak homotopy equivalence of simplicial sets.*

Moreover, the weak S-homotopy model category $\mathbf{Flow}(\mathbf{Top}, \times)$ *of* [11] *is Quillen equivalent to the model category* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$.

Proof. The class of weak equivalences clearly satisfies the two-out-of-three axiom. Any object of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ is small since any simplicial set is small by [16], Lemma 3.1.1. So we only have to check that $\mathbf{cell}(\mathcal{J}) \subset \mathbf{cof}(\mathcal{I}) \cap \mathcal{W}$ where $\mathcal{W}$ denotes the class of weak equivalences and that $\mathbf{inj}(\mathcal{I}) = \mathbf{inj}(\mathcal{J}) \cap \mathcal{W}$ by [16], Theorem 2.1.19.

Let $g : X \longrightarrow Y \in \mathbf{inj}(\mathcal{I})$. Then for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{g(\alpha),g(\beta)}Y$ satisfies the RLP with respect to any cofibration of simplicial sets by Lemma 6.7. So $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{g(\alpha),g(\beta)}Y$ is a trivial fibration of simplicial sets. So it satisfies the RLP with respect to any trivial cofibration of simplicial sets. Therefore $f = \mathrm{Glob}(i)$ satisfies the LLP with respect to g by Lemma 6.7. So $\mathcal{J} \subset \mathbf{cof}(\mathcal{I})$. Since $\mathbf{cof}(\mathcal{I})$ is closed under pushouts and transfinite compositions, one deduces the inclusion $\mathbf{cell}(\mathcal{J}) \subset \mathbf{cof}(\mathcal{I})$.

The inclusion $\mathbf{cell}(\mathcal{J}) \subset \mathcal{W}$ is the consequence of several facts. Let $f : U \longrightarrow V$ be a morphism of simplicial sets. Consider the pushout diagram of $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times)$:

$$\begin{array}{ccc} \mathrm{Glob}(U) & \longrightarrow & X \\ {\scriptstyle \mathrm{Glob}(f)} \downarrow & & \downarrow {\scriptstyle g} \\ \mathrm{Glob}(V) & \longrightarrow & Y \end{array}$$

Then by [11], Proposition 15.2, for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{g(\alpha),g(\beta)}Y$ is a transfinite composition of pushouts of morphisms of the form

$$\mathrm{Id} \times \cdots \times \mathrm{Id} \times f \times \mathrm{Id} \times \cdots \times \mathrm{Id}\,.$$

If f is a (generating) trivial cofibration of $\Delta^{op}\mathbf{Set}$, then $\mathrm{Id} \times \cdots \times \mathrm{Id} \times f \times \mathrm{Id} \times \cdots \times \mathrm{Id}$ is a trivial cofibration as well since any object of the monoidal model category $(\Delta^{op}\mathbf{Set}, \times)$ is cofibrant. So the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{g(\alpha),g(\beta)}Y$ is a trivial cofibration of simplicial sets, and in particular a weak homotopy equivalence. Therefore $\mathbf{cell}(\mathcal{J}) \subset \mathcal{W}$.

Let $f : X \longrightarrow Y \in \mathbf{inj}(\mathcal{I})$. Then f^0 satisfies the RLP with respect to both R and C. So f^0 is a bijection of sets by Lemma 6.8. And for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to any cofibration of simplicial sets by Lemma 6.7. So the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a trivial fibration of simplicial sets. And the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to any trivial cofibration of simplicial sets. Hence $\mathbf{inj}(\mathcal{I}) \subset \mathbf{inj}(\mathcal{J}) \cap \mathcal{W}$.

Let $f : X \longrightarrow Y \in \mathbf{inj}(\mathcal{J}) \cap \mathcal{W}$. Then for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to any trivial cofibration of simplicial sets by Lemma 6.7. So the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a fibration of simplicial sets, and a weak homotopy equivalence since $f \in \mathcal{W}$. Therefore for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to any cofibration of simplicial sets. Hence the inclusion $\mathbf{inj}(\mathcal{J}) \cap \mathcal{W} \subset \mathbf{inj}(\mathcal{I})$ by Lemma 6.7.

So far, we have proved that $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times)$ is a cofibrantly generated model category. It remains to prove that the Quillen equivalence $|-| : \Delta^{op}\mathbf{Set} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$ gives rise to a Quillen equivalence $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times) \rightleftarrows \mathbf{Flow}(\mathbf{Top}, \times)$. Since the geometric realization functor commutes with binary products, it gives rise to a well-defined functor from $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times)$ to $\mathbf{Flow}(\mathbf{Top}, \times)$. Since the singular nerve functor is a right adjoint, it commutes with binary products as well. So it gives rise to a well-defined functor from $\mathbf{Flow}(\mathbf{Top}, \times)$ to $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times)$. It is routine to prove that this pair of functors defines an adjunction between $\mathbf{Flow}(\Delta^{op}\mathbf{Set}, \times)$ and $\mathbf{Flow}(\mathbf{Top}, \times)$.

A morphism $f : X \longrightarrow Y$ of either $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ or $\mathbf{Flow}(\mathbf{Top}, \times)$ is a fibration (resp. a trivial fibration) if and only if for any $(\alpha, \beta) \in X^0 \times X^0$, the morphism $\mathbb{P}_{\alpha,\beta}X \longrightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a fibration (resp. a trivial fibration and f^0 is a bijection of sets). So the functor from $\mathbf{Flow}(\mathbf{Top}, \times)$ to $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ preserves fibrations and trivial fibrations. Therefore it is a right Quillen functor and the adjunction is actually a Quillen adjunction between $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ and $\mathbf{Flow}(\mathbf{Top}, \times)$.

It remains to prove that this Quillen adjunction is actually a Quillen equivalence. Let X be a cofibrant object of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$. Let Y be a fibrant object of $\mathbf{Flow}(\mathbf{Top}, \times)$ (that is, any object of $\mathbf{Flow}(\mathbf{Top}, \times)$). One has to prove that $|X| \longrightarrow Y$ is a weak equivalence of $\mathbf{Flow}(\mathbf{Top}, \times)$ if and only if $X \longrightarrow \operatorname{Sing} Y$ is a weak equivalence of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$. Since any weak equivalence in $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ or $\mathbf{Flow}(\mathbf{Top}, \times)$ induces a bijection between the 0-skeletons, it remains to prove that for any $\alpha, \beta \in X^0 = Y^0$, the continuous map $|\mathbb{P}_{\alpha,\beta}X| \longrightarrow \mathbb{P}_{\alpha,\beta}Y$ is a weak homotopy equivalence of topological spaces if and only if the morphism of simplicial sets $\mathbb{P}_{\alpha,\beta}X \longrightarrow \operatorname{Sing} \mathbb{P}_{\alpha,\beta}Y$ is a weak homotopy equivalence of simplicial sets. Since any simplicial set is cofibrant, and since any topological space is fibrant, this follows from the fact that the pair of functors $(| - |, \operatorname{Sing})$ gives rise to a Quillen equivalence between $\Delta^{\mathrm{op}}\mathbf{Set}$ and $\mathbf{Top}$. □

Considering $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ does not prove Theorem 6.2 and does not allow to take in account the T-homotopy equivalences. Indeed, one still has:

THEOREM 6.10. *Let* (Cof, Fib, $\mathcal{W}$) *be a model structure on* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *such that a morphism of* $\mathcal{W}$ *is never a non-trivial pushout of* $R : \{0, 1\} \longrightarrow \{0\}$. *Then:* (1) *$R$ is necessarily a cofibration, and therefore there exists a cofibration which is not a monomorphism;* (2) *any weak equivalence of* $\mathcal{W}$ *induces a bijection of sets between the* 0*-skeletons.*

Proof. The proof goes exactly as the proofs of Lemma 5.3 and Theorem 5.7. □

We are going to use a little bit the theory of locally presentable categories. A good reference is [2].

PROPOSITION 6.11. *The category* $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ *is locally finitely presentable.*

Proof. Let $\mathbb{S}$ be the limit-sketch corresponding to the category of “small categories without identities”. The category of models $\operatorname{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})$ of the sketch $\mathbb{S}$ in the category of simplicial sets is locally finitely presentable by [2], Theorem 1.53 since the category of simplicial sets is locally finitely presentable. That does not complete the proof because the category $\operatorname{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})$ corresponds to flows having a 0-skeleton which is not necessarily a discrete simplicial set anymore.

Now consider the adjunction $\pi_0 : \Delta^{\mathrm{op}}\mathbf{Set} \leftrightarrows \mathbf{Set} : D$ between simplicial sets and sets where π_0 is the path-connected component functor and where for any set S, $D(S)$ is the discrete simplicial set associated to S. Then $D : \mathbf{Set} \longrightarrow \Delta^{\mathrm{op}}\mathbf{Set}$ can be extended in an obvious way to a functor $\widehat{D} : \mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times) \longrightarrow$

$\mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})$. The functor $\widehat{D}$ is limit-preserving and colimit-preserving since any limit and colimit of discrete simplicial sets is a discrete simplicial set.

Let Z be an object of $\mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})$. Then any morphism $Z \longrightarrow \widehat{D}X$ factors as a composite $Z \longrightarrow \widehat{D}T \longrightarrow \widehat{D}X$ where the cardinal of the underlying set of T is lower than the cardinal of the underlying set of Z. Thus there exists a set of solutions which proves that $\widehat{D}$ admits a left adjoint $\widehat{\pi_0} : \mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set}) \longrightarrow \mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ by Freyd's adjoint functor theorem. Since the category $\Delta^{\mathrm{op}}\mathbf{Set}$ (resp. $\mathbf{Set}$) can be viewed as a full subcategory of $\mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})$ (resp. $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$) by the 0-skeleton, $\widehat{\pi_0}(K) = \pi_0(K)$ for any simplicial set K. Moreover, one has $\widehat{\pi_0} \circ \widehat{D} = \mathrm{Id}$.

It is already known that $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ is cocomplete. Let $X \in \mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$. Then $\widehat{D}(X)$ is isomorphic to a directed colimit $\varinjlim X_i$ where the X_i are finitely presentable. Then $X \cong \widehat{\pi_0}(\widehat{D}(X)) \cong \varinjlim \widehat{\pi_0}(X_i)$ since the functor $\widehat{\pi_0}$ is a left adjoint. It then suffices to prove that the $\widehat{\pi_0}(X_i)$ are finitely presentable flows.

Let $\varinjlim Y_j$ be a directed colimit of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$. Then

$$\begin{aligned}
&\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)(\widehat{\pi_0}(X_i), \varinjlim Y_j)\\
&\quad\cong \mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})(X_i, \widehat{D}(\varinjlim Y_j)) \quad \text{by adjunction}\\
&\quad\cong \mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})(X_i, \varinjlim \widehat{D}(Y_j)) \quad \text{since } \widehat{D} \text{ colimit-preserving}\\
&\quad\cong \varinjlim \mathrm{Mod}(\mathbb{S}, \Delta^{\mathrm{op}}\mathbf{Set})(X_i, \widehat{D}(Y_j)) \quad \text{since } X_i \text{ finitely presentable}\\
&\quad\cong \varinjlim \mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)(\widehat{\pi_0}(X_i), Y_j) \quad \text{by adjunction again.}
\end{aligned}$$

□

Proof of Theorem 6.2. By Theorem 6.9, the model category $\mathbf{Flow}(\mathbf{Top}, \times)$ is Quillen equivalent to the locally presentable model category $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$. By Dugger's works [8] and [9], the model category $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ is Quillen equivalent to some Bousfield localization of the model category consisting of the simplicial presheaves over the cofibrant λ-presentable objects of $\mathbf{Flow}(\Delta^{\mathrm{op}}\mathbf{Set}, \times)$ for a regular cardinal λ which must be large enough and equipped with the Bousfield–Kan model structure [4]. The latter turns out to be cellular. Therefore all its cofibrations are effective monomorphisms. □

To write down the proof of Theorem 6.2, we used a locally presentable model category which is Quillen equivalent to the model category of topological spaces. Other choices were possible. Instead of considering the category of simplicial sets, it would have been possible for instance to take the category of Δ-generated topological spaces [10]. This category is the largest full subcategory of $\mathbf{Top}$ such that the full subcategory of n-dimensional simplices Δ^n for $n \geqslant 0$ is dense in it. Since the full subcategory of simplices is small, Vopěnka's principle ensures that this category is locally presentable. Moreover, there exists an unpublished proof of this fact due to Jeff Smith which does not make use of Vopěnka's principle [20].

The weak S-homotopy model category of flows is replaced by another one Quillen equivalent and cellular. What do the flows with empty space become in the new model category? The restriction of the weak S-homotopy model category of flows to the category of sets is the model structure $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W}) = (\underline{\mathrm{All}}, \underline{\mathrm{All}}, \underline{\mathrm{Iso}})$. It is locally presentable and the path-connected component functor $\pi_0 : \Delta^{\mathrm{op}}\mathbf{Set} \longrightarrow \mathbf{Set}$ induces a homotopically surjective morphism of model categories in the sense of [8]. So there exists a set S of morphisms of simplicial sets such that Bousfield-localizing $\Delta^{\mathrm{op}}\mathbf{Set}$ by S makes π_0 a Quillen equivalence. It is actually possible to take $S = \{\mathbf{S}^1 \subset \mathbf{D}^2\}$ since in the latter model structure, a weak equivalence between two simplicial sets is indeed a morphism of simplicial sets inducing a bijection between the path-connected components ([15], Section 1.5 "Postnikov approximations"). So the answer is: the sets are replaced by the simplicial sets and the epimorphism $R : \{0, 1\} \longrightarrow \{0\}$ is replaced by the effective monomorphism $\{0, 1\} \subset [0, 1]$.

7. Concluding Discussion

The main result of this paper is that the category of flows cannot be the underlying category of a model category whose corresponding homotopy types are the flows up to weak dihomotopy. A hint of how the underlying category of such model category could look like is given by Dugger's works on combinatorial model categories. The new candidate for the study of dihomotopy is another model category whose cofibrations are effective monomorphisms. It contains more objects and the same weak S-homotopy types. For example, the objects without path space are the simplicial sets, not only the discrete ones. This way, the problems existing because of the badly-behaved cofibration $R : \{0, 1\} \longrightarrow \{0\}$ do not appear anymore. It remains to see whether the Bousfield localization of this new model category with respect to T-homotopy equivalences has the correct behaviour.

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