T-HOMOTOPY AND REFINEMENT OF OBSERVATION (V) : STRØM MODEL STRUCTURE FOR BRANCHING AND MERGING HOMOLOGIES

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Abstract. We check that there exists a model structure on the category of flows whose weak equivalences are the S-homotopy equivalences. As an application, we prove that the generalized T-homotopy equivalences preserve the branching and merging homology theories of a flow. The method of proof is completely different from the one of the third part of this series of papers.

Contents

1. Introduction

The purpose of this paper is to prove that the new notion of T-homotopy equivalence preserves the branching and merging homologies of a flow. The method of proof is completely different from the one of the third part [Gau05b] of this series of papers. The main theorem of this paper is:

Theorem. *Let $f : X \longrightarrow Y$ be a generalized T-homotopy equivalence. Then for any $n \geqslant 0$, the morphisms of abelian groups $H_n^-(f) : H_n^-(X) \longrightarrow H_n^-(Y)$, $H_n^+(f) : H_n^+(X) \longrightarrow H_n^+(Y)$ are isomorphisms of groups where H_n^- (resp.H_n^+) is the n-th branching (resp. merging) homology group.*

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The main tool to establish this fact is the construction of a model structure on the category of flows which is the analogue of the Strøm model category structure on the category of compactly generated topological spaces [Str72] (after [Str66] [Str68]). The weak equivalences of the latter are the homotopy equivalences, the fibrations are the Hurewicz fibrations, that is the continuous maps satisfying the right lifting property with respect to the inclusions $M \times \{0\} \longrightarrow M \times [0,1]$ for any topological space M and the cofibrations are the NDR pairs. Any space is cofibrant and fibrant for the Strøm model structure. More precisely, we are going to prove that:

Theorem. *There exists a model structure on the category of flows such that the weak equivalences are the S-homotopy equivalences. In this model category, any flow is cofibrant and fibrant. This model structure is called the Cole-Strøm model structure.*

The whole construction of Strøm does not seem to be adaptable to the category of flows because it is unknown how to construct for a given "NDR pair" of flows (X, A) the analogue of the characterization of NDR pairs of topological spaces using a height function $\mu : X \to [0,1]$. It is actually not even known whether the class of cofibrations of the Cole-Strøm model structure is the whole class of morphisms of flows satisfying the S-homotopy extension property, or only a proper subclass.

Cole's work [Col99a] provides a remarkable extension of the results of Strøm to any topological bicomplete category satisfying one additional hypothesis (Theorem 6.4). The category of flows satisfies the axioms of topological bicomplete category only for non-empty connected topological spaces (cf. Proposition 4.6). Moreover, the Cole additional hypothesis involves colimits and the behavior of these latter are very different in the category of flows from their behavior in the category of topological spaces. So a discussion is required to ensure that all Cole's key arguments are still true in this new setting.

I thank very much Peter May for answering my questions and for sending me the unpublished work of Michael Cole [1].

2. Prerequisites and notations

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to any morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to any morphism of $\mathbf{inj}(K)$. This is a purely categorical fact that $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, any morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$. An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex.* If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative K-cell complex, one says that X is a *K-cell complex.*

The initial object (resp. the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp. $\mathbf{1}$).

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the object X of $\mathcal{C}$ such that the canonical morphism $\varnothing \longrightarrow X$ is an object of $\mathbf{cell}(I)$. In other terms, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing \downarrow \mathcal{C}) \cap \mathbf{cell}(I)$.

[1] More explanations about unpublished Cole's work can now be found in [MS04]

It is obviously impossible to read this paper without a strong familiarity with *model categories*. Possible references for model categories are [Hov99], [Hir03] and [DS95]. The original reference is [Qui67] but Quillen's axiomatization is not used in this paper. The Hovey's book axiomatization is preferred. If $\mathcal{M}$ is a *cofibrantly generated* model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$. Any cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \longrightarrow \mathbf{cell}(\mathcal{M})$. For any morphism f of $\mathcal{M}$, the morphism $Q(f)$ is a cofibration, and even an inclusion of subcomplexes.

A *partially ordered set* $(P, \leqslant)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relations $\leqslant$. A poset is *locally finite* if for any $(x, y) \in P \times P$, the set $[x, y] = \{z \in P, x \leqslant z \leqslant y\}$ is finite. A poset $(P, \leqslant)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P \subset [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. Let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element).

Any poset P, and in particular any ordinal, can be viewed as a small category denoted in the same way: the objects are the elements of P and there exists a morphism from x to y if and only if $x \leqslant y$. If λ is an ordinal, a λ-*sequence* in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor X from λ to $\mathcal{C}$. We denote by X_λ the colimit $\varinjlim X$ and the morphism $X_0 \longrightarrow X_\lambda$ is called the *transfinite composition* of the $X_\mu \longrightarrow X_{\mu+1}$.

Let $\mathcal{C}$ be a category. Let α be an object of $\mathcal{C}$. The *latching category* $\partial(\mathcal{C} \downarrow \alpha)$ at α is the full subcategory of $\mathcal{C} \downarrow \alpha$ containing all the objects except the identity map of α. The *matching category* $\partial(\alpha \downarrow \mathcal{C})$ at α is the full subcategory of $\alpha \downarrow \mathcal{C}$ containing all the objects except the identity map of α.

Let $\mathcal{B}$ be a small category. A *Reedy structure* on $\mathcal{B}$ consists of two subcategories $\mathcal{B}_-$ and $\mathcal{B}_+$, a functor $d : \mathcal{B} \longrightarrow \lambda$ called the degree function for some ordinal λ, such that every nonidentity map in $\mathcal{B}_+$ raises the degree, every nonidentity map in $\mathcal{B}_-$ lowers the degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = g \circ h$ with $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. A small category together with a Reedy structure is called a *Reedy category*.

Let $\mathcal{C}$ be a complete and cocomplete category. Let $\mathcal{B}$ be a Reedy category. Let i be an object of $\mathcal{B}$. The *latching space functor* is the composite $L_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(\mathcal{B}_+\downarrow i)} \longrightarrow \mathcal{C}$ where the latter functor is the colimit functor. The *matching space functor* is the composite $M_i : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\partial(i\downarrow\mathcal{B}_-)} \longrightarrow \mathcal{C}$ where the latter functor is the limit functor.

If $\mathcal{C}$ is a small category and of $\mathcal{M}$ is a model category, the notation $\mathcal{M}^{\mathcal{C}}$ is the category of functors from $\mathcal{C}$ to $\mathcal{M}$, i.e. the category of diagrams of objects of $\mathcal{M}$ over the small category $\mathcal{C}$.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. The model categories **Top** and **Flow** (see below) are both left proper.

In this paper, the notation $\hookrightarrow$ means *cofibration*, the notation $\twoheadrightarrow$ means *fibration*, the notation $\simeq$ means *weak equivalence*, and the notation $\cong$ means *isomorphism*.

A categorical adjunction $\mathbb{L} : \mathcal{M} \leftrightarrows \mathcal{N} : \mathbb{R}$ between two model categories is a *Quillen adjunction* if one of the following equivalent conditions is satisfied: 1) $\mathbb{L}$ preserves cofibrations (resp. trivial cofibrations), 2) $\mathbb{R}$ preserves fibrations (resp. trivial fibrations). In that case, $\mathbb{L}$ (resp. $\mathbb{R}$) preserves weak equivalences between cofibrant (resp. fibrant) objects.

If P is a poset, let us denote by $\Delta(P)$ the *order complex* associated to P. Recall that the order complex is a simplicial complex having P as underlying set and having the subsets

$\{x_0, x_1, \dots, x_n\}$ with $x_0 < x_1 < \dots < x_n$ as n-simplices. Such a simplex will be denoted by $(x_0, x_1, \dots, x_n)$. The order complex $\Delta(P)$ can be viewed as a poset ordered by the inclusion, and therefore as a small category. The corresponding category will be denoted in the same way. The opposite category $\Delta(P)^{op}$ is freely generated by the morphisms $\partial_i : (x_0, \dots, x_n) \longrightarrow (x_0, \dots, \widehat{x_i}, \dots, x_n)$ for $0 \leqslant i \leqslant n$ and by the simplicial relations $\partial_i \partial_j = \partial_{j-1}\partial_i$ for any $i < j$, where the notation $\widehat{x_i}$ means that x_i is removed.

If $\mathcal{C}$ is a small category, the *classifying space* of $\mathcal{C}$ is denoted by $B\mathcal{C}$.

The category **Top** of *compactly generated topological spaces* (i.e. of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [Bro88, May99], the appendix of [Lew78] and also the preliminaries of [Gau03]). For the sequel, any topological space will be supposed to be compactly generated. A *compact space* is always Hausdorff.

The category **Top** is equipped with the unique model structure having the *weak homotopy equivalences* as weak equivalences and having the *Serre fibrations* [2] as fibrations.

The time flow of a higher dimensional automaton is encoded in an object called a *flow* [Gau03]. A flow X consists of a set X^0 called the 0-*skeleton* and whose elements correspond to the *states* (or *constant execution paths*) of the higher dimensional automaton. For each pair of states $(\alpha, \beta) \in X^0 \times X^0$, there is a topological space $\mathbb{P}_{\alpha,\beta}X$ whose elements correspond to the *(nonconstant) execution paths* of the higher dimensional automaton *beginning at* α and *ending at* β. If $x \in \mathbb{P}_{\alpha,\beta}X$, let $\alpha = s(x)$ and $\beta = t(x)$. For each triple $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, there exists a continuous map $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ called the *composition law* which is supposed to be associative in an obvious sense. The topological space $\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X$ is called the *path space* of X. The category of flows is denoted by **Flow**. A point α of X^0 such that there are no non-constant execution paths ending to α (resp. starting from α) is called an *initial state* (resp. a *final state*).

The category **Flow** is equipped with the unique model structure having the *weak S-homotopy equivalences* [3] as weak equivalences and having as fibrations the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration [Gau03]. It is cofibrantly generated. The set of generating cofibrations is $I^{gl}_+ = I^{gl} \cup \{R, C\}$ with $I^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geqslant 0\}$ where $\mathbf{S}^n$ is the n-dimensional sphere, where R and C are the set maps $R : \{0, 1\} \longrightarrow \{0\}$ and $C : \varnothing \longrightarrow \{0\}$ and where for any topological space Z, the flow $\mathrm{Glob}(Z)$ is the flow defined by $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = \widehat{0}$ and $t = \widehat{1}$, and a trivial composition law. The set of generating trivial cofibrations is $J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0, 1]), n \geqslant 0\}$.

If X is an object of **cell**(**Flow**), then a presentation of the morphism $\varnothing \longrightarrow X$ as a transfinite composition of pushouts of morphisms of I^{gl}_+ is called a *globular decomposition* of X.

[2] that is a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0, 1]$ for any $n \geqslant 0$ where $\mathbf{D}^n$ is the n-dimensional disk.

[3] that is a morphism of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.

3. Outline of the paper and warnings

The reader must beware of the fact that the two model structures of **Flow** are used in Section 8. The redaction is hopefully clear enough to avoid any confusion ! The main facts that will be used are:

(1) Any cofibration for the weak S-homotopy model structure of **Flow** is a cofibration for the Cole-Strøm model structure of **Flow** .
(2) Any Reedy cofibration for the weak S-homotopy model structure of **Flow** is a Reedy cofibration for the Cole-Strøm model structure of **Flow** .

The notation Q will always mean the cofibrant replacement functor of the weak S-homotopy model structure of **Flow**. The cofibrant replacement functor for the Cole-Strøm model structure of **Flow** exists but it is of course useless.

Section 4 defines the cofibrations, fibrations and weak equivalences of the Cole-Strøm model structure. Then Section 5 (except Lemma 5.15 which can be found in Cole's preprint) follows proofs of [Str66] [Str68] and [Str72] and Section 6 follows the proofs of [Col99a]. Next Section 7 ends up the construction of the Cole-Strøm model structure. During this construction, the weak S-homotopy model structure of **Flow** is not used at all.

4. The classes of cofibrations and fibrations

In this section, we introduce the classes of cofibrations and fibrations of the model structure we are going to construct. They must be compared with the well-known notions of Hurewicz fibration and Hurewicz cofibration of topological spaces.

Notation 4.1. *The space* $\mathbf{FLOW}(X,Y)$ *is the set* $\mathbf{Flow}(X,Y)$ *equipped with the Kelleyfication of the relative topology induced by that of* $\mathbf{TOP}(X^0 \sqcup \mathbb{P}X, Y^0 \sqcup \mathbb{P}Y)$.

Definition 4.2. *Let U be a topological space. Let X be a flow. The flow $\{U,X\}_S$ is defined as follows:*

(1) *The* 0*-skeleton of* $\{U,X\}_S$ *is* X^0.
(2) *For* $\alpha,\beta \in X^0$*, the topological space* $\mathbb{P}_{\alpha,\beta}\{U,X\}_S$ *is* $\mathbf{TOP}(U,\mathbb{P}_{\alpha,\beta}X)$ *with the composition law induced for any* $(\alpha,\beta,\gamma)\in X^0\times X^0\times X^0$ *by the composite map*

$$\mathbf{TOP}(U,\mathbb{P}_{\alpha,\beta}X)\times \mathbf{TOP}(U,\mathbb{P}_{\beta,\gamma}X)\cong \mathbf{TOP}(U,\mathbb{P}_{\alpha,\beta}X\times \mathbb{P}_{\beta,\gamma}X)\to \mathbf{TOP}(U,\mathbb{P}_{\alpha,\gamma}X).$$

Theorem 4.3. *Let U be a topological space. The mapping $X\mapsto \{U,X\}_S$ induces a functor from* **Flow** *to* **Flow**. *The functor $\{U,-\}_S : X\mapsto \{U,X\}_S$ has a left adjoint $U\boxtimes - : X\mapsto U\boxtimes X$. Moreover the flows $\{U,X\}_S$ and $U\boxtimes X$ are also functorial with respect to U and one has the isomorphisms of flows*

$$\begin{aligned}
&\varnothing\boxtimes X = X^0\\
&(U\boxtimes X)^0 = X^0\\
&U\boxtimes\{0\}\cong\{0\}\\
&U\boxtimes \mathrm{Glob}(Z)\cong \mathrm{Glob}(U\times Z)\\
&(U\times V)\boxtimes X\cong U\boxtimes(V\boxtimes X)\\
&\{U\times V,X\}_S\cong\{V,\{U,X\}_S\}_S\cong\{U,\{V,X\}_S\}_S.
\end{aligned}$$

The flow $\{\varnothing, X\}_S$ is the flow having the same 0*-skeleton as X and exactly one non-constant execution path between two points of X^0. At last, there exists a natural isomorphism of sets*

$$\mathbf{Flow}\,(U \boxtimes X, Y) \cong \mathbf{Flow}\,(X, \{U, Y\}_S)\,.$$

The adjunction between $U \boxtimes -$ and $\{U, -\}_S$ implies the obvious:

Proposition 4.4. *Let (Y, B) be a pair of topological spaces. Let $i : A \longrightarrow X$ and $p : E \longrightarrow D$ be two morphisms of flows. Then the following conditions are equivalent:*

1. *The exists a morphism of flows k making commutative the diagram*

$$\begin{tikzcd} (Y \boxtimes A) \sqcup_{B\boxtimes A} (B \boxtimes X) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Y \boxtimes X \arrow[r] \arrow[ur, dashed, "k"] & D. \end{tikzcd}$$

2. *There exists a morphism of flows $\overline{k}$ making commutative the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \{Y, E\}_S \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed, "\overline{k}"] & \{Y, D\}_S \times_{\{B,D\}_S} \{B, E\}_S. \end{tikzcd}$$

Definition 4.5. *Two morphisms of flows f and g from X to Y are* S-homotopic *if and only if there exists*

$$H \in \mathbf{Top}([0, 1], \mathbf{FLOW}(X, Y))$$

such that $H(0) = f$ and $H(1) = g$. We denote this situation by $f \sim_S g$.

One has the:

Proposition 4.6. *Let U be a connected non empty space. Let X and Y be two flows. Then there exists a natural isomorphism of sets*

$$\mathbf{Top}(U, \mathbf{FLOW}(X, Y)) \cong \mathbf{Flow}(U \boxtimes X, Y).$$

The latter proposition is false if U is empty. Indeed, one has $\mathbf{Top}(\varnothing, \mathbf{FLOW}(X, Y)) \cong \{0\}$ and $\mathbf{Flow}(\varnothing \boxtimes X, Y) \cong \mathbf{Flow}(X^0, Y) \cong \mathbf{Set}(X^0, Y^0)$.

Hence, two morphisms of flows f and g from X to Y are S-homotopic if and only if there exists a morphism of flows $H : [0, 1] \boxtimes X \longrightarrow Y$ such that $H(0 \boxtimes x) = f(x)$ and $H(1 \boxtimes x) = g(x)$.

Definition 4.7. *Two flows are* S-homotopy equivalent *if and only if there exist morphisms of flows $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \sim_S \mathrm{Id}_Y$ and $g \circ f \sim_S \mathrm{Id}_X$. One says that f and g are* two inverse S-homotopy equivalences.

Definition 4.8. *Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the left lifting property (LLP) with respect to p (or p has the right lifting property (RLP) with respect to i) if for any commutative square*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there exists g making both triangles commutative.

Definition 4.9. *A morphism of flows $p : E \longrightarrow B$ is a* Hurewicz fibration of flows *if p has the RLP with respect to the morphisms of flows $\{0\} \boxtimes M \longrightarrow [0,1] \boxtimes M$ for any flow M.*

Definition 4.10. *A Hurewicz fibration of flows is* trivial *if it is at the same time a S-homotopy equivalence.*

Definition 4.11. *A morphism of flows $i : A \longrightarrow X$ is a* Hurewicz cofibration of flows *if i satisfies the LLP with respect to any trivial Hurewicz fibration of flows.*

Definition 4.12. *A Hurewicz cofibration of flows is* trivial *if it is at the same time a S-homotopy equivalence.*

5. Preparatory lemmas

Definition 5.1. *Let $i : A \longrightarrow X$ be a morphism of flows. The morphism $i : A \longrightarrow X$ satisfies* the S-homotopy extension property *if for any flow Y and any morphism $f : X \longrightarrow Y$ and any S-homotopy $h : [0,1] \boxtimes A \longrightarrow Y$ such that for any $a \in A$, $h(0 \boxtimes a) = f(i(a))$, there exists a S-homotopy $H : [0,1] \boxtimes X \longrightarrow Y$ such that for any $x \in X$, $H(0 \boxtimes x) = f(x)$ and for any $(t,a) \in [0,1] \times A$, $H(t \boxtimes i(a)) = h(t \boxtimes a)$.*

Definition 5.2. *If $i : A \longrightarrow X$ is a morphism of flows, then the* mapping cylinder Mi *of i is defined by the pushout of flows*

$$\begin{tikzcd} A \arrow[r, "a\mapsto 0\boxtimes a"] \arrow[d, "i"'] & {[0,1]\boxtimes A} \arrow[d] \\ X \arrow[r] & Mi. \end{tikzcd}$$

Lemma 5.3. *([Gau03] Theorem 9.4) Let $i : A \longrightarrow X$ be a morphism of flows. Then the following assertions are equivalent:*

(1) *the morphism i satisfies the S-homotopy extension property*
(2) *the morphism of flows $\psi(i)$ has a retract r, that is to say there exists a morphism of flows*
$$r : [0,1] \boxtimes X \longrightarrow ([0,1] \boxtimes A) \sqcup_{\{0\}\boxtimes A} (\{0\} \boxtimes X)$$
such that $r \circ \psi(i) = \mathrm{Id}_{([0,1]\boxtimes A)\sqcup_{\{0\}\boxtimes A}(\{0\}\boxtimes X)}$.

Lemma 5.4. *Let $i : A \to X$ be a S-homotopy equivalence satisfying the S-homotopy extension property. Then there exists $r : X \to A$ such that $r \circ i = \mathrm{Id}_A$.*

Proof. Let g be a S-homotopic inverse of $i : A \to X$. By hypothesis, there exists $h : [0,1] \boxtimes A \to A$ such that $h(1 \boxtimes a) = a$ and $h(0 \boxtimes a) = g \circ i$. The mappings h and g induce a morphism of flows
$$\overline{h} : ([0,1] \boxtimes A) \sqcup_{\{0\}\boxtimes A} (\{0\} \boxtimes X) \to A$$
Since $i : A \to X$ satisfies the S-homotopy extension property, then the morphism of flows $\overline{h}$ can be extended to a morphism of flows $H : [0,1] \boxtimes X \to A$ by Lemma 5.3. Then $r = H(1 \boxtimes -)$ is the desired retraction. Indeed $r(i(a)) = h(1,a) = a$ and $H(0 \boxtimes x) = g(x)$. □

Lemma 5.5. *Let $p : E \longrightarrow D$ be a trivial Hurewicz fibration of flows. Then there exists a morphism of flows $s : D \longrightarrow E$ such that $p \circ s = \mathrm{Id}_D$.*

Proof. Let $q : D \longrightarrow E$ be a S-homotopic inverse of p. Let $h : [0,1] \boxtimes D \longrightarrow D$ such that $h(0 \boxtimes d) = p(s(d))$ and $h(1 \boxtimes d) = d$. Then consider the commutative square

$$\begin{tikzcd} D \arrow[r, "s"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ {[0,1] \boxtimes D} \arrow[r, "h"] \arrow[ur, dashed, "H"] & D. \end{tikzcd}$$

Then there exists H making the diagram commutative. And $s = H(1 \boxtimes -)$ is a solution. □

Lemma 5.6. *Let X be a flow. Then $X \longrightarrow \mathbf{1}$ (where $\mathbf{1}$ is the terminal flow) is a Hurewicz fibrations of flows.*

Proof. Consider a commutative diagram like

$$\begin{tikzcd} \{0\} \boxtimes M \arrow[r, "u"] \arrow[d] & X \arrow[d] \\ {[0,1] \boxtimes M} \arrow[r] \arrow[ur, dashed, "k"] & \mathbf{1}. \end{tikzcd}$$

Then $k(t \boxtimes m) = u(m)$ is a solution. □

Lemma 5.7. *For any flow X, for any NDR pair (Y, B), the morphism of flows $\{Y, X\}_S \longrightarrow \{B, X\}_S$ is a Hurewicz fibration of flows. Moreover, if (Y, B) is a DR pair, then the morphism of flows $\{Y, X\}_S \longrightarrow \{B, X\}_S$ is a S-homotopy equivalence, that is the morphism of flows $\{Y, X\}_S \longrightarrow \{B, X\}_S$ is a trivial Hurewicz fibration of flows.*

Proof. Consider a commutative diagram of flows like

$$\begin{tikzcd} \{0\} \boxtimes M \arrow[r] \arrow[d] & \{Y, X\}_S \arrow[d] \\ {[0,1] \boxtimes M} \arrow[r] \arrow[ur, dashed, "k"] & \{B, X\}_S \end{tikzcd}$$

is equivalent to considering a morphism of flows

$$(Y \times \{0\} \cup B \times [0,1]) \boxtimes M \longrightarrow X.$$

And finding k making both triangles of the diagram commutative is equivalent to finding $\overline{k}$ such that the diagram

$$\begin{tikzcd} (Y \times \{0\} \cup B \times [0,1]) \boxtimes M \arrow[r] \arrow[d] & X \\ (Y \times [0,1]) \boxtimes M \arrow[ur, dashed, "\overline{k}"'] & \end{tikzcd}$$

is commutative. It then suffices to notice that the inclusion $Y \times \{0\} \cup B \times [0,1] \longrightarrow Y \times [0,1]$ has a retract. □

Lemma 5.8. *Let $E \longrightarrow D$ be a Hurewicz fibration of flows. Then the canonical morphism of flows*

$$q : \{[0,1], E\}_S \longrightarrow \{\{0\}, E\}_S \times_{\{\{0\},D\}_S} \{[0,1], D\}_S$$

is a trivial Hurewicz fibration of flows.

Proof. Let (Y, B) be a pair of topological spaces. Then there exists k making the diagram

$$\begin{tikzcd} \{0\} \boxtimes M \arrow[r] \arrow[d] & \{Y, E\}_S \arrow[d] \\ {[0,1]} \boxtimes M \arrow[r] \arrow[ur, dashed, "k"] & \{B, E\}_S \times_{\{B,D\}_S} \{Y, D\}_S \end{tikzcd}$$

commutative if and only if there exists $\overline{k}$ making the diagram

$$\begin{tikzcd} (Y \boxtimes \{0\} \boxtimes M) \sqcup_{B \boxtimes \{0\} \boxtimes M} (B \boxtimes [0,1] \boxtimes M) \arrow[r] \arrow[d] & E \arrow[d] \\ Y \boxtimes [0,1] \boxtimes M \arrow[r] \arrow[ur, dashed, "\overline{k}"] & D \end{tikzcd}$$

commutative because of the adjunction between $\boxtimes$ and $\{-,-\}_S$. But

$$(Y \boxtimes \{0\} \boxtimes M) \sqcup_{B \boxtimes \{0\} \boxtimes M} (B \boxtimes [0,1] \boxtimes M) \cong (Y \times \{0\} \cup B \times [0,1]) \boxtimes M.$$

In our case, $(Y, B) = ([0,1], \{0\})$. Thus $(Y \times [0,1], Y \times \{0\} \cup B \times [0,1])) \cong ([0,1] \times [0,1], [0,1])$. So one has to find $\overline{k}$ making the diagram

$$\begin{tikzcd} {[0,1]} \boxtimes M \arrow[r] \arrow[d] & E \arrow[d] \\ {[0,1]} \boxtimes [0,1] \boxtimes M \arrow[r] \arrow[ur, dashed, "\overline{k}"] & D \end{tikzcd}$$

commutative. But $E \longrightarrow D$ is a Hurewicz fibration of flows. The fact that q is also a S-homotopy equivalence is obvious. □

Lemma 5.9. *Let $E \longrightarrow D$ be a Hurewicz fibration of flows. Then the canonical morphism of flows $q : \{[0,1], E\}_S \longrightarrow \{\{0,1\}, E\}_S \times_{\{\{0,1\},D\}_S} \{[0,1], D\}_S$ is a Hurewicz fibration of flows.*

Proof. We repeat the proof of Lemma 5.8 with the NDR pair $(Y, B) = ([0,1], \{0,1\})$. Then the pair $(Y \times [0,1], Y \times \{0\} \cup B \times [0,1])) = (|0,1] \times [0,1], [0,1] \times \{0\} \cup \{0,1\} \times [0,1])$ is again isomorphic to $([0,1] \times [0,1], [0,1])$. □

Lemma 5.10. *Let M be a flow. Then $\varnothing \longrightarrow M$ and $M^0 \longrightarrow M$ are Hurewicz cofibrations of flows.*

Proof. The morphism of flows $C : \varnothing \longrightarrow \{0\}$ satisfies the LLP with respect to any trivial fibration of flows because of Lemma 5.5. So it suffices to prove that $M^0 \longrightarrow M$ satisfies the RLP with respect to any trivial fibration. Consider the commutative square

$$\begin{tikzcd} M^0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ M \arrow[r, "\phi"] \arrow[ur, dashed, "k"] & D. \end{tikzcd}$$

Since p is trivial, then $E^0 = D^0$, so the construction of k on M^0 is clear. Then $k = s \circ \phi$ with the s of Lemma 5.5 is a solution. □

Lemma 5.11. *Let $i : A \longrightarrow X$ be a Hurewicz cofibration of flows. Then i satisfies the S-homotopy extension property.*

Proof. Consider a morphism of flows $Mi \longrightarrow Y$. It gives rise to the commutative diagram of flows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \{[0,1], Y\}_S \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "k"] & \{\{0\}, Y\}_S. \end{tikzcd}$$

But the morphism of flows $\{[0,1], Y\}_S \longrightarrow \{\{0\}, Y\}_S$ is a trivial Hurewicz morphism of flows by Lemma 5.7 since $([0,1], \{0\})$ is a DR pair. Therefore k making both triangles commutative exists: this is exactly the S-homotopy extension property. □

Lemma 5.12. *For any flow M, the morphism of flows $\{0,1\} \boxtimes M \longrightarrow [0,1] \boxtimes M$ is a Hurewicz cofibration of flows.*

Proof. Let $p : E \longrightarrow D$ be a trivial Hurewicz fibration of flows. Then $\{0,1\} \boxtimes M \longrightarrow [0,1]\boxtimes M$ satisfies the LLP with respect to p if and only if $\{0,1\}\boxtimes M \sqcup_{\{0,1\}\boxtimes M^0} [0,1]\boxtimes M^0 \longrightarrow [0,1]\boxtimes M$ satisfies the LLP with respect to p. The latter property holds if and only if $M^0 \longrightarrow M$ satisfies the LLP with respect to $\{[0,1], E\}_S \longrightarrow \{[0,1], D\}_S \times_{\{\{0,1\},D\}_S} \{\{0,1\}, E\}_S$. The latter morphism is a Hurewicz fibration by Lemma 5.9, and it is trivial since p is a S-homotopy equivalence. Hence the result by Lemma 5.10. □

Lemma 5.13. *For any morphism of flows $i : A \longrightarrow X$, the morphism of flows $i_1 : A \longrightarrow Mi$ is a Hurewicz cofibration of flows.*

Proof. Consider a commutative diagram of flows as follows:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i_1"'] & E \arrow[d, "p"] \\ Mi \arrow[r, "v"] & D \end{tikzcd}$$

where p is a trivial Hurewicz fibration of flows. One has the isomorphism of flows

$$Mi \cong \left(\left[0, \frac{1}{2}\right] \boxtimes A \sqcup_{\{0\}\boxtimes A} X\right) \sqcup_{\{\frac{1}{2}\}\boxtimes A} \left(\left[\frac{1}{2}, 1\right] \boxtimes A\right).$$

By Lemma 5.5, there exists $s : D \longrightarrow E$ such that $p \circ s = \mathrm{Id}_D$. Let β be the composite

$$\beta : [0, \tfrac{1}{2}] \boxtimes A \sqcup_{\{0\}\boxtimes A} X \longrightarrow Mi \longrightarrow D \xrightarrow{s} E.$$

Then consider the commutative diagram of flows:

$$\begin{tikzcd}
\left\{\frac{1}{2},1\right\} \boxtimes A \arrow[r, "(\beta_{\frac{1}{2}},u)"] \arrow[d] & E \arrow[d, "p"] \\
\left[\frac{1}{2},1\right] \boxtimes A \arrow[r, "v"] \arrow[ur, dashed, "k"] & D.
\end{tikzcd}$$

By Lemma 5.12, there exists k making both triangles commutative. Hence the result by pasting k and β. □

Lemma 5.14. *Let $i : A \longrightarrow X$ be a morphism of flows satisfying the LLP with respect to any Hurewicz fibration of flows. Then i is a S-homotopy equivalence.*

Proof. Since $A \longrightarrow \mathbf{1}$ is a Hurewicz fibration of flows by Lemma 5.6, there exists $r : X \longrightarrow A$ making commutative the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "i"'] & A \arrow[d] \\
X \arrow[r] \arrow[ur, "r"] & \mathbf{1}.
\end{tikzcd}$$

The canonical morphism of flows $\{[0,1],X\}_S \longrightarrow \{\{0,1\},X\}_S$ is a Hurewicz fibration of flows by Lemma 5.7. Then consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f''"] \arrow[d, "i"'] & \{[0,1],X\}_S \arrow[d] \\
X \arrow[r] \arrow[ur, dashed, "k"] & \{\{0,1\},X\}_S
\end{tikzcd}$$

with $f''(a)(t) = a$ and $f'(x) = (x, r(x))$. The morphism of flows k exists since i satisfies the LLP with respect to any Hurewicz fibration of flows. So i is a S-homotopy equivalence. □

Lemma 5.15. *(Cole) A morphism of flows $p : E \longrightarrow B$ is a Hurewicz fibration of flows if and only if there exists $\lambda : [0,1] \boxtimes Np \longrightarrow E$ making commutative the diagram*

$$\begin{tikzcd}
Np \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
{[0,1] \boxtimes Np} \arrow[r] \arrow[ur, dashed, "\lambda"] & D
\end{tikzcd}$$

where Np is defined as the pullback

$$\begin{tikzcd}
Np \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
{\{[0,1],D\}_S} \arrow[r] & D.
\end{tikzcd}$$

Proof. The necessity comes from the definition of a Hurewicz fibration. Let us suppose now that such a λ exists. The morphism of flows $\lambda : [0,1] \boxtimes Np \longrightarrow E$ gives rise to a morphism of flows $\widetilde{\lambda} : Np \longrightarrow \{[0,1],E\}_S$ by adjunction. It is easily seen that $\widetilde{\lambda}$ is a right inverse of the natural morphism of flows $\{[0,1],E\}_S \longrightarrow Np$ coming from the universal property satisfied by Np and from the commutative diagram

$$\begin{tikzcd} \{[0,1],E\}_S \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \{[0,1],D\}_S \arrow[r] & D. \end{tikzcd}$$

Let us consider a commutative diagram like

$$\begin{tikzcd} \{0\} \boxtimes M \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ {[0,1]} \boxtimes M \arrow[r] \arrow[ur, dashed, "k"] & D. \end{tikzcd}$$

Since $\{0\} \boxtimes M \cong \{0\} \boxtimes M \sqcup_{\{0\}\boxtimes\varnothing} ([0,1] \boxtimes \varnothing)$, finding k making the diagram above commutative is equivalent to finding $\overline{k}$ making the following diagram commutative:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & \{[0,1],E\}_S \arrow[d] \\ M \arrow[r] \arrow[ur, dashed, "\overline{k}"] & \{\{0\},E\}_S \times_{\{\{0\},D\}_S} \{[0,1],D\}_S = Np. \end{tikzcd}$$

Hence the result using $\widetilde{\lambda}$. $\square$

6. The Cole-Strøm model structure

Theorem 6.1. *Let $i : A \longrightarrow X$ be a trivial Hurewicz cofibration of flows. Then i satisfies the LLP with respect to any Hurewicz fibration of flows.*

Proof. Let $p : E \longrightarrow D$ be a Hurewicz fibration of flows. Then the trivial fibration of flows

$$q : \{[0,1],E\}_S \longrightarrow \{\{0\},E\}_S \times_{\{\{0\},D\}_S} \{[0,1],D\}_S$$

of Lemma 5.8 satisfies the RLP with respect to i. Or equivalently by adjunction, $k : Mi \longrightarrow [0,1] \boxtimes X$ satisfies the LLP with respect to p. By Lemma 5.11, i satisfies the S-homotopy extension property. Therefore by Lemma 5.4, there exists $r : X \to A$ such that $r \circ i = \mathrm{Id}_A$. So $i : A \longrightarrow X$ is a retract of the morphism of flows k. So i satisfies the LLP with respect to p. $\square$

The following lemmas recall some easy but important facts about pullbacks and colimits of flows:

Lemma 6.2. *Let*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ Y \arrow[r, "k"] & W \end{tikzcd}$$

be a pullback of flows. Then one has the pullback of sets

$$\begin{tikzcd} X^0 \arrow[r, "h^0"] \arrow[d, "f^0"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z^0 \arrow[d, "g^0"] \\ Y^0 \arrow[r, "k^0"] & W^0 \end{tikzcd}$$

and for any $(\alpha, \beta) \in X^0 \times X^0$ *the pullback of topological spaces*

$$\begin{tikzcd} \mathbb{P}_{\alpha,\beta}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}_{h(\alpha),h(\beta)}Z \arrow[d] \\ \mathbb{P}_{f(\alpha),f(\beta)}Y \arrow[r] & \mathbb{P}_{g(h(\alpha)),g(h(\beta))}W. \end{tikzcd}$$

Proof. Obvious. □

Lemma 6.3. *Let* $\zeta_n : Z_n \longrightarrow Z_{n+1}$ *be a morphism of flows for any* $n \geqslant 0$ *such that the continuous map* $\mathbb{P}Z_n \to \mathbb{P}Z_{n+1}$ *is a closed inclusion of topological spaces. Let* $\varinjlim Z_n$ *be the colimit of this* $\aleph_0$*-sequence. Then the natural continuous map* $\varinjlim \mathbb{P}Z_n \longrightarrow \mathbb{P}(\varinjlim Z_n)$ *is a homeomorphism and the natural set map* $\varinjlim Z_n^0 \longrightarrow (\varinjlim Z_n)^0$ *is a bijection.*

Proof. There exists a canonical continuous map

$$\varinjlim_n \mathbb{P}Z_n \longrightarrow \mathbb{P}(\varinjlim_n Z_n).$$

Let $\gamma \in \mathbb{P}(\varinjlim_n Z_n)$. Then $\gamma = \gamma_1 * \cdots * \gamma_p$ for $\gamma_1 \in Z_{n_1}$, $\gamma_2 \in Z_{n_2}$, ..., $\gamma_p \in Z_{n_p}$. So $\gamma \in Z_{\sup(n_1,\dots,n_p)}$. Therefore the continuous map $\mathbb{P}\zeta_n : \mathbb{P}Z_n \longrightarrow \mathbb{P}Z_{n+1}$ is actually a bijection. And since the inclusions $\mathbb{P}\zeta_n : \mathbb{P}Z_n \longrightarrow \mathbb{P}Z_{n+1}$ are closed, the colimit $\varinjlim_n \mathbb{P}Z_n$ in the category of compactly generated topological spaces coincides with the colimit in the category of k-spaces. Therefore $\varinjlim_n \mathbb{P}Z_n$ is equipped with the final topology and one has the homeomorphism $\varinjlim_n \mathbb{P}Z_n \cong \mathbb{P}(\varinjlim_n Z_n)$. □

Theorem 6.4. *(The Cole Hypothesis for* **Flow***) Suppose that for any* $n \geqslant 0$*, we have a cartesian diagram of flows*

$$\begin{tikzcd} X_n \arrow[r, "h_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z_n \arrow[d, "g_n"] \\ Y \arrow[r, "k"] & W \end{tikzcd}$$

and suppose that there are morphisms $\zeta_n : Z_n \longrightarrow Z_{n+1}$ *such that for all* n, $g_{n+1} \circ \zeta_n = g_n$*. If the morphisms* ζ_n *are trivial Hurewicz cofibrations of flows, then the commutative diagram*

$$\begin{tikzcd} \varinjlim_n X_n \arrow[r, "\varinjlim_n h_n"] \arrow[d, "f_n"'] & \varinjlim_n Z_n \arrow[d, "g_n"] \\ Y \arrow[r, "k"] & W \end{tikzcd}$$

is a pullback of flows.

Proof. The first point is that the Cole hypothesis holds for the category of compactly generated topological spaces, as already remarked by Cole himself in [Col99a]. Indeed in the category $k\mathbf{Top}$ of k-spaces, the functor $k\mathbf{Top}/W \longrightarrow k\mathbf{Top}/Y$ induced by the pullback has a right adjoint, and therefore it commutes with all colimits [BB78b] [BB78a] [Lew85]. So in the category of compactly generated topological spaces (i.e. the weak Hausdorff k-spaces), since the colimit $\varinjlim_n Z_n$ is the colimit of a $\aleph_0$-sequence of closed inclusions of topological spaces, the same property holds.

Let us come back now to the category of flows. The underlying topological space of a flow is compactly generated. Since the ζ_n are trivial Hurewicz cofibrations of flows, they induce bijections between the 0-skeletons of Z_n and Z_{n+1}. So $(\varinjlim_n Z_n)^0 = \varinjlim_n Z_n^0 =: Z^0$ by Lemma 6.3. Again since the ζ_n are trivial Hurewicz cofibrations of flows, then they satisfy the S-homotopy extension property by Lemma 5.11 and therefore they induce closed inclusions of topological spaces $\mathbb{P}\zeta_n : \mathbb{P}Z_n \longrightarrow \mathbb{P}Z_{n+1}$ ([Gau03] Theorem 10.6). So $\varinjlim_n \mathbb{P}Z_n$ equipped yields the homeomorphism $\varinjlim_n \mathbb{P}Z_n \cong \mathbb{P}(\varinjlim_n Z_n)$ by Lemma 6.3.

One has the pullbacks of sets

$$\begin{tikzcd} X_n^0 \arrow[r, "h_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z^0 \arrow[d, "g_n"] \\ Y^0 \arrow[r, "k"] & W^0 \end{tikzcd}$$

by Lemma 6.2 and therefore $X_n^0 = (\varinjlim_n X_n)^0 = Y^0 \times_{W^0} Z^0$. And one has the pullbacks of topological spaces

$$\begin{tikzcd} \mathbb{P}X_n \arrow[r, "h_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P}Z_n \arrow[d, "g_n"] \\ \mathbb{P}Y \arrow[r, "k"] & \mathbb{P}W \end{tikzcd}$$

again by Lemma 6.2 so the canonical continuous maps $\mathbb{P}X_n \longrightarrow \mathbb{P}X_{n+1}$ are closed inclusion of topological spaces for any $n \geqslant 0$ since $\mathbb{P}W$ is compactly generated [Lew78]. Therefore one has the homeomorphism $\varinjlim_n \mathbb{P}X_n \cong \mathbb{P}(\varinjlim_n X_n)$. Since the Cole hypothesis holds for compactly generated topological spaces, one obtains the pullback

$$\begin{tikzcd} \varinjlim_n \mathbb{P}X_n \arrow[r, "\varinjlim_n h_n"] \arrow[d, "f_n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \varinjlim_n \mathbb{P}Z_n \arrow[d, "g_n"] \\ \mathbb{P}Y \arrow[r, "k"] & \mathbb{P}W. \end{tikzcd}$$

Hence the result. ☐

Theorem 6.5. *Every morphism of flows can be factored as $p \circ i$ where i is a trivial Hurewicz cofibration of flows and where p is a Hurewicz fibration of flows.*

Proof. We follow Cole's proof [Col99a]. Let $f : X \longrightarrow Y$ be a morphism of flows. Let $Z_0 = X$. Suppose $Z_0 \to \cdots \to Z_n$ constructed for $n \geqslant 0$ where the morphisms of flows $\gamma_i : Z_i \longrightarrow Z_{i+1}$ are trivial Hurewicz cofibrations of flows and such that there exist morphisms of flows $h_p : Z_p \longrightarrow Y$ for any $p \leqslant n$ such that $h_{p+1} \circ \gamma_p = h_p$ for any $p \leqslant n-1$. Let us consider the cocartesian diagram

$$\begin{tikzcd}
Nh_n \arrow[r] \arrow[d, "i_0"'] & Z_n \arrow[d, "\gamma_n"] \arrow[ddr, bend left] & \\
{[0,1] \boxtimes Nh_n} \arrow[r] \arrow[ddr, phantom, ""] \arrow[rrd, bend right] & Z_{n+1} \arrow[dr, dashed, "h_{n+1}"] & \\
 & & Y.
\end{tikzcd}$$

It defines Z_{n+1}, γ_n and h_{n+1}. It remains to prove that γ_n is a trivial Hurewicz cofibration of flows and that $\varinjlim_n h_n : \varinjlim_n Z_n \longrightarrow Y$ is a Hurewicz fibration of flows.

Since $\gamma_n : Z_n \longrightarrow Z_{n+1}$ is a pushout of $i_0 : Nh_n \longrightarrow [0,1] \boxtimes Nh_n$, then γ_n satisfies the LLP with respect to any Hurewicz fibration of flows. So the composition of the $\aleph_0$-sequence of γ_n satisfies the LLP with respect to any Hurewicz fibration of flows since the class of morphisms satisfying the LLP with respect to any Hurewicz fibration is closed under transfinite composition. In particular, this composition satisfies the LLP with respect to the trivial ones. So the composition is by definition a Hurewicz cofibration of flows. And it is also a S-homotopy equivalence by Lemma 5.14.

For any n, one has the pullback of flows

$$\begin{tikzcd}
Nh_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z_n \arrow[d, "h_n"] \\
\{[0,1], Y\}_S \arrow[r] & Y.
\end{tikzcd}$$

Thanks to Theorem 6.4, one obtains the pullback of flows

$$\begin{tikzcd}
\varinjlim_n Nh_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varinjlim_n Z_n \arrow[d, "\varinjlim_n h_n"] \\
\{[0,1], Y\}_S \arrow[r] & Y.
\end{tikzcd}$$

Therefore $N(h) = \varinjlim_n Nh_n$ with $h = \varinjlim_n h_n$. And $[0,1] \boxtimes Nh \cong \varinjlim_n ([0,1] \boxtimes Nh_n)$ since the functor $[0,1] \boxtimes -$ preserves colimits. By Lemma 5.15, h is a Hurewicz fibration of flows if and only if one can find λ making the following diagram commutative:

$$\begin{tikzcd}
\varinjlim_n Nh_n \arrow[r] \arrow[d, "i_0"'] & \varinjlim_n Z_n \arrow[d, "h"] \\
\varinjlim_n ([0,1] \boxtimes Nh_n) \arrow[r] \arrow[ur, dashed, "\lambda"] & Y.
\end{tikzcd}$$

It is clear that λ can be given as the colimit of the compositions

$$[0,1] \boxtimes Nh_n \longrightarrow Z_{n+1} \longrightarrow Z.$$

$\square$

Theorem 6.6. *Every morphism of flows can be factored as $p \circ i$ where i is a Hurewicz cofibration of flows and where p is a trivial Hurewicz fibration of flows.*

Proof. Let $f : X \longrightarrow Y$ be a morphism of flows. Then f factors as

$$f : X \xrightarrow{i_1} Mf \longrightarrow Y.$$

The first arrow i_1 is a Hurewicz cofibration of flows by Lemma 5.13. Then apply Theorem 6.5 to the canonical morphism of flows $Mf \longrightarrow Y$:

$$Mf \xrightarrow{j} Z \xrightarrow{q} Y$$

where j is a trivial Hurewicz cofibration and q a Hurewicz fibration of flows. Since $q \circ j$ is a S-homotopy equivalence, then q is a S-homotopy equivalence as well. Hence the result. □

7. The end of the construction

For any category $\mathcal{C}$, $\mathrm{Map}(\mathcal{C})$ denotes the class of morphisms of $\mathcal{C}$. In a category $\mathcal{C}$, an object x is *a retract* of an object y if there exist $f : x \longrightarrow y$ and $g : y \longrightarrow x$ of $\mathcal{C}$ such that $g \circ f = \mathrm{Id}_x$. A *functorial factorization* (α, β) of $\mathcal{C}$ is a pair of functors from $\mathrm{Map}(\mathcal{C})$ to $\mathrm{Map}(\mathcal{C})$ such that for any f object of $\mathrm{Map}(\mathcal{C})$, $f = \beta(f) \circ \alpha(f)$.

Definition 7.1. *Let $\mathcal{C}$ be a category. A* weak factorization system *is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that the class $\mathcal{L}$ is the class of morphisms having the LLP with respect to $\mathcal{R}$, such that the class $\mathcal{R}$ is the class of morphisms having the RLP with respect to $\mathcal{L}$ and such that any morphism of $\mathcal{C}$ factors as a composite $r \circ \ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. The weak factorization system is* functorial *if the factorization $r \circ \ell$ is a functorial factorization.*

In a weak factorization system $(\mathcal{L}, \mathcal{R})$, the class $\mathcal{L}$ (resp. $\mathcal{R}$) is completely determined by $\mathcal{R}$ (resp. $\mathcal{L}$).

Definition 7.2. [Hov99] *A* model category *is a complete and cocomplete category equipped with three classes of morphisms* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *(resp. called the classes of cofibrations, fibrations and weak equivalences) such that:*

(1) *the class of morphisms $\mathcal{W}$ is closed under retracts and satisfies the two-out-of-three axiom i.e.: if f and g are morphisms of $\mathcal{C}$ such that $g \circ f$ is defined and two of f, g and $g \circ f$ are weak equivalences, then so is the third.*
(2) *the pairs* $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ *and* $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ *are both functorial weak factorization systems.*

The triple $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *is called a* model structure*. An element of* $\mathrm{Cof} \cap \mathcal{W}$ *is called a* trivial cofibration*. An element of* $\mathrm{Fib} \cap \mathcal{W}$ *is called a* trivial fibration.

Theorem 7.3. *There exists a model structure on the category of flows where the cofibrations (resp. fibrations, weak equivalences) are the Hurewicz cofibrations (resp. the Hurewicz fibrations, the S-homotopy equivalences). For this model structure, any flow is fibrant and cofibrant and any Hurewicz cofibration satisfies the S-homotopy extension property. The morphism of flows $R : \{0, 1\} \longrightarrow \{0\}$ is a cofibration. and therefore this model structure is not cellular.*

Proof. We already know that the pairs trivial Hurewicz cofibrations/Hurewicz fibrations and the pair Hurewicz cofibrations/trivial Hurewicz fibrations are weak factorization systems. It remains to verify that these weak factorization systems are both functorial. The functoriality of the first pair is a consequence of the small object argument presented in the proof of Theorem 6.5. The functoriality of the second pair is a consequence of the functoriality of $f \mapsto Mf$. Any trivial Hurewicz fibration is by definition a S-homotopy equivalence. Therefore, any such morphism induces a bijection between the 0-skeletons. Therefore it satisfies the RLP with respect to $R : \{0, 1\} \longrightarrow \{0\}$. So $R : \{0, 1\} \longrightarrow \{0\}$ is a cofibration. □

Definition 7.4. *We call this model structure the Cole-Strøm model structure.*

It is probably not cofibrantly generated because if it was, then the Strøm model structure would be cofibrantly generated as well.

It is still an open question to know whether the class of Hurewicz cofibrations is the whole class of morphisms of flows satisfying the S-homotopy extension property, or only a proper subclass. At last, using the:

Theorem 7.5. *(Cole)* [Col99b] *If* $(\mathrm{Cof}_i \cap \mathcal{W}_i, \mathrm{Fib}_i)$ *for* $i = 1, 2$ *are two model structures on the same category* $\mathcal{A}$ *(where* $\mathcal{W}_i$ *is the class of weak equivalences, where* Cof_i *is the class of cofibrations, and if* Fib_i *is a class of fibrations) such that* $\mathcal{W}_1 \subset \mathcal{W}_2$ *and* $\mathrm{Fib}_1 \subset \mathrm{Fib}_2$*, then there exists a model structure* $(\mathrm{Cof}_m \cap \mathcal{W}_m, \mathrm{Fib}_m)$ *such that* $\mathcal{W}_m = \mathcal{W}_2$ *and* $\mathrm{Fib}_m = \mathrm{Fib}_1$*.*

one deduces the corollary:

Corollary 7.6. *There exists a model structure on the category of flows such that the class of weak equivalences is the class of weak S-homotopy equivalences and such that the class of fibrations is the class of Hurewicz fibrations of flows.*

8. Preservation of the branching and merging homologies

Several proofs as the ones of Lemma 8.8, Proposition 8.9 and Corollary 8.10 are already present in the fourth paper [Gau05c] of this series. They are repeated here to keep the paper self-contained and understandable.

Most of the work to prove the preservation of the branching and merging homology theories by generalized T-homotopy equivalences is already done in the third paper [Gau05b] of this series of paper. It only remains to prove the following crucial fact:

Theorem 8.1. *Let* $\overrightarrow{D}$ *be a full directed ball with initial state* $\widehat{0}$ *and final state* $\widehat{1}$*. Then the topological space* $\mathrm{ho}\mathbb{P}^-_{\widehat{0}}\overrightarrow{D}$ *is contractible.*

After recalling the meaning of each term of the statement above, it will be proved. The notation Q will mean the cofibrant replacement functor of the weak S-homotopy model structure of **Flow**.

Proposition 8.2. *(*[Gau05a] *Proposition 3.1) Let* X *be a flow. There exists a topological space* $\mathbb{P}^-X$ *unique up to homeomorphism and a continuous map* $h^- : \mathbb{P}X \longrightarrow \mathbb{P}^-X$ *satisfying the following universal property:*

(1) *For any* x *and* y *in* $\mathbb{P}X$ *such that* $t(x) = s(y)$*, the equality* $h^-(x) = h^-(x * y)$ *holds.*

(2) *Let $\phi : \mathbb{P}X \longrightarrow Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(x) = \phi(x * y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^{-}X \longrightarrow Y$ such that $\phi = \overline{\phi} \circ h^{-}$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^{-}X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^{-}_{\alpha}X$$

where $\mathbb{P}^{-}_{\alpha}X := h^{-}\left(\bigsqcup_{\beta\in X^0} \mathbb{P}^{-}_{\alpha,\beta}X\right)$. The mapping $X \mapsto \mathbb{P}^{-}X$ yields a functor $\mathbb{P}^{-}$ from **Flow** *to* **Top**.

Definition 8.3. *Let X be a flow. The topological space $\mathbb{P}^{-}X$ is called the* branching space *of the flow X. The functor $\mathbb{P}^{-}$ is called the* branching space functor*.*

Definition 8.4. *The* homotopy branching space $\mathrm{ho}\mathbb{P}^{-}X$ *of a flow X is by definition the topological space $\mathbb{P}^{-}Q(X)$, where Q is the cofibrant replacement functor of the weak S-homotopy model structure of* **Flow**. *If $\alpha \in X^0$, let $\mathrm{ho}\mathbb{P}^{-}_{\alpha}X = \mathbb{P}^{-}_{\alpha}Q(X)$.*

Definition 8.5. *A flow X is* loopless *if for any $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.*

A flow X is loopless if and only if the transitive closure of the set $\{(\alpha, \beta) \in X^0 \times X^0$ such that $\mathbb{P}_{\alpha,\beta}X \neq \varnothing\}$ induces a partial ordering on X^0.

Definition 8.6. *A* full directed ball *is a flow $\overrightarrow{D}$ such that:*

- *the 0-skeleton $\overrightarrow{D}^0$ is finite*
- *$\overrightarrow{D}$ has exactly one initial state $\widehat{0}$ and one final state $\widehat{1}$ with $\widehat{0} \neq \widehat{1}$*
- *each state α of $\overrightarrow{D}^0$ is between $\widehat{0}$ and $\widehat{1}$, that is there exists an execution path from $\widehat{0}$ to α, and another execution path from α to $\widehat{1}$*
- *$\overrightarrow{D}$ is loopless*
- *for any $(\alpha, \beta) \in \overrightarrow{D}^0 \times \overrightarrow{D}^0$, the topological space $\mathbb{P}_{\alpha,\beta}\overrightarrow{D}$ is empty or weakly contractible.*

Let $\overrightarrow{D}$ be a full directed ball. Then the set $\overrightarrow{D}^0$ can be viewed as a finite bounded poset. Conversely, if P is a finite bounded poset, let us consider the *flow $F(P)$ associated to P*: it is of course defined as the unique flow $F(P)$ such that $F(P)^0 = P$ and $\mathbb{P}_{\alpha,\beta}F(P) = \{u_{\alpha,\beta}\}$ if $\alpha < \beta$ and $\mathbb{P}_{\alpha,\beta}F(P) = \varnothing$ otherwise. Then $F(P)$ is a full directed ball and for any full directed ball $\overrightarrow{D}$, the composite morphism $Q(\overrightarrow{D}) \longrightarrow \overrightarrow{D} \longrightarrow F(\overrightarrow{D}^0)$ is a weak S-homotopy equivalence.

The problem already mentioned in [Gau05b] to prove Theorem 8.1 is that the flow $F(P)$ is in general not cofibrant for the weak S-homotopy model structure of **Flow** constructed in [Gau03]. For instance, the flow $F(P)$ associated with the poset of Figure 1 is not cofibrant because the composition law contains relations, for instance $u_{\widehat{0},A} * u_{A,\widehat{1}} = u_{\widehat{0},C} * u_{C,\widehat{1}}$.

If $F(P)$ was cofibrant, the calculation of $\mathrm{ho}\mathbb{P}^{-}_{\widehat{0}}F(P) \simeq \mathbb{P}^{-}_{\widehat{0}}F(P)$ would be trivial. So the principle of the proof consists of using a model structure on **Flow** such that any flow is cofibrant. This is the case for the Cole-Strøm model structure constructed in this paper.

Let P be a finite bounded poset with lower element $\widehat{0}$ and with top element $\widehat{1}$. Let us denote by $\Delta^{ext}(P)$ the full subcategory of $\Delta(P)$ consisting of the simplices $(\alpha_0, \ldots, \alpha_p)$

$$\begin{tikzcd}
& A \arrow[r, "\leqslant"] & B \arrow[dr, "\leqslant"] & \\
\widehat{0} \arrow[ur, "\leqslant"] \arrow[dr, "\leqslant"'] & & & \widehat{1} \\
& C \arrow[urr, "\leqslant"'] & &
\end{tikzcd}$$

FIGURE 1. Example of finite bounded poset

such that $\widehat{0} = \alpha_0$ and $\widehat{1} = \alpha_p$. If $P = \{\widehat{0} < A < B < \widehat{1}, \widehat{0} < C < \widehat{1}\}$ is the poset of Figure 1, the small category $\Delta^{ext}(P)^{op}$ looks as follows:

$$\begin{tikzcd}
(\widehat{0}, A, B, \widehat{1}) \arrow[d] \arrow[dr] & & (\widehat{0}, C, \widehat{1}) \arrow[ddl, bend left] \\
(\widehat{0}, A, \widehat{1}) \arrow[dr] & (\widehat{0}, B, \widehat{1}) \arrow[d] & \\
& (\widehat{0}, \widehat{1}) &
\end{tikzcd}$$

The simplex $(\widehat{0}, \widehat{1})$ is always a terminal object of $\Delta^{ext}(P)^{op}$.

Notation 8.7. *Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let (α, β) be a 1-simplex of $\Delta(X^0)$. We denote by $\ell(\alpha, \beta)$ the maximum of the set of integers*

$$\big\{p \geqslant 1, \exists (\alpha_0, \dots, \alpha_p) \text{ } p\text{-simplex of } \Delta(X^0) \text{ s.t. } (\alpha_0, \alpha_p) = (\alpha, \beta)\big\}$$

One always has $1 \leqslant \ell(\alpha, \beta) \leqslant \operatorname{card}(]\alpha, \beta])$.

Lemma 8.8. *Let X be a loopless flow such that $(X^0, \leqslant)$ is locally finite. Let (α, β, γ) be a 2-simplex of $\Delta(X^0)$. Then one has*

$$\ell(\alpha, \beta) + \ell(\beta, \gamma) \leqslant \ell(\alpha, \gamma).$$

Proof. Let $\alpha = \alpha_0 < \dots < \alpha_{\ell(\alpha,\beta)} = \beta$. Let $\beta = \beta_0 < \dots < \beta_{\ell(\beta,\gamma)} = \gamma$. Then

$$(\alpha_0, \dots, \alpha_{\ell(\alpha,\beta)}, \beta_1, \dots, \beta_{\ell(\beta,\gamma)})$$

is a simplex of $\Delta(X^0)$ with $\alpha = \alpha_0$ and $\beta_{\ell(\beta,\gamma)} = \gamma$. So $\ell(\alpha, \beta) + \ell(\beta, \gamma) \leqslant \ell(\alpha, \gamma)$. $\square$

Proposition 8.9. *Let P be a finite bounded poset. Let*

$$d(\alpha_0, \dots, \alpha_p) = \ell(\alpha_0, \alpha_1)^2 + \dots + \ell(\alpha_{p-1}, \alpha_p)^2$$

where ℓ is the function of Notation 8.7. Then d yields a functor $\Delta^{ext}(P)^{op} \longrightarrow \mathbb{N}$ making $\Delta^{ext}(P)^{op}$ a direct category, that is a Reedy category with $\Delta^{ext}(P)^{op}_+ := \Delta^{ext}(P)^{op}$ and $\Delta^{ext}(P)^{op}_- = \varnothing$.

Proof. Let $\partial_i : (\alpha_0, \dots, \alpha_p) \longrightarrow (\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p)$ be a morphism of $\Delta^{ext}(P)^{op}$ with $p \geqslant 2$ and $0 < i < p$. Then

$$d(\alpha_0, \dots, \alpha_p) = \ell(\alpha_0, \alpha_1)^2 + \dots + \ell(\alpha_{p-1}, \alpha_p)^2$$
$$d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) = \ell(\alpha_0, \alpha_1)^2 + \dots + \ell(\alpha_{i-1}, \alpha_{i+1})^2 + \dots + \ell(\alpha_{p-1}, \alpha_p)^2.$$

So one obtains

$$d(\alpha_0, \dots, \alpha_p) - d(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p) = \ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 - \ell(\alpha_{i-1}, \alpha_{i+1})^2.$$

By Lemma 8.8, one has

$$(\ell(\alpha_{i-1}, \alpha_i) + \ell(\alpha_i, \alpha_{i+1}))^2 \leqslant \ell(\alpha_{i-1}, \alpha_{i+1})^2$$

and one has

$$\ell(\alpha_{i-1}, \alpha_i)^2 + \ell(\alpha_i, \alpha_{i+1})^2 < (\ell(\alpha_{i-1}, \alpha_i) + \ell(\alpha_i, \alpha_{i+1}))^2$$

since $2\ell(\alpha_{i-1}, \alpha_i)\ell(\alpha_i, \alpha_{i+1}) \geqslant 2$. So any morphism of $\Delta^{ext}(P)^{op}$ raises the degree. $\square$

Corollary 8.10. *Let P be a finite bounded poset. Then the colimit functor*

$$\varinjlim : \mathbf{Flow}^{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}} \longrightarrow \mathbf{Flow}$$

is a left Quillen functor if the category of diagrams $\mathbf{Flow}^{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}}$ is equipped with the Reedy model structure and if **Flow** *is equipped with the Cole-Strøm model structure.*

Of course, the corollary above is also true if **Flow** is equipped with the weak S-homotopy model structure. It is stated like this because it will be used with the category of flows equipped with the Cole-Strøm model structure.

The fact that the colimit functor is a left Quillen functor will be indeed applied for $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$. Recall that the pair $(\widehat{0},\widehat{1})$ is a terminal object of $\Delta^{ext}(P)^{op}$. Therefore it is not very interesting to calculate a colimit of diagram of flows over the whole category $\Delta^{ext}(P)^{op}$. Notice also that one has the isomorphism of small categories

$$\boxed{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\} \cong \partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}_+ {\downarrow} (\widehat{0},\widehat{1}))}.$$

Proof. The Reedy structure on $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$ provides a model structure on the category $\mathbf{Flow}^{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}}$ of diagrams of flowss over $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$ such that a morphism of diagram $f : D \longrightarrow E$ is

(1) a weak equivalence if and only if for any object $\underline{\alpha}$ of $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}}$ is a S-homotopy equivalence of **Flow** (we will use the term objectwise S-homotopy equivalence to describe this situation)
(2) a cofibration if and only if for any object $\underline{\alpha}$ of $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$, the morphism $D_{\underline{\alpha}} \sqcup_{L_{\underline{\alpha}}D} L_{\underline{\alpha}}E \longrightarrow E_{\underline{\alpha}}$ is a Hurewicz cofibration of **Flow**
(3) a fibration if and only if for any object $\underline{\alpha}$ of $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$, the morphism $D_{\underline{\alpha}} \longrightarrow E_{\underline{\alpha}} \times_{M_{\underline{\alpha}}E} M_{\underline{\alpha}}D$ is a Hurewicz fibration of **Flow**.

For any object $\underline{\alpha}$ of $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$, the matching category $\partial(\underline{\alpha} {\downarrow} \Delta^{ext}(P)^{op}_-)$ is empty. So for any object A of $\mathbf{Flow}^{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}}$ and any object $\underline{\alpha}$ of $\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}$, one has $M_{\underline{\alpha}}A \cong \mathbf{1}$. So a Reedy fibration is an objectwise Hurewicz fibration. Therefore the diagonal functor Diag of the adjunction $\varinjlim : \mathbf{Flow}^{\Delta^{ext}(P)^{op}\backslash\{(\widehat{0},\widehat{1})\}} \leftrightarrows \mathbf{Flow} : \mathrm{Diag}$ is a right Quillen functor. $\square$

Definition 8.11. *Let X be a flow. Let A be a subset of X^0. Then the restriction $X\upharpoonright_A$ of X over A is the unique flow such that $(X\upharpoonright_A)^0 = A$, such that $\mathbb{P}_{\alpha,\beta}(X\upharpoonright_A) = \mathbb{P}_{\alpha,\beta}X$ for any $(\alpha,\beta) \in A \times A$ and such that the inclusions $A \subset X^0$ and $\mathbb{P}(X\upharpoonright_A) \subset \mathbb{P}X$ induces a morphism of flows $X\upharpoonright_A \longrightarrow X$.*

Proposition 8.12. *Let $\overrightarrow{D}$ be a full directed ball. Let (α,β) be a simplex of $\Delta(\overrightarrow{D}^0)$. Then $\overrightarrow{D}\upharpoonright_{[\alpha,\beta]}$ is a full directed ball.*

Proof. Obvious. $\square$

Notation 8.13. *Let $\overrightarrow{D}$ and $\overrightarrow{D'}$ be two full directed balls. Then the flow $\overrightarrow{D} * \overrightarrow{D'}$ is obtained by identifying the final state $\widehat{1}$ of $\overrightarrow{D}$ with the initial state $\widehat{0}$ of $\overrightarrow{D'}$.*

Proposition 8.14. *Let $\overrightarrow{D}$ be a full directed ball. There exists one and only one functor*

$$\mathcal{G}\overrightarrow{D} : \Delta^{ext}(\overrightarrow{D}^0)^{op} \longrightarrow \mathbf{Flow}$$

satisfying the following conditions:

(1) *for any object $(\alpha_0,\dots,\alpha_p)$ of $\Delta^{ext}(\overrightarrow{D}^0)^{op}$, let*

$$\mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\alpha_p) = \overrightarrow{D}\upharpoonright_{[\alpha_0,\alpha_1]} * \dots * \overrightarrow{D}\upharpoonright_{[\alpha_{p-1},\alpha_p]}$$

(2) *the unique morphism $\mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\alpha_p) \longrightarrow \mathcal{G}\overrightarrow{D}(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_p)$ for $0 < i < p$ is induced by the composition law $\overrightarrow{D}\upharpoonright_{[\alpha_{i-1},\alpha_i]} * \overrightarrow{D}\upharpoonright_{[\alpha_i,\alpha_{i+1}]} \longrightarrow \overrightarrow{D}\upharpoonright_{[\alpha_{i-1},\alpha_{i+1}]}$.*

Notice that $\overrightarrow{D}\upharpoonright_{[\widehat{0},\widehat{1}]} = \overrightarrow{D}$.

Proof. This comes from the associativity of the composition law of a flow. $\square$

Proposition 8.15. *If $f : X \longrightarrow Y$ is a relative I^{gl}_+-cell subcomplexes and if f induces a bijection $X^0 \cong Y^0$, then $f : X \longrightarrow Y$ is a relative I^{gl}-cell subcomplexes.*

Proof. Any pushout of R appearing in the presentation of f as a relative I^{gl}_+-cell complex is trivial. And C cannot appear in the presentation since $X^0 \cong Y^0$. $\square$

Theorem 8.16. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of flows $\mathcal{G}Q(\overrightarrow{D})$ (where Q is the cofibrant replacement functor of the weak S-homotopy model structure of* **Flow***) is Reedy cofibrant if* **Flow** *is equipped with the Cole-Strøm model structure.*

Proof. It is already proved in [Gau05c] that the diagram of flows $\mathcal{G}Q(\overrightarrow{D})$ is Reedy cofibrant if **Flow** is equipped with the weak S-homotopy model structure. We repeat the proof here because it is not too long.

The flow $Q(\overrightarrow{D})$ is an object of $\mathbf{cell}(I^{gl}_+)$. Therefore, the canonical morphism of flows $\overrightarrow{D}^0 \longrightarrow Q(\overrightarrow{D})$ is a transfinite composition of pushouts of elements of I^{gl} by Proposition 8.15. So there exists an ordinal λ and a λ-sequence $M : \lambda \longrightarrow \mathbf{Flow}$ such that $M_0 = \overrightarrow{D}^0$, $M_\lambda = Q(\overrightarrow{D})$ and for any $\mu < \lambda$, the morphism of flows $M_\mu \longrightarrow M_{\mu+1}$ is a pushout of the inclusion of flows $e_\mu : \mathrm{Glob}(\mathbf{S}^{n_\mu-1}) \subset \mathrm{Glob}(\mathbf{D}^{n_\mu})$ for some $n_\mu \geqslant 0$, that is one has the pushout diagram of flows:

$$\begin{tikzcd} \mathrm{Glob}(\mathbf{S}^{n_\mu-1}) \arrow[r, "\phi_\mu"] \arrow[d] & M_\mu \arrow[d] \\ \mathrm{Glob}(\mathbf{D}^{n_\mu}) \arrow[r] & M_{\mu+1}. \end{tikzcd}$$

Let $(\alpha_0,\dots,\alpha_p)$ be a simplex of $\Delta^{ext}(\overrightarrow{D}^0)^{op}$. For a given ordinal $\mu < \lambda$, if the segment $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})]$ of $\overrightarrow{D}^0$ is not included in none of the segments $[\alpha_i, \alpha_{i+1}]$ with $0 \leqslant i < p$, then one has the homeomorphism $\mathbb{P}_{\alpha,\beta} M_\mu \cong \mathbb{P}_{\alpha,\beta} M_{\mu+1}$ for any $(\alpha,\beta) \in \overrightarrow{D}^0 \times \overrightarrow{D}^0$ such that $\{\alpha,\beta\} \subset [\alpha_0,\alpha_1] \cup \dots \cup [\alpha_{p-1},\alpha_p]$. So the canonical morphism of flows $\overrightarrow{D}^0 \longrightarrow \mathcal{G}\overrightarrow{D}_{(\alpha_0,\dots,\alpha_p)}$ is the transfinite composition of the inclusion of flows e_μ such that $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})] \subset [\alpha_0,\alpha_1] \cup \dots \cup [\alpha_{p-1},\alpha_p]$. In other terms, the relative I^{gl}-cell complex $\overrightarrow{D}^0 \longrightarrow \mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$ is a relative I^{gl}-cell subcomplex which is the union of the cells e_μ such that $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})] \subset [\alpha_0,\alpha_1] \cup \dots \cup [\alpha_{p-1},\alpha_p]$ [4]. We also deduce that any morphism of the diagram $\mathcal{G}Q(\overrightarrow{D})$ is an inclusion of relative I^{gl}-cell subcomplexes. So the canonical morphism of flows $L_{(\alpha_0,\dots,\alpha_p)}\mathcal{G}Q(\overrightarrow{D}) \longrightarrow \mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$ is an inclusion of relative I^{gl}-cell subcomplexes as well. More precisely, the transfinite composition of the inclusion of flows $\mathrm{Glob}(\mathbf{S}^{n_\mu-1}) \subset \mathrm{Glob}(\mathbf{D}^{n_\mu})$ such that $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})] \subset [\alpha_0,\alpha_1] \cup \dots \cup [\alpha_{p-1},\alpha_p]$ and such that there exists a state α such that $[\phi_\mu(\widehat{0}), \phi_\mu(\widehat{1})] \subset [\alpha_0,\alpha_1] \cup \dots \cup [\alpha_i,\alpha] \cup [\alpha,\alpha_{i+1}] \cup \dots \cup [\alpha_{p-1},\alpha_p]$ and $\alpha_i < \alpha < \alpha_{i+1}$.

Now we can end up the proof of the statement. Since any cofibration of the weak S-homotopy model structure is a cofibration for the Cole-Strøm model structure, the diagram $\mathcal{G}Q(\overrightarrow{D})$ is Reedy cofibrant if **Flow** is equipped with the Cole-Strøm model structure. □

Theorem 8.17. *Let $\overrightarrow{D}$ be a full directed ball. Then the diagram of flows $\mathcal{G}F(\overrightarrow{D}^0)$ is Reedy cofibrant if* **Flow** *is equipped with the Cole-Strøm model structure.*

Notice that in general, the diagram of flows $\mathcal{G}F(\overrightarrow{D}^0)$ is *not* Reedy cofibrant if **Flow** is equipped with the weak S-homotopy model structure.

Proof. Any flow is cofibrant for the Cole-Strøm model structure. Let $\alpha < \beta < \gamma$ be three elements of $\overrightarrow{D}^0$. Then one has the bijection of sets

$$\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right)^0 \cong \left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\gamma]}\right)^0$$

and for any $x, y \in [\alpha,\gamma]$, one has (the four cases below are not mutually exclusive):

(1) $\mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right) \cong \mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]}\right)$ if $x < y \leqslant \beta$

(2) $\mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right) \cong \mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right)$ if $\beta \leqslant x < y$

[4] A I^{gl}-cell subcomplex is characterized by its cells since any morphism of I^{gl} is an effective monomorphism of flows by [Gau03] Theorem 10.6.

(3) $\mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right) \cong \mathbb{P}_{x,\beta}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]}\right) \times \mathbb{P}_{\beta,y}\left(F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right)$ if $x \leqslant \beta \leqslant y$

(4) $\varnothing$ otherwise.

So the space $\mathbb{P}_{x,y}\left(F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]}\right)$ is a singleton if $x < y$, and the empty space otherwise. Therefore one has the isomorphism of flows

$$F(\overrightarrow{D}^0)\restriction_{[\alpha,\beta]} * F(\overrightarrow{D}^0)\restriction_{[\beta,\gamma]} \cong F(\overrightarrow{D}^0)\restriction_{[\alpha,\gamma]} .$$

One obtains for any simplex $\underline{\alpha}$ of $\Delta^{ext}(\overrightarrow{D}^0)^{op}$ the isomorphism $\mathcal{G}F(\overrightarrow{D}^0)_{\underline{\alpha}} \cong F(\overrightarrow{D}^0)_{(\widehat{0},\widehat{1})}$. In other terms, the diagram of flows $\mathcal{G}F(\overrightarrow{D}^0)$ is the constant diagram associated to $F(\overrightarrow{D}^0)_{(\widehat{0},\widehat{1})}$. Therefore for any simplex $\underline{\alpha}$ of $\Delta^{ext}(\overrightarrow{D}^0)^{op}$, the canonical morphism of flows

$$L_{\underline{\alpha}}\mathcal{G}F(\overrightarrow{D}^0) \longrightarrow \mathcal{G}F(\overrightarrow{D}^0)_{\underline{\alpha}}$$

is an isomorphism, and also a cofibration for the Cole-Strøm model structure. □

Theorem 8.18. *Let $\overrightarrow{D}$ be a full directed ball. Then the unique morphism of flows $Q(\overrightarrow{D}) \longrightarrow F(\overrightarrow{D}^0)$ is a S-homotopy equivalence.*

Let us repeat that we already know that this morphism is a weak S-homotopy equivalence.

Proof. We are going to prove this result by induction on the cardinal $\mathrm{card}(\overrightarrow{D}^0)$ of the 0-skeleton of $\overrightarrow{D}$.

If $\overrightarrow{D}^0 = \{\widehat{0},\widehat{1}\}$, let $\overrightarrow{D} = \mathrm{Glob}(Z)$ where Z is a weakly contractible topological space. Then $Q(\overrightarrow{D}) = \mathrm{Glob}(Z^{cof})$ where $Z^{cof} \longrightarrow Z$ is a cofibrant replacement for the Quillen model structure of **Top**. And $F(\overrightarrow{D}^0) = \overrightarrow{I}$. The canonical continuous map $Z^{cof} \longrightarrow \{0\}$ is a homotopy equivalence since Z^{cof} and $\{0\}$ are both cofibrant for the Quillen model structure of **Top**. This homotopy equivalence yields a S-homotopy equivalence from $Q(\overrightarrow{D}) = \mathrm{Glob}(Z^{cof})$ to $F(\overrightarrow{D}^0) = \overrightarrow{I}$.

Now suppose that $\overrightarrow{D}^0\backslash\{\widehat{0},\widehat{1}\} \neq \varnothing$. Let us suppose that the theorem is proved for any full directed ball whose 0-skeleton has a cardinal strictly lower than the cardinal of the 0-skeleton of $\overrightarrow{D}$. Let us consider the following diagram of flows:

$$\begin{tikzcd}
L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) \arrow[r] \arrow[d] & L_{(\widehat{0},\widehat{1})}\mathcal{G}F(\overrightarrow{D}^0) \cong F(\overrightarrow{D}^0) \\
Q(\overrightarrow{D}). &
\end{tikzcd}$$

The diagram $\mathcal{G}Q(\overrightarrow{D})$ and $\mathcal{G}F(\overrightarrow{D}^0)$ are both Reedy cofibrant for the Cole-Strøm model structure by Theorem 8.16 and Theorem 8.17. So their restriction to the full subcategory $\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}_{+}\downarrow(\widehat{0},\widehat{1})) \cong \Delta^{ext}(\overrightarrow{D}^0)^{op}\backslash\{(\widehat{0},\widehat{1})\}$ of $\Delta^{ext}(\overrightarrow{D}^0)^{op}$ is Reedy cofibrant for the Cole-Strøm model structure of **Flow** as well. Thus by Corollary 8.10, one has

$$L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) \cong \varinjlim_{\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}\downarrow(\widehat{0},\widehat{1}))} \mathcal{G}Q(\overrightarrow{D}) \cong \underrightarrow{\mathrm{holim}}_{\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}\downarrow(\widehat{0},\widehat{1}))} \mathcal{G}Q(\overrightarrow{D}) \tag{1}$$

and

$$L_{(\widehat{0},\widehat{1})}\mathcal{G}F(\overrightarrow{D}^0) \cong \varinjlim_{\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}\downarrow(\widehat{0},\widehat{1}))} \mathcal{G}F(\overrightarrow{D}^0) \cong \operatorname{\underrightarrow{holim}}_{\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}\downarrow(\widehat{0},\widehat{1}))} \mathcal{G}F(\overrightarrow{D}^0). \tag{2}$$

For any simplex $(\alpha_0,\dots,\alpha_p)$ of the latching category $\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}_+\downarrow(\widehat{0},\widehat{1}))$, one has

$$\mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)} = \mathcal{G}Q(\overrightarrow{D})\restriction_{[\alpha_0,\alpha_1]} * \dots * \mathcal{G}Q(\overrightarrow{D})\restriction_{[\alpha_{p-1},\alpha_p]}$$

and

$$\mathcal{G}F(\overrightarrow{D}^0)_{(\alpha_0,\dots,\alpha_p)} = \mathcal{G}F(\overrightarrow{D}^0)\restriction_{[\alpha_0,\alpha_1]} * \dots * \mathcal{G}F(\overrightarrow{D}^0)\restriction_{[\alpha_{p-1},\alpha_p]}.$$

For any $0 \leqslant i < p$, one has $\operatorname{card}([\alpha_i,\alpha_{i+1}]) < \operatorname{card}(\overrightarrow{D}^0)$. Therefore, by induction hypothesis, for any $0 \leqslant i < p$, the morphism of diagrams $\mathcal{G}Q(\overrightarrow{D}) \longrightarrow \mathcal{G}F(\overrightarrow{D}^0)$ induces a S-homotopy equivalence $\mathcal{G}Q(\overrightarrow{D})_{[\alpha_i,\alpha_{i+1}]} \longrightarrow \mathcal{G}F(\overrightarrow{D}^0)_{[\alpha_i,\alpha_{i+1}]}$.

The flow $\mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$ is the colimit of the diagram of flows

$$\begin{tikzcd}
\mathcal{G}Q(\overrightarrow{D})\restriction_{[\alpha_0,\alpha_1]} & & \cdots & & \mathcal{G}Q(\overrightarrow{D})\restriction_{[\alpha_{p-1},\alpha_p]} \\
& \{0\} \arrow[ul] \arrow[ur] & & \{0\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The flow $\mathcal{G}F(\overrightarrow{D}^0)_{(\alpha_0,\dots,\alpha_p)}$ is the colimit of the diagram of flows

$$\begin{tikzcd}
\mathcal{G}F(\overrightarrow{D}^0)\restriction_{[\alpha_0,\alpha_1]} & & \cdots & & \mathcal{G}F(\overrightarrow{D}^0)\restriction_{[\alpha_{p-1},\alpha_p]} \\
& \{0\} \arrow[ul] \arrow[ur] & & \{0\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

One can put on the underlying category of the two diagrams of flows above the Reedy structure

$$\begin{tikzcd}
1 & & \cdots & & 1 \\
& 0 \arrow[ul] \arrow[ur] & & 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Then the colimit functor from the category of diagrams of flows over this underlying category equipped with the Reedy model structure to the Cole-Strøm model structure of the category of flows is a left Quillen functor for the same reason as in Corollary 8.10. So the S-homotopy equivalences $\mathcal{G}Q(\overrightarrow{D})_{[\alpha_i,\alpha_{i+1}]} \longrightarrow \mathcal{G}F(\overrightarrow{D}^0)_{[\alpha_i,\alpha_{i+1}]}$ with $0 \leqslant i < p$ induces a S-homotopy equivalence from $\mathcal{G}Q(\overrightarrow{D})_{(\alpha_0,\dots,\alpha_p)}$ to $\mathcal{G}F(\overrightarrow{D}^0)_{(\alpha_0,\dots,\alpha_p)}$. Thus the morphism of diagrams $\mathcal{G}Q(\overrightarrow{D}) \longrightarrow \mathcal{G}F(\overrightarrow{D}^0)$ restricted to $\partial(\Delta^{ext}(\overrightarrow{D}^0)^{op}_+\downarrow(\widehat{0},\widehat{1}))$ is a objectwise S-homotopy equivalence. Therefore the morphism of flows $L_{(\widehat{0},\widehat{1})}\mathcal{G}\overrightarrow{D} \longrightarrow L_{(\widehat{0},\widehat{1})}\mathcal{G}F(\overrightarrow{D}^0) \cong F(\overrightarrow{D}^0)$ is a S-homotopy equivalence as well by Equation (1) and Equation (2).

In another hand, the morphism of flows $L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) \longrightarrow Q(\overrightarrow{D})$ induces the equalities

$$\mathbb{P}_{\alpha,\beta}L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) = \mathbb{P}_{\alpha,\beta}Q(\overrightarrow{D})$$

for any $(\alpha,\beta) \neq (\widehat{0},\widehat{1})$ and a cofibration (for both the Quillen model structure or the Strøm model structure of **Top**) $\mathbb{P}_{\widehat{0},\widehat{1}}L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D}) \longrightarrow \mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D})$. In other terms, the latter continuous map is a DR pair which has a retract $r : \mathbb{P}_{\widehat{0},\widehat{1}}Q(\overrightarrow{D}) \longrightarrow \mathbb{P}_{\widehat{0},\widehat{1}}L_{(\widehat{0},\widehat{1})}\mathcal{G}Q(\overrightarrow{D})$. By composition, we deduce a morphism of flows $Q(\overrightarrow{D}) \longrightarrow F(\overrightarrow{D}^0)$ which is a S-homotopy equivalence. □

Proof of Theorem 8.1. The branching space functor is badly behaved with respect to the weak S-homotopy equivalence. But by [Gau05a] Corollary B.7 and of course by Theorem 8.18, the branching space of $Q(\overrightarrow{D})$ and of $F(\overrightarrow{D}^0)$ are homotopy equivalent ! The calculation of the branching space of $F(\overrightarrow{D}^0)$ is trivial. This completes the proof. □

9. Conclusion

Using an analogue of the Strøm model structure on the category of compactly generated topological spaces, we are able to prove that the branching and merging theory homologies are let unchanged by the generalized T-homotopy equivalences.

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