Semantic Subtyping: Challenges, Perspectives, and Open Problems

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Outline

1. Motivations and goals.

2. Semantic subtyping.

3. λ-calculus.

4. π-calculus.

5. Some Perspectives.

6. Conclusion
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1 Motivations and goals.

2 Semantic subtyping.

3 $\lambda$-calculus.

4 $\pi$-calculus.

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1. Motivations and goals.
2. Semantic subtyping.
3. $\lambda$-calculus.
4. $\pi$-calculus.
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6. Conclusion
The goal is to show how to take your favourite type constructors

\times, \to, \{\ldots\}, \text{chan}(\ldots)

and add boolean combinators:

\lor, \land, \lnot

so that they behave set-theoretically w.r.t. \leq

**WHY?**

Short answer: they are convenient and you need them to program XML in a typed language with pattern matching.
The goal is to show how to take your favourite type constructors

\[ \times, \to, \{ \ldots \}, \text{chan}() , \ldots \]

and add boolean combinators:

\[ \lor, \land, \neg \]

so that they behave set-theoretically w.r.t. \( \leq \)

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so that they behave set-theoretically w.r.t. \( \leq \)

**WHY?**

Short answer: they are convenient and you need them to program XML in a typed language with **pattern matching**.
Why it is difficult?

\[
t ::= B \mid t \times t \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1
\]

- Handling subtyping without combinators is easy: constructors do not mix, e.g.
  \[
  s_1 \leq s_1 \quad t_1 \leq t_2 \\
  s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2 \\
  s_1 \times t_1 \leq s_2 \times t_2
  \]

- With combinators is much harder: combinators distribute over constructors, e.g.
  \[
  (s \lor s) \rightarrow t \leq \exists \chi t \rightarrow (s_1 \rightarrow t \land (s_2 \rightarrow t))
  \]

- Without a clear semantics, subtyping is hard to define.
Why it is difficult?

\[
t ::= B \mid t \times t \mid t \rightarrow t \mid t \vee t \mid t \wedge t \mid \neg t \mid 0 \mid 1
\]

- Handling subtyping without combinators is easy: constructors do not mix, e.g.:
  \[
  s_2 \leq s_1 \quad t_1 \leq t_2 \quad s_1 \leq s_2 \quad t_1 \leq t_2
  \]
  \[
  s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2 \quad s_1 \times t_1 \leq s_2 \times t_2
  \]

- With combinators is much harder: combinators distribute over constructors, e.g.:
  \[
  (s \vee s_2) \rightarrow t \leq s_1 \rightarrow (t \vee (s \rightarrow t))
  \]

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Why it is difficult?

\[ t ::= \text{ } B \mid t \times t \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \]

Handling subtyping without combinators is easy: constructors do not mix, e.g.:

\[ s_2 \leq s_1 \quad t_1 \leq t_2 \quad s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2 \]

\[ s_1 \times t_1 \leq s_2 \times t_2 \]

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\[ (s \lor s) \rightarrow t \leq \neg s \rightarrow t \lor \neg s \rightarrow t \]

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  \[
  \frac{s_2 \leq s_1 \quad t_1 \leq t_2}{s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2}
  \]
  \[
  \frac{s_1 \leq s_2 \quad t_1 \leq t_2}{s_1 \times t_1 \leq s_2 \times t_2}
  \]

- With combinators is much harder: combinators distribute over constructors, e.g.
  \[
  (s_1 \lor s_2) \rightarrow t \quad \geq \quad (s_1 \rightarrow t) \land (s_2 \rightarrow t)
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  \[
  \begin{align*}
  s_2 \leq s_1 & \quad t_1 \leq t_2 \\
  s_1 \rightarrow t_1 & \leq s_2 \rightarrow t_2
  \end{align*}
  \]

- With combinators is much harder: combinators distribute over constructors, e.g.
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  (s_1 \lor s_2) \rightarrow t \geq (s_1 \rightarrow t) \land (s_2 \rightarrow t)
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Handling subtyping without combinators is easy: constructors do not mix, e.g.:

\[
\frac{s_2 \leq s_1}{s_1 \rightarrow t_1} \leq \frac{s_2 \rightarrow t_2}{s_1 \rightarrow t_1} \leq \frac{s_1 \leq s_2}{s_1 \times t_1} \leq \frac{t_1 \leq t_2}{s_2 \times t_2}
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With combinators is much harder: combinators distribute over constructors, e.g.

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(s_1 \lor s_2) \rightarrow t \geq (s_1 \rightarrow t) \land (s_2 \rightarrow t)
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Without a clear semantics, subtyping is hard to define, e.g.

\[
ch^+(s) \land ch^-(t) \leq ch^-(s) \lor ch^+(t)
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Why it is difficult?

\[ t ::= B \mid t \times t \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \]

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  \[ ch^+(s) \land ch^-(t) \quad \leq \quad ch^-(s) \lor ch^+(t) \quad ??? \]
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MAIN IDEA

Instead of defining the subtyping relation so that it conforms to the semantic of types, define the semantics of types and derive the subtyping relation.

- Handling subtyping without combinators is easy: constructors do not mix, e.g.:
  \[ s_2 \leq s_1 \quad t_1 \leq t_2 \quad s_1 \to t_1 \leq s_2 \to t_2 \]

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Instead of defining the subtyping relation so that it conforms to the semantic of types, define the semantics of types and derive the subtyping relation.

- Not a particularly new idea. Many attempts (e.g. Aiken&Wimmers, Damm,..., Hosoya&Pierce).

- None fully satisfactory. (no negation, or no function types, or restrictions on unions and intersections, ...)

- Starting point of what follows: the approach of Hosoya&Pierce.
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- **Starting point of what follows: the approach of Hosoya&Pierce.**
Semantic subtyping
Define a set-theoretic semantics of the types:

\[ [\cdot] : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

Define the subtyping relation as follows:

\[ s \leq t \overset{\text{def}}{\iff} [s] \subseteq [t] \]

**KEY OBSERVATION 1:**

The *model of types* may be independent from a *model of terms*

Hosoya and Pierce use the model of values:

\[ [t]_\gamma = \{ v \mid \vdash v : t \} \]

Works because XML documents are the only XDuce values and for them \([t]_\gamma\) can be defined independently from the typing relation
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Works because XML documents are the only XDuce values and for them \(\llbracket t \rrbracket_V\) can be defined independently from the typing relation.
For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined.
1. Motivations

2. Semantic subtyping

3. $\lambda$-calculus

4. $\pi$-calculus

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ICTCS '05 Invited Talk

Circularity

Model of values

\[
t \leq s \iff [t]_\forall \subseteq [s]_\forall \quad \text{where} \quad [t]_\forall = \{ v \mid \vdash v : t \}
\]

For instance, it does not work with arrow types: values are $\lambda$-abstractions and need (sub)typing to be defined.
Circularity

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where

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For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined.

\[ [t] \subseteq [s] \]

A similar circularity holds for \( \pi \)-calculus channels, as well.
Bootstrap

Model of values

\[ t \leq s \iff [t]_\mathcal{V} \subseteq [s]_\mathcal{V} \text{ where } [t]_\mathcal{V} = \{ v \mid \vdash v : t \} \]

For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined.
Model of values

\[ t \leq s \iff \llbracket t \rrbracket \subseteq \llbracket s \rrbracket \]
where \( \llbracket t \rrbracket = \{ v \mid \vdash v : t \} \)

For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined.

\[ t \leq t \]
\[ \llbracket t \rrbracket \]
\[ \vdash e : t \]
\[ \vdash v : t \]
Model of values

\[ t \leq s \iff \llbracket t \rrbracket_V \subseteq \llbracket s \rrbracket_V \quad \text{where} \quad \llbracket t \rrbracket_V = \{ v \mid \vdash v : t \} \]

For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined

\[ \llbracket t \rrbracket_D \]

\[ t \leq t \quad \llbracket t \rrbracket_V \]

\[ \vdash e : t \quad \vdash v : t \]
Model of values

\[ t \leq s \iff [t]_\forall \subseteq [s]_\forall \quad \text{where} \quad [t]_\forall = \{ v \mid \vdash v : t \} \]

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\[ \vdash e : t \quad \vdash v : t \]
Model of values

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For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined

\[ [t] \triangleleft s \]

\[ t \leq t \]

\[ \vdash e : t \]

\[ \vdash v : t \]
Model of values

\[ t \leq s \iff \llbracket t \rrbracket \subseteq \llbracket s \rrbracket \quad \text{where} \quad \llbracket t \rrbracket = \{ v \mid \vdash v : t \} \]

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Model of values

\[ t \leq s \iff \llbracket t \rrbracket \psi \subseteq \llbracket s \rrbracket \psi \quad \text{where} \quad \llbracket t \rrbracket \psi = \{ v \mid \vdash v : t \} \]

For instance, it does not work with arrow types: values are \( \lambda \)-abstractions and need (sub)typing to be defined.
Semantic subtyping in five steps:

1. **Add boolean combinators: ∨, ∧, ¬** to your favourite type constructors (e.g., →, ×, ch( ), . . .)

2. **Define a set-theoretic semantics:** \[ ] : Types → \( \mathcal{P}(D) \)

   \[ [s ∧ t] \triangleq [s] ∩ [t] \]
   \[ [s ∨ t] \triangleq [s] ∪ [t] \]
   \[ [¬t] \triangleq D \setminus [t] \]

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model.

4. **Define a language and type it** by using \( s \leq_D t \):

   \[ \Gamma \vdash_D e : s \leq_D t \]

5. **Close the circle:** define the types-as-set-of-values semantics \([t]_V = \{ v \in V \mid \vdash_v D \vdash t \} \) and check \( s \leq_D t \iff s \leq_V t \)

The rest of the story is standard: subject reduction.
Semantic subtyping in five steps:

1. **Add boolean combinators: ∨, ∧, ¬**
   to your favourite type *constructors* (e.g., →, ×, ch( ), ...)

2. **Define a set-theoretic semantics:** \[ \llbracket \cdot \rrbracket : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]
   \[
   \begin{align*}
   \llbracket s \land t \rrbracket & = \llbracket s \rrbracket \cap \llbracket t \rrbracket \\
   \llbracket s \lor t \rrbracket & = \llbracket s \rrbracket \cup \llbracket t \rrbracket \\
   \llbracket \neg t \rrbracket & = \mathcal{D} \setminus \llbracket t \rrbracket
   \end{align*}
   \]

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model

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   \[ \llbracket t \rrbracket_V = \{ v \in V | \Gamma \vdash_D v : t \} \]
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Semantic subtyping in five steps:

1. **Add boolean combinators**: \( \lor, \land, \neg \)
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\llbracket \llbracket s \lor t \rrbracket_D = \llbracket s \rrbracket_D \cup \llbracket t \rrbracket_D,
\llbracket s \land t \rrbracket_D = \llbracket s \rrbracket_D \cap \llbracket t \rrbracket_D,
\llbracket \neg t \rrbracket_D = D \setminus \llbracket t \rrbracket_D
\]
   \( s \leq_D t \iff \llbracket s \rrbracket_D \subseteq \llbracket t \rrbracket_D \)

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model \( \llbracket \rrbracket_D \).

4. **Define a language and type it** by using \( s \leq_D t \):
   \[
   \Gamma \vdash D \colon s \leq_D t
   \]

5. **Close the circle**: define the types-as-set-of-values semantics
   \[
   \llbracket t \rrbracket_V = \{ v \in V \mid \Gamma \vdash D v : t \}\]

   and check \( s \leq_D t \iff s \leq_V t \)

The rest of the story is standard: subject reduction.
Semantic subtyping in five steps:

1. **Add boolean combinator**: $\lor$, $\land$, $\neg$
   to your favourite type *constructors* (e.g., $\rightarrow$, $\times$, $ch(\ )$, $\ldots$)

2. **Define a set-theoretic semantics**: $\llbracket \cdot \rrbracket_\mathcal{D} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D})$
   
   $([s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}, \quad [s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D}, \quad [\neg t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D})$

   $s \leq_\mathcal{D} t \overset{\text{def}}{\iff} [s]_\mathcal{D} \subseteq [t]_\mathcal{D}$

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model
   [optional but advisable]

4. **Define a language and type it by using** $s \leq_\mathcal{D} t$:

   \[
   \Gamma \vdash D_e : s \leq_\mathcal{D} t
   \]

5. **Close the circle**: define the types-as-set-of-values semantics
   $\llbracket t \rrbracket_V = \{ v \in V | \vdash D_v : t \}$ and check
   $s \leq_\mathcal{D} t \iff s \leq_V t$

The rest of the story is standard: subject reduction, \ldots
Semantic subtyping in five steps:

1. **Add boolean combinators: ∨, ∧, ¬** to your favourite type *constructors* (e.g., →, ×, ch(), ...)

2. **Define a set-theoretic semantics:** \[\llbracket \cdot \rrbracket_D : \text{Types} \to \mathcal{P}(D)\]

   \[
   \begin{align*}
   \llbracket s \land t \rrbracket_D &= \llbracket s \rrbracket_D \cap \llbracket t \rrbracket_D, \\
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   \end{align*}
   \]

   \[s \leq_D t \overset{\text{def}}{\iff} \llbracket s \rrbracket_D \subseteq \llbracket t \rrbracket_D\]

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model [optional but advisable]

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5. **Close the circle:** define the types-as-set-of-values semantics \[\llbracket t \rrbracket_V = \{v \in V | \exists \Gamma \vdash D_v : t\}\] and check \[s \leq_D t \iff s \leq_V t\]

The rest of the story is standard: subject reduction...
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   to your favourite type *constructors* (e.g., $\to$, $\times$, $ch(\ )$, ...)

2. **Define a set-theoretic semantics:** $\llbracket \cdot \rrbracket_D : \text{Types} \rightarrow \mathcal{P}(D)$
   
   $([s \land t]_D = [s]_D \cap [t]_D$,  
   $[s \lor t]_D = [s]_D \cup [t]_D$,  
   $[\neg t]_D = D \setminus [t]_D$)

   $$s \leq_D t \quad \overset{\text{def}}{\iff} \quad [s]_D \subseteq [t]_D$$

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model  
   [optional but advisable]

4. **Define a language and type it** by using $s \leq_D t$:

5. **Close the circle**: define the types-as-set-of-values semantics

   $$\llbracket t \rrbracket_V = \{ v \in V | \Gamma \vdash D_v : t \}$$  
   and check $s \leq_D t \iff s \leq_V t$

The rest of the story is standard: subject reduction, ...
Semantic subtyping in five steps:

1. **Add boolean combinators**: `∨`, `∧`, `¬` to your favourite type *constructors* (e.g., `→`, `×`, `ch()`, …)

2. **Define a set-theoretic semantics**: \( [\ ]_\mathcal{D} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \)
   \[
   [s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}, \quad [s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D}, \quad [\neg t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D}
   \]
   \[
   s \leq_\mathcal{D} t \iff [s]_\mathcal{D} \subseteq [t]_\mathcal{D}
   \]

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model [optional but advisable]

4. **Define a language and type it** by using \( s \leq_\mathcal{D} t \):
   \[
   \Gamma \vdash_\mathcal{D} e : s \quad s \leq_\mathcal{D} t \quad \Gamma \vdash_\mathcal{D} e : t
   \]

5. **Close the circle**: define the *types-as-set-of-values semantics* \( [t]_\mathcal{V} = \{ v \in \mathcal{V} \mid \Gamma \vdash_\mathcal{D} v : t \} \) and check
   \[
   s \leq_\mathcal{D} t \iff [s]_\mathcal{D} \subseteq [t]_\mathcal{D}
   \]

The rest of the story is standard: subject reduction, …
Semantic subtyping in five steps:

1. **Add boolean combinators:** $\lor$, $\land$, $\neg$
   to your favourite type *constructors* (e.g., $\to$, $\times$, $ch(\ )$, ...)

2. **Define a set-theoretic semantics:** $\llbracket \cdot \rrbracket : \text{Types} \to \mathcal{P}(\mathcal{D})$
   
   \[
   \llbracket s \land t \rrbracket \mathcal{D} = \llbracket s \rrbracket \mathcal{D} \cap \llbracket t \rrbracket \mathcal{D},
   \llbracket s \lor t \rrbracket \mathcal{D} = \llbracket s \rrbracket \mathcal{D} \cup \llbracket t \rrbracket \mathcal{D},
   \llbracket \neg t \rrbracket \mathcal{D} = \mathcal{D} \setminus \llbracket t \rrbracket \mathcal{D}
   \]

   \[s \preceq \mathcal{D} t \iff \llbracket s \rrbracket \mathcal{D} \subseteq \llbracket t \rrbracket \mathcal{D}\]

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model
   [optional but advisable]

4. **Define a language and type it** by using $s \preceq \mathcal{D} t$:

   \[\Gamma \vdash \mathcal{D} e : s \quad s \preceq \mathcal{D} t \]

   \[\Gamma \vdash \mathcal{D} e : t\]

5. **Close the circle:** define the types-as-set-of-values semantics
   $\llbracket t \rrbracket \mathcal{V} = \{ v \in \mathcal{V} \mid \vdash \mathcal{D} v : t \}$ and check
   
   \[s \preceq \mathcal{D} t \iff \llbracket s \rrbracket \mathcal{V} \subseteq \llbracket t \rrbracket \mathcal{V}\]

The rest of the story is standard: subject reduction, ...

Giuseppe Castagna  Semantic Subtyping: Challenges, Perspectives, and Open Problems 8/39
Semantic subtyping in five steps:

1. Add boolean combinators: \( \lor, \land, \neg \) to your favourite type constructors (e.g., \( \to, \times, \text{ch}() \), ...)

2. Define a set-theoretic semantics: \( \llbracket \cdot \rrbracket_D : \text{Types} \to \mathcal{P}(D) \)
   \[
   \begin{align*}
   \llbracket s \land t \rrbracket_D &= \llbracket s \rrbracket_D \cap \llbracket t \rrbracket_D, \\
   \llbracket s \lor t \rrbracket_D &= \llbracket s \rrbracket_D \cup \llbracket t \rrbracket_D, \\
   \llbracket \neg t \rrbracket_D &= D \setminus \llbracket t \rrbracket_D
   \end{align*}
   \]

3. Find a subtyping algorithm by using the set-theoretic properties of the model

4. Define a language and type it by using \( s \leq_D t \): 
   \[
   \Gamma \vdash_D e : s \quad s \leq_D t \\
   \Gamma \vdash_D e : t
   \]

5. Close the circle: define the types-as-set-of-values semantics
   \[
   \llbracket t \rrbracket_{\mathcal{V}} = \{ v \in \mathcal{V} \mid \Gamma \vdash_D v : t \}
   \]

The rest of the story is standard: subject reduction.
Semantic subtyping in five steps:

1. **Add boolean combinators:** $\lor$, $\land$, $\neg$
   to your favourite type *constructors* (e.g., $\to$, $\times$, $ch()$, ...)

2. **Define a set-theoretic semantics:** $\llbracket \cdot \rrbracket_D : \text{Types} \rightarrow \mathcal{P}(D)$
   
   $(\llbracket s \land t \rrbracket_D = [s]_D \cap [t]_D)$, $(\llbracket s \lor t \rrbracket_D = [s]_D \cup [t]_D)$, $(\llbracket \neg t \rrbracket_D = D \setminus [t]_D)$

   $$s \leq_D t \iff [s]_D \subseteq [t]_D$$

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model
   [optional but advisable]

4. **Define a language and type it** by using $s \leq_D t$:
   $$\Gamma \vdash_D e : s \quad s \leq_D t \quad \Gamma \vdash_D e : t$$

5. **Close the circle:** define the types-as-set-of-values semantics
   $$\llbracket t \rrbracket_\mathcal{V} = \{ v \in \mathcal{V} \mid \Gamma \vdash_D v : t \}$$ and check
   $$s \leq_D t \iff s \leq_\mathcal{V} t$$

The rest of the story is standard: subject reduction...
Semantic subtyping in five steps:

1. **Add boolean combinators:** ∨, ∧, ¬
   to your favourite type *constructors* (e.g., →, ×, ch( ), . . .)

2. **Define a set-theoretic semantics:** \([ \cdot ] \mathcal{D} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D})\)
   
   \(\begin{align*}
   ([s \land t] \mathcal{D} &= [s] \mathcal{D} \cap [t] \mathcal{D}, \\
   [s \lor t] \mathcal{D} &= [s] \mathcal{D} \cup [t] \mathcal{D}, \\
   [\neg t] \mathcal{D} &= \mathcal{D} \setminus [t] \mathcal{D}
   \end{align*}\)

   \(s \leq \mathcal{D} t \iff [s] \mathcal{D} \subseteq [t] \mathcal{D}\)

3. **Find a subtyping algorithm** by using the set-theoretic properties of the model
   [optional but advisable]

4. **Define a language and type it** by using \(s \leq \mathcal{D} t\):

   \[
   \frac{\Gamma \vdash \mathcal{D} e : s \quad s \leq \mathcal{D} t}{\Gamma \vdash \mathcal{D} e : t}
   \]

5. **Close the circle:** define the types-as-set-of-values semantics

   \([t] \mathcal{V} = \{ v \in \mathcal{V} \mid \vdash \mathcal{D} v : t \}\) and check

   \(s \leq \mathcal{D} t \iff s \leq \mathcal{V} t\)

The rest of the story is standard: subject reduction, . . .
\lambda\text{-calculus.}
STEP 1: types for $\lambda$

**Types** $t ::= b$ basic types

| $t \times t$ product type |
| $t \rightarrow t$ function type |
| $0$ empty type |
| $1$ top type |
| $\neg t$ negation type |
| $t \lor t$ union type |
| $t \land t$ intersection type |

(type constructors)

(type combinators)
STEP 1: types for $\lambda$

**Types**

$t ::= \begin{align*}
& b & \text{basic types} \\
& t \times t & \text{product type} \\
& t \rightarrow t & \text{function type} \\
& 0 & \text{empty type} \\
& 1 & \text{top type} \\
& \neg t & \text{negation type} \\
& t \lor t & \text{union type} \\
& t \land t & \text{intersection type}
\end{align*}

\{ \text{type constructors} \}

\{ \text{type combinators} \}
STEP 1: types for $\lambda$

$\textbf{Types} \quad t ::= b \quad \text{basic types}
\quad \mid t \times t \quad \text{product type}
\quad \mid t \rightarrow t \quad \text{function type}
\quad \mid 0 \quad \text{empty type}
\quad \mid 1 \quad \text{top type}
\quad \mid \neg t \quad \text{negation type}
\quad \mid t \lor t \quad \text{union type}
\quad \mid t \land t \quad \text{intersection type}$

\{ type constructors \}
\{ type combinators \}
STEP 2: set-theoretic model

\[ \boxed{\llbracket - \rrbracket : \text{Types} \rightarrow \mathcal{P}(D)} \]

Easy part:

\[ \llbracket s \land t \rrbracket = \llbracket s \rrbracket \cap \llbracket t \rrbracket \]
\[ \llbracket s \lor t \rrbracket = \llbracket s \rrbracket \cup \llbracket t \rrbracket \]
\[ \llbracket \neg t \rrbracket = D \setminus \llbracket t \rrbracket \]
\[ \llbracket s \times t \rrbracket = \llbracket s \rrbracket \times \llbracket t \rrbracket \]

Impossible since it requires \( \mathcal{P}(D^2) \subseteq D \)

KEY OBSERVATION 2:

Use any \( \llbracket - \rrbracket \) that behaves w.r.t. \( \subseteq \) as if equation (\( \ast \)) held, namely

\[ \llbracket t_1 \rightarrow s_1 \rrbracket \subseteq \llbracket t_2 \rightarrow s_2 \rrbracket \iff \mathcal{P}(\llbracket t_1 \rrbracket \times \llbracket s_1 \rrbracket) \subseteq \mathcal{P}(\llbracket t_2 \rrbracket \times \llbracket s_2 \rrbracket) \]
STEP 2: set-theoretic model

\[ 
\llbracket \text{D} \rrbracket : \text{Types} \rightarrow \mathcal{P}(\text{D}) 
\]

Easy part:

\[ 
\begin{align*}
\llbracket s \land t \rrbracket &= \llbracket s \rrbracket \cap \llbracket t \rrbracket \\
\llbracket \neg t \rrbracket &= \text{D} \setminus \llbracket t \rrbracket \\
\llbracket s \lor t \rrbracket &= \llbracket s \rrbracket \cup \llbracket t \rrbracket \\
\llbracket s \times t \rrbracket &= \llbracket s \rrbracket \times \llbracket t \rrbracket 
\end{align*} 
\]

Impossible since it requires \( \mathcal{P}(\text{D}^2) \subseteq \text{D} \)

KEY OBSERVATION 2:

Use any \( \llbracket \rrbracket \) that behaves w.r.t. \( \subseteq \) as if equation (*) held, namely

\[ 
\llbracket t_1 \rightarrow s_1 \rrbracket \subseteq \llbracket t_2 \rightarrow s_2 \rrbracket \iff \mathcal{P}(\llbracket t_1 \rrbracket \times \llbracket s_1 \rrbracket) \subseteq \mathcal{P}(\llbracket t_2 \rrbracket \times \llbracket s_2 \rrbracket) 
\]
STEP 2: set-theoretic model

\[ \mathcal{G} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

**Easy part:**

\[ [s \wedge t]_{\mathcal{G}} = [s]_{\mathcal{G}} \cap [t]_{\mathcal{G}} \]
\[ [s \vee t]_{\mathcal{G}} = [s]_{\mathcal{G}} \cup [t]_{\mathcal{G}} \]
\[ [\neg t]_{\mathcal{G}} = \mathcal{D} \setminus [t]_{\mathcal{G}} \]
\[ [s \times t]_{\mathcal{G}} = [s]_{\mathcal{G}} \times [t]_{\mathcal{G}} \]

**Hard part:**

\[ [t \rightarrow s] = ??? \]

Impossible since it requires \( \mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D} \)

**KEY OBSERVATION 2:**

Use any \([\ ]\) that behaves w.r.t. \(\subseteq\) as if equation (\(\ast\)) held, namely

\[ [t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \iff \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2]) \]
STEP 2: set-theoretic model

\[ [ ]_\mathcal{D} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

Easy part:
- \([s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}\)
- \([s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D}\)
- \([-t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D}\)
- \([s \times t]_\mathcal{D} = [s]_\mathcal{D} \times [t]_\mathcal{D}\)

Hard part:
- \([t \rightarrow s] = \{\text{functions from } [t] \text{ to } [s]\}\)

Impossible since it requires \(\mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D}\)

KEY OBSERVATION 2:

Use any \([ ]\) that behaves w.r.t. \(\subseteq\) as if equation (*) held, namely

\([t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \quad \iff \quad \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2])\)
STEP 2: set-theoretic model

\[ [[ ]] : \text{Types} \rightarrow \mathcal{P}(D) \]

**Easy part:**
\[
\begin{align*}
[s \land t]_D &= [s]_D \cap [t]_D \\
[s \lor t]_D &= [s]_D \cup [t]_D \\
[\neg t]_D &= D \setminus [t]_D \\
[s \times t]_D &= [s]_D \times [t]_D
\end{align*}
\]

**Hard part:**
\[
[t \rightarrow s] = \{ f \subseteq D^2 \mid \forall (d_1, d_2) \in f. \ d_1 \in [t] \Rightarrow d_2 \in [s] \}
\]
Impossible since it requires \( \mathcal{P}(D^2) \subseteq D \)

**KEY OBSERVATION 2:**

Use any \( [[ ]] \) that behaves w.r.t. \( \subseteq \) as if equation (*) held, namely
\[
[t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \iff \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2])
\]
STEP 2: set-theoretic model

\[ \llbracket \cdot \rrbracket_{\mathcal{D}} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

**Easy part:**
\[
\begin{align*}
\llbracket s \land t \rrbracket_{\mathcal{D}} &= \llbracket s \rrbracket_{\mathcal{D}} \cap \llbracket t \rrbracket_{\mathcal{D}} \\
\llbracket \neg t \rrbracket_{\mathcal{D}} &= \mathcal{D} \setminus \llbracket t \rrbracket_{\mathcal{D}} \\
\llbracket s \lor t \rrbracket_{\mathcal{D}} &= \llbracket s \rrbracket_{\mathcal{D}} \cup \llbracket t \rrbracket_{\mathcal{D}} \\
\llbracket s \times t \rrbracket_{\mathcal{D}} &= \llbracket s \rrbracket_{\mathcal{D}} \times \llbracket t \rrbracket_{\mathcal{D}}
\end{align*}
\]

**Hard part:**
\[ \llbracket t \rightarrow s \rrbracket = \mathcal{P}(\llbracket t \rrbracket \times \llbracket s \rrbracket) \quad (*) \]

Impossible since it requires \( \mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D} \)

**KEY OBSERVATION 2:**

Use any \( \llbracket \cdot \rrbracket \) that behaves w.r.t. \( \subseteq \) as if equation \((*)\) held, namely

\[ \llbracket t_1 \rightarrow s_1 \rrbracket \subseteq \llbracket t_2 \rightarrow s_2 \rrbracket \quad \iff \quad \mathcal{P}(\llbracket t_1 \rrbracket \times \llbracket s_1 \rrbracket) \subseteq \mathcal{P}(\llbracket t_2 \rrbracket \times \llbracket s_2 \rrbracket) \]
STEP 2: set-theoretic model

\[ \llbracket \_ \rrbracket_D : \text{Types} \rightarrow \mathcal{P}(D) \]

**Easy part:**
- \([s \land t]_D = [s]_D \cap [t]_D\)
- \([s \lor t]_D = [s]_D \cup [t]_D\)
- \([-t]_D = D \setminus [t]_D\)
- \([s \times t]_D = [s]_D \times [t]_D\)

**Hard part:**
- \([t \rightarrow s] = \mathcal{P}([t] \times [s])\) (\(*\))

Impossible since it requires \(\mathcal{P}(D^2) \subseteq D\)

**KEY OBSERVATION 2:**
Use any \(\llbracket \_ \rrbracket\) that behaves w.r.t. \(\subseteq\) as if equation \((\ast)\) held, namely

\[ [t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \iff \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2]) \]
STEP 2: set-theoretic model

\[ [\text{Types}]_\mathcal{D} : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

**Easy part:**
- \([s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}\)
- \([s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D}\)
- \([-t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D}\)
- \([s \times t]_\mathcal{D} = [s]_\mathcal{D} \times [t]_\mathcal{D}\)

**Hard part:**
- \([t \rightarrow s] = \mathcal{P}([t] \times [s])\) \(\text{(*)}\)

Impossible since it requires \(\mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D}\)

**KEY OBSERVATION 2:**
We need the model to state how types are related rather than what the types are

Use any \([\ ]\) that behaves w.r.t. \(\subseteq\) as if equation (\(\ast\)) held, namely

\([t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \quad \iff \quad \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2])\)
STEP 2: set-theoretic model

\[ [\square] : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \]

Easy part:

\[ [s \land t] = [s] \cap [t] \]
\[ \lnot t = \mathcal{D} \setminus [t] \]
\[ [s \lor t] = [s] \cup [t] \]
\[ [s \times t] = [s] \times [t] \]

Hard part:

\[ [t \rightarrow s] = \mathcal{P}([t] \times [s]) \] (\(\ast\))

Impossible since it requires \(\mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D}\)

KEY OBSERVATION 2:

We need the model to state how types are related rather than what the types are.

Use any \([\square]\) that behaves w.r.t. \(\subseteq\) as if equation \((\ast)\) held, namely

\[ [t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \iff \mathcal{P}([t_1] \times [s_1]) \subseteq \mathcal{P}([t_2] \times [s_2]) \]
STEP 2: set-theoretic model

$$[\Box] \vDash \text{Types} \to \mathcal{P}(\mathcal{D})$$

Easy part:

$$[s \land t] \vDash = [s] \vDash \cap [t] \vDash$$
$$[s \lor t] \vDash = [s] \vDash \cup [t] \vDash$$
$$[\neg t] \vDash = \mathcal{D} \setminus [t] \vDash$$
$$[s \times t] \vDash = [s] \vDash \times [t] \vDash$$

Hard part:

$$[t \rightarrow s] = \mathcal{P}(\overline{[t] \times [s]})$$

Impossible since it requires $$\mathcal{P}(\mathcal{D}^2) \subseteq \mathcal{D}$$

KEY OBSERVATION 2:

We need the model to state how types are related rather than what the types are.

Use any $$[\Box]$$ that behaves w.r.t. $$\subseteq$$ as if equation (\ast) held, namely

$$[t_1 \rightarrow s_1] \subseteq [t_2 \rightarrow s_2] \iff \mathcal{P}(\overline{[t_1] \times [s_1]}) \subseteq \mathcal{P}(\overline{[t_2] \times [s_2]})$$
STEP 2: set-theoretic model

Solution: \[ [t \rightarrow s] = \mathcal{P}_f ([t] \times [s]) \] (*)

Subtyping is completely characterised by type emptiness

Indeed: \( s \leq t \iff [s] \subseteq [t] \iff [s] \cap [\overline{t}] = \emptyset \iff [s \land \neg t] = \emptyset \)
\[
\mathcal{P}_f (X) = \emptyset \iff \mathcal{P} (X) = \emptyset
\]

Therefore, (*) induces the same subtyping relation subtyping as
\[
[t \rightarrow s] = \mathcal{P} ([t] \times [s])
\]

Bootstrap model

\( \mathcal{D} \) least solution of \( X = X^2 + \mathcal{P}_f (X^2) \)
\[
[0]_{\mathcal{D}} = \emptyset \quad [1]_{\mathcal{D}} = \mathcal{D} \quad [s \lor t]_{\mathcal{D}} = [s]_{\mathcal{D}} \cup [t]_{\mathcal{D}} \quad [s \land t]_{\mathcal{D}} = [s]_{\mathcal{D}} \cap [t]_{\mathcal{D}}
\]
\[
[\neg t]_{\mathcal{D}} = \mathcal{D} \setminus [t]_{\mathcal{D}} \quad [s \times t]_{\mathcal{D}} = [s] \times [t] \quad [t \rightarrow s]_{\mathcal{D}} = \mathcal{P}_f ([t]_{\mathcal{D}} \times [s]_{\mathcal{D}})
\]
STEP 2: set-theoretic model

**Solution:**

\[ [t \rightarrow s] = \mathcal{P}_f([t] \times [s]) \]  

(\ast)

Subtyping is completely characterised by type emptiness

Indeed:

\[ s \leq t \iff [s] \subseteq [t] \iff [s] \cap \overline{[t]} = \emptyset \iff [s \land \neg t] = \emptyset \]

\[ \mathcal{P}_f(X) = \emptyset \iff \mathcal{P}(X) = \emptyset \]

Therefore, (\ast) induces the same subtyping relation subtyping as

\[ [t \rightarrow s] = \mathcal{P}([t] \times [s]) \]

Bootstrap model

\[ \emptyset \text{ least solution of } X = X^2 + \mathcal{P}_f(X^2) \]

\[ [0]_{\emptyset} = \emptyset \quad [1]_{\emptyset} = \emptyset \quad [s \lor t]_{\emptyset} = [s]_{\emptyset} \cup [t]_{\emptyset} \quad [s \land t]_{\emptyset} = [s]_{\emptyset} \cap [t]_{\emptyset} \]

\[ [\neg t]_{\emptyset} = \emptyset \setminus [t]_{\emptyset} \quad [s \times t]_{\emptyset} = [s]_{\emptyset} \times [t]_{\emptyset} \quad [t \rightarrow s]_{\emptyset} = \mathcal{P}_f([t]_{\emptyset} \times [s]_{\emptyset}) \]
**STEP 2: set-theoretic model**

**Solution:**

\[
[t → s] = P_f([t] × [s])
\]  

(Subtyping) is completely characterised by type **emptiness**

Indeed:

\[
s ≤ t \iff [s] ≤ [t] \iff [s] ∩ [t] = \emptyset \iff [s ∧ ¬t] = \emptyset
\]

\[
P_f(X) = \emptyset \iff P(X) = \emptyset
\]

Therefore, (*) induces the same subtyping relation subtyping as

\[
[t → s] = P([t] × [s])
\]

**Bootstrap model**

\(D\) least solution of \(X = X^2 + P_f(X^2)\)

\[
\begin{align*}
[0]_D &= \emptyset \\
[1]_D &= D \\
[s ∨ t]_D &= [s]_D ∪ [t]_D \\
[s ∧ t]_D &= [s]_D ∩ [t]_D \\
¬[t]_D &= D \setminus [t]_D \\
[s ∘ t]_D &= [s]_D × [t]_D \\
[t → s]_D &= P_f([t]_D × [s]_D)
\end{align*}
\]
Solution: \[ [t \rightarrow s] = \mathcal{P}_f(\overline{[t] \times [s]}) \] \hspace{1cm} (\ast)

**Subtyping** is completely characterised by type **emptiness**

Indeed: \( s \leq t \iff [s] \subseteq [t] \iff [s] \cap \overline{[t]} = \emptyset \iff [s \land \neg t] = \emptyset \)

\[ \mathcal{P}_f(X) = \emptyset \iff \mathcal{P}(X) = \emptyset \]

Therefore, (\ast) induces the same subtyping relation subtyping as

\[ [t \rightarrow s] = \mathcal{P}(\overline{[t] \times [s]}) \]

**Bootstrap model**

\( \mathcal{D} \) least solution of \( X = X^2 + \mathcal{P}_f(X^2) \)

\[
\begin{align*}
[0]_\mathcal{D} &= \emptyset \\
[1]_\mathcal{D} &= \mathcal{D} \\
[s \lor t]_\mathcal{D} &= [s]_\mathcal{D} \cup [t]_\mathcal{D} \\
[s \land t]_\mathcal{D} &= [s]_\mathcal{D} \cap [t]_\mathcal{D} \\
\neg t]_\mathcal{D} &= \mathcal{D} \setminus [t]_\mathcal{D} \\
[s \times t]_\mathcal{D} &= [s] \times [t] \\
[t \rightarrow s]_\mathcal{D} &= \mathcal{P}_f(\overline{[t]_\mathcal{D} \times [s]_\mathcal{D}})
\end{align*}
\]
**Solution:**

\[ [t \rightarrow s] = \mathcal{P}_f([t] \times [s]) \]  

(*)

**Subtyping** is completely characterised by type *emptiness*

Indeed:

\[ s \leq t \iff [s] \subseteq [t] \iff [s] \cap \overline{[t]} = \emptyset \iff [s \land \neg t] = \emptyset \]

\[ \mathcal{P}_f(X) = \emptyset \iff \mathcal{P}(X) = \emptyset \]

Therefore, (*) induces the same subtyping relation subtyping *as*

\[ [t \rightarrow s] = \mathcal{P}([t] \times [s]) \]

**Bootstrap model**

\[ D \text{ least solution of } X = X^2 + \mathcal{P}_f(X^2) \]

\[
\begin{align*}
[0]_D &= \emptyset \\
[1]_D &= D \\
[s \lor t]_D &= [s]_D \cup [t]_D \\
[s \land t]_D &= [s]_D \cap [t]_D \\
[\neg t]_D &= D \setminus [t]_D \\
[s \times t]_D &= [s] \times [t] \\
[t \rightarrow s]_D &= \mathcal{P}_f([t]_D \times [s]_D)
\end{align*}
\]
**STEP 2: set-theoretic model**

**Solution:**

\[
[t \rightarrow s] = \mathcal{P}_f([t] \times [s])
\]

\( \star \)

**Subtyping** is completely characterised by type **emptiness**

Indeed:

\[
s \leq t \Leftrightarrow [s] \subseteq [t] \Leftrightarrow [s] \cap [t] = \emptyset \Leftrightarrow [s \land \neg t] = \emptyset
\]

\[
\mathcal{P}_f(X) = \emptyset \iff \mathcal{P}(X) = \emptyset
\]

Therefore, \( \star \) induces the same subtyping relation subtyping *as*_

\[
[t \rightarrow s] = \mathcal{P}([t] \times [s])
\]

**Bootstrap model**

\( \mathcal{D} \) least solution of \( X = X^2 + \mathcal{P}_f(X^2) \)

\[
[0]_\mathcal{D} = \emptyset \quad [1]_\mathcal{D} = \mathcal{D} \quad [s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D} \quad [s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}
\]

\[
[\neg t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D} \quad [s \times t]_\mathcal{D} = [s]_\mathcal{D} \times [t]_\mathcal{D} \quad [t \rightarrow s]_\mathcal{D} = \mathcal{P}_f([t]_\mathcal{D} \times [s]_\mathcal{D})
\]
STEP 2: set-theoretic model

Solution: \([t \rightarrow s] = \mathcal{P}_f([t] \times [s])\)  

**Subtyping** is completely characterised by type emptiness

Indeed: \(s \leq t \iff [s] \subseteq [t] \iff [s] \cap [t] = \emptyset \iff [s \land \neg t] = \emptyset\) \(\iff \mathcal{P}_f(X) = \emptyset \iff \mathcal{P}(X) = \emptyset\)

Therefore, \((\ast)\) induces the same subtyping relation subtyping as \([t \rightarrow s] = \mathcal{P}([t] \times [s])\)

Bootstrap model

\(\mathcal{D}\) least solution of \(X = X^2 + \mathcal{P}_f(X^2)\)

\([0]_\mathcal{D} = \emptyset\quad [1]_\mathcal{D} = \mathcal{D}\quad [s \lor t]_\mathcal{D} = [s]_\mathcal{D} \cup [t]_\mathcal{D}\quad [s \land t]_\mathcal{D} = [s]_\mathcal{D} \cap [t]_\mathcal{D}\)

\([\neg t]_\mathcal{D} = \mathcal{D} \setminus [t]_\mathcal{D}\quad [s \land t]_\mathcal{D} = [s]_\mathcal{D} \times [t]_\mathcal{D}\quad [t \rightarrow s]_\mathcal{D} = \mathcal{P}_f([t]_\mathcal{D} \times [s]_\mathcal{D})\)
STEP 3: Subtyping algorithm

Define:

\[ s \leq t \overset{\text{def}}{\iff} [s] \subseteq [t] \]

Use it to deduce some subtyping relations, e.g.

\[(t_1 \lor t_2) \rightarrow (s_1 \land s_2) \leq (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \leq (t_1 \lor t_2) \rightarrow (s_1 \lor s_2)\]

How to decide \( s \leq t \) in general?
STEP 3: Subtyping algorithm

- Define:

\[ s \leq t \iff [s] \subseteq [t] \]

- Use it to deduce some subtyping relations, e.g.

\[ (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \preceq (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \preceq (t_1 \lor t_2) \rightarrow (s_1 \lor s_2) \]

- How to decide \( s \leq t \) in general?
STEP 3: Subtyping algorithm

Define:

\[ s \leq t \overset{\text{def}}{\iff} \llbracket s \rrbracket \subseteq \llbracket t \rrbracket \]

Use it to deduce some subtyping relations, e.g.

\[(t_1 \lor t_2) \rightarrow (s_1 \land s_2) \not\leq (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \not\leq (t_1 \lor t_2) \rightarrow (s_1 \lor s_2)\]

How to decide \( s \leq t \) in general?
STEP 3: Subtyping algorithm

Some ugly formulae:

\[ \bigwedge_{i \in I} t_i \times s_i \leq \bigvee_{i \in J} t_i \times s_i \]

\[ \iff \forall J' \subseteq J. \left( \bigwedge_{i \in I} t_i \leq \bigvee_{i \in J'} t_i \right) \text{ or } \left( \bigwedge_{i \in I} s_i \leq \bigvee_{i \in J \setminus J'} s_i \right) \]

\[ \bigwedge_{i \in I} t_i \rightarrow s_i \leq \bigvee_{i \in J} t_i \rightarrow s_i \]

\[ \iff \exists j \in J. \forall I' \subseteq I. \left( t_j \leq \bigvee_{i \in I'} t_i \right) \text{ or } \left( I' \neq I \text{ et } \bigwedge_{i \in I \setminus I'} s_i \leq s_j \right) \]
STEP 3: Subtyping algorithm

$s \leq t$?

Recall that:

$s \leq t \iff \llbracket s \rrbracket \cap \llbracket t \rrbracket = \emptyset \iff \llbracket s \land \neg t \rrbracket = \emptyset \iff s \land \neg t = 0$

- Consider $s \land \neg t$
- Put it in canonical form

 Decide (coinductively) whether the two summands are both empty by applying the ugly formulae of the previous slide.
STEP 3: Subtyping algorithm

$s \leq t$?

Recall that:

$s \leq t \iff \llbracket s \rrbracket \cap \llbracket t \rrbracket = \emptyset \iff \llbracket s \land \lnot t \rrbracket = \emptyset \iff s \land \lnot t = 0$

1. Consider $s \land \lnot t$
2. Put it in canonical form

$$\bigvee_{(P,N) \in \Pi} \left( \bigwedge_{s \times t \in P} (s \times t) \land \bigwedge_{s \times t \in N} \lnot (s \times t) \right) \bigvee_{(P,N) \in \Sigma} \left( \bigwedge_{s \rightarrow t \in P} (s \rightarrow t) \land \bigwedge_{s \rightarrow t \in N} \lnot (s \rightarrow t) \right)$$

3. Decide (coinductively) whether the two summands are both empty by applying the ugly formulae of the previous slide.
**STEP 3: Subtyping algorithm**

$s \leq t$?

Recall that:

$$s \leq t \iff \llbracket s \rrbracket \cap \llbracket t \rrbracket = \emptyset \iff \llbracket s \land \neg t \rrbracket = \emptyset \iff s \land \neg t = 0$$

1. **Consider** $s \land \neg t$
2. **Put it in canonical form**

$$\bigvee_{(P,N) \in \Pi} \bigwedge_{s \times t \in P} s \times t \bigwedge_{s \times t \in N} \neg(s \times t)$$

$$\bigvee_{(P,N) \in \Sigma} \bigwedge_{s \rightarrow t \in P} s \rightarrow t \bigwedge_{s \rightarrow t \in N} \neg(s \rightarrow t)$$

3. **Decide (coinductively)** whether the two summands are both empty by applying the ugly formulae of the previous slide.
STEP 3: Subtyping algorithm

$s \leq t$?

Recall that:

$s \leq t \iff \llbracket s \rrbracket \cap \llbracket t \rrbracket = \emptyset \iff \llbracket s \land \neg t \rrbracket = \emptyset \iff s \land \neg t = \emptyset$

1. Consider $s \land \neg t$
2. Put it in canonical form

$$\bigvee_{(P,N) \in \Pi} (\bigwedge s \times t \in P \land (\bigwedge s \times t \neg (s \times t))) \bigvee_{s \times t \in N} (\bigwedge s \times t \neg (s \times t))$$

$$\bigvee_{(P,N) \in \Sigma} (\bigwedge s \rightarrow t \in P \land (\bigwedge s \rightarrow t \neg (s \rightarrow t))) \bigvee_{s \rightarrow t \in N} (\bigwedge s \rightarrow t \neg (s \rightarrow t))$$

3. Decide (coinductively) whether the two summands are both empty by applying the ugly formulae of the previous slide.
STEP 3: Subtyping algorithm

$s \leq t$?

Recall that:

$s \leq t \iff \llbracket s \rrbracket \cap \llbracket t \rrbracket = \emptyset \iff \llbracket s \land \neg t \rrbracket = \emptyset \iff s \land \neg t = 0$

1. Consider $s \land \neg t$
2. Put it in canonical form

$$\bigvee_{(P,N) \in \Pi} \big( \bigwedge s \times t \in P \big) \bigwedge \neg \big(s \times t\big) \big) \bigwedge \bigvee_{s \times t \in N} \big( \bigwedge s \times t \in P \big) \bigwedge \neg \big(s \times t\big) \big)$$

$$\bigvee_{(P,N) \in \Sigma} \big( \bigwedge s \rightarrow t \in P \big) \bigwedge \neg \big(s \rightarrow t\big) \big) \bigwedge \bigvee_{s \rightarrow t \in N} \big( \bigwedge s \rightarrow t \in P \big) \bigwedge \neg \big(s \rightarrow t\big) \big)$$

3. Decide (coinductively) whether the two summands are both empty by applying the ugly formulae of the previous slide.
STEP 4: The Language

Lambda-abstractions: $\lambda^{i \in I} s_i \rightarrow t_i . e$

Overloading:

$$\llbracket (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \rrbracket \not\subset \llbracket (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \rrbracket$$
STEP 4: The Language

Lambda-abstractions: $\lambda^{i \in I} s_i \rightarrow t_i x. e$

$$t \equiv (\bigwedge_{i=1}^{n} s_i \rightarrow t_i) \land (\bigwedge_{j=1}^{m} \neg (s_j' \rightarrow t_j')) \neq 0$$

(abstr) \hspace{1cm} \frac{(\forall i) \quad \Gamma, x : s_i \vdash e : t_i}{\Gamma \vdash D^\land_{i \in I} s_i \rightarrow t_i x. e : t}$

Overloading:

$$\llbracket (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \rrbracket \not\subset \llbracket (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \rrbracket$$
STEP 4: The Language

Lambda-abstractions: $\lambda^{\prod_{i \in I} s_i \rightarrow t_i} x. e$

\[
\begin{align*}
\text{Overloading:} & \\
\llbracket (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \rrbracket & \not\subseteq \llbracket (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \rrbracket
\end{align*}
\]
**STEP 4: The Language**

Lambda-abstractions: \( \lambda^{i \in I} s_i \rightarrow t_i \ x \ . \ e \)

\[
t \equiv (\bigwedge_{i=1}^{n} s_i \rightarrow t_i) \land (\bigwedge_{j=1}^{m} \neg (s'_j \rightarrow t'_j)) \neq \emptyset
\]

(abstr) \[
(\forall i) \quad \Gamma, \ x : s_i \vdash_{\mathcal{D}} e : t_i
\]

\[
\Gamma \vdash_{\mathcal{D}} \lambda^{i \in I} s_i \rightarrow t_i \ x \ . \ e : t
\]

Overloading:

\[
\llbracket (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \rrbracket \nsubseteq \llbracket (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \rrbracket
\]
Lambda-abstractions: \( \lambda^{i \in I} s_i \rightarrow t_i x . e \)

\[
t \equiv \left( \bigwedge_{i=1}^n s_i \rightarrow t_i \right) \land \left( \bigwedge_{j=1}^m s'_j \rightarrow t'_j \right) \neq \emptyset
\]

(abstr) \[
\frac{(\forall i) \quad \Gamma, x : s_i \vdash \phi \quad e : t_i}{\Gamma \vdash \phi \quad \lambda^{i \in I} s_i \rightarrow t_i x . e : t}
\]

Overloading:

\[
\llbracket (t_1 \lor t_2) \rightarrow (s_1 \land s_2) \rrbracket \not\subseteq \llbracket (t_1 \rightarrow s_1) \land (t_2 \rightarrow s_2) \rrbracket
\]
**STEP 5: Close the circle**

Let $[t]_\mathcal{V} = \{ v \in \mathcal{V} \mid \vdash_D v : t \}$, then:

$$s \leq_D t \iff s \leq_{\mathcal{V}} t$$  \hspace{1cm} (1)

Equation (1) (actually, $\Rightarrow$) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with “enough points”

$$s \not\leq_D t \iff \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t$$

In particular, thanks to negative arrows in (abstr) rule, the following two types:

$$\bigwedge_{i=1..k} s_i \to t_i \not\leq t$$

are distinguished by $\lambda^{i=1..k}s_i \to t_i \ x.e$ which inhabits their difference.

For practice try the Cduce programming language (www.cduce.org)
STEP 5: Close the circle

Let $[t]_\mathcal{V} = \{ v \in \mathcal{V} | \vdash_D v : t \}$, then:

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In particular, thanks to negative arrows in (abstr) rule, the following two types:

$$\bigwedge_{i=1..k} s_i \rightarrow t_i \not\leq t$$

are distinguished by $\lambda^{i=1..k}s_i \rightarrow t_i \mathcal{x} . e$ which inhabits their difference.

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STEP 5: Close the circle

Let \( [t]_\mathcal{V} = \{ v \in \mathcal{V} | \vdash \mathcal{D} v : t \} \), then:

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s \leq_{\mathcal{D}} t \iff s \leq_{\mathcal{V}} t
\]

Equation (1) (actually, \( \Rightarrow \)) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with “enough points”

\[
s \not\leq_{\mathcal{D}} t \iff \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t
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STEP 5: Close the circle

Let \([t]_\mathcal{V} = \{ v \in \mathcal{V} \mid \vdash_D v : t \}\), then:

\[
s \leq_D t \iff s \leq_{\mathcal{V}} t\tag{1}
\]

Equation (1) (actually, \(\Rightarrow\)) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with “enough points”

\[s \not\leq_D t \iff \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t\]

In particular, thanks to negative arrows in (abstr) rule, the following two types:

\[
\bigwedge_{i=1..k} s_i \rightarrow t_i \not\leq t
\]

are distinguished by \(\lambda^{i=1..k}s_i \rightarrow t_i \ x.e\) which inhabits their difference.

For practice try the CDuce programming language (www.cduce.org)
\(\pi\)-calculus.
STEP 1: types for $\pi$

Types $t ::= b$ basic types

$| ch^+(t)$ input channel type

$| ch^-(t)$ output channel type

$| ch(t)$ I/O channel type

$| 0$

$| 1$

$| \neg t$

$| t \lor t$

$| t \land t$

boolean combinators
STEP 1: types for $\pi$

Types $t ::=$

- $b$ basic types
- $ch^+(t)$ input channel type
- $ch^-(t)$ output channel type
- $ch(t)$ I/O channel type
- 0
- 1
- $\neg t$ boolean combinators
- $t \lor t$
- $t \land t$
STEP 2: set-theoretic model

- A channel is like a box with a particular shape.
  The box can contain only objects that fit that shape.
- A type $t$ denotes the set of objects of type $t$.
- $\llbracket ch(t) \rrbracket = \{\text{channels whose shape fits objects of type } t\}$

Channels are identified by the objects they transport.
A channel is like a box with a particular shape. The box can contain only objects that fit that shape.

A type $t$ denotes the set of objects of type $t$.

$\llbracket ch(t) \rrbracket = \{\text{channels whose shape fits objects of type } t\}$

Channels are identified by the objects they transport.

Channels are their shape.
STEP 2: set-theoretic model

- A channel is like a box with a particular shape.
  The box can contain only objects that fit that shape.
- A type \( t \) denotes the set of objects of type \( t \).
- \([ch(t)] = \{\text{channels whose shape fits objects of type } t\}\)

Channels are identified by the objects they transport.

- Channels are the set of objects that fit their shape.
STEP 2: set-theoretic model

- A channel is like a box with a particular shape
  The box can contain only objects that fit that shape
- A type \( t \) denotes the set of objects of type \( t \).
- \( \llbracket ch(t) \rrbracket = \{ \text{channels whose shape fits objects of type } t \} \)

Channels are identified by the objects they transport

- \( \llbracket ch(t) \rrbracket = \{ \text{set of objects of type } t \} \)
STEP 2: set-theoretic model

- A channel is like a box with a particular shape
  The box can contain only objects that fit that shape
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- $\llbracket ch(t) \rrbracket = \{\text{channels whose shape fits objects of type } t\}$

Channels are identified by the objects they transport

- $\llbracket ch(t) \rrbracket = \llbracket [t] \rrbracket$

Invariance of channel types
STEP 2: set theoretic model

\[ ch^+(t) \] types all channels on which I expect to read a \( t \)-message

\[ [ch^+(t)] = \{ [t'] | t' \leq t \} \]

Covariance of input types

\[ ch^-(t) \] types all channels on which I’m allowed to write a \( t \)-message

\[ [ch^-(t)] = \{ [t'] | t' \geq t \} \]

Contravariance of output types
**STEP 2: set theoretic model**

\( ch^+(t) \) types all channels on which I expect to read a \( t \)-message

- \( \llbracket ch^+(t) \rrbracket = \{ \llbracket t' \rrbracket \mid t' \leq t \} \)

Covariance of input types

\( ch^-(t) \) types all channels on which I'm allowed to write a \( t \)-message

- \( \llbracket ch^-(t) \rrbracket = \{ \llbracket t' \rrbracket \mid t' \geq t \} \)

Contravariance of output types
STEP 2: set theoretic model

\( \text{ch}^+(t) \) types all channels on which I expect to read a \( t \)-message

\[
\llbracket \text{ch}^+(t) \rrbracket = \{ \llbracket t' \rrbracket \mid \llbracket t' \rrbracket \subseteq \llbracket t \rrbracket \}
\]

Covariance of input types

\( \text{ch}^-(t) \) types all channels on which I’m allowed to write a \( t \)-message

\[
\llbracket \text{ch}^-(t) \rrbracket = \{ \llbracket t' \rrbracket \mid \llbracket t' \rrbracket \supseteq \llbracket t \rrbracket \}
\]

Contravariance of output types
STEP 2: set-theoretic model

Define $[-] : \text{Types} \rightarrow \mathcal{P}(\mathcal{D})$:

- $\llbracket t_1 \lor t_2 \rrbracket = \llbracket t_1 \rrbracket \cup \llbracket t_2 \rrbracket$, $\llbracket t_1 \land t_2 \rrbracket = \llbracket t_1 \rrbracket \cap \llbracket t_2 \rrbracket$, ...
- $\llbracket \text{ch}^+(t) \rrbracket = \{ \llbracket t' \rrbracket | \llbracket t' \rrbracket \subseteq \llbracket t \rrbracket \}$
- $\llbracket \text{ch}^-(t) \rrbracket = \{ \llbracket t' \rrbracket | \llbracket t' \rrbracket \supseteq \llbracket t \rrbracket \}$

Some induced equations:

\[
\begin{align*}
\text{ch}^-(t) \land \text{ch}^+(t) &= \text{ch}(t) \\
\text{ch}^-(s) \land \text{ch}^-(t) &= \text{ch}^-(s \lor t) \\
\text{ch}^-(s) \lor \text{ch}^-(t) &\leq \text{ch}^-(s \land t)
\end{align*}
\]

Can be checked graphically
STEP 2: set-theoretic model

Define \( [-] : \text{Types} \rightarrow \mathcal{P}(\mathcal{D}) \):

- \([t_1 \lor t_2] = [t_1] \cup [t_2], \ [t_1 \land t_2] = [t_1] \cap [t_2], \ldots\)
- \([ch^+(t)] = \{[t'] | [t'] \subseteq [t]\}\)
- \([ch^-(t)] = \{[t'] | [t'] \supseteq [t]\}\)

We need that \(\mathcal{D} = \mathcal{B} + [\mathcal{D}]\) (not straightforward)

Some induced equations:

\[
ch^-(t) \land ch^+(t) = ch(t)
\]
\[
ch^-(s) \land ch^-(t) = ch^-(s \lor t)
\]
\[
ch^-(s) \lor ch^-(t) \leq ch^-(s \land t)
\]

Can be checked graphically
STEP 2: set-theoretic model

Define \([\cdot] : \mathit{Types} \to \mathcal{P}(\mathcal{D})\):

- \([t_1 \lor t_2] = [t_1] \cup [t_2], [t_1 \land t_2] = [t_1] \cap [t_2], \ldots\)
- \([ch^+(t)] = \{[t'] \mid [t'] \subseteq [t]\}\)
- \([ch^-(t)] = \{[t'] \mid [t'] \supseteq [t]\}\)

Then define\[ t \preceq \emptyset t' \iff [t] \subseteq [t'] \]

Some induced equations:

\[
\begin{align*}
ch^-(t) \land ch^+(t) &= ch(t) \\
ch^-(s) \land ch^-(t) &= ch^-(s \lor t) \\
ch^-(s) \lor ch^-(t) &\preceq ch^-(s \land t)
\end{align*}
\]

Can be checked graphically.
STEP 2: set-theoretic model

Define $\llbracket - \rrbracket : \text{Types} \to \mathcal{P}(\mathcal{D})$:

- $\llbracket t_1 \lor t_2 \rrbracket = \llbracket t_1 \rrbracket \cup \llbracket t_2 \rrbracket$, $\llbracket t_1 \land t_2 \rrbracket = \llbracket t_1 \rrbracket \cap \llbracket t_2 \rrbracket$, …
- $\llbracket \chi^+(t) \rrbracket = \{ \llbracket t' \rrbracket | \llbracket t' \rrbracket \subseteq \llbracket t \rrbracket \}$
- $\llbracket \chi^-(t) \rrbracket = \{ \llbracket t' \rrbracket | \llbracket t' \rrbracket \supseteq \llbracket t \rrbracket \}$

Then define

$$t \leq_\emptyset t' \iff \llbracket t \rrbracket \subseteq \llbracket t' \rrbracket$$

Some induced equations:

$$\chi^-(t) \land \chi^+(t) = \chi(t)$$

$$\chi^-(s) \land \chi^-(t) = \chi^-(s \lor t)$$

$$\chi^-(s) \lor \chi^-(t) \leq \chi^-(s \land t)$$

Can be checked graphically
STEP 2: set-theoretic model

Define \([\cdot] : \text{Types} \to \mathcal{P}(\mathcal{D})\):

- \([t_1 \lor t_2] = [t_1] \cup [t_2], [t_1 \land t_2] = [t_1] \cap [t_2], \ldots\)
- \([\text{ch}^+(t)] = \{[t'] | [t'] \subseteq [t]\}\)
- \([\text{ch}^-(t)] = \{[t'] | [t'] \supseteq [t]\}\)

Then define

\[ t \leq \emptyset t' \iff [t] \subseteq [t'] \]

Some induced equations:

\[ \text{ch}^-(t) \land \text{ch}^+(t) = \text{ch}(t) \]

\[ \text{ch}^-(s) \land \text{ch}^-(t) = \text{ch}^-(s \lor t) \]

\[ \text{ch}^-(s) \lor \text{ch}^-(t) \leq \text{ch}^-(s \land t) \]

Can be checked graphically
STEP 2: set-theoretic model

Define $[-] : \text{Types} \rightarrow \mathcal{P}(\mathcal{D})$:

- $[t_1 \lor t_2] = [t_1] \cup [t_2], \quad [t_1 \land t_2] = [t_1] \cap [t_2], \ldots$
- $[ch^+(t)] = \{ [t'] | [t'] \subseteq [t] \}$
- $[ch^-(t)] = \{ [t'] | [t'] \supseteq [t] \}$

Then define

$$t \leq_{\emptyset} t' \iff [t] \subseteq [t']$$

Some induced equations:

$$ch^-(t) \land ch^+(t) = ch(t)$$
$$ch^-(s) \land ch^-(t) = ch^-(s \lor t)$$
$$ch^-(s) \lor ch^-(t) \leq ch^-(s \land t)$$

Can be checked graphically
$$\text{ch}^{-}(s) \land \text{ch}^{-}(t) = \text{ch}^{-}(s \lor t)$$
\[ ch^{-}(s) \land ch^{-}(t) = ch^{-}(s \lor t) \]
\[ ch^-(s) \land ch^-(t) = ch^-(s \lor t) \]
\[ ch^-(s) \land ch^-(t) = ch^-(s \lor t) \]
\[ ch^-(s) \lor ch^-(t) \preceq ch^-(s \land t) \]
\[ ch^-(s) \lor ch^-(t) \preceq ch^-(s \land t) \]
\[ ch^-(s) \lor ch^-(t) \preceq ch^-(s \land t) \]
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \lnot t = \emptyset \]

Put \( s \land \lnot t \) in disjunctive normal form.
A disjunction is empty if all the summands are empty.

\[ \bigwedge_{i \in P} t_i \land \bigwedge_{j \in N} \lnot t'_j = \emptyset ? \]

Equivalently
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \neg t = \emptyset \]

Put \( s \land \neg t \) in disjunctive normal form
A disjunction is empty if all the summands are empty

\[ \bigwedge_{i \in P} t_i \land \bigwedge_{j \in N} \neg t'_j = \emptyset? \]

Equivalently
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \lnot t = 0 \]

Put \( s \land \lnot t \) in disjunctive normal form

A disjunction is empty is all the summands are empty

\[ \bigwedge_{i \in P} \bigwedge_{j \in N} (t_i \land \lnot t'_j) = 0? \]

Equivalently
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \neg t = \emptyset \]

Put \( s \land \neg t \) in disjunctive normal form

A disjunction is empty is all the summands are empty

\[ \bigwedge_{i \in P} t_i \land \bigwedge_{j \in N} \neg t'_j = \emptyset? \]

Equivalently

\[ \bigwedge_{i \in P} t_i \leq \bigvee_{j \in N} t'_j ? \]
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \neg t = 0 \]

Put \( s \land \neg t \) in disjunctive normal form

A disjunction is empty if all the summands are empty

\[ \bigwedge_{i \in P} t_i \land \bigwedge_{j \in N} \neg t'_j = 0? \]

Equivalently

\[ \bigwedge_{i \in I} ch^+(t^i_1) \land \bigwedge_{j \in J} ch^-(t^j_2) \leq \bigvee_{h \in H} ch^+(t^h_3) \lor \bigvee_{k \in K} ch^-(t^k_4) \]
STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

\[ s \leq t \iff s \land \neg t = 0 \]

Put \( s \land \neg t \) in disjunctive normal form
A disjunction is empty is all the summands are empty

\[ \bigwedge_{i \in P} t_i \land \bigwedge_{j \in N} \neg t'_j = 0? \]

Equivalently

\[ ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t_3^h) \lor \bigvee_{k \in K} ch^-(t_4^k) \]
$$ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t_3^h) \lor \bigvee_{k \in K} ch^-(t_4^k)$$
\[ ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t_3^h) \lor \bigvee_{k \in K} ch^-(t_4^k) \]
\[ ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t^h_3) \lor \bigvee_{k \in K} ch^-(t^k_4) \]
\[ \text{\textit{ch}}^+(t_1) \wedge \text{\textit{ch}}^-(t_2) \leq \bigvee_{h \in H} \text{\textit{ch}}^+(t_3^h) \bigvee_{k \in K} \text{\textit{ch}}^-(t_4^k) \]
\[ ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} \bigvee_{k \in K} ch^+(t^h_3) \lor ch^-(t^k_4) \]
\[ \text{ch}^+(t_1) \land \text{ch}^-(t_2) \leq \bigvee_{h \in H} \text{ch}^+(t_h^3) \lor \bigvee_{k \in K} \text{ch}^-(t_k^4) \]
\[ ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t_3^h) \lor \bigvee_{k \in K} ch^-(t_4^k) \]
Atoms

In some cases the condition to check involves atoms
Types with a singleton interpretation

Consider:

- two types $t_1 \neq t_2$
- $t_2 \leq t_1$
- question: is $ch^+(t_1) \land ch^-(t_2) \leq ch^+(t_2) \lor ch^-(t_1)$?
\( ch^+(t_1) \land ch^-(t_2) \)  \( ch^+(t_2) \lor ch^-(t_1) \)
\[ ch^+(t_1) \land ch^-(t_2) \quad \text{and} \quad ch^+(t_2) \lor ch^-(t_1) \]
\[ ch^+(t_1) \land ch^-(t_2) \quad ch^+(t_2) \lor ch^-(t_1) \]
\[ ch^+(t_1) \land ch^-(t_2) \leq ch^+(t_2) \lor ch^-(t_1) \]
The two sets have these two points in common, namely $\text{ch}(t_1)$ and $\text{ch}(t_2)$. 

$$ch^+(t_1) \land ch^-(t_2) \leq ch^+(t_2) \lor ch^-(t_1)$$
\[ ch^+(t_1) \land ch^-(t_2) \leq ch^+(t_2) \lor ch^-(t_1) \]

The disequation holds if there are no other points in the left hand side.
\[ ch^+(t_1) \land ch^-(t_2) \leq ch^+(t_2) \lor ch^-(t_1) \]

It depends on whether \( t_1 \land \neg t_2 \) is atomic: that is whether there is nothing between \( t_1 \) and \( t_2 \)
STEP 4: The Language

Which kind of $\pi$-calculus?

Consider again

$$
ch^+(t_1) \lor ch^+(t_2) \preceq ch^+(t_1 \lor t_2)
$$

- Containment is strict, so we want programs that distinguish these two types.
- We must be able to dynamically check the type of messages arriving a channel.
- Use type-case in read actions.
STEP 4: The Language

Which kind of $\pi$-calculus?

Consider again

$$\text{ch}^+(t_1) \lor \text{ch}^+(t_2) \not\subseteq \text{ch}^+(t_1 \lor t_2)$$

- Containment is strict, so we want programs that distinguish these two types
- We must be able to dynamically check the type of messages arriving a channel
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- We must be able to dynamically check the type of messages arriving a channel
- Use `type-case` in read actions.
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$$ch^+(t_1) \lor ch^+(t_2) \preceq ch^+(t_1 \lor t_2)$$

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- We must be able to dynamically check the type of messages arriving a channel
- Use type-case in read actions.
STEP 4: The Language

Channels \( \alpha \) ::= \( x \) \hspace{1cm} \text{variables} \\
| \( c^t \) \hspace{1cm} \text{channel constants}

Processes \( P \) ::= \( \overline{\alpha}(\alpha) \) \hspace{1cm} \text{output} \\
| \( \sum_{i \in I} \alpha(x : t_i).P_i \) \hspace{1cm} \text{guarded input} \\
| \( P_1 \parallel P_2 \) \hspace{1cm} \text{parallel} \\
| (\nu c^t)P \hspace{1cm} \text{restriction} \\
| !P \hspace{1cm} \text{replication}

Reduction

\[ \overline{c}_1^s(c_2^t) \parallel \sum_{i \in I} c_1^s(x : t_i)P_i \rightarrow P_j[c_2^t/x] \text{ if } ch(t) \leq t_j \]
STEP 4: The Language

Channels \( \alpha ::= x \) variables
\quad | \quad c^t \) channel constants

Processes \( P ::= \overline{\alpha}(\alpha) \) output
\quad | \quad \sum_{i \in I} \alpha(x:t_i).P_i \) guarded input
\quad | \quad P_1 \parallel P_2 \) parallel
\quad | \quad (\nu c^t)P \) restriction
\quad | \quad !P \) replication

Reduction

\[ \overline{c}_1^s(c_2^t) \parallel \sum_{i \in I} c_1^s(x:t_i)P_i \rightarrow P_j[c_2^t/x] \quad \text{if} \ ch(t) \leq t_j \]
STEP 4: The Language

\[ \text{Channels} \quad \alpha ::= \begin{array}{l}
\begin{array}{l}
\text{x \quad \text{variables}} \\
\mid \quad \text{c}^t \quad \text{channel constants}
\end{array}
\end{array} \]

\[ \text{Processes} \quad P ::= \begin{array}{l}
\begin{array}{l}
\overline{\alpha}(\alpha) \quad \text{output} \\
\mid \quad \sum_{i \in I} \alpha(x : t_i).P_i \quad \text{guarded input} \\
\mid \quad P_1 \parallel P_2 \quad \text{parallel} \\
\mid \quad (\nu c^t)P \quad \text{restriction} \\
\mid \quad !P \quad \text{replication}
\end{array}
\end{array} \]

\[ \text{Reduction} \]

\[ \overline{c}^s_1(c^t_2) \parallel \sum_{i \in I} c^s_1(x : t_i)P_i \quad \rightarrow \quad P_j[c^t_2/x] \quad \text{if} \quad \text{ch}(t) \leq t_j \]

\[ \text{Type case} \]
STEP 4: The Language

Channels \( \alpha ::= x \) \quad \text{variables}
| \( c^t \) \quad \text{channel constants}

Processes \( P ::= \) \( \overline{\alpha}(\alpha) \) \quad \text{output}
| \( \sum_{i \in I} \alpha(x:t_i).P_i \) \quad \text{guarded input}
| \( P_1 || P_2 \) \quad \text{parallel}
| \( (\nu c^t)P \) \quad \text{restriction}
| \( !P \) \quad \text{replication}

Reduction

\[ \overline{c}_1^s(c_2^t) || \sum_{i \in I} c_1^s(x:t_i)P_i \rightarrow P_j[c_2^t/x] \quad \text{if } ch(t) \leq t_j \]

Type case
STEP 4: The Language

Channels \( \alpha ::= x \) \( \alpha \) variables
\( | c^t \) \( \alpha \) channel constants

Processes \( P ::= \overline{\alpha}(\alpha) \) \( \alpha \) output
\( | \sum_{i \in I} \alpha(x:t_i).P_i \) \( \alpha \) guarded input
\( | P_1 \parallel P_2 \) \( \alpha \) parallel
\( | (\nu c^t)P \) \( \alpha \) restriction
\( | !P \) \( \alpha \) replication

Reduction

\[ \overline{c_1^s}(c_2^t) \parallel \sum_{i \in I} c_1^s(x:t_i)P_i \rightarrow P_j[c_2^t/x] \text{ if } ch(t) \leq t_j \]

Type case call-by-value
STEP 4: The Language

\[ \text{Channels } \alpha ::= \begin{array}{c} x \quad \text{variables} \\ c^t \quad \text{channel constants} \end{array} \]

\[ \text{Processes } P ::= \begin{array}{c} \alpha(\alpha) \quad \text{output} \\ \sum_{i \in I} \alpha(x:t_i).P_i \quad \text{guarded input} \\ P_1 \parallel P_2 \quad \text{parallel} \\ (\nu c^t)P \quad \text{restriction} \\ !P \quad \text{replication} \end{array} \]

Reduction

\[ \bar{c}_1^s(c_2^t) \parallel \sum_{i \in I} c_1^s(x : t_i)P_i \rightarrow P_j[c_2^t/x] \text{ if } \text{ch}(t) \leq t_j \]

Type case call-by-value
STEP 4: The Language

(Typing rules)

\[ \Gamma \vdash c^t : ch(t) \] (chan)
\[ \Gamma \vdash x : \Gamma(x) \] (var)
\[ \Gamma \vdash \alpha : t \] (subsum)
\[ \Gamma \vdash P \] (new)
\[ \Gamma \vdash (\nu c^t)P \] (new)
\[ \Gamma \vdash P \] (repl)
\[ \Gamma \vdash !P \] (repl)
\[ \Gamma \vdash P_1 \quad \Gamma \vdash P_2 \] (para)

Matching is exhaustive
\[ t \leq \bigvee_{i \in I} t_i \quad \land \quad t \neq 0 \]
\[ \Gamma \vdash \alpha : ch^+(t) \quad \Gamma, x:t_i \vdash P_i \] (input)
\[ \Gamma \vdash \sum_{i \in I} \alpha(x:t_i).P_i \]

No useless branch
\[ \Gamma \vdash \beta : t \quad \Gamma \vdash \alpha : ch^-(t) \] (output)
\[ \Gamma \vdash \overline{\alpha}(\beta) \]
STEP 4: The Language

(Typing rules)

\[ \Gamma \vdash c^t : ch(t) \quad \text{(chan)} \]
\[ \Gamma \vdash x : \Gamma(x) \quad \text{(var)} \]
\[ \Gamma \vdash \alpha : s \leq \emptyset t \quad \text{(subsum)} \]
\[ \Gamma \vdash \beta : t \quad \Gamma \vdash \alpha : ch^- (t) \quad \text{(output)} \]
\[ \Gamma \vdash \alpha : t \quad \text{(subsum)} \]

Matching is exhaustive
\[ t \leq \bigvee_{i \in I} t_i \quad t_i \land t \neq \emptyset \]

No useless branch
\[ \Gamma \vdash \sum_{i \in I} \alpha(x:t_i).P_i \quad \text{(input)} \]
**STEP 4: The Language**

(Typing rules)

\[ \Gamma \vdash c^t : ch(t) \]  \quad (\text{chan})

\[ \Gamma \vdash x : \Gamma(x) \]  \quad (\text{var})

\[ \Gamma \vdash \alpha : s \leq t \]  \quad (\text{subsum})

\[ \Gamma \vdash P \]  \quad (\text{new})

\[ \Gamma \vdash (\nu c^t)P \]  \quad (\text{new})

\[ \Gamma \vdash P \]  \quad (\text{repl})

\[ \Gamma \vdash \alpha \vdash P_1 \]  \quad (\text{para})

\[ \Gamma \vdash P_1 \parallel P_2 \]

Matching is exhaustive

\[ t \leq \bigvee_{i \in I} t_i \]

\[ t_i \wedge t \neq 0 \]

\[ \Gamma \vdash \alpha : ch^+(t) \]

\[ \Gamma, x : t_i \vdash P_i \]

\[ \Gamma \vdash \sum_{i \in I} \alpha(x : t_i).P_i \]  \quad (\text{input})

No useless branch

\[ \Gamma \vdash \beta : t \]

\[ \Gamma \vdash \alpha : ch^-(t) \]  \quad (\text{output})

\[ \Gamma \vdash \bar{\alpha}(\beta) \]
**STEP 4: The Language**

(Typing rules)

\[
\begin{align*}
\Gamma \vdash c^t : ch(t) \quad \text{(chan)} & \quad \Gamma \vdash x : \Gamma(x) \quad \text{(var)} & \quad \Gamma \vdash \alpha : s \leq \emptyset t \quad \text{(subsum)} \\
\Gamma \vdash P \quad \text{(new)} & \quad \Gamma \vdash P \quad \text{(repl)} & \quad \Gamma \vdash P_1 \quad \Gamma \vdash P_2 \quad \text{(para)} \\
\end{align*}
\]

Matching is exhaustive

\[
\begin{align*}
\Gamma \vdash \alpha : ch^+(t) \quad \Gamma, x : t_i \vdash P_i \quad \Gamma \vdash \sum_{i \in I} \alpha(x : t_i).P_i \quad \text{(input)} \\
\end{align*}
\]

No useless branch

\[
\begin{align*}
\Gamma \vdash \beta : t \quad \Gamma \vdash \alpha : ch^-(t) \quad \Gamma \vdash \overline{\alpha}(\beta) \quad \text{(output)} \\
\end{align*}
\]
STEP 5: Closing the circle

As usual for

\[ [t]_{\nu} = \{ v \mid \vdash v : t \} \]

One has

\[ s \leq_{\emptyset} t \iff s \leq_{\nu} t \]

Note that we did not use negated types in inference rules
STEP 5: Closing the circle

As usual for

\[[t]_\gamma = \{v \mid \vdash v : t\}\]

One has

\(s \leq \emptyset t \iff s \leq \gamma t\)

Note that we did not use negated types in inference rules
Some Perspectives.
Atomic types

In \(\mathbb{C}_\pi\) subtyping check requires atomicity check

The same happens in \(\lambda\)-calculus as soon as we extend it by polymorphic types:

\[
t_1 \leq t_2 \quad \overset{\text{def}}{\iff} \quad \forall s. [t_1[s/X]] \subseteq [t_2[s/X]]
\]

Consider

\[
(t \times X) \leq (t \times \neg t) \lor (X \times t)
\]

holds true iff \(t\) is atomic \(\text{(i.e., for all } X, \text{ either } t \leq X \text{ or } t \leq \neg X)\)

Relation between atomicity and semantic subtyping?

Consider systems where atomic types are not denotable?
Atomic types

In \( C_\pi \) subtyping check requires atomicity check

The same happens in \( \lambda \)-calculus as soon as we extend it by polymorphic types:

\[
\begin{align*}
t_1 \leq t_2 & \iff \forall s. [t_1[s/X]] \subseteq [t_2[s/X]] \\
\end{align*}
\]

Consider

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(t \times X) \leq (t \times \neg t) \lor (X \times t)
\]

holds true iff \( t \) is atomic (ie, for all \( X \), either \( t \leq X \) or \( t \leq \neg X \))

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$$ t_1 \leq t_2 \iff \forall s. [t_1[s/X]] \subseteq [t_2[s/X]] $$

Consider

$$ (t \times X) \leq (t \times \neg \neg t) \lor (X \times t) $$

holds true iff $t$ is atomic (ie, for all $X$, either $t \leq X$ or $t \leq \neg X$)

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Atomic types

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t_1 \leq t_2 \iff \forall s. \llbracket t_1[s/X] \rrbracket \subseteq \llbracket t_2[s/X] \rrbracket
\]

Consider

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\]

holds true iff $t$ is atomic \hspace{1cm} (ie, for all $X$, either $t \leq X$ or $t \leq \neg X$)
Atomic types

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t_1 \leq t_2 \iff \forall s. [t_1[s/X]] \subseteq [t_2[s/X]]
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Consider

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(t \times X) \leq (t \times \lnot t) \lor (X \times t)
\]

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**Relation between atomicity and semantic subtyping?**

Consider systems where atomic types are not denotable?
Recursive types and models

In some cases, a set-theoretic model does not exist:

\[ t = \text{int} \lor (\text{ch(int)} \land \text{ch}(t)) \]

Is \( t \) equal to \( \text{int} \)?

\[ t = \text{int} \implies (\text{ch(int)} \land \text{ch}(t)) = \emptyset \implies t \neq \text{int} \]
\[ t \neq \text{int} \implies (\text{ch(int)} \land \text{ch}(t)) \neq \emptyset \implies t = \text{int} \]

Same problem in \( \lambda \)-calculus when we add reference types

What the non-existence of a set-theoretic model means?

Mathematical limit or approach’s limit?
In some cases, a set-theoretic model does not exist:

$$t = \text{int} \lor (\text{ch(int)} \land \text{ch(t)})$$

Is $t$ equal to $\text{int}$?

- If $t = \text{int}$, then $(\text{ch(int)} \land \text{ch(t)}) = \emptyset \Rightarrow t \neq \text{int}$
- If $t \neq \text{int}$, then $(\text{ch(int)} \land \text{ch(t)}) \neq \emptyset \Rightarrow t = \text{int}$

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\[
\begin{align*}
  t = \text{int} & \Rightarrow (ch(\text{int}) \land ch(t)) = \emptyset \Rightarrow t \neq \text{int} \\
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  t = \text{int} & \Rightarrow (\text{ch} (\text{int}) \land \text{ch} (t)) = \emptyset \Rightarrow t \neq \text{int} \\
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In some cases, a set-theoretic model does not exists:

\[ t = \text{int} \lor (\text{ch}(\text{int}) \land \text{ch}(t)) \]

Is \( t \) equal to \( \text{int} \)?

\[ t = \text{int} \Rightarrow (\text{ch}(\text{int}) \land \text{ch}(t)) = \emptyset \Rightarrow t \neq \text{int} \]

\[ t \neq \text{int} \Rightarrow (\text{ch}(\text{int}) \land \text{ch}(t)) \neq \emptyset \Rightarrow t = \text{int} \]

Same problem in \( \lambda \)-calculus when we add reference types

What the non-existence of a set-theoretic model means?

Mathematical limit or approach’s limit?
Full recursion is possible in *local* version of $\mathbb{C}_\pi$.

- received channels cannot be used in input
- the type $ch^+(\cdot)$ is not needed: we can have full recursion

\[
t ::= b \mid ch^+(t) \mid ch^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t
\]

\[
\alpha ::= x \mid c^t
\]

\[
P ::= \overline{\alpha}(\alpha) \mid \sum_{i \in I} \alpha(x : t_i).P_i \mid P_1 || P_2 \mid (\nu c^t)P \mid !P
\]

- known to be expressive enough (encodes $\lambda$)
- decidability (even with arrow types) is straightforward
Enough points and inference of negation

**Full recursion is possible in local version of $C_{\pi}$.**

- received channels cannot be used in input
- the type $ch^+(\cdot)$ is not needed: we can have full recursion

\[
t ::= \ b \mid ch^+(t) \mid ch^-(t) \mid 0 \mid 1 \mid \lnot t \mid t \lor t \mid t \land t
\]

\[
\alpha ::= \ x \mid c^t
\]

\[
P ::= \overline{\alpha}(\alpha) \mid \sum_{i \in I} \alpha(x:t_i).P_i \mid P_1 \parallel P_2 \mid (\nu c^t)P \mid !P
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$$t ::= b \mid ch^+(t) \mid ch^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t$$

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$$
\begin{align*}
t & ::= b \mid ch^+(t) \mid ch^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t \\
\alpha & ::= x \mid c^t \\
P & ::= \overline{\alpha}({\alpha}) \mid \sum_{i \in I} c^t(x:t_i).P_i \mid P_1 \parallel P_2 \mid (\nu c^t)P \mid !P
\end{align*}
$$

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  \[
  ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} ch^+(t_h^3) \lor \bigvee_{k \in K} ch^-(t_k^4)
  \]
Full recursion is possible in local version of $\mathbb{C}_\pi$.

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t ::= b \mid ch^+(t) \mid ch^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t$$

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$$ch^+(t_1) \land ch^-(t_2) \leq \bigvee_{h \in H} \neg ch^+(t_3^h) \lor \bigvee_{k \in K} ch^-(t_4^k)$$
Enough points and inference of negation

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- received channels cannot be used in input
- the type $ch^+(\cdot)$ is not needed: we can have full recursion
  \[
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  \]
- known to be expressive enough (encodes $\lambda$)
- decidability (even with arrow types) is straightforward
  \[
  ch^-(t) \leq \bigvee_{k \in K} ch^-(t^k) \iff \exists k \in K . t^k \leq t
  \]
Enough points and inference of negation

Full recursion is possible in local version of $\mathbb{C}$.π.

- received channels cannot be used in input
- the type $ch^+(\cdot)$ is not needed: we can have full recursion

\[ t ::= b \mid ch^-(t) \mid ch^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t \]

\[ \alpha ::= x \mid c^t \]

\[ P ::= \overline{\alpha}(\alpha) \mid \sum_{i \in I} c^t(x{:}t_i).P_i \mid P_1 \parallel P_2 \mid (\nu c^t)P \mid !P \]

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Enough points and inference of negation

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- received channels cannot be used in input
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$$t ::= b \mid ch^-(t) \mid ch^+(t) \mid 0 \mid 1 \mid \neg t \mid t\lor t \mid t\land t$$

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$$ch^-(t) \leq \bigvee_{k \in K} ch^-(t^k) \iff \exists k \in K . t^k \leq t$$

Unfortunately

$$\forall c^{\text{int}} : \neg ch^-(\text{bool})$$

(here $c^{\text{int}} : ch^-(\text{int})$ instead of $ch(\text{int})$)
Enough points and inference of negation

Full recursion is possible in \textit{local} version of $\mathbb{C}_\pi$.

- received channels cannot be used in input
- the type $ch^+(\cdot)$ is not needed: we can have full recursion

\[
\begin{align*}
t & ::= \, b \mid \text{ch}_x(t) \mid \text{ch}^-(t) \mid 0 \mid 1 \mid \neg t \mid t \lor t \mid t \land t \\
\alpha & ::= \, x \mid c^t \\
P & ::= \, \overline{\alpha}(\alpha) \mid \sum_{i \in I} c^t(x:t_i).P_i \mid P_1 \parallel P_2 \mid (\nu c^t)P \mid !P
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Unfortunately

$\forall c^{\text{int}} : \neg \text{ch}^-(\text{bool})$ (here $c^{\text{int}} : \text{ch}^-(\text{int})$ instead of $\text{ch}(\text{int})$)
Enough points and inference of negation

Solution: use the same techniques as (abstr):

\[
\frac{t_i \not\preceq t}{\Gamma \vdash c^t : \neg ch^-(t) \land \neg ch^-(t_1) \land \ldots \land \neg ch^-(t_n)} \quad \text{(chan)}
\]

As for (abstr) the rule (chan) inhabits non-empty types so that values form a set-theoretic model.

Such rules are problematic: no intuition and no minimum typing property.

Avoidable by “schemata” but intuition is not recovered and values no longer yield a model.

Inference of negations?

Have these rules a mathematical meaning?
Is it possible to find for λ a “trick” as local-\(\text{C}\)-\(\pi\)?
Enough points and inference of negation

Solution: use the same techniques as (abstr):

\[ t_i \not\leq t \]

\[ \Gamma \vdash c^t : ch^-t \wedge \neg ch^-t_1 \wedge \ldots \wedge \neg ch^-t_n \] (chan)

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Is it possible to find for \( \lambda \) a “trick” as local-\( \overline{\Pi} \)?
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Solution: use the same techniques as (abstr):

\[ t_i \not\subseteq t \]
\[ \Gamma \vdash c^t : \neg \neg ch^-(t) \land \neg \neg ch^-(t_1) \land \ldots \land \neg \neg ch^-(t_n) \] (chan)

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Solution: use the same techniques as (abstr):

\[
\frac{t_i \not\leq t}{\Gamma \vdash c^t : ch^-\left(t\right) \land \neg ch^-\left(t_1\right) \land \ldots \land \neg ch^-\left(t_n\right)} \quad \text{(chan)}
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**Inference of negations?**

Have these rules a mathematical meaning?
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Some further issues

- **Polymorphic types**
  A lot of work carried on (implicit/explicit, parametricity, …) especially in the field of programming language for XML

- **Type cases**
  How much related is semantic subtyping to the presence of typecases? Not strictly necessary but make things easier (PL viewpoint) of worst (mathematical viewpoint)

- **Dependent types**
  An unexplored country.

- **Relating semantic-\(\lambda\) and semantic-\(\pi\)**
  Fascinating as it goes deep into the relation among overloading, sequentiality, and concurrency.

**Main question**

Is semantic subtyping just a definition technique or something more?
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Conclusion
If you have a strong semantic intuition of your favourite language and you want to add set-theoretic $\lor$, $\land$, $\neg$ types then:

1. Define a set-theoretic interpretation $\llbracket \cdot \rrbracket$ for your type constructors so that it matches your semantic intuition.

2. Use the subtyping relation induced by the model to type your language: if the intuition was right then the set of values is also a model, otherwise tweak it.

3. Use the set-theoretic properties of the model to decompose the emptiness test for your type constructors, and hence derive a subtyping algorithm.

4. Enjoy.
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La morale de l’histoire est . . .

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Enjoy.
If you have a strong semantic intuition of your favourite language and you want to add set-theoretic ∨, ∧, ¬ types then:

1. **Define a set-theoretic interpretation** \([\llbracket \rrbracket]\) for your type constructors so that it matches your semantic intuition [may be not easy/possible]

2. Use the subtyping relation induced by the model to type your language: if the intuition was right then the set of values is also a model, otherwise **tweak it.** [may be not easy/possible]

3. Use the set-theoretic properties of the model to decompose the emptiness test for your type constructors, and hence **derive a subtyping algorithm.** [may be not easy/possible]

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If you have a strong semantic intuition of your favourite language and you want to add set-theoretic \(\lor\), \(\land\), \(\neg\) types then:

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2. Use the subtyping relation induced by the model to type your language: if the intuition was right then the set of values is also a model, otherwise **tweak it**.

3. Use the set-theoretic properties of the model to decompose the emptiness test for your type constructors, and hence **derive a subtyping algorithm**.

4. **Enjoy.**