REGULAR DIRECTED PATH AND MOORE FLOW

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ABSTRACT. Using the notion of tame regular d-path of the topological n-cube, we introduce the regular realization of a precubical set as a multipointed d-space. Its execution paths correspond to the nonconstant tame regular d-paths in the geometric realization of the precubical set. The associated Moore flow gives rise to a functor from precubical sets to Moore flows which is weakly equivalent in the h-model structure to a colimit-preserving functor. The two functors coincide when the precubical set is spatial, and in particular proper. As a consequence, it is given a model category interpretation of the known fact that the space of tame regular d-paths of a precubical set is homotopy equivalent to a CW-complex.

CONTENTS

1. INTRODUCTION

Presentation. It is described in [16] a way of realizing a precubical set as a flow without any non-canonical choice of any cofibrant replacement in the construction, by introducing a natural realization functor. It is an improvement of what is done in [9, Section 4]. The latter terminology comes from the fact that it uses the notion of natural d-path of the topological n-cube. This work is devoted to the study of a way of realizing a precubical set as a multipointed d-space without any non-canonical choice in the construction either, unlike what is done in [9, Section 5]. Precubical sets are a model for concurrency theory [6]. The n-cube represents the concurrent execution of n actions. This idea is even further developped in [17] in which it is established that the non-positively curved precubical sets contain most of the useful examples. However, we do not need this concept and we keep working with general precubical sets like in [16]. The d-paths of a precubical set, i.e. in the geometric realization of the precubical set, are the continuous paths which are concatenations of continuous paths included in cubes of the geometric realization which are

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locally nondecreasing with respect to each axis of coordinates. These particular continuous paths represent the possible execution paths of the concurrent process modelled by the precubical set. The local nondecreasingness represents time irreversibility. The notion of regular d-path of a precubical set is introduced in [5, Definition 1.1]. The terminology comes from elementary differential geometry where a differentiable maps $\gamma : [0,1] \to \mathbb{R}^n$ is called regular if $p'(t) \neq 0$ for all $t \in]0,1[$. Intuitively, it is a d-path without zero speed point.

By starting from the topological n-cube equipped with the set of nonconstant tame regular d-paths of the topological n-cube, it is introduced the regular realization $|K|_{reg}$ of a precubical set K as a multipointed d-space in Definition 2.14. Since the concatenation of two tame regular d-paths is still tame regular, all execution paths of this multipointed d-space are tame regular d-paths. However, the regular realization of a precubical set does not contain as execution paths all regular d-paths between two vertices of the precubical set, but only the nonconstant tame ones, i.e. the nonconstant tame regular d-paths which are concatenations of d-paths going from the initial state to the final state of a cube of the precubical set. Thanks to the tamification theorem [29, Theorem 6.1], this point of view is not restrictive.

The relevant information for concurrency theory is contained in the homotopy type of the space of execution paths. It is not contained in the topology of the underlying space. It therefore suffices to consider the Moore flow $\mathbb{M}^{\mathcal{G}}|K|_{reg}$ (see Theorem 4.8). The latter is obtained by forgetting the underlying topological space of the multipointed d-space $|K|_{reg}$ and by keeping only the execution paths and the information about their reparametrization.

The point is that the functor $K \mapsto \mathbb{M}^{\mathcal{G}}|K|_{reg}$ is the composite of a left adjoint $K \mapsto |K|_{reg}$ from precubical sets to multipointed d-spaces and a right adjoint $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ from multipointed d-spaces to Moore flows. The latter does not commute with colimits in general. The main result of this paper is that the composite functor $K \mapsto \mathbb{M}^{\mathcal{G}}|K|_{reg}$ from precubical sets to Moore flows can be replaced up to a natural weak equivalence of the h-model structure of Moore flows by a colimit-preserving functor. This replacement up to a weak equivalence of the h-model structure means that the spaces of execution paths are replaced by homotopy equivalent ones. Moreover, the image of this new functor is included in the class of m-cofibrant Moore flows.

Theorem. *(Theorem 6.6) There exists a colimit-preserving functor*

$$[-]_{reg} : \square^{op}\mathbf{Set} \longrightarrow \mathcal{G}\mathbf{Flow}$$

from precubical sets to Moore flows and a natural map of Moore flows

$$[K]_{reg} \longrightarrow \mathbb{M}^{\mathcal{G}}|K|_{reg}$$

which is a weak equivalence of the h-model structure of Moore flows for all precubical sets K. Moreover, the above natural map of Moore flows is an isomorphism if and only if K is spatial.

As examples of spatial precubical sets, there are all proper precubical sets by [16, Proposition 7.5], and in particular the geometric and non-positively curved ones. In other terms, the precubical sets coming from a lot of real concurrent systems by [17, Proposition 1.29].

Theorem. *(Theorem 6.8) For all precubical sets K, the Moore flow $[K]_{reg}$ is m-cofibrant.*

As any right adjoint between locally presentable category, the functor $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ is accessible. However it is not finitely accessible. This is due to the fact that the convenient category of topological spaces we are working with is locally $(2^{\aleph_0})^+$-presentable and not locally finitely presentable. However, it does commute with some particular transfinite compositions [12, Theorem 6.14] [13, Theorem 5.7]. The right adjoint $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ is also part of a Quillen equivalence $\mathbb{M}^{\mathcal{G}}_! \dashv \mathbb{M}^{\mathcal{G}}$ between the q-model structures of multipointed d-spaces and Moore flows by [13, Theorem 8.1] which satisfies a very specific property: the unit and the counit of this Quillen equivalence induce isomorphisms between the q-cofibrant objects [13, Theorem 7.6 and Corollary 7.9]. These facts together with the isomorphism of Theorem 6.6 when K is a spatial precubical set is an illustration of the following informal observation: the right adjoint $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ commutes with good enough colimits.

It is then deduced that the space of tame regular d-paths of a precubical set is homotopy equivalent to a CW-complex. We obtain a purely model category interpretation of this known fact.

Theorem. *(Corollary 6.9) For every precubical set K, the space of tame regular d-paths is homotopy equivalent to a CW-complex.*

The notion of natural d-path of a precubical set enables us in [16] to find a way to realize any precubical set as an m-cofibrant flow without using any non-canonical choice of any cofibrant replacement functor on the category of flows. In this paper, the notion of tame regular d-path of a precubical set enables us to obtain a realization of any precubical set as a m-cofibrant Moore flow without using any non-canonical choice of any cofibrant replacement functor on the category of multipointed d-spaces or on the category of Moore flows either.

In both cases, we obtain much better results than in [9] because all constructions become canonical thanks to the geometric properties of the cubes. However, the associated Moore flow of the regular realization is m-cofibrant, and not q-cofibrant. It is the counterpart of these improvements: unlike in [9], we have to deal with accessible model categories which are unlikely to be combinatorial.

We want to conclude this presentation by a very last observation. It is possible to build a colimit-preserving functor from precubical sets to multipointed d-spaces by starting from *all* tame d-paths of the n-cubes for all $n \geqslant 0$, i.e. by removing from Proposition 2.13 the adjective regular and by adapting Definition 2.14 accordingly. To develop the same theory as the one of this paper for this new realization functor, the reparametrization category $\mathcal{G}$ defined at the beginning of Section 4 must be replaced by the reparametrization category $\mathcal{M}$ [1] of [12, Proposition 4.11]. In particular, it is necessary to work with $\mathcal{M}$-flows instead of with $\mathcal{G}$-flows (the latter are called Moore flows in this paper). This subject will be

[1]The small category $\mathcal{G}$ is exactly the subcategory of $\mathcal{M}$ with the same set of objects $]0, +\infty[$ and containing only the invertible maps of $\mathcal{M}$.

addressed with other ones concerning the reparametrization category $\mathcal{M}$ in a subsequent paper.

Outline of the paper. Section 2 introduces the regular realization of a precubical set as a multipointed d-space. It is built using the notion of nonconstant tame regular d-path of the topological n-cube. The execution paths of the regular realization are exactly the nonconstant tame regular d-paths in the geometric realization of the precubical set. Section 3 is devoted to the important notion of L_1-arc length of a d-path. Some results coming from [24] are adapted to Δ-generated spaces. Theorem 3.12 is a slightly improved statement of a similar statement in [24]: a homotopy equivalence is replaced by a homeomorphism. It is then proved that Theorem 3.12 implies that the flow associated to the regular realization as a multipointed d-space is the tame concrete realization of a precubical set as recalled in Definition 3.14. In Section 4, we construct the $\{q, m, h\}$-model structures of Moore flows. It is an easy adaptation of the case of flows treated in [15, Theorem 7.4]. Some other reminders about Moore flows are also put in Section 4 to help to understand the two next sections. Section 5 makes the link between the ideas of this paper and Ziemiański notion of cube chain. It is an adaptation of [16, Section 6] in the setting of Moore flows. The idea is to adapt Ziemiański's cube chains initially developed for the closed monoidal category of topological spaces $(\mathbf{Top}, \times)$ to the biclosed semimonoidal category of $\mathcal{G}$-spaces $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$. It culminates with Theorem 5.8. Finally, Section 6 after recalling the notion of spatial precubical set establishes the main results of this paper, namely Theorem 6.6, Theorem 6.8 and Corollary 6.9 expounded in the introduction.

Prerequisites and notations. We refer to [1] for locally presentable categories, to [26] for combinatorial model categories. We refer to [20] and to [19] for more general model categories, and to [7, 18, 27] for accessible model categories. The initial object of a category is denoted by $\varnothing$. The terminal object of a category is denoted by $\mathbf{1}$. The set of maps from X to Y in a category $\mathcal{C}$ is denoted by $\mathcal{C}(X, Y)$. Id_X denotes the identity map of X. We work with the category $\mathbf{Top}$ of Δ-*generated spaces* or of Δ-*Hausdorff* Δ-*generated spaces* (cf. [14, Section 2 and Appendix B]). The inclusion functor from the full subcategory of Δ-generated spaces to the category of general topological spaces together with the continuous maps has a right adjoint called the Δ-kelleyfication functor. The latter functor does not change the underlying set. The category $\mathbf{Top}$ is locally presentable and cartesian closed. The internal hom $\mathbf{TOP}(X, Y)$ is given by taking the Δ-kelleyfication of the compact-open topology on the set $\mathbf{Top}(X, Y)$. The category $\mathbf{Top}$ is equipped with its q-model structure. The m-model structure [4] and the h-model structure [2] of $\mathbf{Top}$ are also used in various places of the paper. Compact means quasicompact Hausdorff (French convention).

2. The regular realization of a precubical set

2.1. **Notation.** *The notations $\ell, \ell', \ell_i, L, \dots$ mean a strictly positive real number unless specified something else.*

2.2. **Notation.** *Let $\ell > 0$. Let $\mu_\ell : [0, \ell] \to [0, 1]$ be the homeomorphism defined by $\mu_\ell(t) = t/\ell$.*

2.3. **Notation.** *The notation* $[0,\ell_1] \cong^+ [0,\ell_2]$ *means a nondecreasing homeomorphism from* $[0,\ell_1]$ *to* $[0,\ell_2]$ *with* $\ell_1,\ell_2 > 0$. *Let*

$$\mathcal{G}(\ell_1,\ell_2) = \{[0,\ell_1] \cong^+ [0,\ell_2]\}$$

for $\ell_1,\ell_2 > 0$.

The sets $\mathcal{G}(\ell_1,\ell_2)$ are equipped with the Δ-kelleyfication of the compact-open topology, which coincides with the compact-open topology, and with the pointwise convergence topology by [13, Proposition 2.5].

2.4. **Definition.** *Let* γ_1 *and* γ_2 *be two continuous maps from* $[0,1]$ *to some topological space such that* $\gamma_1(1) = \gamma_2(0)$. *The composite defined by*

$$(\gamma_1 *_N \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is called the normalized composition.

A *multipointed d-space* X [10] is a triple $(|X|, X^0, \mathbb{P}^{\mathcal{G}}X)$ where

- The pair $(|X|, X^0)$ is a multipointed space. The space $|X|$ is called the *underlying space* of X and the set X^0 the *set of states* of X.
- The set $\mathbb{P}^{\mathcal{G}}X$ is a set of continous maps from $[0,1]$ to $|X|$ called the *execution paths*, satisfying the following axioms:
  - For any execution path γ, one has $\gamma(0), \gamma(1) \in X^0$.
  - Let γ be an execution path of X. Then any composite $\gamma\phi$ with $\phi : [0,1] \cong^+ [0,1]$ is an execution path of X.
  - Let γ_1 and γ_2 be two composable execution paths of X; then the normalized composition $\gamma_1 *_N \gamma_2$ is an execution path of X.

A map $f : X \to Y$ of multipointed d-spaces is a map of multipointed spaces from $(|X|, X^0)$ to $(|Y|, Y^0)$ such that for any execution path γ of X, the map $\mathbb{P}^{\mathcal{G}}f : \gamma \mapsto f.\gamma$ is an execution path of Y. The category of multipointed d-spaces is denoted by $\mathcal{G}$**dTop**. It is locally presentable by [10, Theorem 3.5]. The subset of execution paths from α to β is the set of $\gamma \in \mathbb{P}^{\mathcal{G}}X$ such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$; it is denoted by $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X$: α is called the *initial state* and β the *final state* of such a γ. The set $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X$ is equipped with the Δ-kelleyfication of the relative topology induced by the inclusion $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \subset \mathbf{TOP}([0,1],|X|)$.

2.5. **Notation.** *Let* $[0] = \{()\}$ *and* $[n] = \{0,1\}^n$ *for* $n \geqslant 1$. *By convention, one has* $\{0,1\}^0 = [0] = \{()\}$. *The set* $[n]$ *is equipped with the product ordering* $\{0 < 1\}^n$. *Let* $0_n = (0,\dots,0) \in \{0,1\}^n$ *and* $1_n = (1,\dots,1) \in \{0,1\}^n$

Let $\delta_i^\alpha : [n-1] \to [n]$ be the coface map defined for $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$ by $\delta_i^\alpha(\epsilon_1,\dots,\epsilon_{n-1}) = (\epsilon_1,\dots,\epsilon_{i-1},\alpha,\epsilon_i,\dots,\epsilon_{n-1})$. The small category $\square$ is by definition the subcategory of the category of sets with the set of objects $\{[n], n \geqslant 0\}$ and generated by the morphisms δ_i^α. They satisfy the cocubical relations $\delta_j^\beta\delta_i^\alpha = \delta_i^\alpha\delta_{j-1}^\beta$ for $i < j$ and for all $(\alpha,\beta) \in \{0,1\}^2$. If $p > q \geqslant 0$, then the set of morphisms $\square([p],[q])$ is empty. If $p = q$, then the set $\square([p],[p])$ is the singleton $\{\mathrm{Id}_{[p]}\}$. For $0 \leqslant p \leqslant q$, all maps of $\square$ from $[p]$ to $[q]$ are one-to-one. A good reference for presheaves is [22].

Let K be a precubical set. There exists a functor $\square(K) : (\square{\downarrow}K) \to \square^{op}\mathbf{Set}$ which takes the map of precubical sets $\square[n] \to K$ to $\square[n]$. It is a general property of presheaves that $K = \varinjlim \square(K)$, and the latter colimit is denoted by

$$\varinjlim_{\square[n]\to K} \square[n].$$

2.6. **Definition.** *[3] The category of presheaves over $\square$, denoted by $\square^{op}\mathbf{Set}$, is called the category of* precubical sets. *A precubical set K consists of a family of sets $(K_n)_{n\geqslant 0}$ and of set maps $\partial_i^\alpha : K_n \to K_{n-1}$ with $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$ satisfying the cubical relations $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ for any $\alpha, \beta \in \{0,1\}$ and for $i < j$. An element of K_n is called a n-*cube. *An element of K_0 is also called a vertex of K. Let* $\dim(x) = n$ *if $x \in K_n$. Let*

$$K_{\leqslant n} = \varinjlim_{\substack{\square[p]\to K\\ p\leqslant n}} \square[p].$$

Let $\square[n] = \square(-,[n])$. The boundary of $\square[n]$ is the precubical set $\partial\square[n] = \square[n]_{\leqslant n-1}$. In particular, one has $\partial\square[0] = \varnothing$.

2.7. **Definition.** *A* cocubical object *of a category $\mathcal{C}$ is a functor $\square \to \mathcal{C}$.*

2.8. **Notation.** *Let $\mathcal{C}$ be a cocomplete category. Let $X : \square \to \mathcal{C}$ be a cocubical object of $\mathcal{C}$. It gives rise to a small diagram $(\square{\downarrow}K) \to \mathcal{C}$ for all precubical sets K. Its colimit is denoted by*

$$\widehat{X}(K) = \varinjlim_{\square[n]\to K} X(\square[n]).$$

2.9. **Proposition.** *[9, Proposition 2.3.2] Let $\mathcal{C}$ be a cocomplete category. The mapping $X \mapsto \widehat{X}$ induces an equivalence of categories between the category of cocubical objects of $\mathcal{C}$ and the colimit-preserving preserving functors from $\square^{op}\mathbf{Set}$ to $\mathcal{C}$.*

2.10. **Definition.** *Let U be a topological space. A* (Moore) path *in U consists of a continuous map $\gamma : [0,\ell] \to U$ with $\ell > 0$. The real number $\ell > 0$ is called the* length *of the path. It can be extended to a continuous map $\gamma : [0,+\infty[\to U$ such that $\gamma(t) = \gamma(\ell)$ for all $t \geqslant \ell$.*

The notions of stop interval and regular path appear in [5, Definition 1.1].

2.11. **Definition.** *Let U be a Hausdorff topological space* [2]*. Let $\gamma : [0,\ell] \to U$ be a Moore path in U with $\ell > 0$. A* γ-stop interval *is an interval $[a,b] \subset [0,\ell]$ with $a < b$ such that the restriction $\gamma \restriction_{[a,b]}$ is constant and such that $[a,b]$ is maximal for this property. The set of γ-stop intervals is denoted by Δ_γ. The path γ is* regular *if $\Delta_\gamma = \varnothing$.*

2.12. **Definition.** *Let $\gamma_1 : [0,\ell_1] \to U$ and $\gamma_2 : [0,\ell_2] \to U$ be two paths in the topological space U such that $\gamma_1(\ell_1) = \gamma_2(0)$. The* Moore composition $\gamma_1 * \gamma_2 : [0,\ell_1+\ell_2] \to U$ *is the Moore path defined by*

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0,\ell_1] \\ \gamma_2(t-\ell_1) & \text{for } t \in [\ell_1, \ell_1+\ell_2]. \end{cases}$$

The Moore composition of Moore paths is strictly associative. The Moore composition of two regular paths of a Hausdorff space is regular.

[2] This definition will be used only in the case of U being the geometric realization of a precubical set which is Hausdorff: see Notation 3.1; the Hausdorff condition is used here only to ensure that the point of U are closed, which implies that the γ-stop intervals are closed segments.

There is the obvious proposition:

2.13. **Proposition.** *Let $n \geqslant 1$. The following data assemble into a multipointed d-space $|\square[n]|_{reg}$ called the* regular n-cube*:*

- *The underlying space is the topological n-cube $[0,1]^n$.*
- *The set of states is $\{0,1\}^n \subset [0,1]^n$.*
- *The set of execution paths from $\underline{a}$ to $\underline{b}$ with $\underline{a} < \underline{b} \in \{0 < 1\}^n$ is the set of regular paths $[0,1] \to [0,1]^n$ from $\underline{a}$ to $\underline{b}$ which are nondecreasing with respect to each axis of coordinates.*
- *The set of execution paths from $\underline{a}$ to $\underline{b}$ with $\underline{a} \geqslant \underline{b}$ is empty.*

Let $|\square[0]|_{reg} = \{()\}$. Let $\delta_i^\alpha : [0,1]^{n-1} \to [0,1]^n$ be the continuous map defined for $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$ by $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$. By convention, let $[0,1]^0 = \{()\}$. The mapping $[n] \mapsto |\square[n]|_{reg}$ yields a well-defined cocubical multipointed d-space.

Using Proposition 2.9, this leads to the definition:

2.14. **Definition.** *Let K be a precubical set. The* regular realization *of K is the multipointed d-space*

$$|K|_{reg} = \varinjlim_{\square[n] \to K} |\square[n]|_{reg}.$$

It yields a colimit-preserving functor from precubical sets to multipointed d-spaces.

3. L_1-arc length of d-paths of precubical sets

3.1. **Notation.** *The cocubical topological space $[n] \mapsto [0,1]^n$ gives rise to a colimit-preserving functor from precubical sets to topological spaces denoted by*

$$|K|_{geom} = \varinjlim_{\square[n] \to K} [0,1]^n.$$

Since the underlying space functor from multipointed d-spaces to spaces is colimit-preserving, the topological space $|K|_{geom}$ is the underlying space of the multipointed d-space $|K|_{reg}$. The topological space $|K|_{geom}$ is a CW-complex. It is equipped with the final topology which is Δ-generated and Hausdorff. We introduce the following definition:

3.2. **Definition.** *Let K be a precubical set. Let $(\alpha, \beta) \in K_0 \times K_0$. Let $\ell > 0$. Let $\overrightarrow{R}^\ell_{\alpha,\beta}(K)$ be the subspace of continuous maps from $[0,\ell]$ to $|K|_{geom}$ defined by*

$$\overrightarrow{R}^\ell_{\alpha,\beta}(K) = \{t \mapsto \gamma\mu_\ell \mid \gamma \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}\}.$$

Its elements are called the tame regular d-paths of K (of length ℓ) *from α to β. Let*

$$\overrightarrow{R}^\ell(K) = \coprod_{(\alpha,\beta) \in K_0 \times K_0} \overrightarrow{R}^\ell_{\alpha,\beta}(K).$$

Note that it is supposed that $\ell > 0$: it implies that the paths of $\overrightarrow{R}^\ell_{\alpha,\beta}(K)$ are nonconstant. The definition of $\overrightarrow{R}^\ell_{\alpha,\beta}(K)$ is not restrictive. Indeed, we have:

3.3. **Proposition.** *Let K be a precubical set. Let $\phi : [0,\ell] \cong^+ [0,\ell]$. Let $\gamma \in \overrightarrow{R}^\ell_{\alpha,\beta}(K)$. Then $\gamma\phi \in \overrightarrow{R}^\ell_{\alpha,\beta}(K)$.*

Proof. By definition of $\overrightarrow{R}^{\ell}_{\alpha,\beta}(K)$, there exists $\overline{\gamma} \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$ such that $\gamma = \overline{\gamma}\mu_\ell$. We obtain $\gamma\phi = \overline{\gamma}\mu_\ell\phi\mu_\ell^{-1}\mu_\ell$. Since $\mu_\ell\phi\mu_\ell^{-1} \in \mathcal{G}(1,1)$, we deduce that $\overline{\gamma}\mu_\ell\phi\mu_\ell^{-1} \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$ and that $\gamma\phi \in \overrightarrow{R}^{\ell}_{\alpha,\beta}(K)$. □

3.4. **Notation.** *Let* $\underline{x} = (x_1, \dots, x_n)$ *and* $\underline{x}' = (x'_1, \dots, x'_n)$ *be two elements of* $[0,1]^n$ *with* $n \geqslant 1$. *Let*

$$d_1(\underline{x}, \underline{x}') = \sum_{i=1}^{n} |x_i - x'_i|.$$

An important feature shared by all d-paths (regular or not, tame or not) of a precubical set K is that they have a well-defined L_1-arc length [24, Section 2.2.1] [25, Section 2.2]. It is defined as follows. Consider a d-path $\gamma : [0,\ell] \to [0,1]^n$. Define the L_1-arc length between $\gamma(t)$ and $\gamma(t')$ by $d_1(\gamma(t), \gamma(t'))$. The L_1-arc length of a d-path of $[0,1]^n$ between 0_n an 1_n is therefore n. The L_1-arc length of a Moore composition of d-paths is defined by adding the L_1-arc length of each d-path.

3.5. **Remark.** *The length of a nonconstant d-path* $\gamma : [0,\ell] \to [0,1]^n$ *between two vertices of* $\{0,1\}^n$ *is* ℓ, *whereas its* L_1*-arc length is an integer belonging to* $\{1, \dots, n\}$.

3.6. **Proposition.** *Let* K *be a precubical set. Let* $(\alpha, \beta) \in K_0 \times K_0$. *Two execution paths of* $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$ *which are in the same path-connected component have the same* L_1*-arc length.*

Proof. Write $(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg})_{co}$ for the underlying set of $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$ endowed with the compact-open topology. Let $\overrightarrow{P}_{\alpha,\beta}(K)_{co}$ be the set of *all* d-paths of K from α to β equipped with the compact-open topology. By [25, Proposition 2.2], a composite map of the form $[0,1] \to (\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg})_{co} \subset \overrightarrow{P}_{\alpha,\beta}(K)_{co} \to \mathbb{R}$ is constant, where the right-hand map is the L_1-arc length. The Δ-kelleyfication functor does not change the path-connected components. Hence the proof is complete. □

Due to the variety of the terminologies used in the mathematical literature, we recall the following definition for the convenience of the reader.

3.7. **Definition.** *A* pseudometric space (X, d) *is a set* X *equipped with a map* $d : X \times X \to [0, +\infty]$ *called a* pseudometric *such that:*

- $\forall x \in X, d(x,x) = 0$
- $\forall (x,y) \in X \times X, d(x,y) = d(y,x)$
- $\forall (x,y,z) \in X \times X \times X, d(x,y) \leqslant d(x,z) + d(z,y)$.

A map $f : (X,d) \to (Y,d)$ *of pseudometric spaces is a set map* $f : X \to Y$ *which does not increase distance, i.e.* $\forall (x,y) \in X \times X, d(f(x), f(y)) \leqslant d(x,y)$.

Every metric space is a pseudometric space. The interest of this category for Proposition 3.8 lies in the following two facts. At first, it is bicomplete, being a coreflective subcategory of the bicomplete category of Lawvere metric spaces [3] [21]. Secondly, the family of balls $B(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\})$ with $x \in X$ and $\epsilon > 0$ generates a topology called the underlying topology of (X,d). This gives rise to a functor from pseudometric spaces to general topological spaces.

[3] which are pseudometric spaces with the symmetry condition dropped

3.8. **Proposition.** *Let K be a precubical set. Let $\ell > 0$. The set map*

$$L : \overrightarrow{R}^\ell(K) \times [0,\ell] \longrightarrow [0,+\infty[$$

which takes (γ,t) to the L_1-arc length between $\gamma(0)$ and $\gamma(t)$ is continuous.

Proof. By equipping each topological n-cube $[0,1]^n$ for $n \geqslant 0$ with the metric d_1, we obtain a cocubical pseudometric space, and by Proposition 2.9, a colimit-preserving functor from precubical sets to pseudometric spaces denoted by $K \mapsto |K|_{d_1}$. Since there is a homeomorphism $|\square[n]|_{geom} \cong |\square[n]|_{d_1}$ for all $n \geqslant 0$, the set of tame regular d-paths in $|K|_{geom}$ coincides with the set of tame regular d-paths in $|K|_{d_1}$ and the universal property of the colimit proves that the identity induces a continuous map $|K|_{geom} \to |K|_{d_1}$. Let $\overrightarrow{P}(K)_{d_1}$ be the set of *all* d-paths of K equipped with the compact-open topology associated with the underlying topological space of $|K|_{d_1}$. Using [24, Lemma 2.13], the set map $L : \overrightarrow{P}(K)_{d_1} \times [0,+\infty[\longrightarrow [0,+\infty[$ taking a pair (γ,t) to the L_1-arc length between $\gamma(0)$ and $\gamma(t)$ is continuous. Let $(\overrightarrow{R}^\ell(K))_{d_1}$ ($(\overrightarrow{R}^\ell(K))_{co}$ resp.) be the underlying set of the space $\overrightarrow{R}^\ell(K)$ equipped with the compact-open topology associated with the underlying topological space of $|K|_{d_1}$ (associated with the topological space $|K|_{geom}$ resp.). Since the identity induces a continuous map $|K|_{geom} \to |K|_{d_1}$, we obtain a continuous map

$$L : (\overrightarrow{R}^\ell(K))_{co} \times [0,\ell] \to (\overrightarrow{R}^\ell(K))_{d_1} \times [0,\ell] \subset \overrightarrow{P}(K)_{d_1} \times [0,+\infty[\to [0,+\infty[.$$

Finally, take the image by the Δ-kelleyfication functor. The latter is a right adjoint therefore it preserves binary products. Besides, $[0,\ell]$ and $[0,+\infty[$ are already Δ-generated. Hence the proof is complete. □

3.9. **Notation.** *By adjunction, we obtain a continuous map*

$$\widehat{L} : \overrightarrow{R}^\ell(K) \longrightarrow \mathbf{TOP}([0,\ell],[0,+\infty[).$$

3.10. **Lemma.** *For all $r \in \overrightarrow{R}^\ell(K)$, for all $\phi \in \mathcal{G}(\ell',\ell)$, and for all $t \in [0,\ell']$, one has*

$$\widehat{L}(r\phi)(t) = \widehat{L}(r)(\phi(t)).$$

Proof. The L_1-arc length between $r(0) = r(\phi(0))$ and $r(\phi(t))$ for the d-path r is equal to the L_1-arc length between $(r\phi)(0)$ and $(r\phi)(t)$ for the d-path $r\phi$. □

Intuitively, the natural d-paths are the d-paths whose speed corresponds to the L_1-arc length.

3.11. **Definition.** *[24, Definition 2.14] Let K be a precubical set. Let $(\alpha,\beta) \in K_0 \times K_0$. A d-path γ of $\overrightarrow{R}^\ell_{\alpha,\beta}(K)$ is* natural *if $\widehat{L}(\gamma)(t) = t$ for all $t \in [0,\ell]$. This implies that ℓ is an integer (greater than or equal to 1). The subset of natural d-paths of length $n \geqslant 1$ from α to β equipped with the Δ-kelleyfication of the relative topology is denoted by $\overrightarrow{N}^n_{\alpha,\beta}(K)$. Let*

$$\overrightarrow{N}^\ell(K) = \coprod_{(\alpha,\beta)\in K_0\times K_0} \overrightarrow{N}^\ell_{\alpha,\beta}(K).$$

The following theorem is an improvement and an adaptation in our topological setting of [24, Proposition 2.16]: the homotopy equivalence is replaced by a homeomorphism thanks to the spaces $\mathcal{G}(\ell,n)$.

3.12. **Theorem.** *Let K be a precubical set. Let $(\alpha,\beta) \in K_0 \times K_0$. There is a homeomorphism*

$$\Psi^\ell : \overrightarrow{R}^\ell_{\alpha,\beta}(K) \xrightarrow{\cong} \coprod_{n\geqslant 1} \mathcal{G}(\ell,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K).$$

Proof. By Proposition 3.6, the space $\overrightarrow{R}^\ell_{\alpha,\beta}(K)$ is the direct sum of the subspaces $\overrightarrow{R}^{\ell,n}_{\alpha,\beta}(K)$ of tame regular d-paths of L_1-arc length n for $n \geqslant 1$. It then suffices to prove the homeomorphism $\overrightarrow{R}^{\ell,n}_{\alpha,\beta}(K) \cong \mathcal{G}(\ell,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$ for all $n \geqslant 1$. Consider the continuous map $\Phi^\ell : \mathcal{G}(\ell,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K) \to \overrightarrow{R}^{\ell,n}_{\alpha,\beta}(K)$ defined by

$$\Phi^\ell(\phi,\gamma) = \gamma\phi.$$

Since $\widehat{L}(\gamma\phi) = \widehat{L}(\gamma)\phi = \phi$, the first equality by Lemma 3.10 and the second equality since γ is natural, the d-path $\gamma\phi$ has no stop-interval and it is therefore regular. We want to define a continuous map $\Psi^\ell : \overrightarrow{R}^{\ell,n}_{\alpha,\beta}(K) \to \mathcal{G}(\ell,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$. Let $r \in \overrightarrow{R}^{\ell,n}_{\alpha,\beta}(K)$. Then $\widehat{L}(r) : [0,\ell] \to [0,n]$ is surjective and nondecreasing. Let $t,t' \in [0,1]$ such that $\widehat{L}(r)(t) = \widehat{L}(r)(t')$. Since r is regular, it has no stop interval. It implies that $t = t'$. Thus, $\widehat{L}(r) \in \mathcal{G}(\ell,n)$. Let

$$\Psi^\ell(r) = (\widehat{L}(r), r\widehat{L}(r)^{-1}).$$

The map Ψ^ℓ is continuous by Proposition 3.8 and by [13, Lemma 6.2]. Then one has

$$\widehat{L}(r\widehat{L}(r)^{-1})(t) = \widehat{L}(r)(\widehat{L}(r)^{-1}(t)) = t$$

for all $0 \leqslant t \leqslant n$, the first equality by Lemma 3.10 and the second equality by algebraic simplification. It means that $r\widehat{L}(r)^{-1} \in \overrightarrow{N}^n_{\alpha,\beta}(K)$: the path $r\widehat{L}(r)^{-1}$ is called the naturalization of r in [24, Definition 2.14]. One has

$$\Phi^\ell\Psi^\ell(r) = \Phi^\ell(\widehat{L}(r), r\widehat{L}(r)^{-1}) = r\widehat{L}(r)^{-1}\widehat{L}(r) = r,$$

the first equality by definition of Ψ^ℓ, the second equality by definition of Φ^ℓ and the last equality by algebraic simplification. Let $(\phi,\gamma) \in \mathcal{G}(\ell,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$. Since $\widehat{L}(\gamma\phi) = \phi$, we obtain

$$\Psi^\ell\Phi^\ell(\phi,\gamma) = \Psi^\ell(\gamma\phi) = (\widehat{L}(\gamma\phi), \gamma\phi\widehat{L}(\gamma\phi)^{-1}) = (\phi,\gamma),$$

the first equality by definition of Φ^ℓ, the second equality by definition of Ψ^ℓ, and the last equality by algebraic simplification. □

Theorem 3.12 can be applied as follows. We take this opportunity to recall the definition of a flow, notion which will be briefly used in Section 5.

3.13. **Definition.** *[8, Definition 4.11] A* flow *is a small semicategory enriched over the closed monoidal category* $(\mathbf{Top},\times)$*. The corresponding category is denoted by* **Flow***.*

A *flow* X consists of a topological space $\mathbb{P}X$ of *execution paths*, a discrete space X^0 of *states*, two continuous maps s and t from $\mathbb{P}X$ to X^0 called the source and target map respectively, and a continuous and associative map $* : \{(x,y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$ such that $s(x*y) = s(x)$ and $t(x*y) = t(y)$. Let $\mathbb{P}_{\alpha,\beta}X = \{x \in \mathbb{P}X \mid s(x) = \alpha \text{ and } t(x) = \beta\}$: it is the space of execution paths from α to β, α is called the initial state and β is called the final state. Note that the composition is denoted by $x*y$, not by $y \circ x$. The category **Flow** is locally presentable by [12, Theorem 6.11].

3.14. **Definition.** *[16, Definition 7.1] Let K be a precubical set. The* tame concrete realization *of K is the flow $|K|_{tc}$ defined as follows:*

$$|K|_{tc}^0 = K_0$$
$$\forall (\alpha,\beta) \in K_0 \times K_0, \mathbb{P}_{\alpha,\beta}|K|_{tc} = \coprod_{n\geqslant 1} \overrightarrow{N}^n_{\alpha,\beta}(K)$$

The composition of execution paths is induced by the Moore composition.

As noticed after [16, Theorem 7.7], the tame concrete realization functor is not colimit-preserving in general. It is therefore not a realization functor for flows in the sense of [16, Definition 3.6]. It means that it is difficult to calculate for a general precubical set. However, [16, Theorem 7.7] proves that it coincides with a colimit-realization functor when the precubical set is spatial in the sense of Definition 6.5. Since most of the concrete examples coming from concurrency theory are spatial precubical sets (see the comment after Definition 6.5), the tame concrete realization of a precubical set is easily calculable for this kind of examples.

Let X be a multipointed d-space. Consider for every $(\alpha,\beta) \in X^0 \times X^0$ the coequalizer of spaces

$$\mathbb{P}_{\alpha,\beta}X = \varinjlim \left(\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \times \mathcal{G}(1,1) \rightrightarrows \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X\right)$$

where the two maps are $(c,\phi) \mapsto c$ and $(c,\phi) \mapsto c.\phi$. Let $[-]_{\alpha,\beta} : \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \to \mathbb{P}_{\alpha,\beta}X$ be the canonical map.

3.15. **Theorem.** *[10, Theorem 7.2] Let X be a multipointed d-space. Then there exists a flow* $\mathrm{cat}(X)$ *with* $\mathrm{cat}(X)^0 = X^0$, $\mathbb{P}_{\alpha,\beta}\,\mathrm{cat}(X) = \mathbb{P}_{\alpha,\beta}X$ *and the composition law $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$ is for every triple $(\alpha,\beta,\gamma) \in X^0 \times X^0 \times X^0$ the unique map making the following diagram commutative:*

$$\begin{array}{ccc}
\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \times \mathbb{P}^{\mathcal{G}}_{\beta,\gamma}X & \xrightarrow{*_N} & \mathbb{P}^{\mathcal{G}}_{\alpha,\gamma}X \\
\downarrow{\scriptstyle [-]_{\alpha,\beta}\times[-]_{\beta,\gamma}} & & \downarrow{\scriptstyle [-]_{\alpha,\gamma}} \\
\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X & \longrightarrow & \mathbb{P}_{\alpha,\gamma}X
\end{array}$$

*where $*_N$ is the normalized composition (cf. Definition 2.4). The mapping $X \mapsto cat(X)$ induces a functor from $\mathcal{G}$**dTop** to **Flow**.*

3.16. **Definition.** *The functor* $\mathrm{cat} : \mathcal{G}\mathbf{dTop} \to \mathbf{Flow}$ *is called the* categorization functor.

3.17. **Proposition.** *For every precubical set K, there is the natural isomorphism of flows*

$$\mathrm{cat}(|K|_{reg}) \cong |K|_{tc}.$$

Proof. By Theorem 3.12, there is the homeomorphism

$$\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg} = \overrightarrow{R}^1_{\alpha,\beta}(K) \xrightarrow{\cong} \coprod_{n\geqslant 1} \mathcal{G}(1,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$$

for all $(\alpha,\beta) \in K_0 \times K_0$. The coequalizer above identifies $(\psi,\gamma) \in \mathcal{G}(1,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$ with $(\psi\phi,\gamma) \in \mathcal{G}(1,n) \times \overrightarrow{N}^n_{\alpha,\beta}(K)$ for all $\phi \in \mathcal{G}(1,1)$. The proof is complete by Definition 3.14. □

4. The {q,h,m}-model structures of Moore flows

The enriched small category $\mathcal{G}$ is defined as follows:

- The set of objects is the open interval $]0,+\infty[$.
- The space $\mathcal{G}(\ell_1,\ell_2)$ is the set $\{[0,\ell_1] \cong^+ [0,\ell_2]\}$ for all $\ell_1,\ell_2 > 0$ equipped with the Δ-kelleyfication of the relative topology induced by the set inclusion $\mathcal{G}(\ell_1,\ell_2) \subset \mathbf{TOP}([0,\ell_1],[0,\ell_2])$. By [13, Proposition 2.5], this topology coincides with the compact-open topology and the pointwise topology.
- For every $\ell_1,\ell_2,\ell_3 > 0$, the composition map
$$\mathcal{G}(\ell_1,\ell_2) \times \mathcal{G}(\ell_2,\ell_3) \to \mathcal{G}(\ell_1,\ell_3)$$
is induced by the composition of continuous maps.

The enriched category $\mathcal{G}$ is an example of a reparametrization category in the sense of [12, Definition 4.3] which is different from the terminal category. It is introduced in [12, Proposition 4.9].

We want to recall some basic facts and notations in Definition 4.1, Proposition 4.2, Theorem 4.3, Proposition 4.4 and Proposition 4.5 for the ease of the reader.

4.1. **Definition.** *The enriched category of enriched presheaves from $\mathcal{G}$ to* **Top** *is denoted by $[\mathcal{G}^{op},\mathbf{Top}]$. The underlying set-enriched category of enriched maps of enriched presheaves is denoted by $[\mathcal{G}^{op},\mathbf{Top}]_0$. The objects of $[\mathcal{G}^{op},\mathbf{Top}]_0$ are called the $\mathcal{G}$-*spaces. *Let*
$$\mathbb{F}^{\mathcal{G}^{op}}_{\ell}U = \mathcal{G}(-,\ell)\times U \in [\mathcal{G}^{op},\mathbf{Top}]_0$$
where U is a topological space and where $\ell > 0$.

The category $[\mathcal{G}^{op},\mathbf{Top}]_0$ is locally presentable by [11, Proposition 5.1].

4.2. **Proposition.** *[11, Proposition 5.3 and Proposition 5.5] The category $[\mathcal{G}^{op},\mathbf{Top}]_0$ is a full reflective and coreflective subcategory of $\mathbf{Top}^{\mathcal{G}^{op}_0}$. For every $\mathcal{G}$-space $F:\mathcal{G}^{op}\to\mathbf{Top}$, every $\ell>0$ and every topological space X, we have the natural bijection of sets*
$$[\mathcal{G}^{op},\mathbf{Top}]_0(\mathbb{F}^{\mathcal{G}^{op}}_{\ell}X,F) \cong \mathbf{Top}(X,F(\ell)).$$

4.3. **Theorem.** *([12, Theorem 5.14]) Let D and E be two $\mathcal{G}$-spaces. Let*
$$D\otimes E = \int^{(\ell_1,\ell_2)} \mathcal{G}(-,\ell_1+\ell_2)\times D(\ell_1)\times E(\ell_2).$$
The pair $([\mathcal{G}^{op},\mathbf{Top}]_0,\otimes)$ has the structure of a biclosed semimonoidal category.

4.4. **Proposition.** *([12, Proposition 5.16]) Let U,U' be two topological spaces. Let $\ell,\ell'>0$. There is the natural isomorphism of $\mathcal{G}$-spaces*
$$\mathbb{F}^{\mathcal{G}^{op}}_{\ell}U\otimes \mathbb{F}^{\mathcal{G}^{op}}_{\ell'}U' \cong \mathbb{F}^{\mathcal{G}^{op}}_{\ell+\ell'}(U\times U').$$

4.5. **Proposition.** *[12, Proposition 5.18] Let D and E be two $\mathcal{G}$-spaces. Then there is a natural homeomorphism*
$$\varinjlim(D\otimes E)\cong \varinjlim D\times\varinjlim E.$$

By [23, Theorem 6.5(ii)], since all topological spaces are fibrant and since $(\mathbf{Top},\times,\{0\})$ is a locally presentable base by [11, Corollary 3.3], the category of $\mathcal{G}$-spaces $[\mathcal{G}^{op},\mathbf{Top}]_0$ can be endowed with the projective model structure associated with one of the three model structures $\mathbf{Top}_q, \mathbf{Top}_m, \mathbf{Top}_h$. They are called the projective q-model structure (m-model structure, h-model structure resp.) and denoted by $[\mathcal{G}^{op}, \mathbf{Top}_q]_0^{proj}$ ($[\mathcal{G}^{op}, \mathbf{Top}_m]_0^{proj}$, $[\mathcal{G}^{op}, \mathbf{Top}_h]_0^{proj}$ resp.). The three model structures are accessible. The fibrations are the objectwise fibrations of the corresponding model structure of $\mathbf{Top}$. They are called the projective q-fibrations (projective m-fibrations, projective h-fibrations resp.). The weak equivalences are the objectwise weak equivalence of the corresponding model structure of $\mathbf{Top}$. Since the projective m-fibrations of spaces are the projective h-fibrations of spaces, and since the weak equivalences of the projective m-model structure of $\mathcal{G}$-spaces are the weak equivalences of the projective q-model structure of $\mathcal{G}$-spaces, it implies that the projective m-model structure of $\mathcal{G}$-spaces is the mixing in the sense of [4, Theorem 2.1] of the projective q-model structure of $\mathcal{G}$-spaces and the projective h-model structure of $\mathcal{G}$-spaces. All $\mathcal{G}$-spaces are fibrant for these three model structures.

When the reparametrization category $\mathcal{G}$ is replaced by the terminal category, these model structures coincide with the q-model structure, the m-model structure and the h-model structure of $\mathbf{Top}$.

4.6. **Definition.** *[12, Definition 6.2] A* Moore flow *is a small semicategory enriched over the biclosed semimonoidal category* $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$ *of Theorem 4.3. The corresponding category is denoted by* $\mathcal{G}\mathbf{Flow}$.

Note that by replacing $\mathcal{G}$ by the terminal category, we recover Definition 3.13.

4.7. **Notation.** *Let* $D : \mathcal{G}^{op} \to \mathbf{Top}$ *be a* $\mathcal{G}$*-space. We denote by* $\mathrm{Glob}(D)$ *the Moore flow defined as follows:*

$$\begin{aligned}
&\mathrm{Glob}(D)^0 = \{0,1\}\\
&\mathbb{P}_{0,0}\mathrm{Glob}(D) = \mathbb{P}_{1,1}\mathrm{Glob}(D) = \mathbb{P}_{1,0}\mathrm{Glob}(D) = \varnothing\\
&\mathbb{P}_{0,1}\mathrm{Glob}(D) = D.
\end{aligned}$$

There is no composition law. This construction yields a functor

$$\mathrm{Glob} : [\mathcal{G}^{op}, \mathbf{Top}]_0 \to \mathcal{G}\mathbf{Flow}.$$

The category $\mathcal{G}\mathbf{Flow}$ is locally presentable by [12, Theorem 6.11]. Let X be a multipointed d-space. Let $\mathbb{P}^{\ell}_{\alpha,\beta}X$ be the subspace of continuous maps from $[0,\ell]$ to $|X|$ defined by $\mathbb{P}^{\ell}_{\alpha,\beta}X = \{t \mapsto \gamma\mu_\ell \mid \gamma \in \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X\}$. By [13, Theorem 4.12], there exists a Moore flow $\mathbb{M}^{\mathcal{G}}(X)$ such that:

- The set of states X^0 of X
- For all $\alpha, \beta \in X^0$ and all real numbers $\ell > 0$, $\mathbb{P}^{\ell}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X) = \mathbb{P}^{\ell}_{\alpha,\beta}X$.
- For all maps $[0,\ell] \cong^+ [0,\ell']$, a map $f : [0,\ell'] \to |X|$ of $\mathbb{P}^{\ell'}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X)$ is mapped to the map $[0,\ell] \cong^+ [0,\ell'] \xrightarrow{f} |X|$ of $\mathbb{P}^{\ell}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X)$
- For all $\alpha, \beta, \gamma \in X^0$ and all real numbers $\ell, \ell' > 0$, the composition maps
$$* : \mathbb{P}^{\ell}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}(X) \times \mathbb{P}^{\ell'}_{\beta,\gamma}\mathbb{M}^{\mathcal{G}}(X) \to \mathbb{P}^{\ell+\ell'}_{\alpha,\gamma}\mathbb{M}^{\mathcal{G}}(X)$$
is the Moore composition.

4.8. **Theorem.** *([13, Theorem 4.12 and Appendix B]) The mapping above induces a functor* $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ *which is a right adjoint.*

4.9. **Definition.** *A* $[\mathcal{G}^{op}, \mathbf{Top}]_0$-graph X *consists of a pair* $(X^0, (\mathbb{P}_{\alpha,\beta}X)_{(\alpha,\beta)\in X^0\times X^0})$ *such that* X^0 *is a set and such that each* $\mathbb{P}_{\alpha,\beta}X$ *is a* $\mathcal{G}$*-space. A map of* $[\mathcal{G}^{op}, \mathbf{Top}]_0$*-graphs* $f : X \to Y$ *consists of a set map* $f^0 : X^0 \to Y^0$ *(called the* underlying set map*) together with a map* $\mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f^0(\alpha),f^0(\beta)}Y$ *of* $\mathcal{G}$*-spaces for all* $(\alpha, \beta) \in X^0 \times X^0$*. The composition is defined in an obvious way. The corresponding category is denoted by* $\mathrm{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$.

4.10. **Theorem.** *Let* $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ *be either the projective q-model structure, or the projective m-model structure, or the projective h-model structure of* $\mathcal{G}$*-spaces. Then the category Moore flows can be endowed with an accessible model structure characterized as follows:*

- *A map of Moore flows* $f : X \to Y$ *is a weak equivalence if and only if* $f^0 : X^0 \to Y^0$ *is a bijection and* $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ *belongs to* $\mathcal{W}$*.*
- *A map of Moore flows* $f : X \to Y$ *is a fibration if and only if* $f^0 : X^0 \to Y^0$ *is a bijection and* $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ *belongs to* $\mathcal{F}$*.*

All objects are fibrant. These three model structures are called the q-model structure*, the* m-model structure *and the* h-model structure *of* $\mathcal{G}\mathbf{Flow}$*. The m-model structure of Moore flows is the mixing in the sense of [4, Theorem 2.1] of the q-model structure and the h-model structure of Moore flows. The q-model structure coincides with the one of [12, Theorem 8.8]. Every q-cofibration of Moore flows is an m-cofibration of Moore flows and every m-cofibration of Moore flows is an h-cofibration of Moore flows.*

Sketch of proof. This theorem is already proved in [15, Theorem 7.4] if the reparametrization category $\mathcal{G}$ is replaced by the terminal category. It is the reason why the proof is only sketched. Choose $(\mathcal{C}, \mathcal{F}, \mathcal{W})$. By [15, Corollary 5.6], there exists a unique model structure on the category $\mathrm{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$ such that the weak equivalences are the maps of enriched graphs which induce a bijection between the sets of vertices and which are objectwise weak equivalences and such that the fibrations are the objectwise fibrations. This model structure is accessible and all enriched graphs are fibrant. There is a forgetful functor

$$\mathrm{Gph} : \mathcal{G}\mathbf{Flow} \longrightarrow \mathrm{Gph}([\mathcal{G}^{op}, \mathbf{Top}]_0)$$

which forgets the composition law. This functor is a right-adjoint. The left adjoint is the free Moore flow generated by the enriched graph. The semicategorical description of Moore flows is crucial: see [15, Proposition 7.3] for a description of the left adjoint. Like in [15, Theorem 7.4], the model structure on enriched graphs of $\mathcal{G}$-spaces can be right-lifted along this right adjoint by using [15, Theorem 2.1], namely the Quillen path object argument in model categories where all objects are fibrant, with the path functor $\mathrm{Path} : \mathcal{G}\mathbf{Flow} \to \mathcal{G}\mathbf{Flow}$ defined on objects by $\mathrm{Path}(X)^0 = X^0$, and for all $(\alpha, \beta) \in X^0 \times X^0$ and for all $\ell > 0$ by $\mathbb{P}^\ell_{\alpha,\beta}\mathrm{Path}(X) = \mathbf{TOP}([0, 1], \mathbb{P}^\ell_{\alpha,\beta}X)$ with an obvious definition of the composition law. The hypotheses of [15, Theorem 2.1] are satisfied because they are satisfied objectwise. Indeed, for all $(\alpha, \beta) \in X^0 \times X^0$ and for all $\ell > 0$, the diagonal of the topological space $\mathbb{P}^\ell_{\alpha,\beta}X$ factors as a composite

$$\mathbb{P}^\ell_{\alpha,\beta}X \longrightarrow \mathbf{TOP}([0, 1], \mathbb{P}^\ell_{\alpha,\beta}X) \longrightarrow \mathbb{P}^\ell_{\alpha,\beta}X \times \mathbb{P}^\ell_{\alpha,\beta}X$$

where the left-hand map is a homotopy equivalence and the right-hand map is a r-fibration for $r \in \{q, m, h\}$. The q-model structure of this theorem coincides with the one of [12, Theorem 8.8] because the two model structures have the same class of fibrations and weak equivalences. The last sentence is a general fact about mixed model structures (see [4, Theorem 2.1 and Proposition 3.6]). ☐

4.11. **Notation.** *These three model categories are denoted by* $\mathcal{G}\mathbf{Flow}_q, \mathcal{G}\mathbf{Flow}_m, \mathcal{G}\mathbf{Flow}_h$.

5. Cube chains and regular realization

5.1. **Definition.** *The cocubical Moore flow* $\mathbb{M}^{\mathcal{G}}|\square[*]|_{reg}$ *gives rise by Proposition 2.9 to a colimit-preserving functor* $[-]_{reg} : \square^{op}\mathbf{Set} \to \mathcal{G}\mathbf{Flow}$ *defined by*

$$[K]_{reg} = \varinjlim_{\square[n]\to K} \mathbb{M}^{\mathcal{G}}|\square[n]|_{reg}$$

The following notations coincide with [16, Definition 4.8 and Definition 4.12] by [16, Proposition 4.10 and Proposition 4.14].

5.2. **Notation.** *Let* $N_n = \overrightarrow{N}^n_{0_n,1_n}(\square[n])$ *and* $\partial N_n = \overrightarrow{N}^n_{0_n,1_n}(\partial\square[n])$ *for* $n \geqslant 0$. *These spaces are first countable,* Δ*-generated and* Δ*-Hausdorff.*

The map $[0,1]^{m_1}\sqcup[0,1]^{m_2} \to [0,1]^{m_1+m_2}$ defined by taking $(t_1,\dots,t_{m_1})$ to $(t_1,\dots,t_{m_1},0_{m_2})$ and $(t'_1,\dots,t'_{m_2})$ to $(1_{m_1},t'_1,\dots,t'_{m_2})$ induces a continuous map $N_{m_1}\times N_{m_2} \to N_{m_1+m_2}$ by using the fact that the Moore composition of two natural d-paths is still a natural d-path.

Cube chains are a powerful notion introduced by Ziemiański. We use the presentation given in [29, Section 7]. Let $\mathrm{Seq}(n)$ be the set of sequences of positive integers $\underline{n} = (n_1,\dots,n_p)$ with $n_1+\dots+n_p = n$. Let $\underline{n} = (n_1,\dots,n_p) \in \mathrm{Seq}(n)$. Then $|\underline{n}| = n$ is the length of $\underline{n}$ and $\ell(\underline{n}) = p$ is the number of elements of $\underline{n}$. Let K be a precubical set and $A = a_1 < \dots < a_k \subset \{1,\dots,n\}$ and $\epsilon \in \{0,1\}$. The *iterated face map* is defined by

$$\partial^\epsilon_A = \partial^\epsilon_{a_1}\partial^\epsilon_{a_2}\dots\partial^\epsilon_{a_k}.$$

5.3. **Definition.** *Let* $\underline{n} \in \mathrm{Seq}(n)$. *The* $\underline{n}$-cube *is the precubical set*

$$\square[\underline{n}] = \square[n_1]*\dots*\square[n_p]$$

where the notation $*$ *means that the final state* 1_{n_i} *of the precubical set* $\square[n_i]$ *is identified with the initial state* $0_{n_{i+1}}$ *of the precubical set* $\square[n_{i+1}]$ *for* $1 \leqslant i \leqslant p-1$.

Let K be a precubical set. Let $\alpha,\beta \in K_0$. Let $n \geqslant 1$. The small category $\mathrm{Ch}_{\alpha,\beta}(K,n)$ is defined as follows. The objects are the maps of precubical sets $\square[\underline{n}] \to K$ with $|\underline{n}| = n$ where the initial state of $\square[n_1]$ is mapped to α and the final state of $\square[n_p]$ is mapped to β. Let $A \sqcup B = \{1,\dots,m_1+m_2\}$ be a partition with the cardinal of A equal to $m_1 > 0$ and the cardinal of B equal to $m_2 > 0$. Let

$$\phi_{A,B} : \square[m_1]*\square[m_2] \longrightarrow \square[m_1+m_2]$$

be the unique map of precubical sets such that

$$\phi_{A,B}(\mathrm{Id}_{[m_1]}) = \partial^0_B(\mathrm{Id}_{[m_1+m_2]}),$$
$$\phi_{A,B}(\mathrm{Id}_{[m_2]}) = \partial^1_A(\mathrm{Id}_{[m_1+m_2]}).$$

For $i \in \{1,\dots,\ell(\underline{n})\}$ and a partition $A \sqcup B = \{1,\dots,n_i\}$, let

$$\delta_{i,A,B} = \mathrm{Id}_{\square[n_1]}*\dots*\mathrm{Id}_{\square[n_{i-1}]}*\phi_{A,B}*\mathrm{Id}_{\square[n_{i+1}]}*\dots*\mathrm{Id}_{\square[n_{\ell(\underline{n})}]}.$$

The morphisms are the commutative diagrams

$$\begin{array}{ccc} \square[\underline{n}_a] & \xrightarrow{a} & K \\ \downarrow & & \Vert \\ \square[\underline{n}_b] & \xrightarrow{b} & K \end{array}$$

where the left vertical map is a composite of maps of precubical sets of the form $\delta_{i,A,B}$.

We recall the definition of the functor $K \mapsto ||K||$ from precubical sets to flows introduced in [16, Section 6]. The set of states of $||K||$ is K_0. Consider the small diagram of spaces

$$\mathcal{D}_{\alpha,\beta}(K,n) : \mathrm{Ch}_{\alpha,\beta}(K,n) \longrightarrow \mathbf{Top}$$

defined by on objects by

$$\mathcal{D}_{\alpha,\beta}(K,n)(\square[\underline{n}] \to K) = N_{n_1} \times \ldots \times N_{n_p}$$

with $\underline{n} = (n_1, \ldots, n_p)$ and $\sum_i n_i = n$ and on morphisms by using the maps

$$\mathbb{P}|\phi_{A,B}|_{nat} : \mathbb{P}(\square[m_1] * \square[m_2]) \to \mathbb{P}(\square[m_1 + m_2])$$

which induce maps $N_{m_1} \times N_{m_2} \to N_{m_1+m_2}$ given by the Moore composition of tame natural d-paths. The space of execution spaces $\mathbb{P}_{\alpha,\beta}||K||$ is defined as follows:

$$\mathbb{P}_{\alpha,\beta}||K|| = \coprod_{n\geqslant 1} \varinjlim \mathcal{D}_{\alpha,\beta}(K,n).$$

The concatenation of tuples induces functors

$$\mathcal{D}_{\alpha,\beta}(K,m_1) \times \mathcal{D}_{\beta,\gamma}(K,m_2) \to \mathcal{D}_{\alpha,\gamma}(K,m_1+m_2),$$

and, using [14, Proposition A.4], continuous maps

$$\varinjlim \mathcal{D}_{\alpha,\beta}(K,m_1) \times \varinjlim \mathcal{D}_{\beta,\gamma}(K,m_2) \to \varinjlim \mathcal{D}_{\alpha,\gamma}(K,m_1+m_2)$$

for all $m_1, m_2 \geqslant 1$. We obtain an associative composition map

$$\mathbb{P}_{\alpha,\beta}||K|| \times \mathbb{P}_{\beta,\gamma}||K|| \to \mathbb{P}_{\alpha,\gamma}||K||$$

for all $(\alpha,\beta,\gamma) \in K_0 \times K_0 \times K_0$.

We want to define a Moore flow $||K||^{\mathcal{G}}$ by mimicking the above construction of $||K||$ by using the following two rules: 1) any occurrence of the topological space N_k is replaced by the $\mathcal{G}$-space $\mathbb{F}^{\mathcal{G}^{op}}_k N_k$ for all integers $k \geqslant 1$; 2) any product of spaces of the form $N_{n_1} \times \ldots \times N_{n_p}$ is replaced by the tensor products of $\mathcal{G}$-spaces $\mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \ldots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p}$. The idea is to work with the biclosed semimonoidal category $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$ instead of with the biclosed (semi)monoidal category $(\mathbf{Top}, \times)$.

Let $(\alpha,\beta) \in K_0 \times K_0$. Let

$$\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n) : \mathrm{Ch}_{\alpha,\beta}(K,n) \longrightarrow [\mathcal{G}^{op}, \mathbf{Top}]_0$$

be the functor defined on objects by

$$\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n)(\square[\underline{n}] \to K) = \mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \ldots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p}$$

with $\underline{n} = (n_1, \ldots, n_p)$ and $\sum_i n_i = n$ and on morphisms by taking the map

$$\phi_{A,B} : \square[m_1] * \square[m_2] \to \square[m_1 + m_2]$$

to the composite map of $\mathcal{G}$-spaces

$$\mathbb{F}^{\mathcal{G}^{op}}_{m_1} N_{m_1} \otimes \mathbb{F}^{\mathcal{G}^{op}}_{m_1} N_{m_2} \cong \mathbb{F}^{\mathcal{G}^{op}}_{m_1+m_2}(N_{m_1} \times N_{m_2}) \to \mathbb{F}^{\mathcal{G}^{op}}_{m_1+m_2} N_{m_1+m_2}$$

where the isomorphism is given by Proposition 4.4 and where the map $N_{m_1} \times N_{m_2} \to N_{m_1+m_2}$ is given by the Moore composition of tame natural d-paths. The $\mathcal{G}$-space of execution spaces $\mathbb{P}_{\alpha,\beta}||K||^{\mathcal{G}}$ is defined as follows:

$$\mathbb{P}_{\alpha,\beta}||K||^{\mathcal{G}} = \coprod_{n\geqslant 1} \varinjlim \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n).$$

There is the obvious proposition:

5.4. **Proposition.** *Let $D_i : I_i \to [\mathcal{G}^{op}, \mathbf{Top}]_0$ be two small diagrams of $\mathcal{G}$-spaces with $i = 1, 2$. Then the mappings (*Mor(I) *means the class of morphisms of a small category I)*

$$(i_1, i_2) \in I_1 \times I_2 \mapsto D_1(i_1) \otimes D_2(i_2)$$
$$(f, g) \in \mathrm{Mor}(I_1) \times \mathrm{Mor}(I_2) \mapsto f \otimes g$$

yield a well defined small diagram of $\mathcal{G}$-spaces denoted by

$$D_1 \otimes D_2 : I_1 \times I_2 \longrightarrow [\mathcal{G}^{op}, \mathbf{Top}]_0.$$

Proposition 5.5 is the main fact which enables us to define the composition law of the Moore flow $||K||^{\mathcal{G}}$.

5.5. **Proposition.** *Let K be a precubical set. Let $\alpha, \beta, \gamma \in K_0$. Let $m_1, m_2 \geqslant 1$. There is the isomorphism of $\mathcal{G}$-spaces*

$$\varinjlim \Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1) \otimes \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)\Big) \cong \Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)\Big) \otimes \Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)\Big).$$

Proof. The semimonoidal category $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$ being biclosed [12, Theorem 5.14], write

$$\text{(L)} \qquad [\mathcal{G}^{op}, \mathbf{Top}]_0(D, \{E, F\}_L) \cong [\mathcal{G}^{op}, \mathbf{Top}]_0(D \otimes E, F),$$
$$\text{(R)} \qquad [\mathcal{G}^{op}, \mathbf{Top}]_0(E, \{D, F\}_R) \cong [\mathcal{G}^{op}, \mathbf{Top}]_0(D \otimes E, F),$$

D, E, F being three $\mathcal{G}$-spaces. Then we obtain the sequence of bijections

$$\begin{aligned}
&[\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)\Big) \otimes \Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)\Big), F\Big)\\
&\cong [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1), \Big\{\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2), F\Big\}_L\Big)\\
&\cong \varprojlim_{a_1} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1), \Big\{\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2), F\Big\}_L\Big)\\
&\cong \varprojlim_{a_1} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1) \otimes \Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)\Big), F\Big)\\
&\cong \varprojlim_{a_1} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\varinjlim \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2), \Big\{\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1), F\Big\}_R\Big)\\
&\cong \varprojlim_{a_1} \varprojlim_{a_2} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)(a_2), \Big\{\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1), F\Big\}_R\Big)\\
&\cong \varprojlim_{a_1} \varprojlim_{a_2} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1) \otimes \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)(a_2), F\Big)\\
&\cong \varprojlim_{(a_1,a_2)} [\mathcal{G}^{op}, \mathbf{Top}]_0\Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K, m_1)(a_1) \otimes \mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K, m_2)(a_2), F\Big)
\end{aligned}$$

$$\cong [\mathcal{G}^{op},\mathbf{Top}]_0\Big(\varinjlim\Big(\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,m_1)\otimes\mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K,m_2)\Big),F\Big)$$

for all $\mathcal{G}$-spaces F, the first and third bijections by (L), the second and fifth and eighth bijections by the universal property of the colimit, the fourth and sixth bijections by (R) and finally the seventh bijection because limits commute with each other. The proof is complete thanks to the Yoneda lemma. $\square$

We then consider the category of *all* small diagrams of $\mathcal{G}$-spaces over *all* small categories, denoted by $\mathcal{D}([\mathcal{G}^{op},\mathbf{Top}]_0)$, defined as follows. An object is a functor $F : I \to [\mathcal{G}^{op},\mathbf{Top}]_0$ from a small category I to $[\mathcal{G}^{op},\mathbf{Top}]_0$. A morphism from $F : I_1 \to [\mathcal{G}^{op},\mathbf{Top}]_0$ to $G : I_2 \to [\mathcal{G}^{op},\mathbf{Top}]_0$ is a pair $(f : I_1 \to I_2, \mu : F \Rightarrow G.f)$ where f is a functor and μ is a natural transformation. If (g,ν) is a map from $G : I_2 \to [\mathcal{G}^{op},\mathbf{Top}]_0$ to $H : \underline{K} \to [\mathcal{G}^{op},\mathbf{Top}]_0$, then the composite $(g,\nu).(f,\mu)$ is defined by $(g.f,(\nu.f)\odot\mu)$, $\odot$ meaning the composition of natural transformations. The identity of $F : I_1 \to [\mathcal{G}^{op},\mathbf{Top}]_0$ is the pair $(\mathrm{Id}_{I_1},\mathrm{Id}_F)$. It is well-known that this defines an associative composition law (e.g. see [14, Appendix A]). The colimit construction induces a functor

$$\varinjlim : \mathcal{D}([\mathcal{G}^{op},\mathbf{Top}]_0) \longrightarrow [\mathcal{G}^{op},\mathbf{Top}]_0$$

by [14, Proposition A.2]. We define a map of $\mathcal{D}([\mathcal{G}^{op},\mathbf{Top}]_0)$

$$(f^{m_1,m_2}_{\alpha,\beta,\gamma},\mu^{m_1,m_2}_{\alpha,\beta,\gamma}) : \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,m_1)\otimes\mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K,m_2) \longrightarrow \mathcal{D}^{\mathcal{G}}_{\alpha,\gamma}(K,m_1+m_2)$$

for all $m_1,m_2 \geqslant 1$ and all $(\alpha,\beta,\gamma)\in K_0\times K_0\times K_0$ as follows. The functor

$$f^{m_1,m_2}_{\alpha,\beta,\gamma} : \mathrm{Ch}_{\alpha,\beta}(K,m_1)\times\mathrm{Ch}_{\beta,\gamma}(K,m_2)\to\mathrm{Ch}_{\alpha,\gamma}(K,m_1+m_2)$$

takes a pair $(\square[\underline{m_1}]\to K,\square[\underline{m_2}]\to K)$ to the map of precubical sets $\square[\underline{m_1}]*\square[\underline{m_2}]\to K$ and the natural transformation

$$\mu^{m_1,m_2}_{\alpha,\beta,\gamma} : \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,m_1)\otimes\mathcal{D}^{\mathcal{G}}_{\beta,\gamma}(K,m_2) \Longrightarrow \mathcal{D}^{\mathcal{G}}_{\alpha,\gamma}(K,m_1+m_2)f^{m_1,m_2}_{\alpha,\beta,\gamma}$$

is the identity. Using Proposition 5.5, we obtain a map of $\mathcal{G}$-spaces

$$\mathbb{P}_{\alpha,\beta}||K||^{\mathcal{G}}\otimes\mathbb{P}_{\beta,\gamma}||K||^{\mathcal{G}}\longrightarrow\mathbb{P}_{\alpha,\gamma}||K||^{\mathcal{G}}$$

for all $(\alpha,\beta,\gamma)\in K_0\times K_0\times K_0$. It is strictly associative since $([\mathcal{G}^{op},\mathbf{Top}]_0,\otimes)$ is semi-monoidal [12, Proposition 5.11]. We have obtained a well-defined Moore flow $||K||^{\mathcal{G}}$.

5.6. **Proposition.** *The mapping $(\phi\gamma)\mapsto\gamma\phi$ yields the homemomorphisms*

$$\mathcal{G}(\ell,n)\times N_n\cong\mathbb{P}^{\ell}_{0_n,1_n}\mathbb{M}^{\mathcal{G}}|\square[n]|_{reg}$$
$$\mathcal{G}(\ell,n)\times\partial N_n\cong\mathbb{P}^{\ell}_{0_n,1_n}\mathbb{M}^{\mathcal{G}}|\partial\square[n]|_{reg}$$

for all $n\geqslant 1$ and all $\ell>0$. In particular, for $\ell=1$, we obtain the homemomorphisms

$$\mathcal{G}(1,n)\times N_n\cong\mathbb{P}^{\mathcal{G}}_{0_n,1_n}|\square[n]|_{reg}$$
$$\mathcal{G}(1,n)\times\partial N_n\cong\mathbb{P}^{\mathcal{G}}_{0_n,1_n}|\partial\square[n]|_{reg}$$

for all $n\geqslant 1$. Note that for $n=1$, $\partial N_1=\varnothing=\mathbb{P}^{\mathcal{G}}_{0_n,1_n}|\partial\square[n]|_{reg}$.

Proof. The first part is a consequence of Theorem 3.12 applied to the regular d-paths of length ℓ and of L_1-arc length n of the precubical sets $\square[n]$ and $\partial\square[n]$. The particular case $\ell=1$ is a consequence of the definition of the functor $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop}\to\mathcal{G}\mathbf{Flow}$ $\square$

5.7. **Corollary.** *For all $n \geqslant 1$, there are the isomorphisms of $\mathcal{G}$-spaces*

$$\mathbb{F}^{\mathcal{G}^{op}}_n N_n \cong \mathbb{P}_{0_n,1_n}\mathbb{M}^{\mathcal{G}}|\square[n]|_{reg},$$
$$\mathbb{F}^{\mathcal{G}^{op}}_n \partial N_n \cong \mathbb{P}_{0_n,1_n}\mathbb{M}^{\mathcal{G}}|\partial\square[n]|_{reg}.$$

Proof. It is a consequence of Proposition 5.6, of the fact that all regular d-paths from 0_n to 1_n of $|\square[n]|$ are of L_1-arc length n, and of the definition of the functor $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$. □

5.8. **Theorem.** *There is a natural isomorphism of Moore flows*

$$[K]_{reg} \cong ||K||^{\mathcal{G}}$$

for all precubical sets K.

The proof of Theorem 5.8 is roughly speaking the proof of [16, Proposition 6.2] and [16, Theorem 6.3] by working with the biclosed semimonoidal category $([\mathcal{G}^{op}, \mathbf{Top}]_0, \otimes)$ instead of with the biclosed (semi)monoidal category $(\mathbf{Top}, \times)$. Some details slightly change due to the fact that we work with Moore composition of paths which have a length.

Proof. First of all, we prove that there is an isomorphism of cocubical Moore flows $[\square[*]]_{reg} = \mathbb{M}^{\mathcal{G}}|\square[*]|_{reg} \cong ||\square[*]||^{\mathcal{G}}$. Let $d_1(\alpha,\beta) = m$. For $\alpha < \beta \in \{0,1\}^n$ for the product order, the small category $\mathrm{Ch}_{\alpha,\beta}(\square[n],p)$ is empty if $p \neq m$ and it has a terminal object $\square[m] \to \square[n]$ corresponding to the subcube from α to β of $\square[n]$ if $p = m$. We deduce the isomorphisms of $\mathcal{G}$-spaces

$$\begin{aligned}
\mathbb{P}_{\alpha,\beta}||\square[n]||^{\mathcal{G}} &= \varinjlim_{\substack{\underline{n}=(n_1,\dots,n_p),\ell(\underline{n})=m\\ \square[\underline{n}]\to\square[n]\in\mathrm{Ch}_{\alpha,\beta}(\square[n],m)}} \mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \dots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p}\\
&\cong \varinjlim_{\substack{\underline{n}=(n_1,\dots,n_p),\ell(\underline{n})=m\\ \square[\underline{n}]\to\square[n]\in\mathrm{Ch}_{\alpha,\beta}(\square[n],m)}} \mathbb{F}^{\mathcal{G}^{op}}_{m} (N_{n_1} \times \dots \times N_{n_p})\\
&\cong \mathbb{F}^{\mathcal{G}^{op}}_{m}\left(\varinjlim_{\substack{\underline{n}=(n_1,\dots,n_p),\ell(\underline{n})=m\\ \square[\underline{n}]\to\square[n]\in\mathrm{Ch}_{\alpha,\beta}(\square[n],m)}} (N_{n_1} \times \dots \times N_{n_p})\right)\\
&\cong \mathbb{F}^{\mathcal{G}^{op}}_{m} N_m\\
&\cong \mathbb{P}_{\alpha,\beta}[\square[n]]_{reg},
\end{aligned}$$

the first equality by definition of $||\square[n]||^{\mathcal{G}}$, the first isomorphism by Proposition 4.4, the second isomorphism by Proposition 4.2, the third isomorphism because of the unique map $\underline{c} : \square[m] \to \square[n]$ which is the terminal object of $\mathrm{Ch}_{\alpha,\beta}(\square[n],m)$, and the last isomorphism by Corollary 5.7 and by definition of the Moore flow $[\square[n]]_{reg}$. The universal property of the colimit yields a natural map of Moore flows $[K]_{reg} \to ||K||^{\mathcal{G}}$, the functor $[-]_{reg}$ being colimit-preserving. Let $\underline{n} = (n_1,\dots,n_p) \in \mathrm{Seq}(n)$. Every map of precubical sets $\square[\underline{n}] \to K$ gives rise to a map of Moore flows $[\square[\underline{n}]]_{reg} \to [K]_{reg}$. Let $\alpha_0,\alpha_1,\dots,\alpha_p \in K_0$ the images by $0_{n_1}, 1_{n_1} = 0_{n_2}, \dots, 1_{n_p}$ by this map of precubical sets. The are maps of Moore flows $[\square[n_i]]_{reg} \to [K]_{reg}$ inducing maps of $\mathcal{G}$-spaces

$$\mathbb{F}^{\mathcal{G}^{op}}_{n_i} N_{n_i} \cong \mathbb{P}_{0_{n_i},1_{n_i}}[\square[n_i]]_{reg} \longrightarrow \mathbb{P}_{\alpha_{i-1},\alpha_i}[K]_{reg}$$

for $i \in \{1, \dots, p\}$. Using the composition law of the Moore flow $[K]_{reg}$, one obtains a map of $\mathcal{G}$-spaces

$$\mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \dots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p} \longrightarrow \mathbb{P}[K]_{reg}.$$

Consequently, we obtain a cocone

$$(\mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \dots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p})_{\substack{\square[\underline{n}]\to K \\ \in \mathrm{Ch}_{\alpha,\beta}(K,n)}} \stackrel{\bullet}{\longrightarrow} \mathbb{P}[K]_{reg}$$

and then a map of Moore flows $||K||^{\mathcal{G}} \to [K]_{reg}$ which is bijective on states. The composite map of Moore flows $[K]_{reg} \to ||K||^{\mathcal{G}} \to [K]_{reg}$ is the identity of $[K]_{reg}$ because it is the identity for $K = \square[n]$ for all $n \geqslant 0$. Consequently, for all $(\alpha,\beta) \in K_0 \times K_0$, the composite continuous map $\mathbb{P}^1_{\alpha,\beta}[K]_{reg} \to \mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}} \to \mathbb{P}^1_{\alpha,\beta}[K]_{reg}$ is the identity of $\mathbb{P}^1_{\alpha,\beta}[K]_{reg}$: it means that the left-hand map $\mathbb{P}^1_{\alpha,\beta}[K]_{reg} \to \mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}}$ is one-to-one and that the right-hand map $\mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}} \to \mathbb{P}^1_{\alpha,\beta}[K]_{reg}$ is onto. Consider an element $\gamma \in \mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}}$. It has a representative of the form $\overline{\gamma} \in \mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \dots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p}$ for some map $\square[\underline{n}] \to K$ with $\underline{n} = (n_1, \dots, n_p) \in \mathrm{Seq}(n)$. The same argument as above yields an element of $\mathbb{P}^1_{\alpha,\beta}[K]_{reg}$. It means that the left-hand map $\mathbb{P}^1_{\alpha,\beta}[K]_{reg} \to \mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}}$ is onto. It implies that the map $\mathbb{P}^1_{\alpha,\beta}[K]_{reg} \to \mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}}$ is a homeomorphism, thus the map $\mathbb{P}^\ell_{\alpha,\beta}[K]_{reg} \to \mathbb{P}^\ell_{\alpha,\beta}||K||^{\mathcal{G}}$ is a homeomorphism for all $\ell > 0$. The proof is complete. □

6. Space of tame regular d-paths and m-cofibrancy

Consider a Moore flow X. For all $\alpha,\beta \in X^0$, let $X_{\alpha,\beta} = \varinjlim \mathbb{P}_{\alpha,\beta} X$. Let (α,β,γ) be a triple of states of X. The composition law of the Moore flow X induces, using Proposition 4.5, a continuous map $X_{\alpha,\beta} \times X_{\beta,\gamma} \cong \varinjlim(\mathbb{P}_{\alpha,\beta}X \otimes \mathbb{P}_{\beta,\gamma}X) \to \varinjlim \mathbb{P}_{\alpha,\gamma}X \cong X_{\alpha,\gamma}$ which is associative. We obtain a flow $\mathbb{M}_!(X)$ such that the set of states is X^0, such that for all $\alpha,\beta \in X^0$, one has $\mathbb{P}_{\alpha,\beta}\mathbb{M}_! X = X_{\alpha,\beta}$ and for all $\alpha,\beta,\gamma \in X^0$, the composition law is the map $X_{\alpha,\beta} \times X_{\beta,\gamma} \to X_{\alpha,\gamma}$ above defined. This construction yields a well-defined functor

$$\mathbb{M}_! : \mathcal{G}\mathbf{Flow} \longrightarrow \mathbf{Flow}$$

which is a left Quillen equivalence by [12, Theorem 10.9]. We recall the theorem:

6.1. **Theorem.** *([13, Theorem 8.11]) There is the isomorphism of functors*

$$\mathrm{cat} \cong \mathbb{M}_!.\mathbb{M}^{\mathcal{G}}$$

where $\mathrm{cat} : \mathcal{G}\mathbf{dTop} \to \mathbf{Flow}$ *from multipointed d-spaces to flows is the functor of Definition 3.16.*

Some additional information about cube chains is required before proving Theorem 6.6.

6.2. **Proposition.** *Let K be a precubical set. Let $(\alpha,\beta) \in K_0 \times K_0$. Then there is the isomorphism of $\mathcal{G}$-spaces*

$$\mathbb{F}^{\mathcal{G}^{op}}_{n}\Big(\varinjlim \mathcal{D}_{\alpha,\beta}(K,n)\Big) \cong \varinjlim \mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n)$$

for all integers $n \geqslant 1$.

Proof. Let $\underline{n} = (n_1, \dots, n_p)$ and $\sum_i n_i = n$. Then

$$\begin{aligned}\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n)(\square[\underline{n}] \to K) &= \mathbb{F}^{\mathcal{G}^{op}}_{n_1} N_{n_1} \otimes \dots \otimes \mathbb{F}^{\mathcal{G}^{op}}_{n_p} N_{n_p} \\ &\cong \mathbb{F}^{\mathcal{G}^{op}}_{n}(N_{n_1} \times \dots \times N_{n_p})\end{aligned}$$

$$= \mathbb{F}_n^{\mathcal{G}^{op}} \mathcal{D}_{\alpha,\beta}(K,n)(\square[\underline{n}] \to K),$$

the first equality by definition of $\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n)$, the isomorphism by Proposition 4.4 and the last equality by definition of $\mathcal{D}^{\mathcal{G}}_{\alpha,\beta}(K,n)$. The conclusion follows from Proposition 4.2. ☐

6.3. **Proposition.** *For all precubical sets K, there is the natural isomorphism of flows*

$$\mathbb{M}_!(||K||^{\mathcal{G}}) \cong ||K||.$$

Proof. Let $(\alpha,\beta) \in K_0 \times K_0$. There is the sequence of homeomorphisms

$$\begin{aligned}
\mathbb{P}_{\alpha,\beta}\mathbb{M}_!(||K||^{\mathcal{G}}) &\cong \varinjlim \left(\mathbb{P}_{\alpha,\beta}||K||^{\mathcal{G}}\right) \\
&\cong \coprod_{n\geqslant 1} \varinjlim \left(\mathbb{F}_n^{\mathcal{G}^{op}}\left(\varinjlim \mathcal{D}_{\alpha,\beta}(K,n)\right)\right) \\
&\cong \coprod_{n\geqslant 1} \varinjlim \mathcal{D}_{\alpha,\beta}(K,n) \\
&\cong \mathbb{P}_{\alpha,\beta}||K||,
\end{aligned}$$

the first homeomorphism by definition of $\mathbb{M}_!$, the second homeomorphism by Proposition 6.2 and since colimits commute with coproducts, the third homeomorphism by [11, Proposition 5.8], and the last homeomorphism by definition of the flow $||K||$. ☐

6.4. **Notation.** *[16, Notation A.1] Let $n \geqslant 3$. Let $\mathcal{B}_n$ be the set of precubical sets A such that $A \subset \partial\square[n]$ and such that $|A|_{geom} \subset [0,1]^n$ contains a d-path of $\overrightarrow{N}^n_{0_n,1_n}(\square[n])$ which does not intersect $\{0,1\}^n\backslash\{0_n,1_n\}$. In particular, it means that $0_n, 1_n$ are two vertices of A. One has $\partial\square[n] \in \mathcal{B}_n$.*

6.5. **Definition.** *A precubical set is* spatial *if it is orthogonal to the set of maps of precubical sets*

$$\left\{\square[n] \sqcup_A \square[n] \longrightarrow \square[n] \mid n \geqslant 3 \text{ and } A \in \mathcal{B}_n\right\}.$$

Every proper precubical set in the sense of [28, page 499] is spatial by [16, Proposition 7.5]. In particular, for all $n \geqslant 0$, the precubical sets $\partial\square[n]$ and $\square[n]$ are spatial, as well as all geometric precubical sets in the sense of [17, Definition 1.18] and all non-positively curved precubical sets in the sense of [17, Definition 1.28], since they are proper. Also every 2-dimensional precubical set is spatial by [16, Corollary A.3].

6.6. **Theorem.** *For all precubical sets K, there is a natural weak equivalence of the h-model structure of Moore flows*

$$[K]_{reg} \longrightarrow \mathbb{M}^{\mathcal{G}}(|K|_{reg}).$$

Moreover, the weak equivalence above is an isomorphism of Moore flows if and only if K is spatial.

Proof. From the cocone

$$\left(\mathbb{M}^{\mathcal{G}}|\square[n]|_{reg}\right)_{\square[n]\to K} \stackrel{\bullet}{\longrightarrow} \mathbb{M}^{\mathcal{G}}|K|_{reg}$$

we deduce the natural map of Moore flows $g : [K]_{reg} \longrightarrow \mathbb{M}^{\mathcal{G}}|K|_{reg}$. It is bijective on states. Let $(\alpha,\beta) \in K_0 \times K_0$. Consider the following commutative diagram of topological spaces

$$\begin{array}{ccc}
\coprod_{n\geqslant 1} \mathcal{G}(1,n)\times \varinjlim \mathcal{D}_{\alpha,\beta}(K,n) & \xrightarrow{h_3} & \coprod_{n\geqslant 1} \mathcal{G}(1,n)\times \overrightarrow{N}^n_{\alpha,\beta}(K) \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle \Phi^1} \\
\mathbb{P}^1_{\alpha,\beta}||K||^{\mathcal{G}} \xrightarrow{h_2} \mathbb{P}^1_{\alpha,\beta}[K]_{reg} & \xrightarrow{\mathbb{P}^1_{\alpha,\beta}g} & \mathbb{P}^1_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}|K|_{reg} = \overrightarrow{R}^1_{\alpha,\beta}(K)
\end{array}$$

where 1) the map Φ^1 is the homeomorphism of Theorem 3.12, 2) the map h_1 is the homeomorphism of Proposition 6.2, 3) the map h_2 is the homeomorphism given by Theorem 5.8, and 4) the map h_3 is induced by the continuous map $\mathcal{D}_{\alpha,\beta}(K,n) \to \overrightarrow{N}^n_{\alpha,\beta}(K)$ given by the Moore composition of tame natural d-paths defined as follows. Let $\underline{n} = (n_1,\dots,n_p)$, $n = \sum_i n_i$ and consider a map $\square[\underline{n}] \to K \in \mathrm{Ch}_{\alpha,\beta}(K,n)$. It gives rise to a sequence of cubes $(c_1,\dots,c_p)$ of K. The continuous map $\mathcal{D}_{\alpha,\beta}(K,n) \to \overrightarrow{N}^n_{\alpha,\beta}(K)$ takes

$$(\gamma_1,\dots,\gamma_p) \in \mathcal{D}_{\alpha,\beta}(K,n)(\square[\underline{n}] \to K) = N_{n_1} \times \dots \times N_{n_p}$$

to the Moore composition of tame natural d-paths

$$(|c_1|_{geom}\gamma_1) * \dots * (|c_p|_{geom}\gamma_p) \in \overrightarrow{N}^n_{\alpha,\beta}(K)$$

which is denoted by $[c_1;\gamma_1] * \dots [c_p;\gamma_p]$ in [16]. By [16, Theorem 7.7], the map h_3 is a homeomorphism when K is spatial. For a general precubical set K, the map h_3 is a homotopy equivalence by [16, Theorem 7.8]. Hence the first part of the proof is complete.

Conversely, suppose that the natural map $[K]_{reg} \to \mathbb{M}^{\mathcal{G}}(|K|_{reg})$ is an isomorphism of Moore flows. Then there is the sequence of isomorphisms of flows

$$||K|| \cong \mathbb{M}_!||K||^{\mathcal{G}} \cong \mathbb{M}_![K]_{reg} \xrightarrow{\cong} \mathbb{M}_!\mathbb{M}^{\mathcal{G}}(|K|_{reg}) \cong \mathrm{cat}(|K|_{reg}) \cong |K|_{tc},$$

the first isomorphism by Proposition 6.3, the second isomorphism by Theorem 5.8, the third isomorphism by hypothesis, the fourth isomorphism by Theorem 6.1 and the last isomorphism by Proposition 3.17. This isomorphism of flows from $||K||$ to $|K|_{tc}$ is the identity of K_0 on states, and for all $(\alpha,\beta) \in K_0 \times K_0$, it takes an element of $\mathbb{P}_{\alpha,\beta}||K|| \cong \coprod_{n\geqslant 1} \varinjlim \mathcal{D}_{\alpha,\beta}(K,n)$ to an element of $\mathbb{P}_{\alpha,\beta}|K|_{tc} = \coprod_{n\geqslant 1} \overrightarrow{N}^n_{\alpha,\beta}(K)$ as described above. By [16, Theorem A.2], the precubical set K is spatial. $\square$

A by-product of the calculation made in the proof of Theorem 6.6 is the following fact. The image by the functor $\mathbb{M}_! : \mathcal{G}\mathbf{Flow} \to \mathbf{Flow}$ of the map $[K]_{reg} \longrightarrow \mathbb{M}^{\mathcal{G}}(|K|_{reg})$ is the map of flows $||K|| \to |K|_{tc}$ which is induced by the maps $N_{n_1} \times \dots \times N_{n_p} \to \overrightarrow{N}^n_{\alpha,\beta}(K)$ for $\square[\underline{n}] \to K \in \mathrm{Ch}_{\alpha,\beta}(K,n)$ with $\underline{n} = (n_1,\dots,n_p)$ and $n = \sum_i n_i$ as described in the core of the proof. It is a weak equivalence of the h-model structure of flows, and an isomorphism if and only if K is spatial by [16, Theorem 7.6 and Theorem 7.7].

6.7. **Proposition.** *For all $n \geqslant 1$, there is the pushout diagram of Moore flows*

$$\begin{array}{ccc}
\mathrm{Glob}(\mathbb{F}^{\mathcal{G}^{op}}_n \partial N_n) & \longrightarrow & [\partial\square[n]]_{reg} \\
\downarrow & & \downarrow \\
\mathrm{Glob}(\mathbb{F}^{\mathcal{G}^{op}}_n N_n) & \longrightarrow & [\square[n]]_{reg}
\end{array}$$

Proof. By Corollary 5.7 and $\square[n]$ and $\partial\square[n]$ being spatial for all $n \geqslant 0$, we obtain using Theorem 6.6 the commutative diagram of $\mathcal{G}$-spaces

$$\begin{array}{ccc} \mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n,1_n}[\partial\square[n]]_{reg} \\ \downarrow & & \downarrow \\ \mathbb{F}_n^{\mathcal{G}^{op}} N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n,1_n}[\square[n]]_{reg} \end{array}$$

where the two horizontal maps are isomorphisms of $\mathcal{G}$-spaces. This commutative diagram of $\mathcal{G}$-spaces is therefore a pushout diagram and the proof is complete. $\square$

6.8. **Theorem.** *For all precubical sets K, the Moore flow $[K]_{reg}$ is m-cofibrant.*

Proof. A map of Moore flows $f : X \to Y$ satisfies the RLP with respect to the map of Moore flows $\mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \to \mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$ if and only if for each $(\alpha, \beta) \in K_0 \times K_0$, the map of $\mathcal{G}$-spaces $f : \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to the map of $\mathcal{G}$-spaces $\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n \to \mathbb{F}_n^{\mathcal{G}^{op}} N_n$. Therefore, by Proposition 4.2, a map of Moore flows $f : X \to Y$ satisfies the RLP with respect to the map of Moore flows $\mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \to \mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$ if and only the map of topological spaces $\mathbb{P}^n_{\alpha,\beta}X \to \mathbb{P}^n_{f(\alpha),f(\beta)}Y$ satisfies the RLP with respect to the map of topological spaces $\partial N_n \to N_n$. By [16, Theorem 5.9], the latter map is an m-cofibration of spaces. By Theorem 4.10, the trivial fibrations of the m-model structures of Moore flows are objectwise. We then deduce that the map of Moore flows $\mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} \partial N_n) \to \mathrm{Glob}(\mathbb{F}_n^{\mathcal{G}^{op}} N_n)$ is an m-cofibration of Moore flows for all $n \geqslant 1$. Using Proposition 6.7, and since $[\square[0]]_{reg} = \{0\}$ (the Moore flow without execution paths and one state 0) is m-cofibrant, we deduce that for all precubical sets K, the Moore flow $[K]_{reg}$ is m-cofibrant. $\square$

6.9. **Corollary.** *Let K be a precubical set. Let $(\alpha, \beta) \in K_0 \times K_0$. The space of tame regular d-paths from α to β in the geometric realization of K is homotopy equivalent to a CW-complex.*

Proof. The Moore flow $\mathbb{M}^{\mathcal{G}}[K]_{reg}$ is m-cofibrant by Theorem 6.8. Thus it is weakly equivalent in the h-model structure of Moore flows to a q-cofibrant Moore flow X by [4, Corollary 3.7]. The latter has projective q-cofibrant $\mathcal{G}$-spaces of execution paths by [12, Theorem 9.11]. It implies that $\mathbb{P}_{\alpha,\beta}X$ is injective m-cofibrant by [11, Corollary 7.2]. Thus $\mathbb{P}^1_{\alpha,\beta}X$ is an m-cofibrant topological space. Since weak equivalences of the projective h-model structure of $\mathcal{G}$-spaces are objectwise homotopy equivalences, it implies that $\mathbb{P}^1_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}[K]_{reg}$ is homotopy equivalent to $\mathbb{P}^1_{\alpha,\beta}X$. By Theorem 6.6, $\mathbb{P}^1_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}[K]_{reg}$ is homotopy equivalent to $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$. When $\alpha = \beta$, the space of tame regular d-paths from α to β is homeomorphic to the disjoint union of $\{\alpha\}$ and the space of nonconstant tame regular d-paths from α to β. For $\alpha \neq \beta$, the latter remark is pointless. Hence the proof is complete. $\square$

Appendix A. M-cofibrancy of the regular realization of a spatial precubical set

Despite all our efforts, we are unable to prove or to disprove the fact that the multipointed d-space $|K|_{reg}$ is h-cofibrant. The example of non h-cofibrant multipointed

d-space provided in [15, Proposition 6.19] suggests that the h-cofibrant objects are the objects without algebraic relations. According to this intuition, it means that $|K|_{reg}$ is likely to be h-cofibrant. If it is the case, this section then shows that $|K|_{reg}$ is an m-cofibrant multipointed d-space when K is a spatial precubical set. First of all, let us recall the three model structures of multipointed d-spaces.

A.1. **Theorem.** *[15, Theorem 6.14] Let $r \in \{q, m, h\}$. There exists a unique model structure on $\mathcal{G}\mathbf{dTop}$ such that:*

- *A map of multipointed d-spaces $f : X \to Y$ is a weak equivalence if and only if $f^0 : X^0 \to Y^0$ is a bijection and for all $(\alpha, \beta) \in X^0 \times X^0$, the continuous map $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \to \mathbb{P}^{\mathcal{G}}_{f(\alpha),f(\beta)}Y$ is a weak equivalence of the r-model structure of* **Top**.
- *A map of flows $f : X \to Y$ is a fibration if and only if for all $(\alpha, \beta) \in X^0 \times X^0$, the continuous map $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}X \to \mathbb{P}^{\mathcal{G}}_{f(\alpha),f(\beta)}Y$ is a fibration of the r-model structure of* **Top**.

This model structure is accessible and all objects are fibrant.

A.2. **Notation.** *These three model categories are denoted by $\mathcal{G}\mathbf{dTop}_q, \mathcal{G}\mathbf{dTop}_m, \mathcal{G}\mathbf{dTop}_h$.*

A.3. **Theorem.** *The right adjoint $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ yields a right Quillen equivalence between the m-model structures $\mathcal{G}\mathbf{dTop}_m$ and $\mathcal{G}\mathbf{Flow}_m$.*

Proof. The functor $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ takes (trivial resp.) m-fibrations of multipointed d-spaces to (trivial resp.) m-fibrations of Moore flows by definition of the m-model structures. Thus, the functor $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$ is a right Quillen adjoint between the m-model structures. We have the commutative diagram of right Quillen adjoints

$$\begin{array}{ccc} \mathcal{G}\mathbf{dTop}_m & \xrightarrow{\mathbb{M}^{\mathcal{G}}} & \mathcal{G}\mathbf{Flow}_m \\ \downarrow{\scriptstyle \mathrm{Id}_{\mathcal{G}\mathbf{dTop}}} & & \downarrow{\scriptstyle \mathrm{Id}_{\mathcal{G}\mathbf{Flow}}} \\ \mathcal{G}\mathbf{dTop}_q & \xrightarrow[\mathbb{M}^{\mathcal{G}}]{} & \mathcal{G}\mathbf{Flow}_q \end{array}$$

The bottom horizontal arrow is a right Quillen equivalence by [13, Theorem 8.1]. It is a general fact about mixed model structures that the two vertical arrows are right Quillen equivalences. Thus, the top right Quillen adjoint is a right Quillen equivalence. ☐

A.4. **Notation.** *Denote by $\mathbb{M}^{\mathcal{G}}_{!} : \mathcal{G}\mathbf{Flow} \to \mathcal{G}\mathbf{dTop}$ the left adjoint of $\mathbb{M}^{\mathcal{G}} : \mathcal{G}\mathbf{dTop} \to \mathcal{G}\mathbf{Flow}$. An explicit description is expounded in [13, Appendix B].*

A.5. **Theorem.** *Let K be a spatial precubical set. Then the multipointed d-space $|K|_{reg}$ is m-cofibrant if it is h-cofibrant.*

Proof. Using Theorem 6.6, Theorem 6.8 and Theorem A.3, we deduce that the counit map

$$\mathbb{M}^{\mathcal{G}}_{!}\mathbb{M}^{\mathcal{G}}|K|_{reg} \longrightarrow |K|_{reg}$$

is a weak equivalence of the m-model structure of multipointed d-spaces and, $\mathbb{M}^{\mathcal{G}}_{!}$ being a left Quillen adjoint, the multipointed d-space $\mathbb{M}^{\mathcal{G}}_{!}\mathbb{M}^{\mathcal{G}}|K|_{reg}$ is m-cofibrant. For all $(\alpha, \beta) \in K_0 \times K_0$, we deduce the weak homotopy equivalence

$$\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}_{!}\mathbb{M}^{\mathcal{G}}|K|_{reg} \longrightarrow \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}.$$

By [15, Theorem 8.6], the topological space $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}_!\mathbb{M}^{\mathcal{G}}|K|_{reg}$ is m-cofibrant. By Corollary 6.9, the space $\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}$ is an m-cofibrant space. Thus the weak homotopy equivalence

$$\mathbb{P}^{\mathcal{G}}_{\alpha,\beta}\mathbb{M}^{\mathcal{G}}_!\mathbb{M}^{\mathcal{G}}|K|_{reg} \longrightarrow \mathbb{P}^{\mathcal{G}}_{\alpha,\beta}|K|_{reg}.$$

is a weak homotopy equivalence between m-cofibrant spaces. By [4, Corollary 3.4], the latter map is therefore a homotopy equivalence. It means that the counit map $\mathbb{M}^{\mathcal{G}}_!\mathbb{M}^{\mathcal{G}}|K|_{reg} \to |K|_{reg}$ is a weak equivalence of the h-model structure of $\mathcal{G}$**dTop**. It implies by [4, Corollary 3.7] that the multipointed d-space $|K|_{reg}$ is m-cofibrant if it is h-cofibrant. □

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