Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 87404, 20 pages
doi:10.1155/2007/87404

Research Article

T-Homotopy and Refinement of Observation—Part II: Adding New T-Homotopy Equivalences

Philippe Gaucher

Received 11 October 2006; Revised 24 January 2007; Accepted 31 March 2007

Recommended by Monica Clapp

This paper is the second part of a series of papers about a new notion of T-homotopy of flows. It is proved that the old definition of T-homotopy equivalence does not allow the identification of the directed segment with the 3-dimensional cube. This contradicts a paradigm of dihomotopy theory. A new definition of T-homotopy equivalence is proposed, following the intuition of refinement of observation. And it is proved that up to weak S-homotopy, an old T-homotopy equivalence is a new T-homotopy equivalence. The left properness of the weak S-homotopy model category of flows is also established in this part. The latter fact is used several times in the next papers of this series.

1. Outline of the paper

The first part [1] of this series was an expository paper about the geometric intuition underlying the notion of T-homotopy. The purpose of this second paper is to prove that the class of old T-homotopy equivalences introduced in [2, 3] is actually not big enough. Indeed, the only kind of old T-homotopy equivalence consists of the deformations which locally act like in Figure 1.1. So it becomes impossible with this old definition to identify the directed segment of Figure 1.1 with the full 3-cube of Figure 1.2 by a zig-zag sequence of weak S-homotopy and of T-homotopy equivalences preserving the initial state and the final state of the 3-cube since every point of the 3-cube is related to three distinct edges. This contradicts the fact that concurrent execution paths cannot be distinguished by observation. The end of the paper proposes a new definition of T-homotopy equivalence following the paradigm of invariance by refinement of observation. It will be checked that the preceding drawback is then overcome.

$$\widehat{0} \xrightarrow{\quad U \quad} \widehat{1}$$

$$\widehat{0} \xrightarrow{U'} A \xrightarrow{U''} \widehat{1}$$

Figure 1.1. The simplest example of T-homotopy equivalence.

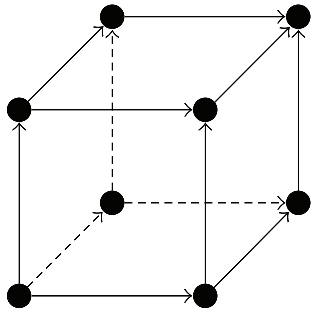

Figure 1.2. The full 3-cube.

This second part gives only a motivation for the new definition of T-homotopy. Further developments and applications are given in [4–6]. The left properness of the model category structure of [7] is also established in this paper. The latter result is used several times in the next papers of this series (e.g., [4, Theorem 11.2], [5, Theorem 9.2]).

Section 4 collects some facts about globular complexes and their relationship with the category of flows. Indeed, it is not known how to establish the limitations of the old form of T-homotopy equivalence without using globular complexes together with a compactness argument. Section 5 recalls the notion of old T-homotopy equivalence of flows which is a kind of morphism between flows coming from globular complexes (the class of flows **cell**(**Flow**)). Section 6 presents elementary facts about relative I_+^{gl}-cell complexes which will be used later in the paper. Section 7 proves that the model category of flows is left proper. This technical fact is used in the proof of the main theorem of the paper, and it was not established in [7]. Section 8 proves the first main theorem of the paper.

Theorem 1.1 (Theorem 8.5). *Let $n \geqslant 3$. There does not exist any zig-zag sequence of S-homotopy equivalences and of old T-homotopy equivalences between the flow associated with the n-cube and the flow associated with the directed segment.*

Finally Section 9 proposes a new definition of T-homotopy equivalence and the second main theorem of the paper is proved.

Theorem 1.2 (Theorem 9.3). *Every T-homotopy in the old sense is the composite of an S-homotopy equivalence with a T-homotopy equivalence in the new sense. (Since a T-homotopy in the old sense is a T-homotopy in the new sense only up to S-homotopy, the terminology "generalized T-homotopy" used in* Section 9 *may not be the best one. However, this terminology is used in the other papers of this series, so it is kept to avoid any confusion.)*

2. Prerequisites and notations

The initial object (resp., the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp., $\mathbf{1}$).

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to any morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to the morphisms of $\mathbf{inj}(K)$. It is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, every morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$ as soon as the domains of K are small relative to $\mathbf{cell}(K)$ (see [8, Corollary 2.1.15]). An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex*. If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \to X$ is a relative K-cell complex, then the object X is called a *K-cell complex*.

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the object X of $\mathcal{C}$ such that the canonical morphism $\varnothing \to X$ is an object of $\mathbf{cell}(I)$. In other terms, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing \downarrow \mathcal{C}) \cap \mathbf{cell}(I)$.

It is obviously impossible to read this paper without a strong familiarity with *model categories*. Possible references for model categories are [8–10]. The original reference is [11] but Quillen's axiomatization is not used in this paper. The axiomatization from Hovey's book is preferred. If $\mathcal{M}$ is a *cofibrantly generated* model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$: this is the full subcategory of *cell complexes* of the model category $\mathcal{M}$. A cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \to \mathbf{cell}(\mathcal{M})$. In all usual model categories which are cellular (see [9, Definition 12.1.1]), all the cofibrations are monomorphisms. Then for every monomorphism f of such a model category $\mathcal{M}$, the morphism $Q(f)$ is a cofibration, and even is an inclusion of subcomplexes (see [9, Definition 10.6.7]) because the cofibrant replacement functor Q is obtained by the small object argument, starting from the identity of the initial object. This is still true in the model category of flows remembered in Section 3 since the class of cofibrations which are monomorphisms is closed under pushout and transfinite composition.

A *partially ordered set* $(P, \leqslant)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relation $\leqslant$. A poset is *locally finite* if for any $(x, y) \in P \times P$, the set $[x, y] = \{z \in P, x \leqslant z \leqslant y\}$ is finite. A poset $(P, \leqslant)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P = [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. For a bounded poset P, let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element). In a poset P, the interval $]\alpha, -]$ (the subposet of elements of P strictly bigger than α) can also be denoted by $P_{>\alpha}$.

A poset P, and in particular an ordinal, can be viewed as a small category denoted in the same way: the objects are the elements of P and there exists a morphism from x to y if and only if $x \leqslant y$. If λ is an ordinal, a *λ-sequence* in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor X from λ to $\mathcal{C}$. We denote by X_λ the colimit $\varinjlim X$ and the morphism $X_0 \to X_\lambda$ is called the *transfinite composition* of the morphisms $X_\mu \to X_{\mu+1}$.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. The model categories **Top** and **Flow** (see below) are both left proper (cf. Theorem 7.4 for **Flow**).

In this paper, the notation $\hookrightarrow$ means *cofibration*, the notation $\twoheadrightarrow$ means *fibration*, the notation $\simeq$ means *weak equivalence*, and the notation $\cong$ means *isomorphism*.

3. Reminder about the category of flows

The category **Top** of *compactly generated topological spaces* (i.e., of weak Hausdorff k-spaces) is complete, cocomplete, and cartesian closed (more details for this kind of topological spaces are in [12, 13], the appendix of [14] and also the preliminaries of [7]). For the sequel, any topological space will be supposed to be compactly generated. A *compact space* is always Hausdorff.

The category **Top** is equipped with the unique model structure having the *weak homotopy equivalences* as weak equivalences and having the *Serre fibrations* (i.e., a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0,1]$ for any $n \geqslant 0$ where $\mathbf{D}^n$ is the n-dimensional disk) as fibrations.

The time flow of a higher-dimensional automaton is encoded in an object called a *flow* [7]. A flow X contains a set X^0 called the 0-*skeleton* whose elements correspond to the *states* (or *constant execution paths*) of the higher-dimensional automaton. For each pair of states $(\alpha,\beta) \in X^0 \times X^0$, there is a topological space $\mathbb{P}_{\alpha,\beta}X$ whose elements correspond to the *(nonconstant) execution paths* of the higher-dimensional automaton *beginning at* α and *ending at* β. For $x \in \mathbb{P}_{\alpha,\beta}X$, let $\alpha = s(x)$ and $\beta = t(x)$. For each triple $(\alpha,\beta,\gamma) \in X^0 \times X^0 \times X^0$, there exists a continuous map $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$ called the *composition law* which is supposed to be associative in an obvious sense. The topological space $\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X$ is called the *path space* of X. The category of flows is denoted by **Flow**. A point α of X^0 such that there are no nonconstant execution paths ending at α (resp., starting from α) is called an *initial state* (resp., a *final state*). A morphism of flows f from X to Y consists of a set map $f^0 : X^0 \to Y^0$ and a continuous map $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ preserving the structure. A flow is therefore "almost" a small category enriched in **Top**. A flow X is *loopless* if for every $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.

Here are four fundamental examples of flows.

(1) Let S be a set. The flow associated with S, still denoted by S, has S as set of states and the empty space as path space. This construction induces a functor **Set** $\to$ **Flow** from the category of sets to that of flows. The flow associated with a set is loopless.

(2) Let $(P,\leqslant)$ be a poset. The flow associated with $(P,\leqslant)$, and still denoted by P, is defined as follows: the set of states of P is the underlying set of P; the space of morphisms from α to β is empty if $\alpha \geqslant \beta$ and equals $\{(\alpha,\beta)\}$ if $\alpha < \beta$ and the composition law is defined by $(\alpha,\beta)*(\beta,\gamma) = (\alpha,\gamma)$. This construction induces a functor **PoSet** $\to$ **Flow** from the category of posets together with the strictly increasing maps to the category of flows. The flow associated with a poset is loopless.

(3) The flow $\mathrm{Glob}(Z)$ defined by

$$\begin{gathered}\mathrm{Glob}(Z)^0 = \{\widehat{0},\widehat{1}\},\\ \mathbb{P}\mathrm{Glob}(Z) = Z \quad \text{with } s(z) = \widehat{0},\ t(z) = \widehat{1},\ \forall z \in Z,\end{gathered} \tag{3.1}$$

and a trivial composition law (cf. Figure 3.1), is called the *globe* of Z.

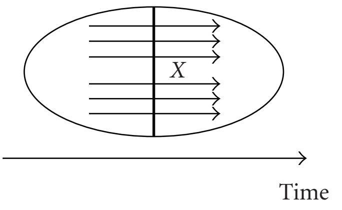

Figure 3.1. Symbolic representation of Glob(Z) for some topological space Z.

(4) The *directed segment* $\vec{I}$ is by definition $\mathrm{Glob}(\{0\}) \cong \{\widehat{0} < \widehat{1}\}$.

The category **Flow** is equipped with the unique model structure such that [7]

(a) the weak equivalences are the *weak S-homotopy equivalences*, that is, the morphisms of flows $f : X \to Y$ such that $f^0 : X^0 \to Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is a weak homotopy equivalence;

(b) the fibrations are the morphisms of flows $f : X \to Y$ such that $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set $I_+^{\mathrm{gl}} = I^{\mathrm{gl}} \cup \{R : \{0,1\} \to \{0\}, C : \varnothing \to \{0\}\}$ with

$$I^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}, \tag{3.2}$$

where $\mathbf{D}^n$ is the n-dimensional disk and $\mathbf{S}^{n-1}$ is the $(n-1)$-dimensional sphere. The set of generating trivial cofibrations is

$$J^{\mathrm{gl}} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geqslant 0\}. \tag{3.3}$$

If X is an object of **cell**(**Flow**), then a presentation of the morphism $\varnothing \to X$ as a transfinite composition of pushouts of morphisms of I_+^{gl} is called a *globular decomposition* of X.

4. Globular complex

The reference is [3]. A *globular complex* is a topological space together with a structure describing the sequential process of attaching *globular cells*. A general globular complex may require an arbitrary long transfinite construction. We restrict our attention in this paper to globular complexes whose globular cells are morphisms of the form $\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1}) \to \mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n)$ (cf. Definition 4.2).

Definition 4.1. A *multipointed topological space* (X, X^0) is a pair of topological spaces such that X^0 is a discrete subspace of X. A morphism of multipointed topological spaces $f : (X, X^0) \to (Y, Y^0)$ is a continuous map $f : X \to Y$ such that $f(X^0) \subset Y^0$. The corresponding category is denoted by $\mathbf{Top}^m$. The set X^0 is called the 0-skeleton of (X,X^0). The space X is called the *underlying topological space* of (X,X^0).

The category of multipointed spaces is cocomplete.

Definition 4.2. Let Z be a topological space. The *globe of* Z, which is denoted by $\mathrm{Glob}^{\mathrm{top}}(Z)$, is the multipointed space

$$(|\mathrm{Glob}^{\mathrm{top}}(Z)|,\{\widehat{0},\widehat{1}\}), \tag{4.1}$$

where the topological space $|\mathrm{Glob}^{\mathrm{top}}(Z)|$ is the quotient of $\{\widehat{0},\widehat{1}\}\sqcup(Z\times[0,1])$ by the relations $(z,0)=(z',0)=\widehat{0}$ and $(z,1)=(z',1)=\widehat{1}$ for any $z,z'\in Z$. In particular, $\mathrm{Glob}^{\mathrm{top}}(\varnothing)$ is the multipointed space $(\{\widehat{0},\widehat{1}\},\{\widehat{0},\widehat{1}\})$.

Notation 4.3. If Z is a singleton, then the globe of Z is denoted by $\vec{I}^{\mathrm{top}}$.

Definition 4.4. Let $I^{\mathrm{gl,top}}:=\{\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n-1})\to\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^n),\ n\geqslant 0\}$. A *relative globular precomplex* is a relative $I^{\mathrm{gl,top}}$-cell complex in the category of multipointed topological spaces.

Definition 4.5. A *globular precomplex* is a λ-sequence of multipointed topological spaces $X:\lambda\to\mathbf{Top}^m$ such that X is a relative globular precomplex and such that $X_0=(X^0,X^0)$ with X^0 a discrete space. This λ-sequence is characterized by a presentation ordinal λ, and for any $\beta<\lambda$, an integer $n_\beta\geqslant 0$ and an attaching map $\phi_\beta:\mathrm{Glob}^{\mathrm{top}}(\mathbf{S}^{n_\beta-1})\to X_\beta$. The family $(n_\beta,\phi_\beta)_{\beta<\lambda}$ is called the *globular decomposition* of X.

Let X be a globular precomplex. The 0-skeleton of $\varinjlim X$ is equal to X^0.

Definition 4.6. A morphism of globular precomplexes $f:X\to Y$ is a morphism of multipointed spaces still denoted by f from $\varinjlim X$ to $\varinjlim Y$.

Notation 4.7. If X is a globular precomplex, then the underlying topological space of the multipointed space $\varinjlim X$ is denoted by $|X|$ and the 0-skeleton of the multipointed space $\varinjlim X$ is denoted by X^0.

Definition 4.8. Let X be a globular precomplex. The space $|X|$ is called the *underlying topological space* of X. The set X^0 is called the 0-*skeleton* of X.

Definition 4.9. Let X be a globular precomplex. A morphism of globular precomplexes $\gamma:\vec{I}^{\mathrm{top}}\to X$ is a *nonconstant execution path* of X if there exists $t_0=0<t_1<\cdots<t_n=1$ such that

(1) $\gamma(t_i)\in X^0$ for any $0\leqslant i\leqslant n$,
(2) $\gamma(]t_i,t_{i+1}[)\subset\mathrm{Glob}^{\mathrm{top}}(\mathbf{D}^{n_{\beta_i}}\backslash\mathbf{S}^{n_{\beta_i}-1})$ for some $(n_{\beta_i},\phi_{\beta_i})$ of the globular decomposition of X,
(3) for $0\leqslant i<n$, there exists $z_\gamma^i\in\mathbf{D}^{n_{\beta_i}}\backslash\mathbf{S}^{n_{\beta_i}-1}$ and a strictly increasing continuous map $\psi_\gamma^i:[t_i,t_{i+1}]\to[0,1]$ such that $\psi_\gamma^i(t_i)=0$ and $\psi_\gamma^i(t_{i+1})=1$, and for any $t\in[t_i,t_{i+1}]$, $\gamma(t)=(z_\gamma^i,\psi_\gamma^i(t))$.

In particular, the restriction $\gamma\restriction_{]t_i,t_{i+1}[}$ of γ to $]t_i,t_{i+1}[$ is one-to-one. The set of nonconstant execution paths of X is denoted by $\mathbb{P}^{\mathrm{top}}(X)$.

Definition 4.10. A morphism of globular precomplexes $f : X \to Y$ is *nondecreasing* if the canonical set map $\mathbf{Top}([0,1],|X|) \to \mathbf{Top}([0,1],|Y|)$ induced by composition by f yields a set map $\mathbb{P}^{\mathrm{top}}(X) \to \mathbb{P}^{\mathrm{top}}(Y)$. In other terms, one has the commutative diagram of sets:

$$\begin{tikzcd}
\mathbb{P}^{\mathrm{top}}(X) \arrow[r] \arrow[d, "\subset"'] & \mathbb{P}^{\mathrm{top}}(Y) \arrow[d, "\subset"'] \\
\mathbf{Top}([0,1],|X|) \arrow[r] & \mathbf{Top}([0,1],|Y|)
\end{tikzcd} \tag{4.2}$$

Definition 4.11. A *globular complex* (resp., a *relative globular complex*) X is a globular precomplex (resp., a relative globular precomplex) such that the attaching maps ϕ_β are nondecreasing. A morphism of globular complexes is a morphism of globular precomplexes which is nondecreasing. The category of globular complexes, together with the morphisms of globular complexes as defined above, is denoted by $\mathbf{glTop}$. The set $\mathbf{glTop}(X,Y)$ of morphisms of globular complexes from X to Y equipped with the Kelleyfication of the compact-open topology is denoted by $\mathbf{glTOP}(X,Y)$.

Definition 4.12. Let X be a globular complex. A point α of X^0 such that there are no nonconstant execution paths ending to α (resp., starting from α) is called *initial state* (resp., *final state*). More generally, a point of X^0 will be sometime called *a state* as well.

Theorem 4.13 (see [3, Theorem III.3.1]). *There exists a unique functor* $\mathrm{cat} : \mathbf{glTop} \to \mathbf{Flow}$ *such that*

(1) *if $X = X^0$ is a discrete globular complex, then* $\mathrm{cat}(X)$ *is the achronal flow* X^0 *("achronal" meaning with an empty path space),*
(2) *if $Z = \mathbf{S}^{n-1}$ or $Z = \mathbf{D}^n$ for some integer $n \geqslant 0$, then* $\mathrm{cat}(\mathrm{Glob}^{\mathrm{top}}(Z)) = \mathrm{Glob}(Z)$,
(3) *for any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any limit ordinal $\beta \leqslant \lambda$, the canonical morphism of flows*

$$\varinjlim_{\alpha<\beta} \mathrm{cat}(X_\alpha) \longrightarrow \mathrm{cat}(X_\beta) \tag{4.3}$$

is an isomorphism of flows,
(4) *for any globular complex X with globular decomposition $(n_\beta, \phi_\beta)_{\beta<\lambda}$, for any $\beta < \lambda$, one has the pushout of flows*

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n_\beta-1}) \arrow[r, "\mathrm{cat}(\phi_\beta)"] \arrow[d] & \mathrm{cat}(X_\beta) \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n_\beta}) \arrow[r] & \mathrm{cat}(X_{\beta+1}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{4.4}$$

The following theorem is important for the sequel.

Theorem 4.14. *The functor* cat *induces a functor still denoted by* cat *from* $\mathbf{glTop}$ *to* $\mathbf{cell}(\mathbf{Flow}) \subset \mathbf{Flow}$ *since its image is contained in* $\mathbf{cell}(\mathbf{Flow})$. *For any flow X of* $\mathbf{cell}(\mathbf{Flow})$, *there exists a globular complex Y such that* $\mathrm{cat}(U) = X$*, which is constructed by using the globular decomposition of* X.

Proof. The construction of U is made in the proof of [3, Theorem V.4.1]. □

5. T-homotopy equivalence

The old notion of T-homotopy equivalence for globular complexes was given in [2]. A notion of T-homotopy equivalence of flows was given in [3] and it was proved in the same paper that these two notions are equivalent.

We first recall the definition of the branching and merging space functors, and then the definition of a T-homotopy equivalence of flows, exactly as given in [3] (see Definition 5.7), and finally a characterization of T-homotopy of flows using globular complexes (see Theorem 5.8).

Roughly speaking, the branching space of a flow is the space of germs of nonconstant execution paths beginning in the same way.

Proposition 5.1 (see [15, Proposition 3.1]). *Let X be a flow. There exists a topological space $\mathbb{P}^-X$ unique up to homeomorphism and a continuous map $h^- : \mathbb{P}X \to \mathbb{P}^-X$ satisfying the following universal property.*

(1) *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^-(x) = h^-(x*y)$ holds.*

(2) *Let $\phi : \mathbb{P}X \to Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(x) = \phi(x*y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^-X \to Y$ such that $\phi = \overline{\phi} \circ h^-$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^-X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^-_\alpha X, \tag{5.1}$$

where $\mathbb{P}^-_\alpha X := h^-(\bigsqcup_{\beta \in X^0} \mathbb{P}^-_{\alpha,\beta} X)$. The mapping $X \mapsto \mathbb{P}^-X$ yields a functor $\mathbb{P}^-$ from **Flow** *to* **Top**.

Definition 5.2. Let X be a flow. The topological space $\mathbb{P}^-X$ is called the *branching space* of the flow X. The functor $\mathbb{P}^-$ is called the *branching space functor*.

Proposition 5.3 (see [15, Proposition A.1]). *Let X be a flow. There exists a topological space $\mathbb{P}^+X$ unique up to homeomorphism and a continuous map $h^+ : \mathbb{P}X \to \mathbb{P}^+X$ satisfying the following universal property.*

(1) *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^+(y) = h^+(x*y)$ holds.*

(2) *Let $\phi : \mathbb{P}X \to Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(y) = \phi(x*y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^+X \to Y$ such that $\phi = \overline{\phi} \circ h^+$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^+X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^+_\alpha X, \tag{5.2}$$

where $\mathbb{P}^+_\alpha X := h^+(\bigsqcup_{\beta \in X^0} \mathbb{P}^+_{\alpha,\beta} X)$. The mapping $X \mapsto \mathbb{P}^+X$ yields a functor $\mathbb{P}^+$ from **Flow** *to* **Top**.

Roughly speaking, the merging space of a flow is the space of germs of nonconstant execution paths ending in the same way.

Definition 5.4. Let X be a flow. The topological space $\mathbb{P}^+X$ is called the *merging space* of the flow X. The functor $\mathbb{P}^+$ is called the *merging space functor*.

Definition 5.5 [3]. Let X be a flow. Let A and B be two subsets of X^0. One says that A is *surrounded* by B (in X) if for any $\alpha \in A$, either $\alpha \in B$ or there exist execution paths γ_1 and γ_2 of $\mathbb{P}X$ such that $s(\gamma_1) \in B$, $t(\gamma_1) = s(\gamma_2) = \alpha$ and $t(\gamma_2) \in B$. Denote this situation by $A \lll B$.

Definition 5.6 [3]. Let X be a flow. Let A be a subset of X^0. Then the *restriction* $X\restriction_A$ of X over A is the unique flow such that $(X\restriction_A)^0 = A$, such that $\mathbb{P}_{\alpha,\beta}(X\restriction_A) = \mathbb{P}_{\alpha,\beta}X$ for any $(\alpha,\beta) \in A \times A$, and such that the inclusions $A \subset X^0$ and $\mathbb{P}(X\restriction_A) \subset \mathbb{P}X$ induce a morphism of flows $X\restriction_A \to X$.

Definition 5.7 [3]. Let X and Y be two objects of **cell**(**Flow**). A morphism of flows $f : X \to Y$ is a *T-homotopy equivalence* if and only if the following conditions are satisfied.

(1) The morphism of flows $f : X \to Y\restriction_{f(X^0)}$ is an isomorphism of flows. In particular, the set map $f^0 : X^0 \to Y^0$ is one-to-one.

(2) For $\alpha \in Y^0 \backslash f(X^0)$, the topological spaces $\mathbb{P}^-_\alpha Y$ and $\mathbb{P}^+_\alpha Y$ are singletons.

(3) $Y^0 \lll f(X^0)$.

We recall the following important theorem for the sequel.

Theorem 5.8 (see [3, Theorem VI.3.5]). *Let X and Y be two objects of* **cell**(**Flow**). *Let U and V be two globular complexes with* $\mathrm{cat}(U) = X$ *and* $\mathrm{cat}(V) = Y$ *(U and V always exist by Theorem 4.14). Then a morphism of flows $f : X \to Y$ is a T-homotopy equivalence if and only if there exists a morphism of globular complexes $g : U \to V$ such that* $\mathrm{cat}(g) = f$ *and such that the continuous map $|g| : |U| \to |V|$ between the underlying topological spaces is a homeomorphism.*

This characterization was actually the first definition of a T-homotopy equivalence proposed in [2] (see [2, Definition 4.10, page 66]).

6. Some facts about relative I^{gl}_+-cell complexes

Recall that $I^{gl}_+ = I^{gl} \cup \{R : \{0,1\} \to \{0\},\ C : \varnothing \to \{0\}\}$ with

$$I^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n),\ n \geqslant 0\}. \tag{6.1}$$

Let $I_g = I^{gl} \cup \{C\}$. Since for any $n \geqslant 0$, the inclusion $\mathbf{S}^{n-1} \subset \mathbf{D}^n$ is a closed inclusion of topological spaces, so an effective monomorphism of the category **Top** of compactly generated topological spaces, every morphism of I_g, and therefore every morphism of **cell**(I_g), is an effective monomorphism of flows as well (cf. also [7, Theorem 10.6]).

Proposition 6.1. *If $f : X \to Y$ is a relative I^{gl}_+-cell complex and if f induces a one-to-one set map from X^0 to Y^0, then $f : X \to Y$ is a relative I_g-cell subcomplex.*

Proof. A pushout of R appearing in the presentation of f cannot identify two elements of X^0 since, by hypothesis, $f^0 : X^0 \to Y^0$ is one-to-one. So either such a pushout is trivial, or it identifies two elements added by a pushout of C. $\square$

Proposition 6.2. *If $f : X \to Y$ is a relative I_+^{gl}-cell complex, then f factors as a composite $g \circ h \circ k$, where $k : X \to Z$ is a morphism of* $\mathbf{cell}(\{R\})$, *where $h : Z \to T$ is a morphism of* $\mathbf{cell}(\{C\})$, *and where $g : T \to Y$ is a relative I^{gl}-cell complex.*

Proof. One can use the small object argument with $\{R\}$ by [7, Proposition 11.8]. Therefore, the morphism $f : X \to Y$ factors as a composite $g \circ h$, where $h : X \to Z$ is a morphism of $\mathbf{cell}(\{R\})$, and where the morphism $Z \to Y$ is a morphism of $\mathbf{inj}(\{R\})$. One deduces that the set map $Z^0 \to Y^0$ is one-to-one. One has the pushout diagram of flows

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, equal] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{6.2}$$

Therefore the morphism $Z \to Y$ is a relative I_+^{gl}-cell complex. Proposition 6.1 implies that the morphism $Z \to Y$ is a relative I_g-cell complex. The morphism $Z \to Y$ factors as a composite $h : Z \to Z \sqcup (Y^0 \backslash Z^0)$ and the inclusion $g : Z \sqcup (Y^0 \backslash Z^0) \to Y$. $\square$

Proposition 6.3. *Let $X = X^0$ be a set viewed as a flow (i.e., with an empty path space). Let Y be an object of* $\mathbf{cell}(\mathbf{Flow})$. *Then any morphism from X to Y is a cofibration.*

Proof. Let $f : X \to Y$ be a morphism of flows. Then f factors as a composite $X = X^0 \to Y^0 \to Y$. Any set map $X^0 \to Y^0$ is a transfinite composition of pushouts of C and R. So any set morphism $X^0 \to Y^0$ is a cofibration of flows. And for any flow Y, the canonical morphism of flows $Y^0 \to Y$ is a cofibration since it is a relative I_g-cell complex. Hence we get the result. $\square$

7. Left properness of the weak S-homotopy model structure of Flow

Proposition 7.1 (see [7, Proposition 15.1]). *Let $f : U \to V$ be a continuous map. Consider the pushout diagram of flows:*

$$\begin{tikzcd} \mathrm{Glob}(U) \arrow[r] \arrow[d, "\mathrm{Glob}(f)"'] & X \arrow[d, "g"] \\ \mathrm{Glob}(V) \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{7.1}$$

Then the continuous map $\mathbb{P}g : \mathbb{P}X \to \mathbb{P}Y$ is a transfinite composition of pushouts of continuous maps of the form a finite product $\mathrm{Id} \times \cdots \times f \times \cdots \times \mathrm{Id}$, *where the symbol* Id *denotes identity maps.*

Proposition 7.2. *Let $f : U \to V$ be a Serre cofibration. Then the pushout of a weak homotopy equivalence along a map of the form a finite product* $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \mathrm{Id}_{X_p}$ *with $p \geqslant 0$ is still a weak homotopy equivalence.*

If the topological spaces X_i for $1 \leqslant i \leqslant p$ are cofibrant, then the continuous map $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \cdots \times \mathrm{Id}_{X_p}$ is a cofibration since the model category of compactly generated topological spaces is monoidal with the categorical product as monoidal structure. So in this case, the result follows from the left properness of this model category (see [9, Theorem 13.1.10]). In the general case, $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \cdots \times \mathrm{Id}_{X_p}$ is not a cofibration anymore. But any cofibration f for the Quillen model structure of **Top** is, an cofibration for the Strøm model structure of **Top** [16–19]. In the latter model structure, any space is cofibrant. Therefore the continuous map $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \cdots \times \mathrm{Id}_{X_p}$ is a cofibration of the Strøm model structure of **Top**, that is a NDR pair. So the continuous map $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \cdots \times \mathrm{Id}_{X_p}$ is a closed T_1-inclusion anyway. This fact will be used below.

Proof. We already know that the pushout of a weak homotopy equivalence along a cofibration is a weak homotopy equivalence. The proof of this proposition is actually an adaptation of the proof of the left properness of the model category of compactly generated topological spaces. Any cofibration is a retract of a transfinite composition of pushouts of inclusions of the form $\mathbf{S}^{n-1} \subset \mathbf{D}^n$ for $n \geqslant 0$. Since the category of compactly generated topological spaces is cartesian closed, the binary product preserves colimits. Thus, we are reduced to considering a diagram of topological spaces like

$$\begin{tikzcd}
X_1 \times \cdots \times \mathbf{S}^{n-1} \times \cdots \times X_p \arrow[r] \arrow[d] & U \arrow[r, "s"] \arrow[d] & X \arrow[d] \\
X_1 \times \cdots \times \mathbf{D}^{n} \times \cdots \times X_p \arrow[r] & \widehat{U} \arrow[r, "\widehat{s}"] & \widehat{X}
\end{tikzcd} \tag{7.2}$$

where s is a weak homotopy equivalence and we have to prove that $\widehat{s}$ is a weak homotopy equivalence as well. By [11, 20], it suffices to prove that $\widehat{s}$ induces a bijection between the path-connected components of $\widehat{U}$ and $\widehat{X}$, a bijection between the fundamental groupoids $\pi(\widehat{U})$ and $\pi(\widehat{X})$, and that for any local coefficient system of Abelian groups A of $\widehat{X}$, one has the isomorphism $\widehat{s}^* : H^*(\widehat{X},A) \cong H^*(\widehat{U},\widehat{s}^*A)$.

For $n = 0$, one has $\mathbf{S}^{n-1} = \varnothing$ and $\mathbf{D}^n = \{0\}$. So $X_1 \times \cdots \times \mathbf{S}^{n-1} \times \cdots \times X_p = \varnothing$ and $X_1 \times \cdots \times \mathbf{D}^n \times \cdots \times X_p = X_1 \times \cdots \times X_p$. So $\widehat{U} \cong U \sqcup (X_1 \times \cdots \times X_p)$ and $\widehat{X} \cong X \sqcup (X_1 \times \cdots \times X_p)$. Therefore, the mapping t is the disjoint sum $s \sqcup \mathrm{Id}_{X_1 \times \cdots \times X_p}$. So it is a weak homotopy equivalence.

Let $n \geqslant 1$. The assertion concerning the path-connected components is clear. Let $\mathbf{T}^n = \{x \in \mathbb{R}^n,\ 0 < |x| \leqslant 1\}$. Consider the diagram of topological spaces:

$$\begin{tikzcd}
X_1 \times \cdots \times \mathbf{S}^{n-1} \times \cdots \times X_p \arrow[r] \arrow[d] & U \arrow[r, "s"] \arrow[d] & X \arrow[d] \\
X_1 \times \cdots \times \mathbf{T}^{n} \times \cdots \times X_p \arrow[r] & \widetilde{U} \arrow[r, "\widetilde{s}"] & \widetilde{X}
\end{tikzcd} \tag{7.3}$$

Since the pair $(\mathbf{T}^n,\mathbf{S}^{n-1})$ is a deformation retract, the three pairs $(X_1 \times \cdots \times \mathbf{T}^n \times \cdots \times X_p, X_1 \times \cdots \times \mathbf{S}^{n-1} \times \cdots \times X_p)$, $(\widetilde{U},U)$, and $(\widetilde{X},X)$ are deformation retracts as well. So the continuous maps $U \to \widetilde{U}$ and $X \to \widetilde{X}$ are both homotopy equivalences. The Seifert-Van-Kampen theorem for the fundamental groupoid (cf. [20] again) then yields the diagram of groupoids:

$$\begin{tikzcd}
\pi(X_1 \times \cdots \times \mathbf{T}^n \times \cdots \times X_p) \arrow[r] \arrow[d] & \pi(\widetilde{U}) \arrow[r, "\pi(\widetilde{s})"] \arrow[d] & \pi(\widetilde{X}) \arrow[d] \\
\pi(X_1 \times \cdots \times \mathbf{D}^n \times X_p) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \pi(\widehat{U}) \arrow[r, "\pi(\widehat{s})"] \arrow[ur, phantom, "\ulcorner", very near end] & \pi(\widehat{X})
\end{tikzcd} \tag{7.4}$$

Since $\pi(\widetilde{s})$ is an isomorphism of groupoids, then so is $\pi(\widehat{s})$.

Let $\mathbf{B}^n = \{x \in \mathbb{R}^n, 0 \leqslant |x| < 1\}$. Then $(\mathbf{B}^n, \widetilde{U})$ is an excisive pair of $\widehat{U}$ and $(\mathbf{B}^n, \widetilde{X})$ is an excisive pair of $\widehat{X}$. The Mayer-Vietoris long exact sequence then yields the commutative diagram of groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(\widehat{X}, A) \arrow[r] \arrow[d] & H^p(X, A) \oplus H^p(\mathbf{B}^n, A) \arrow[r] \arrow[d, "\cong"'] & H^p(\mathbf{B}^n \backslash \{0\}, A) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H^p(\widehat{U}, \widehat{s}^* A) \arrow[r] & H^p(U, s^* A) \oplus H^p(\mathbf{B}^n, s^* A) \arrow[r] & H^p(\mathbf{B}^n \backslash \{0\}, s^* A) \arrow[r] & \cdots
\end{tikzcd} \tag{7.5}$$

A five-lemma argument completes the proof. □

Proposition 7.3. *Let λ be an ordinal. Let $M : \lambda \to$ **Top** and $N : \lambda \to$ **Top** be two λ-sequences of topological spaces. Let $s : M \to N$ be a morphism of λ-sequences which is also an objectwise weak homotopy equivalence. Finally, suppose that for all $\mu < \lambda$, the continuous maps $M_\mu \to M_{\mu+1}$ and $N_\mu \to N_{\mu+1}$ are of the form of a finite product $\mathrm{Id}_{X_1} \times \cdots \times f \times \cdots \times \mathrm{Id}_{X_p}$ with $p \geqslant 0$ and with f a Serre cofibration. Then the continuous map $\varinjlim s : \varinjlim M \to \varinjlim N$ is a weak homotopy equivalence.*

If for all $\mu < \lambda$, the continuous maps $M_\mu \to M_{\mu+1}$ and $N_\mu \to N_{\mu+1}$ are cofibrations, then Proposition 7.3 is a consequence of [9, Proposition 17.9.3] and of the fact that the model category **Top** is left proper. With the same additional hypotheses, Proposition 7.3 is also a consequence of [21, Theorem A.7]. Indeed, the latter states that a homotopy colimit can be calculated either in the usual Quillen model structure of **Top**, or in the Strøm model structure of **Top** [18, 19].

Proof. The principle of the proof is standard. If the ordinal λ is not a limit ordinal, then this is a consequence of Proposition 7.2. Assume now that λ is a limit ordinal. Then $\lambda \geqslant \aleph_0$.

Let $u : \mathbf{S}^n \to \varinjlim N$ be a continuous map. Then u factors as a composite $\mathbf{S}^n \to N_\mu \to \varinjlim N$ since the n-dimensional sphere $\mathbf{S}^n$ is compact and since any compact space is $\aleph_0$-small relative to closed T_1-inclusions (see [8, Proposition 2.4.2]). By hypothesis, there exists a continuous map $\mathbf{S}^n \to M_\mu$ such that the composite $\mathbf{S}^n \to M_\mu \to N_\mu$ is homotopic to $\mathbf{S}^n \to N_\mu$. Hence we have the surjectivity of the set map $\pi_n(\varinjlim M, *) \to \pi_n(\varinjlim N, *)$ (where π_n denotes the n-th homotopy group) for $n \geqslant 0$ and for any base point $*$.

Let $u,v : \mathbf{S}^n \to \varinjlim M$ be two continuous maps such that there exists a homotopy $H : \mathbf{S}^n \times [0,1] \to \varinjlim N$ between $\varinjlim s \circ f$ and $\varinjlim s \circ g$. Since the space $\mathbf{S}^n \times [0,1]$ is compact, the homotopy H factors as a composite $\mathbf{S}^n \times [0,1] \to N_{\mu_0} \to \varinjlim N$ for some $\mu_0 < \lambda$. And again since the space $\mathbf{S}^n$ is compact, the map f (resp., g) factors as a composite $\mathbf{S}^n \to M_{\mu_1} \to \varinjlim M$ (resp., $\mathbf{S}^n \to M_{\mu_2} \to \varinjlim M$) with $\mu_1 < \lambda$ (resp., $\mu_2 < \lambda$). Then $\mu_4 = \max(\mu_0,\mu_1,\mu_2) < \lambda$ since λ is a limit ordinal. And the map $H : \mathbf{S}^n \times [0,1] \to N_{\mu_4}$ is a homotopy between $f : \mathbf{S}^n \to M_{\mu_4}$ and $g : \mathbf{S}^n \to M_{\mu_4}$. So the set map $\pi_n(\varinjlim M, *) \to \pi_n(\varinjlim N, *)$ for $n \geqslant 0$, and for any base point $*$ is one-to-one. □

Theorem 7.4. *The model category* **Flow** *is left proper.*

Proof. Consider the pushout diagram of **Flow**:

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[d, "i"'] & X \arrow[d] \\ V \arrow[r, "t"] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{7.6}$$

where i is a cofibration of **Flow** and s a weak S-homotopy equivalence. We have to check that t is a weak S-homotopy equivalence as well. The morphism i is a retract of a I_+^{gl}-cell complex $j : U \to W$. If one considers the pushout diagram of **Flow**:

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[d, "j"'] & X \arrow[d] \\ W \arrow[r, "u"] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{7.7}$$

then t must be a retract of u. Therefore, it suffices to prove that u is a weak S-homotopy equivalence. So one can suppose that one has a diagram of flows of the form

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "k"'] & U \arrow[r, "s"] \arrow[d, "i"] & X \arrow[d] \\ B \arrow[r] & V \arrow[r, "t"] \arrow[ul, phantom, "\ulcorner", very near start] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{7.8}$$

where $k \in \mathbf{cell}(I_+^{\mathrm{gl}})$. By Proposition 6.2, the morphism $k : A \to B$ factors as a composite $A \to A' \to A'' \to B$ where the morphism $A \to A'$ is an element of $\mathbf{cell}(\{R\})$, where the morphism $A' \to A''$ is an element of $\mathbf{cell}(\{C\})$, and where the morphism $A'' \to B$ is a morphism of $\mathbf{cell}(I^{\mathrm{gl}})$. So we have to treat the cases $k \in \mathbf{cell}(\{R\})$, $k \in \mathbf{cell}(\{C\})$, and $k \in \mathbf{cell}(I^{\mathrm{gl}})$.

The case $k \in \mathbf{cell}(I^{\mathrm{gl}})$ is a consequence of Propositions 7.1, 7.2, and 7.3. The case $k \in \mathbf{cell}(\{C\})$ is trivial.

Let $k \in \mathbf{cell}(\{R\})$. Let $(\alpha,\beta) \in U^0 \times U^0$. Then $\mathbb{P}_{i(\alpha),i(\beta)}V$ (resp., $\mathbb{P}_{i(\alpha),i(\beta)}Y$) is a coproduct of terms of the form $\mathbb{P}_{\alpha,u_0}U \times \mathbb{P}_{v_0,u_1}U \times \cdots \times \mathbb{P}_{v_p,\beta}U$ (resp., $\mathbb{P}_{\alpha,u_0}X \times \mathbb{P}_{v_0,u_1}X \times \cdots \times \mathbb{P}_{v_p,\beta}X$) such that (u_i,v_i) is a pair of distinct elements of $U^0 = X^0$ identified by k. So t is a weak S-homotopy equivalence since a binary product of weak homotopy equivalences is a weak homotopy equivalence. □

8. T-homotopy equivalence and I_+^{gl}-cell complex

The first step to understand the reason why Definition 5.7 is badly behaved is the following theorem which gives a description of the T-homotopy equivalences $f : X \to Y$ such that the 0-skeleton of Y contains exactly one more state than the 0-skeleton of X.

Theorem 8.1. *Let X and Y be two objects of* **cell**(**Flow**). *Let $f : X \to Y$ be a T-homotopy equivalence. Assume that $Y^0 = X^0 \sqcup \{\alpha\}$. Then the canonical morphism $\varnothing \to X$ factors as a composite $\varnothing \to u_f(X) \to v_f(X) \to X$ such that*

(1) *one has the diagram*

$$\begin{tikzcd}
 & \varnothing \arrow[d] & \\
\{\widehat{0},\widehat{1}\} = \mathrm{Glob}(\mathbf{S}^{-1}) \arrow[r] \arrow[d] & u_f(X) \arrow[d] & \\
\vec{I} = \mathrm{Glob}(\mathbf{D}^0) \arrow[r] \arrow[d, "\phi"] & v_f(X) \arrow[r] \arrow[d] & X \arrow[d] \\
\vec{I} * \vec{I} \arrow[r] & \widehat{v}_f(X) \arrow[r] & Y
\end{tikzcd} \tag{8.1}$$

(2) *the morphisms $\varnothing \to u_f(X)$ and $v_f(X) \to X$ are relative I_g-cell complexes.*

By Proposition 6.3, the morphism $\{\widehat{0},\widehat{1}\} = \mathrm{Glob}(\mathbf{S}^{-1}) \to u_f(X)$ is a cofibration. Therefore, the morphism $\vec{I} \to v_f(X)$ is a cofibration as well. The morphism $u_f(X) \to \widehat{v}_f(X)$ is a relative I_g-cell complex as well since it is a pushout of the inclusion $\{\widehat{0},\widehat{1}\} \subset \vec{I} * \vec{I}$ sending $\widehat{0}$ to the initial state of $\vec{I} * \vec{I}$ and $\widehat{1}$ to the final state of $\vec{I} * \vec{I}$.

Proof. By Proposition 6.1, and since Y is an object of **cell**(**Flow**), the canonical morphism of flows $Y^0 \to Y$ is a relative I_g-cell complex. So there exist an ordinal λ and a λ-sequence $\mu \mapsto Y_\mu : \lambda \to$ **Flow** (also denoted by Y) such that $Y = \varinjlim_{\mu<\lambda} Y_\mu$ and such that for any ordinal $\mu < \lambda$, the morphism $Y_\mu \to Y_{\mu+1}$ is a pushout of the form

$$\begin{tikzcd}
\mathrm{Glob}(\mathbf{S}^{n_\mu - 1}) \arrow[r, "\phi_\mu"] \arrow[d] & Y_\mu \arrow[d] \\
\mathrm{Glob}(\mathbf{D}^{n_\mu}) \arrow[r, "\psi_\mu"'] & Y_{\mu+1}
\end{tikzcd} \tag{8.2}$$

of the inclusion of flows $\mathrm{Glob}(\mathbf{S}^{n_\mu}) \to \mathrm{Glob}(\mathbf{D}^{n_\mu+1})$ for some $n_\mu \geqslant 0$.

For any ordinal μ, the morphism of flows $Y_\mu \to Y_{\mu+1}$ induces an isomorphism between the 0-skeletons Y_μ^0 and $Y_{\mu+1}^0$. If $n_\mu \geqslant 1$ for some μ, then for any $\beta, \gamma \in Y^0$, the topological space $\mathbb{P}_{\beta,\gamma} Y_\mu$ is nonempty if and only if the topological space $\mathbb{P}_{\beta,\gamma} Y_{\mu+1}$ is nonempty. Consider the set of ordinals

$$\left\{\mu < \lambda; \bigsqcup_{\beta \in X^0} \mathbb{P}_{\beta,\alpha} Y_\mu \neq \varnothing\right\}. \tag{8.3}$$

It is nonempty since f is a T-homotopy equivalence. Take its smallest element μ_0. Consider the set of ordinals

$$\left\{\mu < \lambda; \bigsqcup_{\beta \in X^0} \mathbb{P}_{\alpha,\beta} Y_\mu \neq \varnothing\right\}. \tag{8.4}$$

Take its smallest element μ_1. Let us suppose for instance that $\mu_0 < \mu_1$.

The ordinal μ_0 cannot be a limit ordinal. Otherwise for any $\mu < \mu_0$, the isomorphisms of flows $Y_\mu = Z_\mu \sqcup \{\alpha\}$ and $Y_{\mu_0} \cong \varinjlim_{\mu<\mu_0} (Z_\mu \sqcup \{\alpha\}) \cong (\varinjlim_{\mu<\mu_0} Z_\mu) \sqcup \{\alpha\}$ would hold, a contradiction. Therefore, $\mu_0 = \mu_2 + 1$ and $n_{\mu_2} = 0$. There does not exist other ordinal μ such that $\phi_\mu(\widehat{1}) = \alpha$, otherwise $\mathbb{P}_\alpha^+ Y$ could not be a singleton anymore.

For a slightly different reason, the ordinal μ_1 cannot be a limit ordinal either. Otherwise if μ_1 was a limit ordinal, then the isomorphism of flows $Y_{\mu_1} \cong \varinjlim_{\mu<\mu_1} Y_\mu$ would hold. The path space of a colimit of flows is in general not the colimit of the path spaces. But any element of $\mathbb{P} Y_{\mu_1}$ is a composite $\gamma_1 * \cdots * \gamma_p$, where the γ_i for $1 \leqslant i \leqslant p$ belong to $\varinjlim_{\mu<\mu_1} \mathbb{P} Y_\mu$. By hypothesis, there exists an execution path $\gamma_1 * \cdots * \gamma_p \in \mathbb{P}_{\alpha,\beta} Y_{\mu_1}$ for some $\beta \in X^0$. So $s(\gamma_1) = \alpha$, which contradicts the definition of μ_1. Therefore, $\mu_1 = \mu_3 + 1$ and necessarily $n_{\mu_3} = 0$. There does not exist any other ordinal μ such that $\phi_\mu(\widehat{0}) = \alpha$, otherwise $\mathbb{P}_\alpha^- Y$ could not be a singleton anymore.

Therefore, one has the following situation: Y_{μ_2} is a flow of the form $Z_{\mu_2} \sqcup \{\alpha\}$. The passage from Y_{μ_2} to Y_{μ_2+1} is as follows:

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{-1}) \arrow[r, "\phi_{\mu_2}"] \arrow[d] & Y_{\mu_2} \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^0) \arrow[r] & Y_{\mu_2+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{8.5}$$

where $\phi_{\mu_2}(\widehat{0}) \in X^0$ and $\phi_{\mu_2}(\widehat{1}) = \alpha$. The morphism of flows $Y_{\mu_2+1} \to Y_{\mu_3}$ is a transfinite composition of pushouts of the inclusion of flows $\mathrm{Glob}(\mathbf{S}^n) \to \mathrm{Glob}(\mathbf{D}^{n+1})$, where $\phi_\mu(\widehat{0})$ and $\phi_\mu(\widehat{1})$ are never equal to α. The passage from Y_{μ_3} to Y_{μ_3+1} is as follows:

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{-1}) \arrow[r, "\phi_{\mu_3}"] \arrow[d] & Y_{\mu_3} \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^0) \arrow[r] & Y_{\mu_3+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{8.6}$$

where $\phi_{\mu_3}(\widehat{0}) = \alpha$ and $\phi_{\mu_3}(\widehat{1}) \in X^0$. The morphism of flows $Y_{\mu_3+1} \to Y_\lambda$ is a transfinite composition of pushouts of the inclusion of flows $\mathrm{Glob}(\mathbf{S}^n) \to \mathrm{Glob}(\mathbf{D}^{n+1})$, where $\phi_\mu(\widehat{0})$ and $\phi_\mu(\widehat{1})$ are never equal to α. Hence we get the result. □

We are now ready to give a characterization of the old T-homotopy equivalences.

Theorem 8.2. *Let X and Y be two objects of* **cell**(**Flow**). *Then a morphism of flows $f : X \to Y$ is a T-homotopy equivalence if and only if there exists a commutative diagram of flows of the form (with $\vec{I}^{*(n+1)} := \vec{I}^{*n} * \vec{I}$ and $\vec{I}^{*1} := \vec{I}$ for $n \geqslant 1$)*

$$\begin{tikzcd}
 & \varnothing \arrow[d] & \\
\bigsqcup_{i\in I}\{0,1\} = \bigsqcup_{i\in I}\mathrm{Glob}(\mathbf{S}^{-1}) \arrow[r] \arrow[d] & u_f(X) \arrow[d] & \\
\bigsqcup_{i\in I}\vec{I} = \bigsqcup_{i\in I}\mathrm{Glob}(\mathbf{D}^0) \arrow[r] \arrow[d, "\bigsqcup_{i\in I} r_i"'] & v_f(X) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\bigsqcup_{i\in I}\vec{I}^{*n_i} \arrow[r] & \widehat{v}_f(X) \arrow[r] & Y
\end{tikzcd} \tag{8.7}$$

where for any $i \in I$, n_i is an integer with $n_i \geqslant 1$ and such that $r_i : \vec{I} \to \vec{I}^{*n_i}$ is the unique morphism of flows preserving the initial and final states and where the morphisms $\varnothing \to u_f(X)$ and $v_f(X) \to X$ are relative I_g-cell complexes.

The pushout above tells us that the copy of $\vec{I}$ corresponding to the indexing $i \in I$ is divided in the concatenation of n_i copies of $\vec{I}$. This intuitively corresponds to a refinement of observation.

Proof. By Theorem 4.14, there exists a globular complex U (resp., V) such that $\mathrm{cat}(U) = X$ (resp., $\mathrm{cat}(V) = Y$). If a morphism of flows $f : X \to Y$ is a T-homotopy equivalence, then by Theorem 5.8, there exists a morphism of globular complexes $g : U \to V$ such that $\mathrm{cat}(g) = f$ and such that the continuous map $|g| : |U| \to |V|$ between the underlying topological spaces is a homeomorphism. So for any pair of points (α,β) of $X^0 \times X^0$, and any morphism $\vec{I} \to X$ appearing in the globular decomposition of X, the set of subdivisions of this segment in Y is finite since Y^0 is discrete and since the segment $[0,1]$ is compact. The result is then established by repeatedly applying Theorem 8.1.

Now suppose that a morphism of flows $f : X = \mathrm{cat}(U) \to Y = \mathrm{cat}(V)$ can be written as a pushout of the form of the statement of the theorem. Then start from a globular decomposition of U which is compatible with the composite $\varnothing \to u_f(X) \to v_f(X) \to X$. Then let us divide each segment of $[0,1]$ corresponding to the copy of $\vec{I}$ indexed by $i \in I$ in n_i pieces. Then one obtains a globular decomposition of V and the identity of U gives rise to a morphism of globular complexes $g : U \to V$ which induces a homeomorphism between the underlying topological spaces and such that $\mathrm{cat}(g) = f$. Hence we get the result. □

Definition 8.3. Let $n \geqslant 1$. The *full n-cube* $\vec{C}_n$ is by definition the flow $Q(\{\widehat{0} < \widehat{1}\}^n)$, where Q is the cofibrant replacement functor.

The flow $\vec{C}_3$ is represented in Figure 1.2.

Lemma 8.4. *If a flow X is loopless, then the transitive closure of the set*

$$\{(\alpha,\beta) \in X^0 \times X^0 \text{ such that } \mathbb{P}_{\alpha,\beta}X \neq \varnothing\} \tag{8.8}$$

induces a partial ordering on X^0.

Proof. If (α,β) and (β,α) with $\alpha \neq \beta$ belong to the transitive closure, then there exists a finite sequence $(x_1,\dots,x_\ell)$ of elements of X^0 with $x_1 = \alpha$, $x_\ell = \alpha$, $\ell > 1$ and with $\mathbb{P}_{x_m,x_{m+1}}X$ nonempty for each m. Consequently, the space $\mathbb{P}_{\alpha,\alpha}X$ is nonempty because of the existence of the composition law of X, a contradiction. □

Theorem 8.5. *Let $n \geqslant 3$. There does not exist any zig-zag sequence*

$$\vec{C}_n = X_0 \xrightarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xleftarrow{f_{2n-1}} X_{2n} = \vec{I}, \tag{8.9}$$

where every X_i is an object of **cell**(**Flow**) *and where every f_i is either an S-homotopy equivalence or a T-homotopy equivalence.*

Proof. By an immediate induction, one sees that each flow X_i is loopless, with a finite 0-skeleton. Moreover by construction, each poset $(X_i^0,\leqslant)$ is bounded, that is, with one bottom element $\widehat{0}$ and one top element $\widehat{1}$. So the zig-zag sequence above gives rise to a zig-zag sequence of posets:

$$\vec{C}_n^0 = X_0^0 = \{\widehat{0} < \widehat{1}\}^n \longrightarrow X_1^0 \longleftarrow X_2^0 \longrightarrow \cdots \longleftarrow X_{2n}^0 = \vec{I}^0 = \{\widehat{0} < \widehat{1}\}, \tag{8.10}$$

where $\{\widehat{0} < \widehat{1}\}^n$ is the product $\{\widehat{0} < \widehat{1}\} \times \cdots \times \{\widehat{0} < \widehat{1}\}$ (n times) in the category of posets. Each morphism of posets is an isomorphism if the corresponding morphism of flows is an S-homotopy equivalence because an S-homotopy equivalence induces a bijection between the 0-skeletons. Otherwise, one can suppose by Theorem 8.1 that the morphism of posets $P_1 \to P_2$ can be described as follows: take a segment $[x,y]$ of P_1 such that $]x,y[= \varnothing$; add a vertex $z \in]x,y[$; then let $P_2 = P_1 \cup \{z\}$ with the partial ordering $x < z < y$. In such a situation, $\min(]z,-[)$ exists and is equal to y, and $\max(]-,z[)$ exists and is equal to x. So by an immediate induction, there must exist $x,y,z \in \{0 < 1\}^n$ with $x < z < y$ and such that $\min(]z,-[) = y$ and $\max(]-,z[) = x$. This situation is impossible in the poset $\{\widehat{0} < \widehat{1}\}^n$ for $n \geqslant 3$. □

9. Generalized T-homotopy equivalence

As explained in the introduction, it is not satisfactory not to be able to identify $\vec{C}_3$, and more generally $\vec{C}_n$ for $n \geqslant 3$, with $\vec{I}$. The following definitions are going to be important for the sequel of the paper, and also for the whole series.

Definition 9.1 (the statement of the definition is slightly different, but is equivalent to the statement given in other parts of this series). A *full directed ball* is a flow $\vec{D}$ such that

(i) $\vec{D}$ is loopless (so by Lemma 8.4, the set $\vec{D}^0$ is equipped with a partial ordering $\leqslant$);
(ii) $(\vec{D}^0, \leqslant)$ is finite bounded;
(iii) for all $(\alpha,\beta) \in \vec{D}^0 \times \vec{D}^0$, the topological space $\mathbb{P}_{\alpha,\beta}\vec{D}$ is weakly contractible if $\alpha < \beta$, and empty otherwise by definition of $\leqslant$.

Let $\vec{D}$ be a full directed ball. Then by Lemma 8.4, the set $\vec{D}^0$ can be viewed as a finite bounded poset. Conversely, if P is a finite bounded poset, let us consider the *flow $F(P)$ associated with P*: it is of course defined as the unique flow $F(P)$ such that $F(P)^0 = P$ and $\mathbb{P}_{\alpha,\beta}F(P) = \{u\}$ if $\alpha < \beta$ and $\mathbb{P}_{\alpha,\beta}F(P) = \varnothing$ otherwise. Then $F(P)$ is a full directed ball and for any full directed ball $\vec{D}$, the two flows $\vec{D}$ and $F(\vec{D}^0)$ are weakly S-homotopy equivalent.

Let $\vec{E}$ be another full directed ball. Let $f : \vec{D} \to \vec{E}$ be a morphism of flows preserving the initial and final states. Then f induces a morphism of posets from $\vec{D}^0$ to $\vec{E}^0$ such that $f(\min \vec{D}^0) = \min \vec{E}^0$ and $f(\max \vec{D}^0) = \max \vec{E}^0$. Hence we have the following definition.

Definition 9.2. Let $\mathcal{T}$ be the class of morphisms of posets $f : P_1 \to P_2$ such that

(1) the posets P_1 and P_2 are finite and bounded;
(2) the morphism of posets $f : P_1 \to P_2$ is one-to-one; in particular, if x and y are two elements of P_1 with $x < y$, then $f(x) < f(y)$;
(3) one has $f(\min P_1) = \min P_2$ and $f(\max P_1) = \max P_2$.

Then a *generalized T-homotopy equivalence* is a morphism of $\mathbf{cof}(\{Q(F(f)), f \in \mathcal{T}\})$, where Q is the cofibrant replacement functor of the model category **Flow**.

It is of course possible to identity $\vec{C}_n$ ($n \geqslant 1$) with $\vec{I}$ by the following zig-zag sequence of S-homotopy and generalized T-homotopy equivalences:

$$\vec{I} \xleftarrow{\simeq} Q(\vec{I}) \xrightarrow{Q(F(g_n))} Q(\{\hat{0} < \hat{1}\}^n) \tag{9.1}$$

where $g_n : \{\hat{0} < \hat{1}\} \longrightarrow \{\hat{0} < \hat{1}\}^n \in \mathcal{T}$.

The relationship between the new definition of T-homotopy equivalence and the old definition is as follows.

Theorem 9.3. *Let X and Y be two objects of* $\mathbf{cell}(\mathbf{Flow})$. *Let $f : X \to Y$ be a T-homotopy equivalence. Then f can be written as a composite $X \to Z \to Y$ where $g : X \to Z$ is a generalized T-homotopy equivalence and where $h : Z \to Y$ is a weak S-homotopy equivalence.*

Proof. By Theorem 8.2, there exists a pushout diagram of flows of the form (with $\vec{I}^{*(n+1)} := \vec{I}^{*n} * \vec{I}$ and $\vec{I}^{*1} := \vec{I}$ for $n \geqslant 1$)

$$\begin{tikzcd} \bigsqcup_{k\in K} \vec{I} \arrow[r, hook] \arrow[d, "\bigsqcup_{k\in K} r_k"'] & X \arrow[d] \\ \bigsqcup_{k\in K} \vec{I}^{*n_k} \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{9.2}$$

where for any $k \in K$, n_k is an integer with $n_k \geqslant 1$ and such that $r_k : \vec{I} \to \vec{I}^{*n_k}$ is the unique morphism of flows preserving the initial and final states. Notice that each $\vec{I}^{*n_k}$ is a full directed ball. Thus one obtains the following commutative diagram:

$$\begin{tikzcd}
\bigsqcup_{k\in K} Q(\vec{I}) \arrow[r, two heads, "\simeq"] \arrow[d] & \bigsqcup_{k\in K} \vec{I} \arrow[r, hook] \arrow[d] & X \arrow[d, hook] \\
\bigsqcup_{k\in K} Q(\vec{I}^{*n_k}) \arrow[r, equal] & \bigsqcup_{k\in K} Q(\vec{I}^{*n_k}) \arrow[r, hook] \arrow[d, "\simeq"'] & Z \arrow[d, "\simeq"'] \arrow[ul, phantom, "\ulcorner", very near start] \\
& \bigsqcup_{k\in K} \vec{I}^{*n_k} \arrow[r, hook] & Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{9.3}$$

Now here are some justifications for this diagram. First of all, a morphism of flows $f : M \to N$ is a fibration of flows if and only if the continuous map $\mathbb{P}f : \mathbb{P}M \to \mathbb{P}N$ is a Serre fibration of topological spaces. Since any coproduct of Serre fibration is a Serre fibration, the morphism of flows $\bigsqcup_{i\in I} Q(\vec{I}) \to \bigsqcup_{k\in K} \vec{I}$ is a trivial fibration of flows. Thus, the underlying set map $\bigsqcup_{k\in K} Q(\vec{I}) \to \bigsqcup_{k\in K} \vec{I}$ is surjective. So the commutative square

$$\begin{tikzcd}
\bigsqcup_{k\in K} Q(\vec{I}) \arrow[r] \arrow[d] & X \arrow[d] \\
\bigsqcup_{k\in K} Q(\vec{I}^{*n_k}) \arrow[r] & Z \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{9.4}$$

is cocartesian and the morphism of flows $X \to Z$ is then a generalized T-homotopy equivalence. It is clear that the morphism $\bigsqcup_{k\in K} Q(\vec{I}^{*n_k}) \to \bigsqcup_{k\in K} \vec{I}^{*n_k}$ is a weak S-homotopy equivalence. The latter morphism is even a fibration of flows, but that does not matter here. So the morphism $Z \to Y$ is the pushout of a weak S-homotopy equivalence along the cofibration $\bigsqcup_{k\in K} Q(\vec{I}^{*n_k}) \to Z$. Since the model category **Flow** is left proper by Theorem 7.4, the proof is complete. □

10. Conclusion

This new definition of T-homotopy equivalence contains the old one up to S-homotopy equivalence. The drawback of the old definition presented in [3] is overcome. It is proved in [4] that this new notion of T-homotopy equivalence does preserve the branching and merging homology theories. And it is proved in [5] that the underlying homotopy type of a flow is also preserved by this new definition of T-homotopy equivalence. Finally, [6] proposes an application of this new notion of dihomotopy, that is, a Whitehead theorem for the full dihomotopy relation.

References

[1] P. Gaucher, "T-homotopy and refinement of observation—I: introduction," to appear in *Electronic Notes in Theoretical Computer Sciences*, http://arxiv.org/abs/math.AT/0505152.

[2] P. Gaucher and E. Goubault, "Topological deformation of higher dimensional automata," *Homology, Homotopy, and Applications*, vol. 5, no. 2, pp. 39–82, 2003.

[3] P. Gaucher, "Comparing globular complex and flow," *New York Journal of Mathematics*, vol. 11, pp. 97–150, 2005.

[4] P. Gaucher, "T-homotopy and refinement of observation—III: invariance of the branching and merging homologies," *New York Journal of Mathematics*, vol. 12, pp. 319–348, 2006.

[5] P. Gaucher, "T-homotopy and refinement of observation—IV: invariance of the underlying homotopy type," *New York Journal of Mathematics*, vol. 12, pp. 63–95, 2006.

[6] P. Gaucher, "Inverting weak dihomotopy equivalence using homotopy continuous flow," *Theory and Applications of Categories*, vol. 16, no. 3, pp. 59–83, 2006.

[7] P. Gaucher, "A model category for the homotopy theory of concurrency," *Homology, Homotopy, and Applications*, vol. 5, no. 1, pp. 549–599, 2003.

[8] M. Hovey, *Model Categories*, vol. 63 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1999.

[9] P. S. Hirschhorn, *Model Categories and Their Localizations*, vol. 99 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2003.

[10] W. G. Dwyer and J. Spaliński, "Homotopy theories and model categories," in *Handbook of Algebraic Topology*, pp. 73–126, North-Holland, Amsterdam, The Netherlands, 1995.

[11] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics, no. 43, Springer, Berlin, Germany, 1967.

[12] R. Brown, *Topology: A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK, 2nd edition, 1988.

[13] J. P. May, *A Concise Course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, Ill, USA, 1999.

[14] L. G. Lewis, *The stable category and generalized Thom spectra*, Ph.D. thesis, University of Chicago, Chicago, Ill, USA, 1978.

[15] P. Gaucher, "Homological properties of non-deterministic branchings of mergings in higher dimensional automata," *Homology, Homotopy, and Applications*, vol. 7, no. 1, pp. 51–76, 2005.

[16] A. Strøm, "Note on cofibrations," *Mathematica Scandinavica*, vol. 19, pp. 11–14, 1966.

[17] A. Strøm, "Note on cofibrations—II," *Mathematica Scandinavica*, vol. 22, pp. 130–142, 1968.

[18] A. Strøm, "The homotopy category is a homotopy category," *Archiv der Mathematik*, vol. 23, pp. 435–441, 1972.

[19] M. Cole, "Many homotopy categories are homotopy categories," *Topology and Its Applications*, vol. 153, no. 7, pp. 1084–1099, 2006.

[20] P. J. Higgins, *Notes on Categories and Groupoids*, Van Nostrand Rienhold Mathematical Studies, no. 32, Van Nostrand Reinhold, London, UK, 1971.

[21] D. Dugger and D. C. Isaksen, "Topological hypercovers and $\mathbb{A}^1$-realizations," *Mathematische Zeitschrift*, vol. 246, no. 4, pp. 667–689, 2004.

Philippe Gaucher: Preuves Programmes et Systèmes, Université Paris 7–Denis Diderot,
Case 7014, 2 Place Jussieu, 75251 Paris Cedex 05, France
Email address: gaucher@pps.jussieu.fr