Semantic Subtyping: Challenges, Perspectives, and Open Problems

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Outline

1. Motivations and goals.
2. Semantic subtyping.
3. λ-calculus.
4. π-calculus.
5. Some Perspectives.
6. Conclusion

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The goal is to show how to take your favourite type constructors

$$\times,\ \rightarrow,\ \{\ldots\},\ \mathtt{chan}(),\ \ldots$$

and add boolean combinators:

$$\vee,\quad \wedge,\quad \neg$$

so that they behave set-theoretically w.r.t. $\leq$

WHY?

Short answer: they are convenient and you need them to program XML in a typed language with **pattern matching**.

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Why it is difficult?

$$t \ ::= \ B \ \mid \ t{\times}t \ \mid \ t{\rightarrow}t \ \mid \ t{\vee}t \ \mid \ t{\wedge}t \ \mid \ \neg t \ \mid \ \mathbb{0} \ \mid \ \mathbb{1}$$

- Handling subtyping without combinators is easy: constructors do not mix, e.g.:

$$\frac{s_2 \leq s_1 \qquad t_1 \leq t_2}{s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2} \qquad\qquad \frac{s_1 \leq s_2 \qquad t_1 \leq t_2}{s_1 \times t_1 \leq s_2 \times t_2}$$

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KEY OBSERVATION 1:

The *model of types* may be independent from a *model of terms*

Hosoya and Pierce use the model of values:

$$[\![t]\!]_{\mathscr{V}} = \{v \mid \vdash v : t\}$$

Works because XML documents are the only XDuce values and for them $[\![t]\!]_{\mathscr{V}}$ can be defined independently from the typing relation

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A similar circularity holds for π-calculus channels, as well

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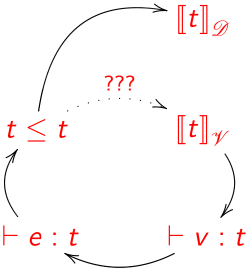

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Semantic subtyping in five steps:

1. **Add boolean combinators:** $\vee$, $\wedge$, $\neg$
   to your favourite type *constructors* (e.g., $\rightarrow$, $\times$, $ch(\,)$, …)
2. **Define a set-theoretic semantics:** $[\![\,]\!]_{\mathscr{D}}$: **Types** $\rightarrow \mathscr{P}(\mathscr{D})$
   $[\![s \wedge t]\!]_{\mathscr{D}} = [\![s]\!]_{\mathscr{D}} \cap [\![t]\!]_{\mathscr{D}}$, $[\![s \vee t]\!]_{\mathscr{D}} = [\![s]\!]_{\mathscr{D}} \cup [\![t]\!]_{\mathscr{D}}$, $[\![\neg t]\!]_{\mathscr{D}} = \mathscr{D} \setminus [\![t]\!]_{\mathscr{D}}$
   $$s \leq_{\mathscr{D}} t \stackrel{def}{\iff} [\![s]\!]_{\mathscr{D}} \subseteq [\![t]\!]_{\mathscr{D}}$$
3. Find a subtyping algorithm by using the set-theoretic properties of the model
4. Define a language and type it by using $s \leq_{\mathscr{D}} t$:
   $$\frac{\Gamma \vdash_{\mathscr{D}} e : s \quad s \leq_{\mathscr{D}} t}{\Gamma \vdash_{\mathscr{D}} e : t}$$
5. Close the circle: define the types-as-set-of-values semantics $[\![t]\!]_{\mathscr{V}} = \{v \in \mathscr{V} \mid \vdash_{\mathscr{D}} v : t\}$ and check
   $$s \leq_{\mathscr{D}} t \iff s \leq_{\mathscr{V}} t$$

The rest of the story is standard: subject reduction, …

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λ-calculus.

STEP 1: types for λ

Types	t	::=	b	basic types	type constructors
		|	$t \times t$	product type	
		|	$t \rightarrow t$	function type	
		|	$\mathbb{0}$	empty type	type combinators
		|	$\mathbb{1}$	top type	
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STEP 2: set-theoretic model

$$[\![\ ]\!]_{\mathscr{D}} : \textbf{Types} \to \mathscr{P}(\mathscr{D})$$

Easy part:

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Impossible since it requires $\mathscr{P}(\mathscr{D}^2) \subseteq \mathscr{D}$

KEY OBSERVATION 2:

Use any $[\![\ ]\!]$ that behaves w.r.t. $\subseteq$ as if equation $(*)$ held, namely

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$$[\![t_1 \to s_1]\!] \subseteq [\![t_2 \to s_2]\!] \iff \mathscr{P}(\overline{[\![t_1]\!] \times \overline{[\![s_1]\!]}}) \subseteq \mathscr{P}(\overline{[\![t_2]\!] \times \overline{[\![s_2]\!]}})$$

STEP 2: set-theoretic model

$$[\![\ ]\!]_{\mathscr{D}} : \textbf{Types} \to \mathscr{P}(\mathscr{D})$$

Easy part:

$$[\![s\wedge t]\!]_{\mathscr{D}} = [\![s]\!]_{\mathscr{D}} \cap [\![t]\!]_{\mathscr{D}} \qquad [\![s\vee t]\!]_{\mathscr{D}} = [\![s]\!]_{\mathscr{D}} \cup [\![t]\!]_{\mathscr{D}}$$

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Hard part: $[\![t\to s]\!] = \mathscr{P}(\overline{[\![t]\!] \times \overline{[\![s]\!]}})$ $\quad (*)$

Impossible since it requires $\mathscr{P}(\mathscr{D}^2) \subseteq \mathscr{D}$

KEY OBSERVATION 2:

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Subtyping is completely characterised by type **emptiness**

Indeed: $s \leq t \Leftrightarrow [\![s]\!] \subseteq [\![t]\!] \Leftrightarrow [\![s]\!] \cap \overline{[\![t]\!]} = \varnothing \Leftrightarrow [\![s{\wedge}\neg t]\!] = \varnothing$

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STEP 3: Subtyping algorithm

- Define:

$$s \leq t \quad \overset{def}{\Longleftrightarrow} \quad [\![s]\!] \subseteq [\![t]\!]$$

- Use it to deduce some subtyping relations, e.g.

$$(t_1 \vee t_2) \rightarrow (s_1 \wedge s_2) \;\lneq\; (t_1 \rightarrow s_1) \wedge (t_2 \rightarrow s_2) \;\lneq\; (t_1 \vee t_2) \rightarrow (s_1 \vee s_2)$$

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Some ugly formulae:

$$\bigwedge_{i\in I} t_i\times s_i \leq \bigvee_{i\in J} t_i\times s_i$$

$$\iff \forall J'\subseteq J.\ \left(\bigwedge_{i\in I} t_i \leq \bigvee_{i\in J'} t_i\right) \text{ or } \left(\bigwedge_{i\in I} s_i \leq \bigvee_{i\in J\setminus J'} s_i\right)$$

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$$\iff \exists j\in J.\forall I'\subseteq I.\ \left(t_j \leq \bigvee_{i\in I'} t_i\right) \text{ or } \left(I'\neq I \text{ et } \bigwedge_{i\in I\setminus I'} s_i \leq s_j\right)$$

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$$s \leq t?$$

Recall that:

$$s \leq t \iff [\![s]\!] \cap \overline{[\![t]\!]} = \varnothing \iff [\![s \wedge \neg t]\!] = \varnothing \iff s \wedge \neg t = \mathbb{0}$$

1. Consider $s \wedge \neg t$
2. Put it in canonical form

$$\bigvee_{(P,N)\in\Pi} \Big(\big(\bigwedge_{s\times t \in P} s\times t\big) \wedge \big(\bigwedge_{s\times t\in N} \neg(s\times t)\big)\Big) \quad \bigvee_{(P,N)\in\Sigma} \Big(\big(\bigwedge_{s\to t \in P} s\to t\big) \wedge \big(\bigwedge_{s\to t\in N} \neg(s\to t)\big)\Big)$$

3. Decide (coinductively) whether the two summands are both empty by applying the ugly formulae of the previous slide.

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STEP 4: The Language

Lambda-abstractions: $\lambda^{\wedge_{i\in I}s_i\to t_i}x.e$

$$t\equiv(\textstyle\bigwedge_{i=1..n} s_i\to t_i)\wedge(\bigwedge_{j=1..m}\neg(s'_j\to t'_j))\neq\mathbb{0}$$

$$\text{(abstr)}\quad \frac{(\forall i)\;\; \Gamma, x : s_i \vdash_{\mathscr{D}} e : t_i}{\Gamma \vdash_{\mathscr{D}} \lambda^{\wedge_{i\in I}s_i\to t_i}x.e : t}$$

Overloading:

$$[\![(t_1\vee t_2)\to(s_1\wedge s_2)]\!] \subsetneq [\![(t_1\to s_1)\wedge(t_2\to s_2)]\!]$$

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STEP 5: Close the circle

Let $[\![t]\!]_{\mathscr{V}} = \{v \in \mathscr{V} \mid \vdash_{\mathscr{D}} v : t\}$, then:

$$s \leq_{\mathscr{D}} t \qquad \Longleftrightarrow \qquad s \leq_{\mathscr{V}} t \tag{1}$$

Equation (1) (actually, $\Rightarrow$) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with "enough points"

$$s \not\leq_{\mathscr{D}} t \Longrightarrow \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t$$

In particular, thanks to negative arrows in (abstr) rule, the following two types:

$$\bigwedge_{i=1..k} s_i \rightarrow t_i \not\leq t$$

are distinguished by $\lambda^{\wedge_{i=1..k} s_i \rightarrow t_i} x.e$ which inhabits their difference.

For practice try the ℂDuce programming language (www.cduce.org)

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Equation (1) (actually, $\Rightarrow$) states that the language is quite rich, since there always exists a value to separate two distinct types; i.e. its set of values is a model of types with "enough points"

$$s \not\leq_{\mathscr{D}} t \Longrightarrow \text{there exists } v \text{ such that } \vdash v : s \text{ and } \not\vdash v : t$$

In particular, thanks to negative arrows in (abstr) rule, the following two types:

$$\bigwedge_{i=1..k} s_i \to t_i \not\leq t$$

are distinguished by $\lambda^{\wedge_{i=1..k} s_i \to t_i} x.e$ which inhabits their difference.

For practice try the $\mathbb{C}$Duce programming language (www.cduce.org)

π-calculus.

STEP 1: types for π

Types	t	::=	b	basic types
		|	$ch^{+}(t)$	input channel type
		|	$ch^{-}(t)$	output channel type
		|	$ch(t)$	I/O channel type
		|	$\mathbb{0}$	boolean combinators
		|	$\mathbb{1}$	
		|	$\neg t$	
		|	$t \vee t$	
		|	$t \wedge t$	

STEP 1: types for π

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		$\mid$	$t \vee t$	
		$\mid$	$t \wedge t$	

STEP 2: set-theoretic model

- A channel is like a box with a particular shape
  The box can contain only objects that fit that shape
- A type t denotes the set of objects of type t.
- $[\![ch(t)]\!] = \{$channels whose shape fits objects of type $t\}$

Channels are identified by the objects they transport

STEP 2: set-theoretic model

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- $[\![ch(t)]\!] = \{[\![t]\!]\}$

Invariance of channel types

STEP 2: set theoretic model

$ch^{+}(t)$ types all channels on which I expect to read a t-message

- $[\![ch^{+}(t)]\!] = \{[\![t']\!] \mid t' \leq t\}$

Covariance of input types

$ch^{-}(t)$ types all channels on which I'm allowed to write a t-message

- $[\![ch^{-}(t)]\!] = \{[\![t']\!] \mid t' \geq t\}$

Contravariance of output types

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Contravariance of output types

STEP 2: set-theoretic model

Define $[\![-]\!] : \mathit{Types} \to \mathscr{P}(\mathscr{D})$:

- $[\![t_1 \vee t_2]\!] = [\![t_1]\!] \cup [\![t_2]\!]$, $[\![t_1 \wedge t_2]\!] = [\![t_1]\!] \cap [\![t_2]\!]$, ...
- $[\![ch^{+}(t)]\!] = \{[\![t']\!] \mid [\![t']\!] \subseteq [\![t]\!]\}$
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Some induced equations:

$$ch^{-}(t) \wedge ch^{+}(t) = ch(t)$$

$$ch^{-}(s) \wedge ch^{-}(t) = ch^{-}(s \vee t)$$

$$ch^{-}(s) \vee ch^{-}(t) \leq ch^{-}(s \wedge t)$$

Can be checked graphically

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We need that $\mathscr{D} = \mathbb{B} + [\![\mathscr{D}]\!]$ (not straightforward)

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$$t \leq_{\mathscr{D}} t' \iff [\![t]\!] \subseteq [\![t']\!]$$

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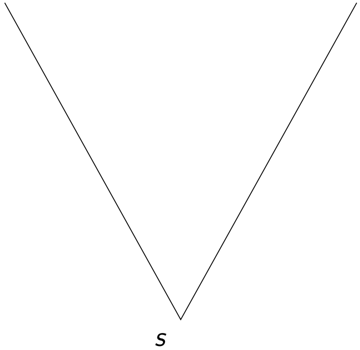

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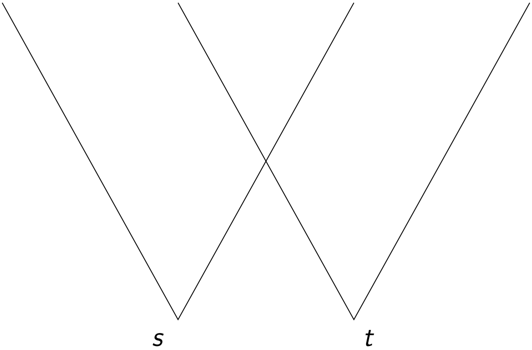

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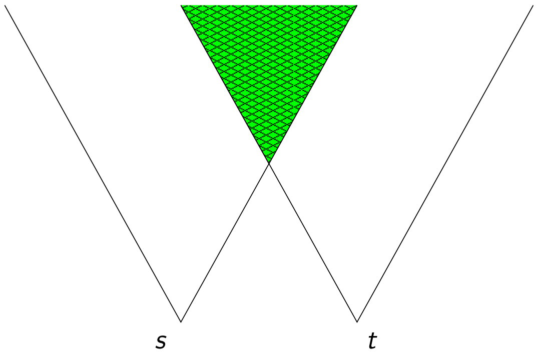

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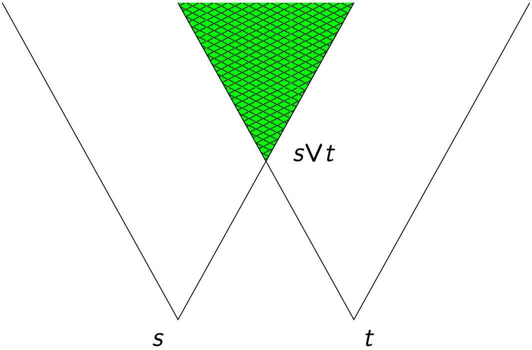

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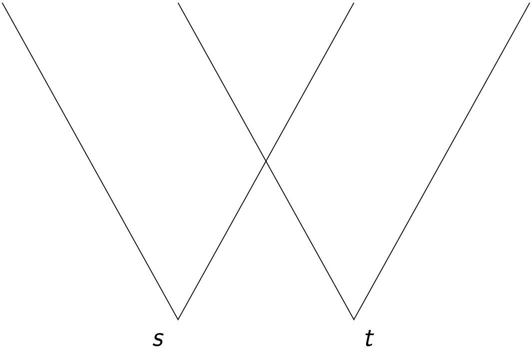

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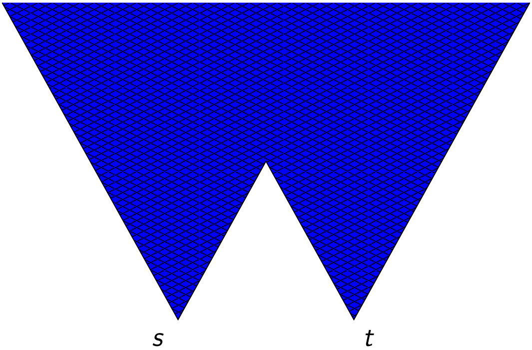

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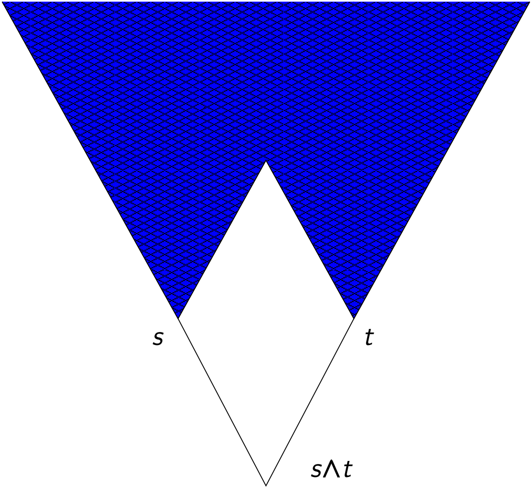

STEP 3: Subtyping Algorithm

It is enough to decide emptiness:

$$s \leq t \iff s \wedge \neg t = \mathbb{0}$$

Put $s \wedge \neg t$ in disjunctive normal form
A disjunction is empty is all the summands are empty

$$\bigwedge_{i \in P} t_i \wedge \bigwedge_{j \in N} \neg t'_j = \mathbb{0}?$$

Equivalently

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Equivalently

$$\bigwedge_{i \in I} ch^{+}(t_1^i) \wedge \bigwedge_{j \in J} ch^{-}(t_2^j) \leq \bigvee_{h \in H} ch^{+}(t_3^h) \vee \bigvee_{k \in K} ch^{-}(t_4^k)$$

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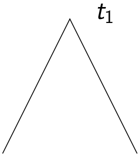

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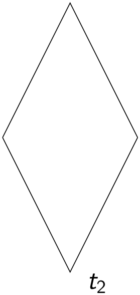

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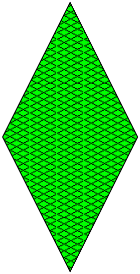

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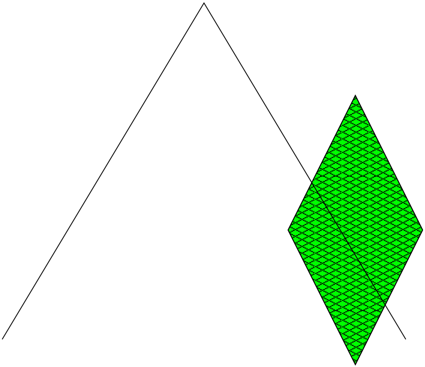

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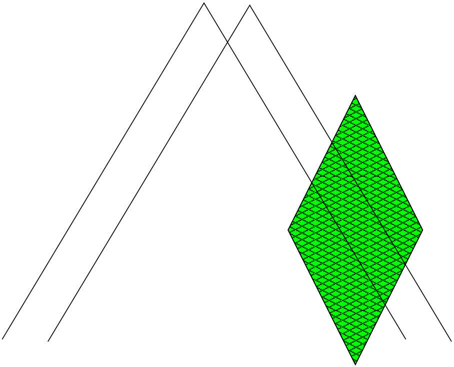

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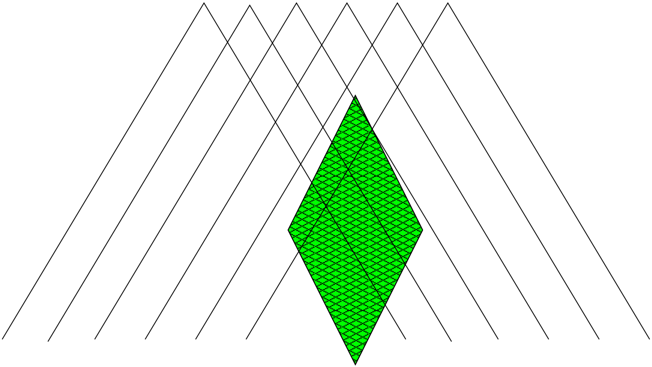

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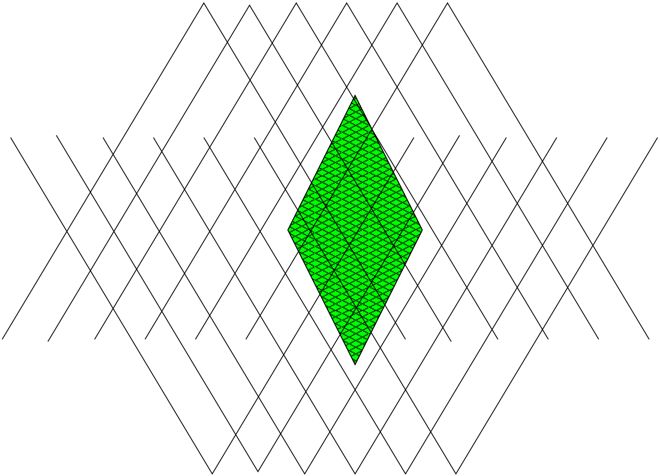


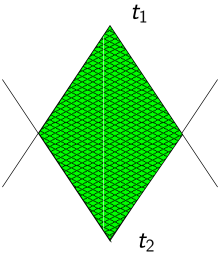
Atoms

In some cases the condition to check involves atoms
Types with a singleton interpretation

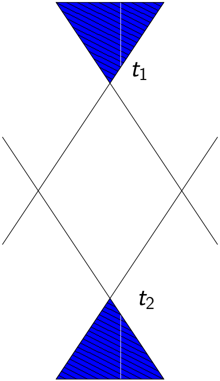
Consider:

- two types $t_1 \neq t_2$
- $t_2 \leq t_1$
- question: is $ch^+(t_1) \wedge ch^-(t_2) \leq ch^+(t_2) \vee ch^-(t_1)$?

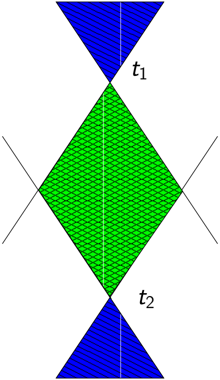
$ch^{+}(t_1) \wedge ch^{-}(t_2)$ $ch^{+}(t_2) \vee ch^{-}(t_1)$



$ch^{+}(t_1) \wedge ch^{-}(t_2)$ $\quad$ $ch^{+}(t_2) \vee ch^{-}(t_1)$



$ch^{+}(t_1) \wedge ch^{-}(t_2)$ $ch^{+}(t_2) \vee ch^{-}(t_1)$



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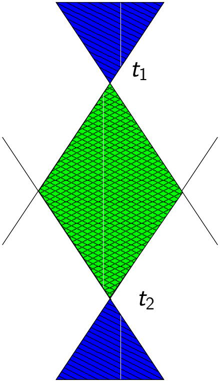

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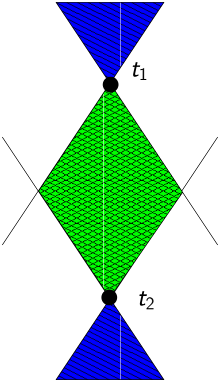
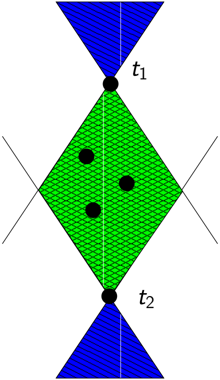


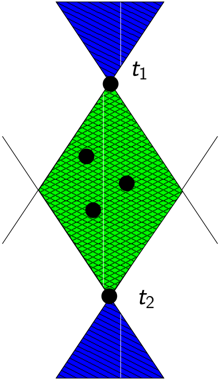
The two sets have these two points in common, namely $ch(t_1)$ and $ch(t_2)$.

$$ch^{+}(t_1) \wedge ch^{-}(t_2) \leq ch^{+}(t_2) \vee ch^{-}(t_1)$$



The disequation holds if there are no other points in the left hand side

$$ch^{+}(t_1) \wedge ch^{-}(t_2) \leq ch^{+}(t_2) \vee ch^{-}(t_1)$$



It depends on whether $t_1 \wedge \neg t_2$ is atomic: that is whether there is nothing between t_1 and t_2

STEP 4: The Language

Which kind of π-calculus?

Consider again

$$ch^{+}(\mathtt{t}_1) \vee ch^{+}(\mathtt{t}_2) \lneq ch^{+}(\mathtt{t}_1 \vee \mathtt{t}_2)$$

- Containment is strict, so we want programs that distinguish these two types
- We must be able to dynamically check the type of messages arriving a channel
- Use type-case in read actions.

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STEP 4: The Language — $\mathbb{C}\pi$-calculus

Channels	α	$::=$	x	variables
		$\mid$	c^t	channel constants
Processes	P	$::=$	$\overline{\alpha}(\alpha)$	output
		$\mid$	$\sum_{i\in I}\alpha(x{:}t_i).P_i$	guarded input
		$\mid$	$P_1 \| P_2$	parallel
		$\mid$	$(\nu c^t)P$	restriction
		$\mid$	$!P$	replication

Reduction

$$\overline{c}_1^s(c_2^t) \parallel \sum_{i\in I} c_1^s(x:t_i)P_i \quad\rightarrow\quad P_j[c_2^t/x] \qquad \text{if } ch(t) \leq t_j$$

STEP 4: The Language $\mathbb{C}\pi$-calculus

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		$\mid$	c^t	channel constants
Processes	P	$::=$	$\overline{\alpha}(\alpha)$	output
		$\mid$	$\sum_{i\in I} \alpha(x{:}t_i).P_i$	guarded input
		$\mid$	$P_1 \| P_2$	parallel
		$\mid$	$(\nu c^t)P$	restriction
		$\mid$	$!P$	replication

Reduction

$$\overline{c}_1^s(c_2^t) \parallel \sum_{i\in I} c_1^s(x : t_i)P_i \quad \rightarrow \quad P_j[c_2^t/x] \qquad \text{if } ch(t) \leq t_j$$

STEP 4: The Language $\mathbb{C}\pi$-calculus

Channels	α	$::=$	x	variables
		$\mid$	c^t	channel constants
Processes	P	$::=$	$\overline{\alpha}(\alpha)$	output
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Type case

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Type case call-by-value

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STEP 4: The Language (Typing rules)

$$\frac{}{\Gamma \vdash c^t : ch(t)}\ (\text{chan}) \qquad \frac{}{\Gamma \vdash x : \Gamma(x)}\ (\text{var}) \qquad \frac{\Gamma \vdash \alpha : s \leq_{\mathscr{D}} t}{\Gamma \vdash \alpha : t}\ (\text{subsum})$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash (\nu c^t)P}\ (\text{new}) \qquad \frac{\Gamma \vdash P}{\Gamma \vdash !P}\ (\text{repl}) \qquad \frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2}{\Gamma \vdash P_1 \| P_2}\ (\text{para})$$

Matching is exhaustive

$$\begin{array}{c} t \leq \bigvee_{i\in I} t_i \\ t_i \wedge t \neq \mathbb{0} \end{array} \quad \frac{\Gamma \vdash \alpha : ch^{+}(t) \qquad \Gamma, x{:}t_i \vdash P_i}{\Gamma \vdash \sum_{i\in I} \alpha(x{:}t_i).P_i}\ (\text{input})$$

No useless branch

$$\frac{\Gamma \vdash \beta : t \qquad \Gamma \vdash \alpha : ch^{-}(t)}{\Gamma \vdash \overline{\alpha}(\beta)}\ (\text{output})$$

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STEP 5: Closing the circle

As usual for

$$[\![t]\!]_{\mathscr{V}} = \{v \mid\ \vdash v : t\}$$

One has

$$s \leq_{\mathscr{D}} t \iff s \leq_{\mathscr{V}} t$$

Note that we did not use negated types in inference rules

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Some Perspectives.

Atomic types

In $\mathbb{C}\pi$ subtyping check requires atomicity check

The same happens in λ-calculus as soon as we extend it by polymorphic types:

$$t_1 \leq t_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \forall s. [\![t_1[s/X]]\!] \subseteq [\![t_2[s/X]]\!]$$

Consider

$$(t \times X) \leq (t \times \neg t) \vee (X \times t)$$

holds true iff t is atomic (ie, for all X, either $t \leq X$ or $t \leq \neg X$)

Relation between atomicity and semantic subtyping?

Consider systems where atomic types are not denotable?

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Recursive types and models

In some cases, a set-theoretic model does not exists:

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Is t equal to $\mathtt{int}$?

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Same problem in λ-calculus when we add reference types

What the non-existence of a set-theoretic model means?

Mathematical limit or approach's limit?

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Enough points and inference of negation

Full recursion is possible in *local* version of $\mathbb{C}\pi$.

- received channels cannot be used in input
- the type $ch^{+}()$ is not needed: we can have full recursion

$$
\begin{array}{lcl}
t & ::= & b \mid ch^{+}(t) \mid ch^{-}(t) \mid \mathbb{0} \mid \mathbb{1} \mid \neg t \mid t \vee t \mid t \wedge t \\
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P & ::= & \overline{\alpha}(\alpha) \mid \sum_{i \in I} \alpha(x{:}t_i).P_i \mid P_1 \| P_2 \mid (\nu c^{t})P \mid\ !P
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As for (abstr) the rule (chan) inhabits non-empty types so that values form a set-theoretic model

Such rules are problematic: no intuition and no minimum typing property

Avoidable by "schemata" but intuition is not recovered and values no longer yield a model

Inference of negations?

Have these rules a mathematical meaning?
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- **Polymorphic types**
  A lot of work carried on (implicit/explicit, parametricity,...) especially in the field of programming language for XML
- **Type cases**
  How much related is semantic subtyping to the presence of typecases? Not strictly necessary but make things easier (PL viewpoint) of worst (mathematical viewpoint)
- **Dependent types**
  An unexplored country.
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Conclusion

La morale de l'histoire est ...

If you have a strong semantic intuition of your favourite language and you want to add set-theoretic $\vee$, $\wedge$, $\neg$ types then:

1. Define a set-theoretic interpretation $[\![\ ]\!]$ for your type constructors so that it matches your semantic intuition
2. Use the subtyping relation induced by the model to type your language: if the intuition was right then the set of values is also a model, otherwise tweak it.
3. Use the set-theoretic properties of the model to decompose the emptiness test for your type constructors, and hence derive a subtyping algorithm.
4. Enjoy.

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