Polymorphic Functions with Set-Theoretic Types

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Based on joint work with
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Outline

1. Motivations and goals.
2. Formal setting.
3. Explicit type-substitutions.
4. Inference of type-substitutions.
5. Efficient evaluation.
6. Conclusion.
Motivations and goals

i.e., why unions, intersections, and negations of types are useful (and not just for XML)
Set-theoretic types for classic data structures

Red-black trees are balanced binary search trees that must satisfy 4 invariants:

1. the root of the tree is black
2. the leaves of the tree are black
3. no red node has a red child
4. every path from root to a leaf contains the same number of black nodes
Set-theoretic types for classic data structures

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The key to implement insert is the function balance which transforms an unbalanced tree, into a valid red tree (as long as a, b, c, and d are valid):
Red-black trees are balanced binary search trees that must satisfy 4 invariants:

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4. every path from root to a leaf contains the same number of black nodes

The key to implement `insert` is the function `balance` which transforms an unbalanced tree, into a valid red tree (as long as a, b, c, and d are valid):

In ML-like languages this yields a simple pattern-matching implementation: [due to Okasaki: *Purely Functional Data Structures*, Cambridge Univ Press, 1998]
The code as written in Okasaki’s book

type $\alpha$RBtree =
  | Leaf
  | Red($\alpha$, RBtree, RBtree)
  | Blk($\alpha$, RBtree, RBtree)

let balance =
  function
    | Blk(z, Red(x, a, Red(y, b, c)), d)
    | Blk(z, Red(y, Red(x, a, b), c), d)
    | Blk(x, a, Red(z, Red(y, b, c), d))
    | Blk(x, a, Red(y, b, Red(z, c, d)))
    -> Red(y, Blk(x, a, b), Blk(z, c, d))
    | x -> x

let insert =
  function (x, t) ->
    let ins =
      function
        | Leaf -> Red(x, Leaf, Leaf)
        | c(y, a, b) as z ->
          if x < y then balance c(y, (ins a), b) else
          if x > y then balance c(y, a, (ins b)) else z
    in let _(y, a, b) = ins t in Blk(y, a, b)
Notice that ML types \emph{do not} enforce the invariants of the previous slide.

\begin{verbatim}
type α RBtree =
  | Leaf
  | Red( α , RBtree , RBtree)
  | Blk( α , RBtree , RBtree)

let balance =
  function
  | Blk( z , Red( x , a , Red(y,b,c) ) , d )
  | Blk( z , Red( y , Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z , Red(y,b,c), d ) )
  | Blk( x , a , Red( y , b , Red(z,c,d) ) )
    -> Red( y , Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert =
  function ( x , t ) ->
  let ins =
    function
      | Leaf -> Red(x,Leaf,Leaf)
      | c(y,a,b) as z ->
        if x < y then balance c( y , (ins a), b ) else
        if x > y then balance c( y , a , (ins b) ) else z
    in let _(y,a,b) = ins t in Blk(y,a,b)
\end{verbatim}
ML needs extra auxiliary functions and GADTs to enforce these invariants.

```ml
type α RBtree =
  | Leaf
  | Red( α , RBtree , RBtree)
  | Blk( α , RBtree , RBtree)

let balance = function
  | Blk( z , Red( x , a , Red( y , b , c ) ) , d )
  | Blk( z , Red( y , Red( x , a , b ) , c ) , d )
  | Blk( x , a , Red( z , Red( y , b , c ) , d ) )
  | Blk( x , a , Red( y , b , Red( z , c , d ) ) )
    -> Red( y , Blk( x , a , b ) , Blk( z , c , d ) )
  | x -> x

let insert = function ( x , t ) ->
  let ins = function
    | Leaf -> Red( x , Leaf , Leaf )
    | c( y , a , b ) as z ->
      if x < y then balance c( y , (ins a) , b )
      else if x > y then balance c( y , a , (ins b) )
      else z
  in let _( y , a , b ) = ins t in Blk( y , a , b )
```
In set-theoretic types these functions are straightforwardly typed as they are

```ocaml
type α RBtree =
  | Leaf
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let balance =
  function
  | Blk( z , Red( x , a , Red(y,b,c) ) , d )
  | Blk( z , Red( y , Red(x,a,b) , c ) , d )
  | Blk( x , a , Red( z , Red(y,b,c) , d ) )
  | Blk( x , a , Red( y , b , Red(z,c,d) ) )
     -> Red( y , Blk(x,a,b) , Blk(z,c,d) )
  | x -> x

let insert =
  function ( x , t ) ->
  let ins =
    function
      | Leaf -> Red(x,Leaf,Leaf)
      | c(y,a,b) as z ->
          if x < y  then balance c( y , (ins a) , b )
           else
           if x > y  then balance c( y , a , (ins b) )
           else z
      in let _(y,a,b) = ins t in Blk(y,a,b)
```
In set-theoretic types these functions are straightforwardly typed as they are:

```ocaml
let balance = function
  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
  | Blk( z , Red( y, Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
  -> Red ( y, Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert = function ( x , t ) ->
  let ins = function
    | Leaf -> Red(x,Leaf,Leaf)
    | c(y,a,b) as z ->
      if x < y then balance c( y, (ins a), b )
      else if x > y then balance c( y, a, (ins b) )
      else z
  in let _(y,a,b) = ins t in Blk(y,a,b)
```

Write the correct type definitions.
In set-theoretic types these functions are straightforwardly typed as they are:

```ocaml
type α RBtree =
  | Leaf
  | Red( α , RBtree , RBtree)
  | Blk( α , RBtree , RBtree)

let balance =
  function
  | Blk( z , Red( x , a , Red(y,b,c) ) , d )
  | Blk( z , Red( y , Red(x,a,b) , c ) , d )
  | Blk( x , a , Red( z , Red(y,b,c) , d ) )
  | Blk( x , a , Red( y , b , Red(z,c,d) ) )
  -> Red( y , Blk(x,a,b) , Blk(z,c,d) )
  | x -> x

let insert =
  function ( x , t ) ->
  let ins =
    function
      | Leaf -> Red(x,Leaf,Leaf)
      | c(y,a,b) as z ->
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        if x > y then balance c( y , a , (ins b) ) else z
    in let _(y,a,b) = ins t in Blk(y,a,b)
```

① Write the correct type definitions.
In set-theoretic types these functions are straightforwardly typed as they are.

① Write the correct type definitions

```plaintext
type αRBtree =
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  | Red( α , RBtree , RBtree)
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let balance =
  function
    | Blk( z , Red( x , a , Red(y,b,c) ) , d )
    | Blk( z , Red( y , Red(x,a,b) , c ) , d )
    | Blk( x , a , Red( z , Red(y,b,c) , d ) )
    | Blk( x , a , Red( y , b , Red(z,c,d) ) )
    -> Red( y , Blk(x,a,b) , Blk(z,c,d) )
    | x -> x

let insert =
  function( x , t ) ->
    let ins =
      function
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        | c(y,a,b) as z ->
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      in let _(y,a,b) = ins t in Blk(y,a,b)
```

② Add type annotations to function definitions

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In set-theoretic types these functions are straightforwardly typed as they are.

```plaintext

type αRBtree =
    | Leaf
    | Red( α , RBtree , RBtree)
    | Blk( α , RBtree , RBtree)

let balance =
    function |
        Blk( z , Red( x, a, Red(y,b,c) ) , d )
        Blk( z , Red( y, Red(x,a,b), c ) , d )
        Blk( x , a , Red( z, Red(y,b,c), d ) )
        Blk( x , a , Red( y, b, Red(z,c,d) ) )
        -> Red ( y, Blk(x,a,b), Blk(z,c,d) )
        x -> x

let insert =
    function ( x , t ) ->
        let ins =
            function |
                Leaf -> Red(x,Leaf,Leaf)
                c(y,a,b) as z ->
                    if x < y then balance c( y, (ins a), b ) else
                    if x > y then balance c( y, a, (ins b) ) else z
        in let _ (y,a,b) = ins t in Blk(y,a,b)
```
type $\alpha$RBtree =
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| Blk( z , Red( x, a, Red(y,b,c) ) , d )
| Blk( z , Red( y, Red(x,a,b), c ) , d )
| Blk( x , a , Red( z, Red(y,b,c), d ) )
| Blk( x , a , Red( y, b, Red(z,c,d) ) )
  -> Red( y, Blk(x,a,b), Blk(z,c,d) )
| x -> x

let insert =
function ( x , t ) ->
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| Leaf -> Red(x,Leaf,Leaf)
| c(y,a,b) as z ->
  if x < y then balance c( y, (ins a), b ) else
  if x > y then balance c( y, a, (ins b) ) else z
in let _(y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

type Rtree = Red(α, Btree, Btree)

type Btree = Blk(α, RBtree, RBtree) | Leaf

type Wrong = Red(α, (Rtree,RBtree) | (RBtree,Rtree))

type Unbal = Blk(α, (Wrong,RBtree) | (RBtree,Wrong))

let balance =
function
| Blk( z , Red( x, a, Red(y,b,c) ) , d )
| Blk( z , Red( y, Red(x,a,b), c ) , d )
| Blk( x , a , Red( z, Red(y,b,c), d ) )
| Blk( x , a , Red( y, b, Red(z,c,d) ) )
 -> Red( y, Blk(x,a,b), Blk(z,c,d) )
| x -> x

let insert =
function ( x , t ) ->
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function
| Leaf -> Red(x,Leaf,Leaf)
| c(y,a,b) as z ->
  if x < y then balance c( y, (ins a), b )
  else if x > y then balance c( y, a, (ins b) )
else z
in let _(y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

type Rtree = Red(\(\alpha\), Btree, Btree)

type Btree = Blk(\(\alpha\), RBtree, RBtree) | Leaf

type Wrong = Red(\(\alpha\), (Rtree,RBtree)| (RBtree,Rtree) )

type Unbal = Blk(\(\alpha\), (Wrong, RBtree)| (RBtree, Wrong) )

let balance =
  function
  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
  | Blk( z , Red( y, Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
  -> Red( y, Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert =
  function ( x , t ) ->
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    | Leaf -> Red(x,Leaf,Leaf)
    | c(y,a,b) as z ->
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    in let _(y,a,b) = ins t in Blk(y,a,b)
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  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
  | Blk( z , Red( y, Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
  -> Red ( y, Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert =
  function ( x , t ) ->
  let ins =
    function
      | Leaf -> Red(x,Leaf,Leaf)
      | c(y,a,b) as z ->
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      in let _ (y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

type Rtree = Red(\alpha, Btree, Btree)

type Btree = Blk(\alpha, RBtree, RBtree) | Leaf

type Wrong = Red(\alpha, (Rtree,RBtree) | (RBtree,Rtree))

type Unbal = Blk(\alpha, (Wrong,RBtree) | (RBtree,Wrong))

let balance: (Unbal \rightarrow Rtree) \& ((\beta \setminus Unbal) \rightarrow (\beta \setminus Unbal)) =

function
| Blk(z, Red(x, a, Red(y,b,c), d)) |
| Blk(z, Red(y, Red(x,a,b), c), d) |
| Blk(x, a, Red(z, Red(y,b,c), d)) |
| Blk(x, a, Red(y, b, Red(z,c,d))) |
| (x) |
| Red(y, Blk(x,a,b), Blk(z,c,d)) |
| x -> x |

let insert: (\alpha, Btree) \rightarrow Btree =

function (x, t) ->

let ins:(Leaf \rightarrow Rtree) \& (Btree \rightarrow RBtree \setminus Leaf) \& (Rtree \rightarrow Rtree | Wrong) =

function
| Leaf -> Red(x,Leaf,Leaf) |
| (c(y,a,b) as z -> |
|  if x < y then balance c(y, (ins a), b) else |
|  if x > y then balance c(y, a, (ins b)) else z |
in let _(y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

type Rtree = Red(α, Btree, Btree)

type Btree = Blk(α, RBtree, RBtree) | Leaf

type Wrong = Red(α, (Rtree,RBtree) | (RBtree,Rtree))

type Unbal = Blk(α, (Wrong,RBtree) | (RBtree,Wrong))

let balance: (Unbal → Rtree) & ((β\Unbal) → (β\Unbal)) =
  function
  | Blk( z , Red( x , a , Red(y,b,c) ) , d )
  | Blk( z , Red( y , Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z , Red(y,b,c), d ) )
  | Blk( x , a , Red( y , b , Red(z,c,d) ) )
  -> Red( y , Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert: (α, Btree) → Btree =
  function ( x , t ) ->
  let ins:(Leaf → Rtree) & (Btree → RBtree\Leaf) & (Rtree → Rtree|Wrong) =
    function
    | Leaf -> Red(x,Leaf,Leaf)
    | c(y,a,b) as z ->
      if x < y then balance c( y , (ins a) , b ) else
      if x > y then balance c( y , a , (ins b) ) else z
    in let _(y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

let balance: (Unbal -> Rtree) & ((β \ Unbal) -> (β \ Unbal)) =

| Blk( z , Red( x , a , Red( y , b , c ) ) , d ) |
| Blk( z , Red( y , Red( x , a , b ) , c ) , d ) |
| Blk( x , a , Red( z , Red( y , b , c ) , d ) ) |
| Blk( x , a , Red( y , b , Red( z , c , d ) ) ) |
| x -> x |

let insert: (α , Btree) -> Btree =

let ins: (Leaf -> Rtree) & (Btree -> RBtree \ Leaf) & (Rtree -> Rtree | Wrong) =

| Leaf -> Red(x,Leaf,Leaf) |
| c(y,a,b) as z -> |
| if x < y then balance c( y , (ins a) , b ) else |
| if x > y then balance c( y , a , (ins b) ) else z |

in let _(y,a,b) = ins t in Blk(y,a,b)
type RBtree = Btree | Rtree

let balance: (Unbal \rightarrow Rtree) \& (Unbal \rightarrow Unbal) =
  function |
  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
  | Blk( z , Red( y, Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
  -> Red ( y, Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert: (\alpha , Btree) \rightarrow Btree =
  function ( x , t ) ->
  let ins:(Leaf \rightarrow Rtree) \& (Btree \rightarrow RBtree \& Leaf) \& (Rtree \rightarrow Rtree | Wrong)=
    function |
    | Leaf -> Red(x,Leaf,Leaf)
    | c(y,a,b) as z ->
      if x < y then balance c( y, (ins a), b ) else
      if x > y then balance c( y, a, (ins b) ) else z
    in let _(y,a,b) = ins t in Blk(y,a,b)
1. Motivations –

```
type RBtree = Btree | Rtree
type Rtree = Red(α, Btree, Btree)
type Btree = Blk(α, RBtree, RBtree) | Leaf

type Wrong = Red(α, (Rtree, RBtree) | (RBtree, Rtree))
type Unbal = Blk(α, (Wrong, RBtree) | (RBtree, Wrong))

let balance: (Unbal → Rtree) & ((β \ Unbal) → (β \ Unbal)) =
  function
  | Blk(z, Red(x, a, Red(y, b, c), d))
  | Blk(z, Red(y, Red(x, a, b), c), d)
  | Blk(x, a, Red(z, Red(y, b, c), d))
  | Blk(x, a, Red(y, b, Red(z, c, d)))
  | -> Red(y, Blk(x, a, b), Blk(z, c, d))
  | x -> x

let insert: (α, Btree) → Btree =
  function (x, t) ->
    let ins: (Leaf → Rtree) & (Btree → RBtree \ Leaf) & (Rtree → Rtree | Wrong) =
      function
        | Leaf -> Red(x, Leaf, Leaf)
        | c(y,a,b) as z ->
          if x < y then balance c(y, (ins a), b) else
          if x > y then balance c(y, a, (ins b)) else z
      in let _(y,a,b) = ins t in Blk(y,a,b)
```
type RBtree = Btree | Rtree

type Rtree = Red(\(\alpha\), Btree, Btree)

type Btree = Blk(\(\alpha\), RBtree, RBtree) | Leaf

type Wrong = Red(\(\alpha\), (Rtree,RBtree)\(\mid\)(RBtree,Rtree))

type Unbal = Blk(\(\alpha\), (Wrong,RBtree)\(\mid\)(RBtree,Wrong))

let balance: (Unbal \(\rightarrow\) Rtree) & ((\(\beta\)\(\backslash\)Unbal) \(\rightarrow\) (\(\beta\)\(\backslash\)Unbal)) =
  function
  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
  | Blk( z , Red( y, Red(x,a,b), c ) , d )
  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
    -> Red( y, Blk(x,a,b), Blk(z,c,d) )
  | x -> x

let insert: (\(\alpha\), Btree) \(\rightarrow\) Btree =
  function ( x , t ) ->
  let ins:(Leaf \(\rightarrow\) Rtree) & (Btree \(\rightarrow\) RBtree\(\backslash\)Leaf) & (Rtree \(\rightarrow\) Rtree|Wrong)=
    function
    | Leaf -> Red(x,Leaf,Leaf)
    | c(y,a,b) as z ->
      if x < y then balance c( y, (ins a), b ) else
      if x > y then balance c( y, a, (ins b) ) else z
    in let _(y,a,b) = ins t in Blk(y,a,b)
A simpler example of the same pattern
A motivating example in Haskell

\[
\begin{align*}
\text{map} & :: (\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta] \\
\text{map } f \ l &= \text{case } l \ \text{of} \\
| \ [] & \rightarrow \ [] \\
| (x : xs) & \rightarrow (f \ x : \text{map } f \ xs)
\end{align*}
\]
A motivating example in Haskell

\[
\text{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta]
\]
\[
\text{map } f \text{ } l = \text{ case } l \text{ of} \\
| [] \to [] \\
| (x : xs) \to (f x : \text{map } f \text{ } xs)
\]

\[
\text{even} :: (\text{Int} \to \text{Bool}) \land ((\alpha \setminus \text{Int}) \to (\alpha \setminus \text{Int}))
\]
\[
\text{even } x = \text{ case } x \text{ of} \\
| \text{Int} \to (x \mod 2) == 0 \\
| \_ \to x
\]
A motivating example in Haskell (almost)

map :: (\alpha \to \beta) \to [\alpha] \to [\beta]
map f l = case l of
| [] -> []
| (x : xs) -> (f x : map f xs)

even :: (Int \to \text{Bool}) \land ((\alpha \setminus \text{Int}) \to (\alpha \setminus \text{Int}))
even x = case x of
| \text{Int} -> (x \mod 2) == 0
| _ -> x
A motivating example in Haskell (almost)

\[
\text{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta]
\]
\[
\text{map } f \ l = \text{case } l \text{ of }
\]
\[
\quad | [] \to []
\quad | (x : xs) \to (f \ x : \text{map } f \ xs)
\]

\[
\text{even} :: (\text{Int} \to \text{Bool}) \land ((\alpha \setminus \text{Int}) \to (\alpha \setminus \text{Int}))
\]
\[
\text{even } x = \text{case } x \text{ of }
\]
\[
\quad | \text{Int} \to (x \ 'mod' \ 2) == 0
\quad | _ \to x
\]

- **Expression:** if the argument is an integer then return the Boolean expression otherwise return the argument
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
  | []    -> []
  | (x : xs) -> (f x : map f xs)

even :: (Int → Bool) ∧ ((α \ Int) → (α \ Int))
even x = case x of
  | Int    -> (x ‘mod‘ 2) == 0
  | _      -> x

- **Expression**: if the argument is an integer then return the Boolean expression otherwise return the argument
- **Type**: when applied to an Int it returns a Bool; when applied to an argument that is not an Int it returns a result of the same type.
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
  | [] -> []
  | (x : xs) -> (f x : map f xs)

even :: (Int → Bool) ∧ ((α \ Int) → (α \ Int))
even x = case x of
  | Int -> (x 'mod' 2) == 0
  | _ -> x

- **Expression**: if the argument is an integer then return the Boolean expression otherwise return the argument
- **Type**: when applied to an `Int` it returns a `Bool`; when applied to an argument that is not an `Int` it returns a result of the same type.
A motivating example in Haskell (almost)

\[
\text{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta]
\]

\[
\text{map} \ f \ \text{ls} = \text{case} \ \text{ls} \ \text{of}
\begin{align*}
| \ [\] & \to \ [\] \\
| (x : \ \text{xs}) & \to (f \ x : \ \text{map} \ f \ \text{xs})
\end{align*}
\]

\[
\text{even} :: (\text{Int} \to \text{Bool}) \land ((\alpha \to \text{Int}) \to (\alpha \to \text{Int}))
\]

\[
\text{even} \ x = \text{case} \ x \ \text{of}
\begin{align*}
| \ \text{Int} & \to (x \ \text{mod} \ 2) = 0 \\
| \_ & \to \ x
\end{align*}
\]

- **Expression**: if the argument is an integer then return the Boolean expression otherwise return the argument.

- **Type**: when applied to an \text{Int} it returns a \text{Bool}; when applied to an argument that is not an \text{Int} it returns a result of the same type.
A motivating example in Haskell (almost)

\[
\text{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta]
\]

\[
\text{map } f \text{ } l = \begin{cases} 
\emptyset & \text{if } l = \emptyset \\
(f \ x : \text{map } f \text{ } xs) & \text{if } l = (x : xs) 
\end{cases}
\]

\[
\text{even} :: (\text{Int} \to \text{Bool}) \wedge ((\alpha \to \text{Int}) \to (\alpha \to \text{Int}))
\]

\[
\text{even } x = \begin{cases} 
\text{if } x \text{ is Int} & \text{(} x \mod 2 \text{) } = 0 \\
\text{else} & x
\end{cases}
\]

- **Expression:** if the argument is an integer then return the Boolean expression otherwise return the argument
- **Type:** when applied to an `Int` it returns a `Bool`; when applied to an argument that is not an `Int` it returns a result of the same type.
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
    | []   -> []
    | (x : xs) -> (f x : map f xs)

even :: (Int → Bool) ∧ ((α \ Int) → (α \ Int))
even x = case x of
    | Int   -> (x 'mod' 2) == 0
    | _     -> x

- **Expression**: if the argument is an integer then return the Boolean expression otherwise return the argument
- **Type**: when applied to an Int it returns a Bool; when applied to an argument that is not an Int it returns a result of the same type.

Common pattern for functional data structures: red-black trees balancing; ZDD operations; XML nodes modification.
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
    | [] -> []
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- **Expression**: if the argument is an integer then return the Boolean expression otherwise return the argument
- **Type**: when applied to an Int it returns a Bool; when applied to an argument that is not an Int it returns a result of the same type.

The combination of type-case and intersections yields statically typed dynamic overloading.
A motivating example in Haskell (almost)

map :: \((\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta]\)
map \ f \ l = \ case \ l \ of
  \ |
  \ | \ [] \rightarrow \ []
  \ | \ (x : xs) \rightarrow (f \ x : \ map \ f \ xs)

even :: \((\text{Int} \rightarrow \text{Bool}) \land ((\alpha \downarrow \text{Int}) \rightarrow (\alpha \downarrow \text{Int}))\)
even \ x = \ case \ x \ of
  \ |
  \ | \ \text{Int} \rightarrow (x \ \text{`mod`} \ 2) == 0
  \ | \ _ \rightarrow x

This example as a yardstick. I want to define a language that:

1. Can define both \text{map} and \text{even}
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
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This example as a yardstick. I want to define a language that:

1. Can define both map and even
2. Can check the types specified in the signature
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
  | [] -> []
  | (x : xs) -> (f x : map f xs)

even :: (Int → Bool) ∧ ((α\Int) → (α\Int))
even x = case x of
  | Int -> (x ‘mod’ 2) == 0
  | _ -> x

This example as a yardstick. I want to define a language that:

1. Can define both `map` and `even`
2. Can *check* the types specified in the signature
3. Can *deduce* the type of the partial application `map even`
A motivating example in Haskell (almost)

map :: \((\alpha \to \beta) \to [\alpha] \to [\beta]\)
map f l = case l of
    | [] -> []
    | (x : xs) -> (f x : map f xs)

even :: (Int \to \text{Bool}) \land ((\alpha \setminus \text{Int}) \to (\alpha \setminus \text{Int}))
even x = case x of
    | Int -> (x \text{mod} 2) == 0
    | _ -> x

This example as a yardstick. I want to define a language that:

1. Can define both \texttt{map} and \texttt{even}
2. Can \textit{check} the types specified in the signature
3. Can deduce the type of the partial application \texttt{map even}
A motivating example in Haskell (almost)

\[
\begin{align*}
\text{map} & : (\alpha \to \beta) \to [\alpha] \to [\beta] \\
\text{map } f \ l & = \text{case } l \ \text{of} \\
& \quad | \ [\] \to [] \\
& \quad | \ (x : xs) \to (f \ x : \text{map } f \ xs)
\end{align*}
\]

\[
\begin{align*}
\text{even} & : (\textsf{Int} \to \textsf{Bool}) \land ((\alpha \land \textsf{Int}) \to (\alpha \land \textsf{Int})) \\
\text{even } x & = \text{case } x \ \text{of} \\
& \quad | \ \textsf{Int} \to (x \ 'mod' \ 2) = 0 \\
& \quad | \ _ \to x
\end{align*}
\]

This example acts as a yardstick. I want to define a language that:

1. Can define both \texttt{map} and \texttt{even}
2. Can check the types specified in the signature
3. Can deduce the type of the partial application \texttt{map even}

\textbf{Tough!}
A motivating example in Haskell (almost)

```
map :: (\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta]
map f l = case l of
  | []    -> []
  | (x : xs) -> (f x : map f xs)

even :: (Int \rightarrow \mathbb{Bool}) \wedge ((\alpha \setminus \text{Int}) \rightarrow (\alpha \setminus \text{Int}))
even x = case x of
  | Int -> (x \ mod\ 2) == 0
  | _    -> x
```

We expect \textbf{map even} to have the following type:

\[
[[\text{Int}] \rightarrow [\text{Bool}]] \wedge

\{[[\alpha \setminus \text{Int}] \rightarrow [[\alpha \setminus \text{Int}]]] \wedge

\{[[\alpha \setminus \text{Int}] \rightarrow [(\alpha \setminus \text{Int}) \vee \text{Bool}]]\}\]
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
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  | _     -> x

We expect \texttt{map even} to have the following type:

\[
\begin{align*}
([\text{Int}] \rightarrow [\text{Bool}]) \land & \quad \text{int lists are transformed into bool lists} \\
([α\text{\textbackslash Int}] \rightarrow [α\text{\textbackslash Int}]) \land & \quad \text{lists w/o ints return the same type} \\
([α\lor\text{Int}] \rightarrow [(α\text{\textbackslash Int})\lor\text{Bool}]) & \quad \text{ints in the arg. are replaced by bools}
\end{align*}
\]
A motivating example in Haskell (almost)

map :: (α → β) → [α] → [β]
map f l = case l of
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even :: (Int → Bool) ∧ ((α\Int) → (α\Int))
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We expect `map even` to have the following type:

\[
\begin{align*}
&([\text{Int}] \rightarrow [\text{Bool}]) \land \\
&([\alpha \backslash \text{Int}] \rightarrow [\alpha \backslash \text{Int}]) \land \\
&([\alpha \backslash \text{Int}] \rightarrow [(\alpha \backslash \text{Int}) \lor \text{Bool}])
\end{align*}
\]

int lists are transformed into bool lists
lists w/o ints return the same type
ints in the arg. are replaced by bools

Difficult because of expansion: needs a set of type substitutions — rather than just one — to unify the domain and the argument types.
Formal framework

i.e., all the gory details you do not want the programmer to ever know
Formal calculus

| Exprs  | $e ::= x | ee | \lambda i \in I . t_i . e | e \in t ? e : e$ |
|--------|----------------------------------------------------------|
| Types  | $t ::= B | t \rightarrow t | t \lor t | t \land t | \neg t | 0 | 1 | \alpha$ |
Expressions include:

\[
\begin{align*}
\text{Exprs} & \quad e & ::= & \ x \ | \ ee \ | \ \lambda^i_{i \in I}s_i \to t_i \ x.e \ | \ e \in t ? e : e \\
\text{Types} & \quad t & ::= & \ B \ | \ t \to t \ | \ t \lor t \ | \ t \land t \ | \ \neg t \ | \ 0 \ | \ 1 \ | \ \alpha
\end{align*}
\]
### Formal calculus

#### Expressions include:

- A type-case:
  - abstracts regular type patterns
  - makes dynamic overloading possible
Expressions include:

A type-case:
- abstracts regular type patterns
- makes dynamic overloading possible

Explicitly-typed functions:
- Needed by the type-case
  \[ \mu f. \lambda x.f(1 \to \text{Int})? \text{true : 42} \]
- More expressive with the result type (parameter type not enough)

\[ \lambda^{i \in I} s_i \to t_i x.e : \text{well typed if for all } i \in I \text{ from } x : s_i \text{ we can deduce } e : t_i. \]
Formal calculus

\[\text{Exprs} \quad e ::= \ x \mid ee \mid \lambda^{i \in I} t_i x.e \mid e \in t? e : e\]

\[\text{Types} \quad t ::= B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha\]

Types may be recursive and have a set-theoretic interpretation:
Formal calculus

<table>
<thead>
<tr>
<th>Exprs</th>
<th>$e ::= x \mid ee \mid \lambda^i \in \exists_i t_i x.e \mid e \in t ? e : e$</th>
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Types may be recursive and have a set-theoretic interpretation:

 Constructors: $[\text{Int}] = \{0, 1, -1, \ldots\}$. $[s \rightarrow t] =$ all $\lambda$-abstractions that applied to arguments in $[s]$ return only results in $[t]$. 
Formal calculus

\[
\text{Exprs} \quad e ::=} x \mid ee \mid \lambda^{i \in I_{s_i} \rightarrow t_i} x.e \mid e \in t \, ? \, e : e
\]

\[
\text{Types} \quad t ::=} B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha
\]

**Types** may be **recursive** and have a **set-theoretic** interpretation:

**Constructors**: \([\textbf{Int}] = \{0, 1, -1, \ldots\}\). \([s \rightarrow t] = \text{all } \lambda\text{-abstractions that applied to arguments in } [s] \text{ return only results in } [t] \).

**Connectives** have the corresponding set-theoretic interpretation:

\([s \lor t] = [s] \cup [t] \quad [s \land t] = [s] \cap [t] \quad [\neg t] = [1] \setminus [t] \)
Formal calculus

\[ \text{Exprs} \quad e ::= x \mid ee \mid \lambda i \in I^s \rightarrow t^i \ x.e \mid e \in t \ ? \ e : e \]

\[ \text{Types} \quad t ::= B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha \]

Types may be recursive and have a set-theoretic interpretation:

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Subtyping:
- it is defined as set-containment: \(s \leq t \iff [s] \subseteq [t] \);
Formal calculus

\[
\text{Exprs} \quad e ::= \ x \ | \ ee \mid \lambda^{i \in I}s_i \mapsto t_i x.e \mid e \in t\ ？
\]

\[
\text{Types} \quad t ::= \ B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha
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\[
[s \lor t] = [s] \cup [t] \quad [s \land t] = [s] \cap [t] \quad [\neg t] = [1] \setminus [t]
\]

Subtyping with type variables:

- it is defined as set-containment: \(s \leq t \iff [s] \subseteq [t]\);
- it is such that for all type-substitutions \(\sigma\): \(s \leq t \Rightarrow s\sigma \leq t\sigma\);
- it is decidable. [ICFP2011].
Formal calculus: **new stuff**

\[
\text{Exprs} \quad e ::= x \mid ee \mid \lambda^{i \in I} s_i \to t_i x.e \mid e \in t \, ? \, e : e
\]

\[
\text{Types} \quad t ::= B \mid t \to t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha
\]

**Polymorphic functions.**
Polymorphic functions: The novelty of this work is that type variables can occur in the interfaces.
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- $\lambda^{\alpha \rightarrow \alpha}x.x$ polymorphic identity
- $\lambda^{(\alpha \rightarrow \beta) \land \alpha \rightarrow \beta}x.xx$ auto-application
## Formal calculus: **new stuff**

**Exprs**

\[
e ::= x \mid ee \mid \lambda^i \in I^{s_i \rightarrow t_i} x.e \mid e \in t \ ? e : e
\]

**Types**

\[
t ::= B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha
\]

**Polymorphic functions:** The novelty of this work is that **type variables** can occur in the **interfaces**.

- \(\lambda^{\alpha \rightarrow \alpha} x.x\) polymorphic identity
- \(\lambda(\alpha \rightarrow \beta)^{\land \alpha \rightarrow \beta} x.xx\) auto-application

**Meaning:** types obtained by subsumption **and by instantiation**

- \(\lambda^{\alpha \rightarrow \alpha} x.x : 0 \rightarrow 1\) subsumption
- \(\lambda^{\alpha \rightarrow \alpha} x.x : \neg \text{Int}\) subsumption
- \(\lambda^{\alpha \rightarrow \alpha} x.x : \text{Int} \rightarrow \text{Int}\) instantiation
- \(\lambda^{\alpha \rightarrow \alpha} x.x : \text{Bool} \rightarrow \text{Bool}\) instantiation
**Problem**

Define an explicitly typed, polymorphic calculus with intersection types and dynamic type-case
Formal calculus: **new stuff**

<table>
<thead>
<tr>
<th>Exprs</th>
<th>$e ::= x$</th>
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**Problem**

Define an **explicitly typed, polymorphic calculus** with **intersection types** and dynamic type-case

Formal calculus: **new stuff**

### Exprs

\[ e ::= x \mid ee \mid \lambda^{i \in I_i \rightarrow t_i} x.e \mid e \in t \ ? \ e : e \]

### Types

\[ t ::= B \mid t \rightarrow t \mid t \lor t \mid t \land t \mid \neg t \mid 0 \mid 1 \mid \alpha \]

**Problem**

Define an explicitly typed, polymorphic calculus with intersection types and dynamic type-case.

---

Four simple points to show why dealing with this blend is quite problematic.
1. Polymorphism needs instantiation:

To apply \( \lambda \alpha \rightarrow \alpha \) to 42, we must use the instance obtained by the type substitution \( \{ \text{Int} / \alpha \} \):

\[
(\lambda \text{Int} \rightarrow \text{Int} \ x \ . \ x) 42
\]

We relabel the function by instantiating its interface.

2. Type-case needs explicit relabeling:

\[
(\lambda \alpha \rightarrow \alpha \rightarrow \alpha \ x \ . \ \lambda \alpha \rightarrow \alpha \ y \ . \ x) 42 \in \text{Int} \rightarrow \text{Int}; \ 
(\lambda \text{Int} \rightarrow \text{Int} \ y \ . \ 42) 42 \ 
(\lambda \alpha \rightarrow \alpha \rightarrow \alpha \ x \ . \ \lambda \alpha \rightarrow \alpha \ y \ . \ x) \text{true} \notin \text{Int} \rightarrow \text{Int}; \ 
(\lambda \text{Bool} \rightarrow \text{Bool} \ y \ . \ \text{true}) 42
\]

Interfaces determine \( \lambda \)-abstractions’s types [intrinsic semantics]

3. Relabeling must be applied also on function bodies:

A "daffy" definition of identity:

\[
(\lambda \alpha \rightarrow \alpha \ x \ . \ (\lambda \alpha \rightarrow \alpha \ y \ . \ x)) x
\]

To apply it to 42, relabeling the outer \( \lambda \) by \( \{ \text{Int} / \alpha \} \) does not suffice:

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\]

is not well typed. The body must be relabeled as well, by applying the \( \{ \text{Int} / \alpha \} \) yielding:

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(\lambda \text{Int} \rightarrow \text{Int} \ y \ . \ 42) 42
\]
1. Polymorphism needs instantiation:
To apply $\lambda^{\alpha \rightarrow \alpha} x.x$ to 42 we must use the instance obtained by the type substitution $\{\text{Int}/\alpha\}$:

$$(\lambda^{\text{Int} \rightarrow \text{Int}} x.x)_{42}$$

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$$(\lambda^{\alpha \rightarrow \alpha \rightarrow \alpha}x.\lambda^{\alpha \rightarrow \alpha}y.x)y.42 \in \text{Int} \rightarrow \text{Int}$$

$$(\lambda^{\alpha \rightarrow \alpha \rightarrow \alpha}x.\lambda^{\alpha \rightarrow \alpha}y.x)\text{true} \notin \text{Int} \rightarrow \text{Int}$$

Interfaces determine $\lambda$-abstractions’s types  

[intrinsic semantics]
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   To apply $\lambda^{\alpha\rightarrow\alpha} x.x$ to 42 we must use the instance obtained by the type substitution $\{\text{Int}/\alpha\}$:

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   Interfaces determine $\lambda$-abstractions’s types

   $\sim \lambda^{\text{Int}\rightarrow\text{Int}} y.42$

   $\sim \lambda^{\text{Bool}\rightarrow\text{Bool}} y.\text{true}$

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Interfaces determine \( \lambda \)-abstractions’s types

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The identity function $\lambda^{\alpha \to \alpha} x. x$ has both these types:

$$\text{Int} \to \text{Int} \quad \text{Bool} \to \text{Bool}$$
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The identity function $\lambda^{\alpha \to \alpha} \ x . x$ has both these types:

$$(\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})$$

So it has their intersection.
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We can feed the identity $\lambda^{\alpha \rightarrow \alpha} x.x$ to a function which expects an argument of the type above. But how do we relabel it?
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We can feed the identity $\lambda^{\alpha \rightarrow \alpha} x.x$ to a function which expects an argument of the type above. But how do we relabel it?

Intuitively: apply $\{\text{Int}/\alpha\}$ and $\{\text{Bool}/\alpha\}$ to the interface and replace it by the intersection of the two instances:
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Intuitively: apply $\{\text{Int}/\alpha\}$ and $\{\text{Bool}/\alpha\}$ to the interface and replace it by the intersection of the two instances:

$$(\lambda^{\alpha \rightarrow \alpha} x. x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] \leadsto \lambda^{(\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})} x. x$$
4. Relabeling the body is not always so straightforward:

1. More than one type-substitution needed
2. Relabeling depends on the dynamic type of the argument

The identity function \( \lambda x. x \) has both these types:

\[
\text{(Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})
\]

So it has their intersection.

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\[
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Consider again the daffy identity \((\lambda \alpha \rightarrow \alpha \ x. (\lambda \alpha \rightarrow \alpha \ y. x)x)\).
It also has type

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\[(\text{Int}\to\text{Int}) \land (\text{Bool}\to\text{Bool})\]

Applying the set of substitutions \([\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]\) both to the interface and the body yields an ill-typed term:

\[(\lambda(\text{Int}\to\text{Int})\land(\text{Bool}\to\text{Bool}) x.(\lambda(\text{Int}\to\text{Int})\land(\text{Bool}\to\text{Bool}) y.x)x)\]
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Let us see why it is not well typed
In order to type

\[(\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})) \cdot x. (\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool}) y. x) x\]

we must check that it has both types of the interface:
In order to type

\((\lambda (\text{Int} \to \text{Int}) \wedge (\text{Bool} \to \text{Bool}))\, x. (\lambda (\text{Int} \to \text{Int}) \wedge (\text{Bool} \to \text{Bool})\, y. x)\, x\) 

we must check that it has both types of the interface:

1. \(x : \text{Int} \vdash (\lambda (\text{Int} \to \text{Int}) \wedge (\text{Bool} \to \text{Bool})\, y. x)\, x : \text{Int}\)

2. \(x : \text{Bool} \vdash (\lambda (\text{Int} \to \text{Int}) \wedge (\text{Bool} \to \text{Bool})\, y. x)\, x : \text{Bool}\)
In order to type

\[
(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \ x. (\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \ y. x) \ x
\]

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Both fail because \( \lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \ y. x \) is not well typed
In order to type

\[(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \, x \cdot (\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, y \cdot x) \, x\]

we must check that it has both types of the interface:

1. \(x : \text{Int} \vdash (\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, y \cdot x) \, x : \text{Int}\)
2. \(x : \text{Bool} \vdash (\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, y \cdot x) \, x : \text{Bool}\)

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\[ \lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, x. (\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, y. \, x) \, x \]

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**Key idea**

The relabeling of the body *must change* according to the type of the parameter
In order to type

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we must check that it has both types of the interface:

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In our example with \((\lambda^\alpha \to^\alpha x. (\lambda^\alpha \to^\alpha y. x) x)\) and \([\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]\):
In order to type

$$(\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool}) \ x. (\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool}) \ y. x) x)$$

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2. $x : \text{Bool} \vdash (\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool}) \ y. x) x : \text{Bool}$

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In our example with \((\lambda \alpha\rightarrow\alpha)x.(\lambda \alpha\rightarrow\alpha\,y.x)x\) and \([\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]\):

- \((\lambda \alpha\rightarrow\alpha\,y.x)\) must be relabeled as \((\lambda \text{Int}\rightarrow\text{Int}\,y.x)\) when \(x : \text{Int}\);
- \((\lambda \alpha\rightarrow\alpha\,y.x)\) must be relabeled as \((\lambda \text{Bool}\rightarrow\text{Bool}\,y.x)\) when \(x : \text{Bool}\).
Observation

This “dependent relabeling” is the stumbling block for the definition of an explicitly-typed $\lambda$-calculus with intersection types.
A new technique

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**Our new technique: “lazy” relabeling of bodies.**

- *Decorate λ-abstractions by sets of type-substitutions:*

  To pass the daffy identity to a function that expects arguments of type \((\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})\)
  
  first “lazily” relabel it as follows:

  \[
  \left( \lambda^\alpha_{\alpha} \left[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\right] \right) \times \left( \lambda^\alpha_{\alpha} y.x \right) x
  \]

  The decoration indicates that the function must be relabeled
  
  The relabeling will be actually propagated to the body of the function at the moment of the reduction (lazy relabeling)
  
  The new decoration is statically used by the type system to ensure soundness.
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- *Decorate $\lambda$-abstractions by sets of type-substitutions:* To pass the daffy identity to a function that expects arguments of type $(\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})$ first “lazily” relabel it as follows:

  $$(\lambda^{\alpha \rightarrow \alpha}_{\{\text{Int} / \alpha\}, \{\text{Bool} / \alpha\}} x . (\lambda^{\alpha \rightarrow \alpha} y . x) x)$$

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  To pass the daffy identity to a function that expects arguments of type $(\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})$ first “lazily” relabel it as follows:

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- The new decoration is statically used by the type system to ensure soundness.
Details follow, but remember we want to program in this language

\[ e ::= x \mid ee \mid \lambda^\land_{i \in I} s_i \rightarrow t_i \cdot x \cdot e \mid e \in t \ ? \ e : e \]

**No decorations:** We do not want to oblige the programmer to write any explicit type substitution.
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The technical development will proceed as follows:

1. Define a calculus with explicit type-substitutions and decorated \( \lambda \)-abstractions.
2. Define an inference system that deduces where to insert explicit type-substitutions in a term of the language above.
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Before proceeding we can already check our first yardstick:

\[
\text{even} = \lambda^{(\text{Int} \rightarrow \text{Bool}) \land (\alpha \setminus \text{Int} \rightarrow \alpha \setminus \text{Int})} \cdot x \cdot (x \in \text{Int} \ ? (x \ \text{mod} \ 2) = 0) \rightarrow x
\]

\[
\text{map} = \mu m^{(\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta]} \cdot f. \lambda^{[\alpha] \rightarrow [\beta]} \cdot \ell \cdot (\ell \in \text{nil} \ ? \ \text{nil} : (f(\pi_1 \ell), m f(\pi_2 \ell)))
\]
A calculus with explicit type-substitutions
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Explicitly pinpoint where sets of type substitutions are applied:

\[ e ::= x \mid ee \mid \lambda i \in I^s \rightarrow t_i x.e \mid e \in t \ ? e : e \]
A calculus with explicit type-substitutions

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\[ e ::= x | ee | \lambda^{\bigwedge_{i \in I} s_i \mapsto t_i} x.e | e : e | e[s_i]_{i \in I} \]
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Explicitly pinpoint where sets of type substitutions are applied:

\[ e ::= x \mid ee \mid \lambda^{s_i \rightarrow t_i}_{i \in I} x.e \mid e \in t \mid e : e \mid e[\sigma_i]_{i \in I} \]

Some examples:

❌ \((\lambda^{\alpha \rightarrow \alpha} x.x)\)42

✅ \((\lambda^{\alpha \rightarrow \alpha} x.x)[\{\text{Int}/\alpha\}]\)42
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- \( \times\) \( (\lambda^{\alpha \rightarrow \alpha} x.x)42 \)
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A calculus with explicit type-substitutions

Explicitly pinpoint where sets of type substitutions are applied:

\[ e ::= x \mid ee \mid \lambda [\sigma_i]_{i \in I} \cdot e \mid e \in t \mid e : e \mid e[\sigma_i]_{i \in I} \]

Some examples:

- \((\lambda^{\alpha \rightarrow \alpha} x.x)42\)
- \((\lambda^{\alpha \rightarrow \alpha} x.x)[\{\text{Int}/\alpha\}]42\)
- \((\lambda^{\alpha \rightarrow \alpha} [\{\text{Int}/\alpha\}]x.x)42\)
- \((\lambda^{\alpha \rightarrow \alpha} x.x)[\{\text{Bool}/\alpha\}]42\)
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Explicitly pinpoint where sets of type substitutions are applied:

\[ e ::= x \mid ee \mid \lambda^{i\in I}x^{s_i\rightarrow t_i} \cdot e \mid e \in t \? e \mid e[\sigma_i]_{i\in I} \]

Some examples:

\( \times \) \( (\lambda^\alpha\rightarrow^\alpha x.x)42 \)
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\( \times \) \( (\lambda^{\alpha\rightarrow\alpha}x.x)[\{\text{Bool/}\alpha}\}]42 \)
\( \times \) \( (\lambda(\text{Int} \rightarrow \text{Int}) \rightarrow \text{Int} y.y3)(\lambda^\alpha\rightarrow^\alpha x.x) \)
\( \checkmark \) \( (\lambda(\text{Int} \rightarrow \text{Int}) \rightarrow \text{Int} y.y3)((\lambda^\alpha\rightarrow^\alpha x.x)[\{\text{Int/}\alpha}\}]) \)
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Some examples:

- \((\lambda^{\alpha \rightarrow \alpha} x.x) 42\)
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- \((\lambda (\text{Int} \rightarrow \text{Int}) \rightarrow \text{Int} y. y 3)(\lambda^{\alpha \rightarrow \alpha} x.x)\)
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Reduction semantics

\[ e ::= x \mid ee \mid \lambda^{\bigwedge i \in I s_i \to t_i} x.e \mid e \in t \ ? e : e \mid e[\sigma_i]_{i \in I} \]

\[ Relabeling \ operation \ e \@[\sigma_j]_{j \in J} \defeq x (\lambda^{\bigwedge i \in I t_i} s_i \to t_i) x. e \@[\sigma_j]_{j \in J} \]

\[ Pushes \ \sigma's \ down \ into \ \lambda's \]

Giuseppe Castagna

Polymorphic Functions with Set-Theoretic Types
Reduction semantics

\[ e ::= x \mid ee \mid \lambda^{\land i \in I s_i \to t_i}_{\sigma_j j \in J} x. e \mid e ? e : e \mid e[\sigma_i]_{i \in I} \]

Relabeling operation \( e@[\sigma_j]_{j \in J} \): *pushes the type substitutions into the decorations of the \( \lambda \)'s inside \( e \)*
Reduction semantics

\[ e ::= x \mid ee \mid \lambda_{i \in I}^{s_i \rightarrow t_i} x.e \mid e \in t \? e : e \mid e[\sigma_i]_{i \in I} \]

**Relabeling operation** \( e@[\sigma_j]_{j \in J} \): [Pushes \( \sigma \)'s down into \( \lambda \)'s]

\[
\begin{align*}
  x@[\sigma_j]_{j \in J} & \quad \text{def} \quad x \\
  (\lambda_{i \in I}^{t_i \rightarrow s_i} x.e)@[\sigma_j]_{j \in J} & \quad \text{def} \quad \lambda_{i \in I}^{t_i \rightarrow s_i} [\sigma_k]_{k \in K} @[\sigma_j]_{j \in J} x.e \\
  (e_1 e_2)@[\sigma_j]_{j \in J} & \quad \text{def} \quad (e_1@[\sigma_j]_{j \in J})(e_2@[\sigma_j]_{j \in J}) \\
  (e \in t \? e_1 : e_2)@[\sigma_j]_{j \in J} & \quad \text{def} \quad e@[\sigma_j]_{j \in J} \in t \? e_1@[\sigma_j]_{j \in J} : e_2@[\sigma_j]_{j \in J} \\
  (e[\sigma_k]_{k \in K})@[\sigma_j]_{j \in J} & \quad \text{def} \quad e@([\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J})
\end{align*}
\]
Reduction semantics

\[ e ::= x \mid ee \mid \lambda^{i \in I} t_i \rightarrow s_i x.e \mid e \in t ? e : e \mid e[\sigma_i]_{i \in I} \]

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- \( x@[\sigma_j]_{j \in J} \quad \text{def} \quad x \)
- \( (\lambda^{i \in I} t_i \rightarrow s_i x.e)@[\sigma_j]_{j \in J} \quad \text{def} \quad \lambda^{i \in I} t_i \rightarrow s_i [\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J} x.e \)
- \( (e_1 e_2)@[\sigma_j]_{j \in J} \quad \text{def} \quad (e_1@[\sigma_j]_{j \in J})(e_2@[\sigma_j]_{j \in J}) \)
- \( (e \in t ? e_1 : e_2)@[\sigma_j]_{j \in J} \quad \text{def} \quad e@[\sigma_j]_{j \in J} \in t ? e_1@[\sigma_j]_{j \in J} : e_2@[\sigma_j]_{j \in J} \)
- \( (e[\sigma_k]_{k \in K})@[\sigma_j]_{j \in J} \quad \text{def} \quad e@([\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J}) \)

**[Pushes \( \sigma \)'s down into \( \lambda \)'s]**
Reduction semantics

\[ e ::= x | ee | \lambda^{i \in I}_{{\sigma_j}_{j \in J}} x.e | e t ? e : e | e[\sigma_i]_{i \in I} \]

Relabeling operation \( e \odot [\sigma_j]_{j \in J} \): [Pushes \( \sigma \)'s down into \( \lambda \)'s]

\[ \begin{align*}
    x \odot [\sigma_j]_{j \in J} & \overset{\text{def}}{=} x \\
    (\lambda^{i \in I}_{{\sigma_k}_{k \in K}} x . e) \odot [\sigma_j]_{j \in J} & \overset{\text{def}}{=} \lambda^{i \in I}_{{\sigma_k}_{k \in K} \circ [\sigma_j]_{j \in J}} x . e \\
    (e_1 e_2) \odot [\sigma_j]_{j \in J} & \overset{\text{def}}{=} (e_1 \odot [\sigma_j]_{j \in J})(e_2 \odot [\sigma_j]_{j \in J}) \\
    (e t ? e_1 : e_2) \odot [\sigma_j]_{j \in J} & \overset{\text{def}}{=} e \odot [\sigma_j]_{j \in J} e_1 \odot [\sigma_j]_{j \in J} : e_2 \odot [\sigma_j]_{j \in J} \\
    (e[\sigma_k]_{k \in K}) \odot [\sigma_j]_{j \in J} & \overset{\text{def}}{=} e \odot ([\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J})
\]
Reduction semantics

\[
e ::= x \mid ee \mid \lambda^{i \in I} t_i \rightarrow s_i \ x.\ e \mid e ? e : e \mid e[\sigma_i]_{i \in I}
\]

Relabeling operation \(e@[\sigma_j]_{j \in J}\): [Pushes \(\sigma\)'s down into \(\lambda\)'s]

\[
\begin{align*}
x@[\sigma_j]_{j \in J} & \overset{\text{def}}{=} x \\
(\lambda^{i \in I} t_i \rightarrow s_i \ x.\ e)@[\sigma_j]_{j \in J} & \overset{\text{def}}{=} \lambda^{i \in I} t_i \rightarrow s_i \ x.\ e \\
(e_1 e_2)@[\sigma_j]_{j \in J} & \overset{\text{def}}{=} (e_1@[\sigma_j]_{j \in J})(e_2@[\sigma_j]_{j \in J}) \\
(e ? e_1 : e_2)@[\sigma_j]_{j \in J} & \overset{\text{def}}{=} e@[\sigma_j]_{j \in J} ? e_1@[\sigma_j]_{j \in J} : e_2@[\sigma_j]_{j \in J} \\
(e[\sigma_k]_{k \in K})@[\sigma_j]_{j \in J} & \overset{\text{def}}{=} e@[\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J}
\end{align*}
\]
Reduction semantics

\[ e ::= x \mid ee \mid \lambda_{i \in I}^{s_i \rightarrow t_i} x.e \mid e \in t \ ? \ e : e \mid e[s_i]_{i \in I} \]

**Relabeling operation** \( e[\sigma_j]_{j \in J} : \) [Pushes \( \sigma \)'s down into \( \lambda \)'s]

\[
\begin{align*}
  x[\sigma_j]_{j \in J} & \overset{\text{def}}{=} x \\
  (\lambda_{k \in K}^{t_i \rightarrow s_i} x.e)[\sigma_j]_{j \in J} & \overset{\text{def}}{=} \lambda_{k \in K}^{t_i \rightarrow s_i} x.e \\\n  (e_1 e_2)[\sigma_j]_{j \in J} & \overset{\text{def}}{=} (e_1[\sigma_j]_{j \in J})(e_2[\sigma_j]_{j \in J}) \\
  (e \in t ? e_1 : e_2)[\sigma_j]_{j \in J} & \overset{\text{def}}{=} e[\sigma_j]_{j \in J} \in t ? e_1[\sigma_j]_{j \in J} : e_2[\sigma_j]_{j \in J} \\
  (e[\sigma_k]_{k \in K})[\sigma_j]_{j \in J} & \overset{\text{def}}{=} e[\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J}
\end{align*}
\]

**Notions of reduction:**

\[
\begin{align*}
  e[\sigma_j]_{j \in J} & \leadsto e[\sigma_j]_{j \in J} \\
  (\lambda_{i \in I}^{t_i \rightarrow s_i} x.e) v & \leadsto (e[\sigma_j]_{j \in P})\{v/x\} \quad P = \{j \in J \mid \exists i \in I, \vdash v : t_i \sigma_j\} \\
  v \in t ? e_1 : e_2 & \leadsto \begin{cases} 
  e_1 & \text{if } \vdash v : t \\
  e_2 & \text{otherwise}
\end{cases}
\end{align*}
\]
Reduction semantics

\[ e ::= x \mid ee \mid \lambda_{i \in I}^{s_i \rightarrow t_i} x.e \mid e \in t \mid e : e \mid e[\sigma_i]_{i \in I} \]

Relabeling operation \( e@[\sigma_j]_{j \in J} \): [Pushes \( \sigma \)'s down into \( \lambda \)'s]

\[
\begin{align*}
\lambda_{k \in K}^{\sigma_k} t_i \rightarrow s_i x.e & \quad \defeq \quad x \\
(\lambda_{k \in K}^{\sigma_k} t_i \rightarrow s_i x.e)[\sigma_j]_{j \in J} & \quad \defeq \quad \lambda_{k \in K \circ \sigma_j}^{\sigma_k} t_i \rightarrow s_i x.e \\
(e_1 e_2)[\sigma_j]_{j \in J} & \quad \defeq \quad (e_1@[\sigma_j]_{j \in J})(e_2@[\sigma_j]_{j \in J}) \\
(e \in t ? e_1 : e_2)[\sigma_j]_{j \in J} & \quad \defeq \quad e@[\sigma_j]_{j \in J \in t ? e_1@[\sigma_j]_{j \in J} : e_2@[\sigma_j]_{j \in J}} \\
(e[\sigma_k]_{k \in K})[\sigma_j]_{j \in J} & \quad \defeq \quad e@([\sigma_k]_{k \in K} \circ [\sigma_j]_{j \in J})
\end{align*}
\]

Notions of reduction:

\[
\begin{align*}
e[\sigma_j]_{j \in J} & \quad \leadsto \quad e@[\sigma_j]_{j \in J} \\
(\lambda_{j \in J}^{\sigma_j} t_i \rightarrow s_i x.e)v & \quad \leadsto \quad (e@[\sigma_j]_{j \in P})\{v/x\} \\
v \in t \mid e_1 : e_2 & \quad \leadsto \quad \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}
\end{align*}
\]

\[ P = \{ j \in J \mid \exists i \in I, \vdash v : t_i \sigma_j \} \]
Reduction semantics

\[ e ::= x \mid ee \mid \lambda^{i \in I} x.e \mid e \in t? e : e \mid e[\sigma]_{i \in I} \]

Relabeling operation \( e[@[\sigma]_{j \in J}] \):

- \( x[@[\sigma]_{j \in J}] \) def \( = x \)
- \( \lambda^{i \in I} x.e[@[\sigma]_{j \in J}] \) def \( = \lambda^{i \in I} x.e \)
- \( (e_1 e_2)@[\sigma]_{j \in J} \)
- \( (e \in t? e_1 : e_2)@[\sigma]_{j \in J} \)
- \( (e[\sigma]_{k \in K})@[\sigma]_{j \in J} \)

Notions of reduction:

- \( e[\sigma]_{j \in J} \) \( \sim \) \( e[@[\sigma]_{j \in J}] \)
- \( (\lambda^{i \in I} x.e)_{\nu} \) \( \sim \) \( (e[@[\sigma]_{j \in P})\{\nu/x\} \)
- \( \nu \in t? e_1 : e_2 \) \( \sim \) \( \{ e_1 \text{ if } \vdash \nu : t \}
\)
- \( e_2 \text{ otherwise} \)

[Pushes \( \sigma \)'s down into \( \lambda \)'s]

Only keep the substitutions that make the type of the argument \( \nu \) match at least one input type of the interface
Example

\[(\lambda \alpha \to \alpha \ x \ . \ (\lambda \alpha \to \alpha \ y \ . \ x) \ x)\]
Example

\[ \lambda(\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool}) \ z.(\lambda \alpha \rightarrow x.(\lambda \alpha \rightarrow y.x)x)z \]
Example

\[ \lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool}) \, z. (\lambda \alpha \to \alpha \, x. (\lambda \alpha \to \alpha \, y. x) x) \{\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\} z \]
Example

\((\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \ z. (\lambda x. (\lambda y. x) x) \ [\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] z) 42\)
Example

\[
(\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})) \, z \, (\lambda \alpha \rightarrow \alpha \, x \, (\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] \, z \, 42
\]

\[
\leadsto \quad (\lambda \alpha \rightarrow \alpha \, x \, (\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] \, 42
\]
Example

\[
\left(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})\right) z \left(\lambda \alpha \to \alpha \ x. (\lambda \alpha \to \alpha \ y. x) x\right) \left[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\right] z \rightarrow^{42} \\
\left(\lambda \alpha \to \alpha \ x. (\lambda \alpha \to \alpha \ y. x) x\right) \left[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\right] 42 \\
\rightarrow^{42} \\
\left(\lambda \alpha \to \alpha \left[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\right] x. (\lambda \alpha \to \alpha \ y. x) x\right) 42
\]
Example

\[
(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \ z. (\lambda \alpha \to \alpha \ x. (\lambda \alpha \to \alpha \ y. x) x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] z) 42
\]

\[
\leadsto (\lambda \alpha \to \alpha \ x. (\lambda \alpha \to \alpha \ y. x) x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] 42
\]

\[
\leadsto (\lambda \alpha \to \alpha \ [\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}] x. (\lambda \alpha \to \alpha \ y. x) x) 42
\]

\[
\leadsto (\lambda \text{Int} \to \text{Int} \ y. 42) 42
\]
Example

\[(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \cdot (\lambda \alpha \to \alpha \cdot (\lambda \alpha \to \alpha y \cdot x) x) [\{\text{Int} / \alpha \}, \{\text{Bool} / \alpha \}] z) 42\]

\[\leadsto (\lambda \alpha \to \alpha \cdot (\lambda \alpha \to \alpha y \cdot x) x) [\{\text{Int} / \alpha \}, \{\text{Bool} / \alpha \}] 42\]

\[\leadsto (\lambda \alpha \to \alpha [\{\text{Int} / \alpha \}, \{\text{Bool} / \alpha \}] x \cdot (\lambda \alpha \to \alpha y \cdot x) x) 42\]

\[\leadsto (\lambda \text{Int} \to \text{Int} y \cdot 42) 42\]

no \text{Bool} here
Example

\[
(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) \quad z.(\lambda \alpha \to \alpha x.(\lambda \alpha \to \alpha y.x)x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]z)42
\]

\[
\leadsto (\lambda \alpha \to \alpha x.(\lambda \alpha \to \alpha y.x)x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]42
\]

\[
\leadsto (\lambda [\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]x.(\lambda \alpha \to \alpha y.x)x)42
\]

\[
\leadsto (\lambda \text{Int} \to \text{Int} y.42)42
\]
Example

\[(\lambda (\text{Int} \to \text{Int}) \land (\text{Bool} \to \text{Bool})) z. (\lambda \alpha \to \alpha x. (\lambda \alpha \to \alpha y.x)x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]z)42\]

\[\leadsto (\lambda \alpha \to \alpha x. (\lambda \alpha \to \alpha y.x)x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]42\]

\[\leadsto (\lambda \alpha \to \alpha x. (\lambda \alpha \to \alpha y.x)x)[\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}]42\]

\[\leadsto (\lambda \text{Int} \to \text{Int} y.42)42\]

\[\equiv (((\lambda \alpha \to \alpha y.x)x)@[\{\text{Int}/\alpha\}])\{42/x\}\]
Example

\[(\lambda (\text{Int} \rightarrow \text{Int}) \land (\text{Bool} \rightarrow \text{Bool})) \, z \, . \, (\lambda \alpha \rightarrow \alpha \, x \, . \, (\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)\{\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\} \, z) \, 42\]

\[\sim \quad (\lambda \alpha \rightarrow \alpha \, x \, . \, (\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)\{\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\} \, 42\]

\[\sim \quad (\lambda \alpha \rightarrow \alpha \, x \, . \, (\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)\{\{\text{Int}/\alpha\}, \{\text{Bool}/\alpha\}\} \, 42\]

\[\sim \quad (\lambda \text{Int} \rightarrow \text{Int} \, y \, . \, 42) \, 42\]

\[\equiv \quad (((\lambda \alpha \rightarrow \alpha \, y \, . \, x) \, x)@\{\{\text{Int}/\alpha\}\}\})\{42/y\}\]

\[\sim \quad 42\]
Type system

\[(\text{subsumption})\]
\[
\Gamma \vdash e : t_1 \quad t_1 \leq t_2
\]
\[
\Gamma \vdash e : t_2
\]

\[(\text{appl})\]
\[
\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1
\]
\[
\Gamma \vdash e_1 e_2 : t_2
\]

\[(\text{inst})\]
\[
\Gamma \vdash e : t
\]
\[
\Gamma \vdash e[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t \sigma_j
\]
\[
\sigma_j \# \Gamma
\]

\[(\text{abstr})\]
\[
\Gamma, x : t_i \sigma_j \vdash e[\sigma_j] : s_i \sigma_j
\]
\[
\Gamma \vdash \lambda^{t_i \sigma_j \rightarrow s_i}_{[\sigma_j]_{j \in J}} x.e : \bigwedge_{i \in I, j \in J} t_i \sigma_j \to s_i \sigma_j
\]
\[
i \in I
\]
\[
j \in J
\]

[plus the rules for type-case and variables]
Type system

(\textit{subsumption})
\[
\Gamma \vdash e : t_1 \quad t_1 \leq t_2 \\
\hline
\Gamma \vdash e : t_2
\]

(\textit{appl})
\[
\Gamma \vdash e_1 : t_1 \rightarrow t_2 \\
\Gamma \vdash e_2 : t_1 \\
\hline
\Gamma \vdash e_1 e_2 : t_2
\]

(\textit{inst})
\[
\Gamma \vdash e : t \\
\hline
\Gamma \vdash e[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t\sigma_j \\
\sigma_j \notin \Gamma
\]

(\textit{abstr})
\[
\Gamma, x : t_i \sigma_j \vdash e[\sigma_j] : s_i \sigma_j \\
\hline
\Gamma \vdash \lambda^{i \in I \; t_i \rightarrow s_i}_{[\sigma_j]_{j \in J}} \; x.\; e : \bigwedge_{i \in I, j \in J} \; t_i \sigma_j \rightarrow s_i \sigma_j \\
i \in I
\]

[plus the rules for type-case and variables]
Type system

(subsumption)
\[ \Gamma \vdash e : t_1 \quad t_1 \leq t_2 \]
\[ \Gamma \vdash e : t_2 \]

(appl)
\[ \Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1 \]
\[ \Gamma \vdash e_1 e_2 : t_2 \]

(inst)
\[ \Gamma \vdash e : t \]
\[ \Gamma \vdash e[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t \sigma_j \]
\[ \sigma_j \not\in \Gamma \]

(abstr)
\[ \Gamma, x : t_i \quad \vdash e : s_i \quad i \in I \]
\[ \Gamma \vdash \lambda^{\bigwedge_{i \in I} t_i \rightarrow s_i} x.e : \bigwedge_{i \in I} t_i \rightarrow s_i \]

[plus the rules for type-case and variables]
Type system

(subsumption)  \[ \Gamma \vdash e : t_1 \quad t_1 \leq t_2 \]
\[ \Gamma \vdash e : t_2 \]

(appl)  \[ \Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1 \]
\[ \Gamma \vdash e_1 e_2 : t_2 \]

(inst)  \[ \Gamma \vdash e : t \]
\[ \Gamma \vdash e[\sigma_j]_{j \in J} : \bigwedge_{j \in J} t \sigma_j \]
\[ \sigma_j \# \Gamma \]

(abstr)  \[ \Gamma, x : t_i \sigma_j \vdash e[\sigma_j] : s_i \sigma_j \]
\[ \Gamma \vdash \lambda^\bigwedge_{i \in I} t_i \rightarrow s_i^{\sigma_j} x.e : \bigwedge_{i \in I, j \in J} t_i \sigma_j \rightarrow s_i \sigma_j \quad i \in I \quad j \in J \]

[plus the rules for type-case and variables]
Properties

Theorem (Subject Reduction)
For every term \( e \) and type \( t \), if \( \Gamma \vdash e : t \) and \( e \leadsto e' \), then \( \Gamma \vdash e' : t \).

Theorem (Progress)
Let \( e \) be a well-typed closed term. If \( e \) is not a value, then there exists a term \( e' \) such that \( e \leadsto e' \).
Properties

Theorem (Subject Reduction)

For every term \( e \) and type \( t \), if \( \Gamma \vdash e : t \) and \( e \rightsquigarrow e' \), then \( \Gamma \vdash e' : t \).

Theorem (Progress)

Let \( e \) be a well-typed closed term. If \( e \) is not a value, then there exists a term \( e' \) such that \( e \rightsquigarrow e' \).

Theorem

Let \( \vdash_{BCD} \) be Barendregt, Coppo, and Dezani, typing, and \( \lceil e \rceil \) the type erasure of \( e \). If \( \vdash_{BCD} a : t \), then \( \exists e \) s.t. \( \vdash e : t \) and \( \lceil e \rceil = a \).
Properties

**Theorem (Subject Reduction)**

For every term \( e \) and type \( t \), if \( \Gamma \vdash e : t \) and \( e \leadsto e' \), then \( \Gamma \vdash e' : t \).

**Theorem (Progress)**

Let \( e \) be a well-typed closed term. If \( e \) is not a value, then there exists a term \( e' \) such that \( e \leadsto e' \).

**Theorem**

Let \( \vdash_{BCD} \) be Barendregt, Coppo, and Dezani, typing, and \( \lceil e \rceil \) the type erasure of \( e \). If \( \vdash_{BCD} a : t \), then \( \exists e \) s.t. \( \vdash e : t \) and \( \lceil e \rceil = a \).

Note that

\[
e := x | ee | \lambda_{[\sigma_j]_{j \in J}}^{\land_{i \in I} s_i \rightarrow t_i} x.e | e \in t ? e : e | e[\sigma_i]_{i \in I}
\]
## Properties

**Theorem (Subject Reduction)**

*For every term* \( e \) *and type* \( t \), *if* \( \Gamma \vdash e : t \) *and* \( e \leadsto e' \), *then* \( \Gamma \vdash e' : t \).

**Theorem (Progress)**

*Let* \( e \) *be a well-typed closed term. If* \( e \) *is not a value, then there exists a term* \( e' \) *such that* \( e \leadsto e' \).

**Theorem**

*Let* \( \vdash_{BCD} \) *be Barendregt, Coppo, and Dezani, typing, and* \( \llbracket e \rrbracket \) *the type erasure of* \( e \). *If* \( \vdash_{BCD} a : t \), *then* \( \exists e \ s.t. \vdash e : t \) *and* \( \llbracket e \rrbracket = a \).

Note that

\[
e ::= x \mid ee \mid \lambda_{[\sigma_j]_{j \in J}}^{\land_{i \in I}s_i \rightarrow t_i} x.e \mid e \in t ? e : e \mid e[\sigma_i]_{i \in I}
\]
Properties

Theorem (Subject Reduction)

For every term $e$ and type $t$, if $\Gamma \vdash e : t$ and $e \leadsto e'$, then $\Gamma \vdash e' : t$.

Theorem (Progress)

Let $e$ be a well-typed closed term. If $e$ is not a value, then there exists a term $e'$ such that $e \leadsto e'$.

Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\lceil e \rceil$ the type erasure of $e$. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\lceil e \rceil = a$.

Note that

$$e ::= x \mid ee \mid \lambda_{\sigma_j : \rightarrow}^{i \in I} \chi \cdot e \mid e \in t \mid e : e$$
Properties

**Theorem (Subject Reduction)**

*For every term* $e$ *and type* $t$, *if* $\Gamma \vdash e : t$ *and* $e \leadsto e'$, *then* $\Gamma \vdash e' : t$.

**Theorem (Progress)**

*Let* $e$ *be a well-typed closed term. If* $e$ *is not a value, then there exists a term* $e'$ *such that* $e \leadsto e'$.

**Theorem**

*Let* $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $[e]$ *the type erasure of* $e$. *If* $\vdash_{BCD} a : t$, *then* $\exists e$ *s.t.* $\vdash e : t$ *and* $[e] = a$.

Note that

$$e ::= x \mid ee \mid \lambda^{\bigwedge_{i \in I} s_i \rightarrow t_i}_{[\sigma_j]_{j \in J}} x.e \mid e \in t \ ? e : e$$

satisfies the above theorem and is closed by reduction.
Properties

Theorem (Subject Reduction)

*For every term* $e$ *and type* $t$, *if* $\Gamma \vdash e : t$ *and* $e \leadsto e'$, *then* $\Gamma \vdash e' : t$.

Theorem (Progress)

*Let* $e$ *be a well-typed closed term. If* $e$ *is not a value, then there exists a term* $e'$ *such that* $e \leadsto e'$.

Theorem

*Let* $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $\lceil e \rceil$ *the type erasure of* $e$. *If* $\vdash_{BCD} a : t$, *then* $\exists e$ *s.t.* $\vdash e : t$ *and* $\lceil e \rceil = a$.

Note that

$$e ::= x \mid ee \mid \lambda_{\sigma_j \in J}^{\bigwedge_{i \in I} s_i \rightarrow t_i} x.e \mid e \mid e : e \mid e$$

satisfies the above theorem and is closed by reduction, too.
Properties

Theorem (Subject Reduction)

For every term $e$ and type $t$, if $\Gamma \vdash e : t$ and $e \leadsto e'$, then $\Gamma \vdash e' : t$.

Theorem (Progress)

Let $e$ be a well-typed closed term. If $e$ is not a value, then there exists a term $e'$ such that $e \leadsto e'$.

Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\llbracket e \rrbracket$ the type erasure of $e$. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\llbracket e \rrbracket = a$.

Note that

$$e ::= x \mid ee \mid \lambda^{i \in I} s_i \rightarrow t_i x.e \mid e \in t ? e : e \mid e[\sigma_i]_{i \in I}$$

The first $n$ terms ($n = 3, 4, 5$) form an explicitly-typed $\lambda$-calculus with intersection types subsuming BCD.
Properties

The definitions we gave:

\[
\text{even} = \lambda (\text{Int} \rightarrow \text{Bool}) \land (\alpha \rightarrow \text{Int}) \cdot \lambda x. x \in \text{Int} \land (x \mod 2) = 0 : x
\]

\[
\text{map} = \mu m (\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta] \cdot f. \\
\lambda [\alpha] \rightarrow [\beta] \cdot \ell. \ell \in \text{nil} \land \text{nil} : (f(\pi_1 \ell), m(f(\pi_2 \ell))
\]

are well typed.
The definitions we gave:

\[\text{even} = \lambda (\text{Int} \to \text{Bool}) \land (\alpha \\rightarrow \alpha \downarrow \text{Int}) \cdot x \cdot x \in \text{Int} ? (x \mod 2) = 0 : x\]

\[\text{map} = \mu m (\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta] \cdot f.\]

\[\lambda [\alpha] \rightarrow [\beta] \ell. \ell \in \text{nil} ? \text{nil} : (f(\pi_1 \ell), m f(\pi_2 \ell))\]

are well typed.

A yardstick for the language

- Can define both \textit{map} and \textit{even}
- Can \textit{check} the types specified in the signature
- Can \textit{deduce} the type of the partial application \textit{map even}
Inference of explicit type-substitutions
Two problems:

1. **Local type-substitution inference:** Given a term of

\[
e ::= x \mid ee \mid \lambda^{i \in I} s_i \mapsto t_i \, x \, . \, e \mid e \in t \, ? \, e : e
\]

find a sound & complete algorithm that, whenever possible, inserts sets of type-substitutions making it a well-typed term of

\[
e ::= x \mid ee \mid \lambda^{i \in I} s_i \mapsto t_i \, x \, . \, e \mid e \in t \, ? \, e : e \mid e[\sigma_j]_{j \in J}
\]
Two problems:

1. **Local type-substitution inference:** Given a term of

   \[ e ::= x \mid ee \mid \lambda^{i \in I} s_i \rightarrow t_i \ x. e \mid e \in t \ ? e : e \]

   find a sound & complete algorithm that, whenever possible, inserts sets of type-substitutions making it a well-typed term of

   \[ e ::= x \mid ee \mid \lambda^{i \in I} s_i \rightarrow t_i \ x. e \mid e \in t \ ? e : e \mid e[\sigma_j]_{j \in J} \]

   (and, yes, the type inferred for `map even` is as expected)
Two problems:

1. **Local type-substitution inference:** Given a term of

   \[ e ::= x \mid ee \mid \lambda^{i \in I_{s_i \rightarrow t_i}} x.e \mid e \in t \ ? e : e \]

   find a sound & complete algorithm that, whenever possible, inserts sets of type-substitutions making it a well-typed term of

   \[ e ::= x \mid ee \mid \lambda^{i \in I_{s_i \rightarrow t_i}} x.e \mid e \in t \ ? e : e \mid e[\sigma_j]_{j \in J} \]

   (and, yes, the type inferred for `map even` is as expected)

2. **Type reconstruction:** Given a term

   \[ \lambda x.e \]

   find, if possible, a set of type-substitutions \([\sigma_j]_{j \in J}\) such that

   \[ \lambda^{\alpha \rightarrow \beta} x.e \]

   is well typed
Local Type-Substitution Inference

Given a term of

\[ e ::= x \mid ee \mid \lambda^{i \in I \Rightarrow t_i} x.e \mid e \in t ? e : e \]

Infer whether it is possible to insert sets of type-substitutions in it to make it a well-typed term of

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\[ \text{No inference for decorations of } \lambda \text{'s} \]

The reason is purely practical:

- \( \lambda^{\alpha \rightarrow \alpha} x.3 \) must return a static type error
- If we infer decorations, then it can be typed: \( \lambda^{\alpha \rightarrow \alpha}_{\{\text{Int}/\alpha\}} x.3 \)
The rule for applications

1. **In the type system:**  [with explicit type-subst.]

   \[(\text{APPL})\]

   \[
   \Gamma \vdash e_1 : s \rightarrow u \quad \Gamma \vdash e_2 : s \\
   \Gamma \vdash e_1 e_2 : u
   \]

   [The type of the function is subsumed to an arrow and the type of the argument is subsumed to the domain of this arrow].
The rule for applications

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\[
\begin{align*}
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\hline
\Gamma \vdash e_1 e_2 : u
\end{align*}
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2. **Subsumption elimination:** [with explicit type-subst.]

\[
\begin{align*}
\text{(Appl-algorithm)} & \quad \Gamma \vdash \mathcal{A} e_1 : t \quad \Gamma \vdash \mathcal{A} e_2 : s \\
\hline
\Gamma \vdash \mathcal{A} e_1 e_2 : \min \{ u \mid t \leq s \to u \} & \quad t \leq 0 \to 1 \\
& \quad s \leq \text{dom}(t)
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   \quad \quad \quad \Gamma \vdash \Delta e_1 e_2 : \min \{u \mid t \leq s \to u\}
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3. **Inference of type substitutions** [w/o explicit type-subst.]

   \[
   (\text{Appl-inference}) \\
   \exists [\sigma_i]_{i \in I}, [\sigma'_{j}]_{j \in J} \quad \Gamma \vdash_I e_1 : t \quad \Gamma \vdash_I e_2 : s \\
   \Gamma \vdash_I e_1 e_2 : \min\{u \mid t[\sigma'_j]_{j \in J} \leq s[\sigma_i]_{i \in I} \rightarrow u\} \\
   \]

   \[
   t[\sigma'_j]_{j \in J} \leq 0 \rightarrow 1 \\
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t[\sigma'_j]_{j \in J} \leq 0 & \rightarrow 1 \\
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The problem of inferring the type of an application is thus to find for $s$ and $t$ given, two sets $[\sigma_i]_{i \in I}, [\sigma'_j]_{j \in J}$ such that:

$$t[\sigma'_j]_{j \in J} \leq 0 \rightarrow 1 \quad \text{and} \quad s[\sigma_i]_{i \in I} \leq \text{dom}(t[\sigma'_j]_{j \in J})$$
Tallying problem

The problem of inferring the type of an application is thus to find for \(s\) and \(t\) given, two sets \([\sigma_i]_{i \in I}, [\sigma'_j]_{j \in J}\) such that:

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This can be reduced to solving a suite of tallying problems:

**Definition (Type tallying)**

Let \(s\) and \(t\) be two types. A type-substitution \(\sigma\) is a solution for the *tallying* of \((s, t)\) iff \(s\sigma \leq t\sigma\).
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**Generally:** let \( C = \{(s_1 \leq t_1), \ldots, (s_n \leq t_n)\} \) a \textit{constraint set}. A type-substitution \( \sigma \) is a solution for the \textit{tallying} of \( C \) iff \( s\sigma \leq t\sigma \) for all \((s \leq t) \in C\).
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**Definition (Type tallying)**

Let $s$ and $t$ be two types. A type-substitution $\sigma$ is a solution for the tallying of $(s, t)$ iff $s\sigma \leq t\sigma$.

**Generally:** let $C = \{(s_1 \leq t_1), ..., (s_n \leq t_n)\}$ a constraint set. A type-substitution $\sigma$ is a solution for the tallying of $C$ iff $s\sigma \leq t\sigma$ for all $(s \leq t) \in C$.

Type tallying is decidable and a sound and complete set of solutions for every tallying problem can be effectively found in three simple steps.
**Step 1: Decompose constraints.**

Use the set-theoretic decomposition rules to transform $C$ into a set of constraint sets whose constraints are of the form $\alpha \leq t$ or $t \leq \alpha$. 
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Example:
1. $\{(s_1 \rightarrow t_1 \leq s_2 \rightarrow t_2)\} \leadsto \{(s_2 \leq \emptyset)\}$ or $\{(s_2 \leq s_1), (t_1 \leq t_2)\}$
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Use the set-theoretic decomposition rules to transform $C$ into a set of constraint sets whose constraints are of the form $\alpha \leq t$ or $t \leq \alpha$.

**Step 2: Merge constraints on the same variable.**
- if $\alpha \leq t_1$ and $\alpha \leq t_2$ are in $C$, then replace them by $\alpha \leq t_1 \land t_2$;
- if $s_1 \leq \alpha$ and $s_2 \leq \alpha$ are in $C$, then replace them by $s_1 \lor s_2 \leq \alpha$;

Possibly decompose the new constraints generated by transitivity.

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Step 3: Transform into a set of equations.
After Step 2 we have constraint-sets of the form
\[
\{ s_i \leq \alpha_i \leq t_i \mid i \in [1..n] \}
\] where $\alpha_i$ are pairwise distinct.

1. select $s \leq \alpha \leq t$ and replace it by $\alpha = (s \lor \beta) \land t$ with $\beta$ fresh.
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2. \{(\text{Int} \leq \alpha), (\text{Bool} \leq \alpha)\} \sim \{(\text{Int} \lor \text{Bool} \leq \alpha)\}
3. \{(\text{Int} \leq \alpha_1 \leq \text{Real}), (\alpha_2 \leq \alpha_1 \land \text{Int})\} \sim \{\alpha_1 = (\text{Int} \lor \beta) \land \text{Real}, (\alpha_2 = \text{Int})\}
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At the end we have a sets of equations $\{ \alpha_i = u_i \mid i \in [1..n] \}$ that (with some care) are contractive. By Courcelle there exists a solution, ie, a substitution for $\alpha_1, ..., \alpha_n$ into (possibly recursive regular) types $t_1, ..., t_n$ (in which the fresh $\beta$’s are free variables).
Example: map even

Start with the following tallying problem:

\[(\alpha_1 \rightarrow \beta_1) \rightarrow [\alpha_1] \rightarrow [\beta_1] \leq s \rightarrow \gamma\]

where \(s = (\text{Int} \rightarrow \text{Bool}) \land (\alpha \backslash \text{Int} \rightarrow \alpha \backslash \text{Int})\) is the type of even
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- The algorithm generates 9 constraint-sets: one is unsatisfiable \((s \leq 0)\); four are implied by the others; remain

\[
\{ \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq 0 \},
\{ \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq \text{Int} , \text{Bool} \leq \beta_1 \},
\{ \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq \alpha \downarrow \text{Int} , \alpha \downarrow \text{Int} \leq \beta_1 \},
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  \[
  \begin{align*}
  \{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq 0 \}, \\
  \{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \text{Int}, \text{Bool} \leq \beta_1 \}, \\
  \{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \alpha \mid \text{Int}, \alpha \mid \text{Int} \leq \beta_1 \}, \\
  \{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \alpha \lor \text{Int}, (\alpha \mid \text{Int}) \lor \text{Bool} \leq \beta_1 \};
  \end{align*}
  \]

- Four solutions for \(\gamma\):

  \[
  \begin{align*}
  \{ & \gamma = [] \rightarrow [] \}, \\
  \{ & \gamma = [\text{Int}] \rightarrow [\text{Bool}] \}, \\
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  \]

- The last two are minimal and we take their intersection:
  \[
  \{\gamma = ([\alpha \rightarrow \text{Int}] \rightarrow [\alpha \rightarrow \text{Int}]) \land ([\alpha \lor \text{Int}] \rightarrow [(\alpha \rightarrow \text{Int}) \lor \text{Bool}])\}
On completeness and decidability

The algorithm produces a set of solutions that is **sound** (it finds only correct solutions) and **complete** (any other solution can be derived from them).
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In a dully execution of the algorithm on `map even` the good solution is the second one.
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In a duly execution of the algorithm on map even the good solution is the second one.

Principality: This raises the problem of the existence of principal types: may an infinite sequence of increasingly general solutions exist?
Type reconstruction

- Solve sets of constraint-sets by the tallying algorithm:

\[ \Gamma, x : \alpha \vdash \text{e} : t \leadsto S \]
\[ \Gamma \vdash \text{x} : \Gamma(x) \leadsto \{\emptyset\} \]
\[ \Gamma \vdash \text{\lambda x.e} : \alpha \rightarrow \beta \leadsto S \sqcap \{\{(t \leq \beta)\}\} \]
\[ \Gamma \vdash \text{e}_1 : t_1 \leadsto S_1 \]
\[ \Gamma \vdash \text{e}_2 : t_2 \leadsto S_2 \]
\[ \Gamma \vdash \text{e}_1 e_2 : \alpha \leadsto S_1 \sqcap S_2 \sqcap \{(t_1 \leq t_2 \rightarrow \alpha)\} \]

+ rule for typecase
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\end{align*}
\]

\[
\Gamma \vdash e_1 : t_1 \leadsto S_1 \quad \Gamma \vdash e_2 : t_2 \leadsto S_2
\]

\[
\Gamma \vdash e_1 e_2 : \alpha \leadsto S_1 \cap S_2 \cap \{\{(t_1 \leq t_2 \rightarrow \alpha)\}\} + \text{rule for typecase}
\]

- **Sound.** It’s a variant: fix interfaces and infer decorations

\[
\lambda^{\alpha \rightarrow \beta} x. e
\]

Not complete: reconstruction is undecidable
Solve sets of constraint-sets by the tallying algorithm:

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It types more than ML

\[
\lambda x.x x : \mu X. (\alpha \land (X \rightarrow \beta)) \rightarrow \beta \\
(\leq \alpha \land (\alpha \rightarrow \beta) \rightarrow \beta)
\]

and for functions typable in ML, it deduces a type at least as good:

\[
\text{map} : ((\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta]) \land ((\emptyset \rightarrow \emptyset) \rightarrow [] \rightarrow [])
\]
Efficient evaluation
Monomorphic language

\begin{align*}
e & ::= c \mid x \mid \lambda^t x. e \mid ee \mid e \in t ? e : e \\
v & ::= c \mid \langle \lambda^t x. e, \mathcal{E} \rangle
\end{align*}
Monomorphistic language

\[ e ::= c \mid x \mid \lambda^t x.e \mid ee \mid e \in t?e : e \]

\[ \nu ::= c \mid \langle \lambda^t x.e, \mathcal{E} \rangle \]

\[ \begin{align*}
(Closure) & \quad \frac{}{\mathcal{E} \vdash_m \lambda^t x.e \downarrow \langle \lambda^t x.e, \mathcal{E} \rangle} \\
(Appl) & \quad \frac{\mathcal{E} \vdash_m e_1 \downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash_m e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash_m e \downarrow \nu}{\mathcal{E} \vdash_m e_1 e_2 \downarrow \nu}
\end{align*} \]
Monomorphic language

\[
e ::= c \mid x \mid \lambda^t x.e \mid ee \mid e \in t? e : e
\]

\[
v ::= c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\]

\[
(Closure) \quad \frac{\mathcal{E} \vdash_m \lambda^t x.e \downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}{\vdash_m \lambda^t x.e \downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}
\]

\[
(Apply) \quad \frac{\mathcal{E} \vdash_m e_1 \downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash_m e_2 \downarrow v_0 \quad \mathcal{E}', x \mapsto v_0 \vdash_m e \downarrow v}{\mathcal{E} \vdash_m e_1 e_2 \downarrow v}
\]
Monomorphic language

\[
e ::= c \mid x \mid \lambda^t x. e \mid ee \mid e \in t? e : e
\]

\[
\nu ::= c \mid \langle \lambda^t x. e, \mathcal{E} \rangle
\]

(Closure) \[
\mathcal{E} \vdash_m \lambda^t x. e \downarrow \langle \lambda^t x. e, \mathcal{E} \rangle
\]

(Apply) \[
\mathcal{E} \vdash_m e_1 \downarrow \langle \lambda^t x. e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash_m e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash_m e \downarrow \nu
\]

\[
\mathcal{E} \vdash_m e_1 e_2 \downarrow \nu
\]
Monomorphic language

\[ e ::= c \mid x \mid \lambda^t x.e \mid ee \mid e \in t \ ? \ e : e \]

\[ \nu ::= c \mid \langle \lambda^t x.e, \mathcal{E} \rangle \]

\[
\begin{align*}
\text{\textsc{(Closure)}} & \quad \mathcal{E} \vdash_m \lambda^t x.e \downarrow \langle \lambda^t x.e, \mathcal{E} \rangle \\
\text{\textsc{(Apply)}} & \quad \mathcal{E} \vdash_m e_1 \downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash_m e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash_m e \downarrow \nu \\
& \quad \mathcal{E} \vdash_m e_1 e_2 \downarrow \nu \end{align*}
\]

\[
\begin{align*}
\text{\textsc{(Typcase True)}} & \quad \mathcal{E} \vdash_m e_1 \downarrow \nu_0 \quad \nu_0 \in_m t \quad \mathcal{E} \vdash_m e_2 \downarrow \nu \\
& \quad \mathcal{E} \vdash_m e_1 \in t \ ? \ e_2 : e_3 \downarrow \nu \end{align*}
\]

\[
\begin{align*}
\text{\textsc{(Typcase False)}} & \quad \mathcal{E} \vdash_m e_1 \downarrow \nu_0 \quad \nu_0 \notin_m t \quad \mathcal{E} \vdash_m e_3 \downarrow \nu \\
& \quad \mathcal{E} \vdash_m e_1 \in t \ ? \ e_2 : e_3 \downarrow \nu \end{align*}
\]

\[
\begin{align*}
c \in_m t & \quad \text{def} \quad \{c\} \leq t \\
\langle \lambda^s x.e, \mathcal{E} \rangle \in_m t & \quad \text{def} \quad s \leq t
\end{align*}
\]
Polymorphic language: naïve implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_l} x.e \mid ee \mid e \in t \ ? e : e \mid e\sigma_l \]

(\(\sigma_l\) short for \([\sigma_i]_{i \in I}\))
Polymorphic language: naive implementation

\[
e ::= c \mid x \mid \lambda_{\sigma_I} x.e \mid ee \mid e \in t \ ? e : e \mid e\sigma_I
\]

\[
\nu ::= c \mid \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle
\]
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma}^t x.e \mid ee \mid e \in t \mathop{?} e : e \mid e\sigma_I \]

\[ v ::= c \mid \langle \lambda_{\sigma}^t x.e, \mathcal{E}, \sigma_I \rangle \]

\[ \text{(Closure)} \quad \frac{\sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma}^t x.e \Downarrow \langle \lambda_{\sigma}^t x.e, \mathcal{E}, \sigma_I \rangle}{\sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma}^t x.e \Downarrow \langle \lambda_{\sigma}^t x.e, \mathcal{E}, \sigma_I \rangle} \]
Polymorphic language: naive implementation

\[ e ::= \textit{c} \mid \textit{x} \mid \lambda_{\sigma_I}^{t} \cdot \textit{x}.e \mid ee \mid e \in t ? e : e \mid e\sigma_I \]

\[ v ::= \textit{c} \mid \langle \lambda_{\sigma_J}^{t} \cdot \textit{x}.e, \mathcal{E}, \sigma_I \rangle \]

(CLOSURE)

\[ \sigma_I, \mathcal{E} \vdash_{p} \lambda_{\sigma_J}^{t} \cdot \textit{x}.e \Downarrow \langle \lambda_{\sigma_J}^{t} \cdot \textit{x}.e, \mathcal{E}, \sigma_I \rangle \]
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda^{\tau}_{\sigma_I} x. e \mid ee \mid e \in t \? e : e \mid e\sigma_I \]

\[ \nu ::= c \mid \langle \lambda^{\tau}_{\sigma_J} x. e, \mathcal{E}, \sigma_I \rangle \]

(Closure)

\[ \sigma_I \mathcal{E} \vdash \lambda^{\tau}_{\sigma_J} x. e \Downarrow \langle \lambda^{\tau}_{\sigma_J} x. e, \mathcal{E}, \sigma_I \rangle \]

(\sigma_I \text{ short for } [\sigma_i]_{i \in I})

(save the environment)

(save current type-substitutions)
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_I} x. e \mid ee \mid e \in t \? e : e \mid e \sigma_I \]

\[ v ::= c \mid \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle \]

\[(\text{Closure}) \quad \frac{\sigma_I; E \vdash_p \lambda_{\sigma_J} x. e \downarrow \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle}{E \vdash_p \lambda_{\sigma_I} x. e \downarrow \langle \lambda_{\sigma_I} x. e, E, \sigma_I \rangle} \]

\[(\text{Instance}) \quad \frac{\sigma_J \circ \sigma_I; E \vdash_p e \downarrow v}{E \vdash_p e \sigma_I \downarrow v} \]
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_I} x.e \mid ee \mid e \in t ? e : e \mid e\sigma_I \]

\[ v ::= c \mid \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle \]

\[(\text{Closure}) \quad \sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma_J} x.e \downarrow \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle \]

\[(\text{Instance}) \quad \sigma_I \circ \sigma_J; \mathcal{E} \vdash_p e \downarrow v \]

\[(\text{Apply}) \quad \sigma_I; \mathcal{E} \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\ell \in L} t_{\ell} x.e, \mathcal{E}', \sigma_H \rangle \quad \sigma_I; \mathcal{E} \vdash_p e_2 \downarrow v_0 \quad \sigma_P; \mathcal{E}', x \mapsto v_0 \vdash_p e \downarrow v \]

\[ \sigma_I; \mathcal{E} \vdash_p e_1 e_2 \downarrow v \]

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in L : v_0 \in_P s_{\ell} \sigma_j \} \)
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_I} x. e \mid ee \mid e \in t \ ? e : e \mid e \sigma_I \]

\[ v ::= c \mid \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle \]

**(Closure)** \[ \sigma_I; E \vdash_p \lambda_{\sigma_J} x. e \downarrow \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle \]

**CLOSURE**

**INSTANCE**

\[ \sigma_I; E \vdash_p e \downarrow v \]

**INSTANCE**

\[ \sigma_I; E \vdash_p e \sigma_J \downarrow v \]

**APPLY**

\[ \sigma_I; E \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\ell \in L \rightarrow t} x. e, E', \sigma_H \rangle \quad \sigma_I; E \vdash_p e_2 \downarrow v_0 \quad \sigma_P; E', x \mapsto v_0 \vdash_p e \downarrow v \]

**APPLY**

\[ \sigma_I; E \vdash_p e_1 e_2 \downarrow v \]

**APPLY**

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in L : v_0 \in_P s_{\ell \sigma_j} \} \)
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_1} x. e \mid ee \mid e \in t \mid e : e \mid e_{\sigma_1} \]

\[ v ::= c \mid \langle \lambda_{\sigma_j} x. e, \mathcal{E}, \sigma_1 \rangle \]

\[ \sigma_1; \mathcal{E} \vdash_p \lambda_{\sigma_j} x. e \downarrow \langle \lambda_{\sigma_j} x. e, \mathcal{E}, \sigma_1 \rangle \]

\[ \sigma_1; \mathcal{E} \vdash_p e_{\sigma_j} \downarrow v \]

\[ \sigma_1; \mathcal{E} \vdash_p e_1 \downarrow \langle \lambda_{\sigma_1} x. e, \mathcal{E}', \sigma_H \rangle \]

\[ \sigma_1; \mathcal{E} \vdash_p e_2 \downarrow v_0 \]

\[ \mathcal{E}, x \mapsto v_0 \vdash_p e \downarrow v \]

\[ \sigma_1; \mathcal{E} \vdash_p e_1 e_2 \downarrow v \]

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in L : v_0 \in_p s_{\ell} \sigma_j \} \)

\[ (\sigma_I \text{ short for } [\sigma_i]_{i \in I}) \]
Polymorphic language: naive implementation

\[ e ::= c \ | \ x \ | \ \lambda_{\sigma_I}x.e \ | \ ee \ | \ e \in t ? e : e \ | \ e\sigma_I \]

\[ v ::= c \ | \ \langle \lambda_{\sigma_J}x.e, E, \sigma_I \rangle \]

(Closure) \[ \sigma_I; E \vdash_p \lambda_{\sigma_J}x.e \downarrow \langle \lambda_{\sigma_J}x.e, E, \sigma_I \rangle \]

(INSTANCE) \[ \sigma_I; E \vdash_p e \downarrow v \]

(Associate) \[ \sigma_I; E \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\land_{\ell\in L}s_{\ell}\rightarrow t_{\ell}}x.e, E', \sigma_H \rangle \quad \sigma_I; E \vdash_p e_2 \downarrow v_0 \]

\[ \sigma_I; E \vdash_p e_1 e_2 \downarrow v \]

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in L : v_0 \in_p s_{\ell} \sigma_j \} \)
Polymorphic language: naive implementation

\[
e := c \mid x \mid \lambda_{\sigma_I}^t x. e \mid ee \mid e \in t \ ? e : e \mid e \sigma_I
\]

\[
\nu := c \mid \langle \lambda_{\sigma_J}^t x. e, \mathcal{E}, \sigma_I \rangle
\]

(Closure) \hspace{1cm} (Instance)
\[
\sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma_J}^t x. e \downarrow \langle \lambda_{\sigma_J}^t x. e, \mathcal{E}, \sigma_I \rangle \quad \sigma_I; \mathcal{E} \vdash_p e \sigma_J \downarrow \nu
\]

(Apply)
\[
\sigma_I; \mathcal{E} \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\ell \in L \sigma_{\ell \rightarrow t_\ell}} x. e, \mathcal{E}', \sigma_H \rangle \quad \sigma_I; \mathcal{E} \vdash_p e_2 \downarrow \nu_0 \quad \sigma_P; \mathcal{E}', x \mapsto \nu_0 \vdash_p e \downarrow \nu
\]
\[
\sigma_I; \mathcal{E} \vdash_p e_1 e_2 \downarrow \nu
\]

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in L : \nu_0 \in_p s_\ell \sigma_j \} \)

Problem:

At every application compute \( \sigma_P \):
Polymorphic language: naive implementation

\( e ::= c | x | \lambda_{\sigma_I} x. e | ee | e \in t ? e : e | e\sigma_I \)

\( \nu ::= c | \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle \)

(Closure) \( \sigma_I; E \vdash_p \lambda_{\sigma_J} x. e \downarrow \langle \lambda_{\sigma_J} x. e, E, \sigma_I \rangle \)

(INSTANCE) \( \sigma_I \circ \sigma_J; E \vdash_p e \downarrow \nu \)

(Appl) \( \sigma_I; E \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\subseteq \ell \in L} \rightarrow t_{\ell} x. e, E', \sigma_H \rangle \)

\( \sigma_I; E \vdash_p e_2 \downarrow \nu_0 \)

\( \sigma_P; E', x \mapsto \nu_0 \vdash_p e \downarrow \nu \)

\( \sigma_I; E \vdash_p e_1 e_2 \downarrow \nu \)

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J | \exists \ell \in L : \nu_0 \in_p s_{\ell} \sigma_j \} \)

Problem:

At every application compute \( \sigma_P \):

1. compose of two sets of type-substitution
Polymorphic language: naive implementation

\[
e ::= c \mid x \mid \lambda_{\sigma_I} x.e \mid ee \mid e \in t ? e : e \mid e\sigma_I
\]

\[
v ::= c \mid \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle
\]

\[\text{(Closure)}\]
\[
\sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma_J} x.e \Downarrow \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle
\]

\[\text{(Instance)}\]
\[
\sigma_I; \mathcal{E} \vdash_p e\sigma_J \Downarrow v
\]

\[\text{(Apply)}\]
\[
\sigma_I; \mathcal{E} \vdash_p e_1 \Downarrow \langle \lambda_{\sigma_K}^{\ell \in L^j} \rightarrow t_\ell x.e, \mathcal{E}', \sigma_H \rangle \quad \sigma_I; \mathcal{E} \vdash_p e_2 \Downarrow v_0 \quad \sigma_P; \mathcal{E}', x \mapsto v_0 \vdash_p e \Downarrow v
\]

where \(\sigma_J = \sigma_H \circ \sigma_K\) and \(P = \{ j \in J \mid \exists \ell \in L : v_0 \in p s_\ell \sigma_J \}\)

Problem:

At every application compute \(\sigma_P\):

1. compose of two sets of type-substitution
2. select the substitutions compatible with the argument \(v_0\)
Polymorphic language: naive implementation

\[ e ::= c \mid x \mid \lambda_{\sigma_l} x.e \mid ee \mid e \in t \cdot e : e \mid e\sigma_l \]

\[ \sigma_l; \mathcal{E} \vdash_p \lambda_{\sigma_j} x. e \downarrow \langle \lambda_{\sigma_j} x. e, \mathcal{E}, \sigma_l \rangle \]

\[ \sigma_l; \mathcal{E} \vdash_p e_1 e_2 \downarrow \nu \]

\[ \sigma_j = \sigma_H \circ \sigma_K \quad \text{and} \quad P = \{ j \in J \mid \exists \ell \in L : \nu_0 \in_p s_\ell \sigma_j \} \]

**Problem:**

At every application compute \( \sigma_P \):

1. **compose** of two sets of type-substitution
2. **select** the substitutions compatible with the argument \( \nu_0 \)
Polymorphic language: naive implementation

\[ \begin{align*}
  e & ::= \ c \ | \ x \ | \ \lambda_{\sigma_I} x.e \ | \ ee \ | \ e \in t ? e : e \ | \ e\sigma_I \\
  \nu & ::= \ c \ | \ \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle \\
\end{align*} \]

\[ \begin{align*}
  \text{(Closure)} & \quad \sigma_I; \mathcal{E} \vdash_p \lambda_{\sigma_J} x.e \downarrow \langle \lambda_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle \\
  \text{(Instance)} & \quad \sigma_I \circ \sigma_J; \mathcal{E} \vdash_p e \downarrow v \\
  \text{(Apply)} & \quad \sigma_I; \mathcal{E} \vdash_p e_1 \downarrow \langle \lambda_{\sigma_K}^{\sum_{\ell \in \mathcal{L}} t_{\ell}} x.e, \mathcal{E}', \sigma_H \rangle \quad \sigma_I; \mathcal{E} \vdash_p e_2 \downarrow v_0 \quad \sigma_{p}; \mathcal{E}', x \mapsto v_0 \vdash_p e \downarrow v \\
\end{align*} \]

where \( \sigma_J = \sigma_H \circ \sigma_K \) and \( P = \{ j \in J \mid \exists \ell \in \mathcal{L} : v_0 \in_p s_\ell \sigma_J \} \)

\[ \sigma_I; \mathcal{E} \vdash_p e_1 e_2 \downarrow v \]

\[ \sigma_I \circ \sigma_J; \mathcal{E} \vdash_p e \downarrow v \]

Solution:

Compute compositions and selections lazily.
Intermediate language as compilation target

\[
e ::= c \mid x \mid \lambda^t x.e \mid ee \mid e \in t \? e : e
\]

\[
\nu ::= c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\]

\[\text{Closure}\]
\[
\mathcal{E} \vdash \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle
\]

\[\text{Apply}\]
\[
\mathcal{E} \vdash e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_2 \Downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash e \Downarrow \nu
\]
\[
\mathcal{E} \vdash e_1 e_2 \Downarrow \nu
\]

\[\text{Typecase True}\]
\[
\mathcal{E} \vdash e_1 \Downarrow \nu_0 \quad \nu_0 \in t \quad \mathcal{E} \vdash e_2 \Downarrow \nu
\]
\[
\mathcal{E} \vdash e_1 \in t ? e_2 : e_3 \Downarrow \nu
\]

\[\text{Typecase False}\]
\[
\mathcal{E} \vdash e_1 \Downarrow \nu_0 \quad \nu_0 \not\in t \quad \mathcal{E} \vdash e_3 \Downarrow \nu
\]
\[
\mathcal{E} \vdash e_1 \in t ? e_2 : e_3 \Downarrow \nu
\]

\[
c \in t \overset{\text{def}}{=} \{c\} \leq t
\]

\[
\langle \lambda^s x.e, \mathcal{E} \rangle \in t \overset{\text{def}}{=} s \leq t
\]
Intermediate language as compilation target

\[ e ::= c \mid x \mid \lambda^t_x e \mid ee \mid e \in t \ ? e : e \]
\[ \nu ::= c \mid \langle \lambda^t_x e, \mathcal{E} \rangle \]
\[ \Sigma ::= \sigma_I \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma) \]  

\text{symbolic substitutions}

**(Closure)**

\[ \mathcal{E} \vdash \lambda^t_x e \downarrow \langle \lambda^t_x e, \mathcal{E} \rangle \]

**(Apply)**

\[ \mathcal{E} \vdash e_1 \downarrow \langle \lambda^t_x e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash e \downarrow \nu \]

\[ \mathcal{E} \vdash e_1 e_2 \downarrow \nu \]

**(Typecase True)**

\[ \mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \in t \quad \mathcal{E} \vdash e_2 \downarrow \nu \]

\[ \mathcal{E} \vdash e_1 \in t \ ? e_2 : e_3 \downarrow \nu \]

**(Typecase False)**

\[ \mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \not\in t \quad \mathcal{E} \vdash e_3 \downarrow \nu \]

\[ \mathcal{E} \vdash e_1 \in t \ ? e_2 : e_3 \downarrow \nu \]

\[ \begin{align*}
  & c \in t \quad \text{def} \quad \{ c \} \leq t \\
  & \langle \lambda^s_x e, \mathcal{E} \rangle \in t \quad \text{def} \quad s \leq t
\end{align*} \]
Intermediate language as compilation target

\[ e ::= c \mid x \mid \lambda^t_x.e \mid ee \mid e \in t \? e : e \]

\[ \nu ::= c \mid \langle \lambda^t_x.e, \mathcal{E} \rangle \]

\[ \Sigma ::= \sigma_1 \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma) \quad \text{symbolic substitutions} \]

\[ (\text{Closure}) \quad \mathcal{E} \vdash \lambda^t_x.e \Downarrow \langle \lambda^t_x.e, \mathcal{E} \rangle \]

\[ (\text{Apply}) \quad \mathcal{E} \vdash e_1 \Downarrow \langle \lambda^t_x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_2 \Downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash e \Downarrow \nu \]

\[ \mathcal{E} \vdash e_1 e_2 \Downarrow \nu \]

\[ (\text{Typecase True}) \quad \mathcal{E} \vdash e_1 \Downarrow \nu_0 \quad \nu_0 \in t \quad \mathcal{E} \vdash e_2 \Downarrow \nu \]

\[ \mathcal{E} \vdash e_1 \in t \? e_2 : e_3 \Downarrow \nu \]

\[ (\text{Typecase False}) \quad \mathcal{E} \vdash e_1 \Downarrow \nu_0 \quad \nu_0 \not\in t \quad \mathcal{E} \vdash e_3 \Downarrow \nu \]

\[ \mathcal{E} \vdash e_1 \in t \? e_2 : e_3 \Downarrow \nu \]

\[ c \in t \quad \text{def} \quad \{c\} \leq t \]

\[ \langle \lambda^s_x.e, \mathcal{E} \rangle \in t \quad \text{def} \quad s \leq t \]
Intermediate language as compilation target

\[
e ::= c \mid x \mid \lambda_{\Sigma}^{t}x.e \mid ee \mid e \in t \,?\, e : e
\]

\[
v ::= c \mid \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle
\]

\[
\Sigma ::= \sigma_{I} \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma) \quad \text{symbolic substitutions}
\]

\[
\text{(Closure)} \quad \frac{\mathcal{E} \vdash \lambda_{\Sigma}^{t}x.e \Downarrow \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle}{\mathcal{E} \vdash e \Downarrow \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle}
\]

\[
\text{(Apply)} \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_{2} \Downarrow v_{0} \quad \mathcal{E}', x \mapsto v_{0} \vdash e \Downarrow v}{\mathcal{E} \vdash e_{1}e_{2} \Downarrow v}
\]

\[
\text{(Typecase True)} \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow v_{0} \quad v_{0} \in t \quad \mathcal{E} \vdash e_{2} \Downarrow v}{\mathcal{E} \vdash e_{1} \in t \,?\, e_{2} : e_{3} \Downarrow v}
\]

\[
\text{(Typecase False)} \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow v_{0} \quad v_{0} \not\in t \quad \mathcal{E} \vdash e_{3} \Downarrow v}{\mathcal{E} \vdash e_{1} \in t \,?\, e_{2} : e_{3} \Downarrow v}
\]

\[
c \in t \quad \text{def} \quad \{c\} \leq t
\]

\[
\langle \lambda^{s}x.e, \mathcal{E} \rangle \in t \quad \text{def} \quad s \leq t
\]
Intermediate language as compilation target

\[ e ::= c \mid x \mid \lambda_{\Sigma}^t x. e \mid ee \mid e \in t \, ? \, e : e \]

\[ \nu ::= c \mid \langle \lambda_{\Sigma}^t x. e, \mathcal{E} \rangle \]

\[ \Sigma ::= \sigma_I \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma) \quad \text{symbolic substitutions} \]

\[
\frac{\text{(Closure)}}{\mathcal{E} \vdash \lambda_{\Sigma}^t x. e \downarrow \langle \lambda_{\Sigma}^t x. e, \mathcal{E} \rangle}
\]

\[
\frac{\mathcal{E} \vdash e_1 \downarrow \langle \lambda_{\Sigma}^t x. e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash e \downarrow \nu}{\mathcal{E} \vdash e_1 e_2 \downarrow \nu}
\]

\[
\frac{\text{(Typecase True)}}{\mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \in t \quad \mathcal{E} \vdash e_2 \downarrow \nu}{\mathcal{E} \vdash e_1 \in t \, ? \, e_2 : e_3 \downarrow \nu}
\]

\[
\frac{\text{(Typecase False)}}{\mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \not\in t \quad \mathcal{E} \vdash e_3 \downarrow \nu}{\mathcal{E} \vdash e_1 \in t \, ? \, e_1 : e_2 \downarrow \nu}
\]

\[
c \in t \quad \langle \lambda^s x. e, \mathcal{E} \rangle \in t \quad \{c\} \subseteq t \quad s \subseteq t
\]
Intermediate language as compilation target

\[
e ::= c \mid x \mid \lambda^t_x e \mid ee \mid e \in t ? e : e
\]

\[
\nu ::= c \mid \langle \lambda^t_x e, \mathcal{E} \rangle
\]

\[
\Sigma ::= \sigma I \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma)
\]

symbolic substitutions

\[(CLOSURE) \quad \frac{}{\mathcal{E} \vdash \lambda^t_x e \downarrow \langle \lambda^t_x e, \mathcal{E} \rangle}
\]

\[(APPLY) \quad \frac{\mathcal{E} \vdash e_1 \downarrow \langle \lambda^t_x e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_2 \downarrow \nu_0 \quad \mathcal{E}', x \mapsto \nu_0 \vdash e \downarrow \nu}{\mathcal{E} \vdash e_1 e_2 \downarrow \nu}
\]

\[(TYPECASE True) \quad \frac{\mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \in t}{\mathcal{E} \vdash e_1 \in t ? e_2 : e_3 \downarrow \nu}
\]

\[(TYPECASE False) \quad \frac{\mathcal{E} \vdash e_1 \downarrow \nu_0 \quad \nu_0 \notin t}{\mathcal{E} \vdash e_1 \in t ? e_2 : e_3 \downarrow \nu}
\]

\[
c \in t \quad \text{def} \quad \{c\} \leq t
\]

\[
\langle \lambda^S_x e, \mathcal{E} \rangle \in t \quad \text{def} \quad s(\text{eval}(\mathcal{E}, \Sigma)) \leq t
\]

Giuseppe Castagna

Polymorphic Functions with Set-Theoretic Types

35/39
Intermediate language as compilation target

\[
e ::= c \mid x \mid \lambda_{\Sigma}^{t}x.e \mid ee \mid e \in t \ ? \ e : e
\]
\[
\nu ::= c \mid \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle
\]
\[
\Sigma ::= \sigma_{I} \mid \text{comp}(\Sigma, \Sigma') \mid \text{sel}(x, t, \Sigma)
\]

symbolic substitutions

\[
(C\text{LOSURE}) \quad \frac{\mathcal{E} \vdash \lambda_{\Sigma}^{t}x.e \Downarrow \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle}{\mathcal{E} \vdash \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle}
\]

\[
(A\text{PLY}) \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow \langle \lambda_{\Sigma}^{t}x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash e_{2} \Downarrow \nu_{0} \quad \mathcal{E}', x \mapsto \nu_{0} \vdash e \Downarrow \nu}{\mathcal{E} \vdash e_{1}e_{2} \Downarrow \nu}
\]

\[
(T\text{YPECASE TRUE}) \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow \nu_{0} \quad \nu_{0} \in t \quad \mathcal{E} \vdash e_{2} \Downarrow \nu}{\mathcal{E} \vdash e_{1} \in t \ ? \ e_{2} : e_{3} \Downarrow \nu}
\]

\[
(T\text{YPECASE FALSE}) \quad \frac{\mathcal{E} \vdash e_{1} \Downarrow \nu_{0} \quad \nu_{0} \not\in t \quad \mathcal{E} \vdash e_{2} \Downarrow \nu}{\mathcal{E} \vdash e_{1} \in t \ ? \ e_{2} : e_{3} \Downarrow \nu}
\]

\[
c \in t \quad \text{def} \quad \{c\} \leq t
\]

\[
\langle \lambda_{\Sigma}^{t}x.e, \mathcal{E} \rangle \in t \quad \text{def} \quad s(\text{eval}(\mathcal{E}, \Sigma)) \leq t
\]
Compilation

Compile into the intermediate language

\[
[x]_\Sigma = x \\
[\lambda^t_\sigma \, x. \, e]_\Sigma = \lambda^t_{\text{comp}(\Sigma, \sigma_1)} \, x \cdot [e]_{\text{sel}(x, t, \text{comp}(\Sigma, \sigma_1))} \\
[e_1 \, e_2]_\Sigma = [e_1]_\Sigma \cdot [e_2]_\Sigma \\
[e \sigma_1]_\Sigma = [e]_{\text{comp}(\Sigma, \sigma_1)} \\
[e_1 \in t \, ? \, e_2 : e_3]_\Sigma = [e_1]_\Sigma \in t \, ? \, [e_2]_\Sigma : [e_3]_\Sigma
\]
Compilation

1. Compile into the intermediate language

\[ \begin{align*}
[x]_\Sigma &= x \\
[\lambda^t_{\sigma_I}x.e]_\Sigma &= \lambda^{t}_{\text{comp}(\Sigma, \sigma_I)x}.[e]_{\text{sel}(x, t, \text{comp}(\Sigma, \sigma_I))} \\
[e_1 e_2]_\Sigma &= [e_1]_\Sigma [e_2]_\Sigma \\
[e_{\sigma_I}]_\Sigma &= [e]_{\text{comp}(\Sigma, \sigma_I)} \\
[e_1 \in t \ ? \ e_2 : e_3]_\Sigma &= [e_1]_\Sigma \in t \ ? [e_2]_\Sigma : [e_3]_\Sigma
\end{align*} \]

2. For \( \langle \lambda^{s}_{\Sigma} x. e, E \rangle \in t \overset{\text{def}}{=} s(\text{eval}(E, \Sigma)) \leq t \) we have \( s(\text{eval}(E, \Sigma)) \neq s \) only if \( \lambda^{s}_{\Sigma} x. e \) results from the partial application of a polymorphic function (\( ie, \) in \( s \) there occur free variables bound in the context).
Compile into the intermediate language

\[
\begin{align*}
[x]_\Sigma & = x \\
[\lambda^t_{\sigma_I} x. e]_\Sigma & = \lambda^t_{\text{comp}(\Sigma, \sigma_I)} x [e]_{\text{sel}(x, t, \text{comp}(\Sigma, \sigma_I))} \\
[e_1 e_2]_\Sigma & = [e_1]_\Sigma [e_2]_\Sigma \\
[e_{\sigma_I}]_\Sigma & = [e]_{\text{comp}(\Sigma, \sigma_I)} \\
[e_1 \in t \ ? \ e_2 : e_3]_\Sigma & = [e_1]_\Sigma e_1 \in t ? [e_2]_\Sigma : [e_3]_\Sigma
\end{align*}
\]

For \( \langle \lambda^s_{\Sigma} x. e, \mathcal{E} \rangle \in t \overset{\text{def}}{=} s(\text{eval}(\mathcal{E}, \Sigma)) \leq t \) we have \( s(\text{eval}(\mathcal{E}, \Sigma)) \neq s \) only if \( \lambda^s_{\Sigma} x. e \) results from the partial application of a polymorphic function (ie, in \( s \) there occur free variables bound in the context).

**Execution may be slowed only** when testing the type of the result of a partial application of a polymorphic function.
Compilation can flag the functions that may require to compute eval:

$$\left[ \lambda^t_i x. e \right]_\Sigma = \begin{cases} 
\lambda^t_x \left[ e \right]_{sel(x, t, \Sigma)} & \text{if } \text{var}(t) \cap \text{dom}(\Sigma) = \emptyset \\
\hat{\lambda}^t_x \left[ e \right]_{sel(x, t, \Sigma)} & \text{otherwise}
\end{cases}$$

and then we evaluate the symbolic substitutions only for marked functions:

$$\langle \lambda^s_x x. e, \mathcal{E} \rangle \in t \iff s \leq t$$

$$\langle \hat{\lambda}^s_x x. e, \mathcal{E} \rangle \in t \iff s(\text{eval}(\mathcal{E}, \Sigma)) \leq t$$
Compilation can flag the functions that may require to compute eval:

\[
\left[ \lambda^t_x \cdot e \right]_\Sigma = \begin{cases} 
\lambda^t_x \cdot \left[ e \right]_{se1(x,t,\Sigma)} & \text{if } \text{var}(t) \cap \text{dom}(\Sigma) = \emptyset \\
\hat{\lambda}^t_x \cdot \left[ e \right]_{se1(x,t,\Sigma)} & \text{otherwise}
\end{cases}
\]

and then we evaluate the symbolic substitutions only for marked functions:

\[
\langle \lambda^s_{\Sigma} x . e, \mathcal{E} \rangle \in t \quad \overset{\text{def}}{\iff} \quad s \leq t
\]

\[
\langle \hat{\lambda}^s_{\Sigma} x . e, \mathcal{E} \rangle \in t \quad \overset{\text{def}}{\iff} \quad s(\text{eval}(\mathcal{E}, \Sigma)) \leq t
\]

This holds also with products (used to encode lists records and XML), whose testing accounts for most of the execution time.
Compilation can flag the functions that may require to compute eval:

$$\left[ \lambda^t_x . e \right]_\Sigma = \begin{cases} \lambda^t_x . [e]_{\text{sel}(x,t,\Sigma)} & \text{if } \text{var}(t) \cap \text{dom}(\Sigma) = \emptyset \\ \hat{\lambda}^t_x . [e]_{\text{sel}(x,t,\Sigma)} & \text{otherwise} \end{cases}$$

and then we evaluate the symbolic substitutions only for marked functions:

$$\langle \lambda^s_x . e, \mathcal{E} \rangle \in t \iff s \leq t$$
$$\langle \hat{\lambda}^s_x . e, \mathcal{E} \rangle \in t \iff s(\text{eval}(\mathcal{E}, \Sigma)) \leq t$$

This holds also with products (used to encode lists records and XML), whose testing accounts for most of the execution time.

**Bottom Line**

The execution is as efficient as in the monomorphic case, apart from a single well identified exception.
Conclusion
Theory: All the theoretical machinery necessary to design and implement a programming language is there. The practical relevance of the open theoretical issues is negligible.
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Languages: The implementation of the polymorphic extension of CDuce is almost done (see git); we intend to study the definition of polymorphic extensions of XQuery and to embed some of this type machinery in ML (e.g., type balance for red-black trees).
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**Languages:** The implementation of the polymorphic extension of CDuce is almost done (see git); we intend to study the definition of polymorphic extensions of XQuery and to embed some of this type machinery in ML (e.g., type `balance` for red-black trees).

**Runtime:** Relabeling cannot be avoided but it is materialized only in case of partial polymorphic applications that end up in type-cases, that is, just when it is needed.
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**Implementation:** Subtyping of polymorphic types require minimal modifications to the implementation. Existing data structures (e.g., binary decision trees with lazy unions) and optimizations mostly transpose smoothly.
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**Runtime:** Relabeling cannot be avoided but it is materialized only in case of partial polymorphic applications that end up in type-cases, that is, just when it is needed.

**Implementation:** Subtyping of polymorphic types require minimal modifications to the implementation. Existing data structures (e.g., binary decision trees with lazy unions) and optimizations mostly transpose smoothly.

**Type reconstruction:** Full usage needs more research, especially about the production of human readable types and helpful error messages, but it is mature enough to use it to type local functions.
Extra slides
References

1. **Subtyping**: Set-theoretic Foundation of Parametric Polymorphism and Subtyping. *ICFP ’11*

2. **Language**: Polymorphic Functions with Set-Theoretic Types. Part 1: Syntax, Semantics, and Evaluation. *POPL ’14*

3. **Language**: Polymorphic Functions with Set-Theoretic Types. Part 2: Local Type Inference and Type Reconstruction. *POPL ’15*. 
The problem of inferring the type of an application is thus to find for $s$ and $t$ given, $[\sigma_i]_{i \in I}, [\sigma'_j]_{j \in J}$ such that:

$$t[\sigma'_j]_{j \in J} \leq 0 \to 1 \quad \text{and} \quad s[\sigma_i]_{i \in I} \leq \text{dom}(t[\sigma'_j]_{j \in J})$$

This can be reduced to solving a suite of *tallying problems*:

**Definition (Type tallying)**

Let $C = \{(s_1, t_1), \ldots, (s_n, t_n)\}$ a *constraint set*. A type-substitution $\sigma$ is a solution for the *tallying* of $C$ iff $s\sigma \leq t\sigma$ for all $(s, t) \in C$. 
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Type tallying is decidable and a sound and complete set of solutions for every tallying problem can be effectively found in three simple steps.
**Step 1: Decompose constraints.**

Use the set-theoretic decomposition rules to transform $C$ into a set of constraint sets whose constraints are of the form $(\alpha, t)$ or $(t, \alpha)$. 

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Step 2: Merge constraints on the same variable.

If $(\alpha, t_1)$ and $(\alpha, t_2)$ are in $C$, then replace them by $(\alpha, t_1 \land t_2)$.

If $(s_1, \alpha)$ and $(s_2, \alpha)$ are in $C$, then replace them by $(s_1 \lor s_2, \alpha)$.

Possibly decompose the new constraints generated by transitivity.

Step 3: Transform into a set of equations.

After Step 2 we have constraint-sets of the form:

$$\left\{ \begin{array}{l} s_i \leq \alpha_i \leq t_i \mid i \in [1..n] \end{array} \right. $$

where $\alpha_i$ are pairwise distinct.

1. Select $s \leq \alpha \leq t$ and replace it by $\alpha = (s \lor \beta) \land t$ with $\beta$ fresh.

2. In all other constraints replace every $\alpha$ by $(s \lor \beta) \land t$.

3. Repeat with another constraint.

At the end we have a sets of equations:

$$\left\{ \alpha_i = u_i \mid i \in [1..n] \right. $$

that (with some care) are contractive. By Courcelle there exists a solution, i.e., a substitution for $\alpha_1, \ldots, \alpha_n$ into (possibly recursive) regular types $t_1, \ldots, t_n$ (in which the fresh $\beta$'s are free variables).
**Step 1: Decompose constraints.**
Use the set-theoretic decomposition rules to transform $C$ into a set of constraint sets whose constraints are of the form $(\alpha, t)$ or $(t, \alpha)$.

**Step 2: Merge constraints on the same variable.**
- if $(\alpha, t_1)$ and $(\alpha, t_2)$ are in $C$, then replace them by $(\alpha, t_1 \land t_2)$;
- if $(s_1, \alpha)$ and $(s_2, \alpha)$ are in $C$, then replace them by $(s_1 \lor s_2, \alpha)$;

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Possibly decompose the new constraints generated by transitivity.

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At the end we have a sets of equations $\{\alpha_i = u_i \mid i \in [1..n]\}$ that (with some care) are *contractive*. By Courcelle there exists a solution, *ie*, a substitution for $\alpha_1, ..., \alpha_n$ into (possibly recursive regular) types $t_1, ..., t_n$ (in which the fresh $\beta$’s are free variables).
The application problem

Definition (Inference application problem)

Given $s$ and $t$ types, find $[\sigma_i]_{i \in I}$ and $[\sigma'_j]_{j \in J}$ such that:

\[
\bigwedge_{i \in I} t \sigma_i \leq 0 \rightarrow 1 \quad \text{and} \quad \bigwedge_{j \in J} s \sigma_j \leq \text{dom}(\bigwedge_{i \in I} t \sigma_i)
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\]

1. Fix the cardinalities of \( I \) and \( J \) (at the beginning both 1);
2. Split each substitution \( \sigma_k \) (for \( k \in I \cup J \)) in two: \( \sigma_k = \rho_k \circ \sigma'_k \)
   where \( \rho_k \) is a renaming substitution mapping each variable of the domain of \( \sigma_k \) into a fresh variable:

\[
\bigwedge_{i \in I} (t\rho_i)\sigma'_i \leq 0 \rightarrow 1 \quad \text{and} \quad \bigwedge_{j \in J} (s\rho_j)\sigma'_j \leq \text{dom} \left( \bigwedge_{i \in I} (t\rho_i)\sigma'_i \right);
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   where $\rho_k$ is a renaming substitution mapping each variable of the domain of $\sigma_k$ into a fresh variable:

   $$(\bigwedge_{i \in I} t \rho_i) \sigma \leq 0 \rightarrow 1 \quad \text{and} \quad (\bigwedge_{j \in J} s \rho_j) \sigma \leq \text{dom}((\bigwedge_{i \in I} t \rho_i) \sigma);$$

3. Solve the tallying problem for

   $$\{(t_1, 0 \rightarrow 1), (t_1, t_2 \rightarrow \gamma)\}$$

   with $t_1 = \bigwedge_{i \in I} t \rho_i$, $t_2 = \bigwedge_{j \in J} s \rho_j$, and $\gamma$ fresh
The application problem

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Given \( s \) and \( t \) types, find \([\sigma_i]_{i \in I}\) and \([\sigma'_j]_{j \in J}\) such that:
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1. Fix the cardinalities of \( I \) and \( J \) (at the beginning both 1);
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   \[
   (\bigwedge_{i \in I} t\rho_i)\sigma \leq 0 \rightarrow 1 \quad \text{and} \quad (\bigwedge_{j \in J} s\rho_j)\sigma \leq \text{dom}(\bigwedge_{i \in I} t\rho_i)\sigma ;
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   \[
   \{(t_1, 0 \rightarrow 1), (t_1, t_2 \rightarrow \gamma)\}
   \]
   with \( t_1 = \bigwedge_{i \in I} t\rho_i \), \( t_2 = \bigwedge_{j \in J} s\rho_j \), and \( \gamma \) fresh
   - if it fails at Step 1, then fail.
   - if it fails at Step 2, then change cardinalities (dove-tail)
The application problem

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Given \( s \) and \( t \) types, find \([\sigma_i]_{i \in I}\) and \([\sigma'_j]_{j \in J}\) such that:

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\[
(\bigwedge_{i \in I} t \rho_i)\sigma \leq 0 \rightarrow 1 \quad \text{and} \quad (\bigwedge_{j \in J} s \rho_j)\sigma \leq \text{dom}\left(\bigwedge_{i \in I} t \rho_i\right);
\]
3. Solve the tallying problem for

\[
\{(t_1, 0 \rightarrow 1), (t_1, t_2 \rightarrow \gamma)\}
\]

with \( t_1 = \bigwedge_{i \in I} t \rho_i \), \( t_2 = \bigwedge_{j \in J} s \rho_j \), and \( \gamma \) fresh

- if it fails at Step 1, then fail.
- if it fails at Step 2, then change cardinalities (dove-tail)

Every solution for \( \gamma \) is a solution for the application.
Example: `map even`

Start with the following tallying problem:

\[\{(\alpha_1 \rightarrow \beta_1) \rightarrow [\alpha_1] \rightarrow [\beta_1] \leq t \rightarrow \gamma\}\]

where \(t = (\text{Int} \rightarrow \text{Bool}) \land (\alpha \setminus \text{Int} \rightarrow \alpha \setminus \text{Int})\) is the type of `even`
**Example: map even**

Start with the following tallying problem:

\[
\{(\alpha_1 \rightarrow \beta_1) \rightarrow [\alpha_1] \rightarrow [\beta_1] \leq t \rightarrow \gamma\}
\]

where \( t = (\text{Int} \rightarrow \text{Bool}) \land (\alpha \downarrow \text{Int} \rightarrow \alpha \downarrow \text{Int}) \) is the type of \texttt{even}.

- At step 2 the algorithm generates 9 constraint-sets: one is unsatisfiable (\( t \leq \emptyset \)); four are implied by the others; remain

  \[
  \begin{align*}
  &\{\gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq 0\}, \\
  &\{\gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \text{Int}, \text{Bool} \leq \beta_1\}, \\
  &\{\gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \alpha \downarrow \text{Int}, \alpha \downarrow \text{Int} \leq \beta_1\}, \\
  &\{\gamma \geq [\alpha_1] \rightarrow [\beta_1], \alpha_1 \leq \alpha \lor \text{Int}, (\alpha \downarrow \text{Int}) \lor \text{Bool} \leq \beta_1\};
  \end{align*}
  \]
Example: map even

Start with the following tallying problem:

\[ \{(\alpha_1 \rightarrow \beta_1) \rightarrow [\alpha_1] \rightarrow [\beta_1] \leq t \rightarrow \gamma\} \]

where \( t = (\text{Int} \rightarrow \text{Bool}) \land (\alpha \downarrow \text{Int} \rightarrow \alpha \downarrow \text{Int}) \) is the type of even

- At step 2 the algorithm generates 9 constraint-sets: one is unsatisfiable \( (t \leq \emptyset) \); four are implied by the others; remain

\[
\begin{align*}
\{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq 0 \}, \\
\{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq \text{Int} , \text{Bool} \leq \beta_1 \}, \\
\{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq \alpha \downarrow \text{Int} , \alpha \downarrow \text{Int} \leq \beta_1 \}, \\
\{ & \gamma \geq [\alpha_1] \rightarrow [\beta_1] , \alpha_1 \leq \alpha \lor \text{Int} , (\alpha \downarrow \text{Int}) \lor \text{Bool} \leq \beta_1 \}; \\
\end{align*}
\]

- Four solutions for \( \gamma \):

\[
\begin{align*}
\{ & \gamma = [] \rightarrow [] \}, \\
\{ & \gamma = [\text{Int}] \rightarrow [\text{Bool}] \}, \\
\{ & \gamma = [\alpha \downarrow \text{Int}] \rightarrow [\alpha \downarrow \text{Int}] \}, \\
\{ & \gamma = [\alpha \lor \text{Int}] \rightarrow [(\alpha \downarrow \text{Int}) \lor \text{Bool}] \}. \\
\end{align*}
\]
Example: map even

Start with the following tallying problem:
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\{(\alpha_1 \to \beta_1) \to [\alpha_1] \to [\beta_1] \leq t \to \gamma\}
\]
where \( t = (\text{Int} \to \text{Bool}) \land (\alpha \land \text{Int} \to \alpha \land \text{Int}) \) is the type of even

- At step 2 the algorithm generates 9 constraint-sets: one is unsatisfiable (\( t \leq \emptyset \)); four are implied by the others; remain
  \[
  \{\gamma \geq [\alpha_1] \to [\beta_1], \alpha_1 \leq 0\},
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  \{\gamma \geq [\alpha_1] \to [\beta_1], \alpha_1 \leq \alpha \lor \text{Int}, \text{Int} \leq \beta_1\},
  \{\gamma \geq [\alpha_1] \to [\beta_1], \alpha_1 \leq \alpha \lor \text{Int}, (\alpha \land \text{Int}) \lor \text{Bool} \leq \beta_1\};
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- Four solutions for \( \gamma \):
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  \{\gamma = [\text{Int}] \to [\text{Bool}]\},
  \{\gamma = [\alpha \land \text{Int}] \to [\alpha \land \text{Int}]\},
  \{\gamma = [\alpha \lor \text{Int}] \to [(\alpha \land \text{Int}) \lor \text{Bool}]\}.
  \]

- The last two are minimal and we take their intersection:
  \[
  \{\gamma = ([\alpha \land \text{Int}] \to [\alpha \land \text{Int}]) \land ([\alpha \lor \text{Int}] \to [(\alpha \land \text{Int}) \lor \text{Bool}]\})
  \]
Type Reconstruction Algorithm

\[ \Gamma \vdash_R c : b_c \leadsto \{ \emptyset \} \]  
\( \text{(R-CONST)} \)

\[ \Gamma \vdash_R x : \Gamma(x) \leadsto \{ \emptyset \} \]  
\( \text{(R-VAR)} \)

\[ \Gamma \vdash_R m_1 : t_1 \leadsto S_1 \quad \Gamma \vdash_R m_2 : t_2 \leadsto S_2 \]  
\[ \Gamma \vdash_R m_1 m_2 : \alpha \leadsto S_1 \cap S_2 \cap \{ \{ (t_1 \leq t_2 \rightarrow \alpha) \} \} \]  
\( \text{(R-APPL)} \)

\[ \Gamma, x : \alpha \vdash_R m : t \leadsto S \]  
\[ \Gamma \vdash_R \lambda x.m : \alpha \rightarrow \beta \leadsto S \cap \{ \{ (t \leq \beta) \} \} \]  
\( \text{(R-ABSTR)} \)

\( \text{(R-CASE)} \)

\[ S = (S_0 \cap \{ \{ (t_0 \leq \emptyset) \} \}) \]  
\[ \sqcup (S_0 \cap S_1 \cap \{ \{ (t_0 \leq t), (t_1 \leq \alpha) \} \}) \]  
\[ \sqcup (S_0 \cap S_2 \cap \{ \{ (t_0 \leq \neg t), (t_2 \leq \alpha) \} \}) \]  
\[ \sqcup (S_0 \cap S_1 \cap S_2 \cap \{ \{ (t_1 \lor t_2 \leq \alpha) \} \}) \]

\[ \Gamma \vdash_R m_0 : t_0 \leadsto S_0 \quad \Gamma \vdash_R m_1 : t_1 \leadsto S_1 \quad \Gamma \vdash_R m_2 : t_2 \leadsto S_2 \]  
\[ \Gamma \vdash_R (m_0 \in t \? m_1 : m_2) : \alpha \leadsto S \]  

where \( \alpha, \alpha_i \) and \( \beta \) in each rule are fresh type variables.
Semantic subtyping with type variables

The subtyping relation is decidable in EXPTIME.
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We can prove relevant relations on infinite types, eg., for the type of generic $\alpha$-lists:

$$
\begin{align*}
\left[\alpha\right] & \overset{\text{def}}{=} \mu z.(\alpha \times z) \lor \text{nil}
\end{align*}
$$
Semantic subtyping with type variables

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$$\}\begin{array}{lll}\left[ \alpha \right] \quad \text{def} & \equiv & \mu z.(\alpha \times z) \lor \text{nil} \\
\end{array}$$

we can prove that it contains both the $\alpha$-lists of even length

$$\mu z.(\alpha \times (\alpha \times z)) \lor \text{nil} \leq \mu z.(\alpha \times z) \lor \text{nil}$$

and the $\alpha$-lists with of odd length

$$\mu z.(\alpha \times (\alpha \times z)) \lor (\alpha \times \text{nil}) \leq \mu z.(\alpha \times z) \lor \text{nil}$$
Semantic subtyping with type variables

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We can prove relevant relations on infinite types, *e.g.*, for the type of generic $\alpha$-lists:

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$\alpha$-lists of even length

$\alpha$-lists

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$$\mu z. (\alpha \times (\alpha \times z)) \lor (\alpha \times \text{nil}) \leq \mu \alpha \times z \lor \text{nil}$$

$\alpha$-lists of odd length

$\alpha$-lists

and that it is itself contained in the union of the two, that is:

$$[\alpha] \sim (\mu z. (\alpha \times (\alpha \times z)) \lor \text{nil}) \lor (\mu z. (\alpha \times (\alpha \times z)) \lor (\alpha \times \text{nil}))$$
Axiomatic properties of intersection types are here *deduced* from the semantic interpretation:

$$(\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma) \sim \alpha \lor \beta \rightarrow \gamma$$
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as well as classic distributivity laws:

\[(\alpha \lor \beta \times \gamma) \sim (\alpha \times \gamma) \lor (\beta \times \gamma)\]
Axiomatic properties of intersection types are here *deduced* from the semantic interpretation:

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as well as classic distributivity laws:

\[(\alpha \lor \beta \times \gamma) \sim (\alpha \times \gamma) \lor (\beta \times \gamma)\]

Most importantly we can use standard set-theoretic laws to show:

- that every type is equivalent to a type in disjunctive normal form
- to *deduce* decomposition rules used in algorithms such as

\[s_1 \times s_2 \leq t_1 \times t_2 \iff (s_1 \leq \emptyset \text{ or } s_2 \leq \emptyset \text{ or } (s_1 \leq t_1 \text{ and } s_2 \leq t_2))\]
Expressions

\[ e ::= x \mid ee \mid \lambda^{i : I_i \rightarrow t_i} \cdot x. e \mid e \in t \Rightarrow e : e \mid (e, e) \mid \pi_i e \]
Expressions

\[
e ::= x \mid ee \mid \lambda^{i \in I} s_i \rightarrow t_i \cdot x \cdot e \mid e \in t \ ? e \ : e \mid (e, e) \mid \pi_i e
\]

Why a type-case:

Why explicitly-typed functions:
Expressions

\[ e ::= x \mid ee \mid \lambda^{i \in I_s \rightarrow t_i} x.e \mid e \in t \mid e : e \mid (e, e) \mid \pi_i e \]

**Why a type-case:**

- Intersection types with “real” overloading vs. coherent one
  [eg, non diverging functions in \((\text{Int} \rightarrow \text{Bool}) \land (\text{Bool} \rightarrow \text{Int})\)]

**Why explicitly-typed functions:**
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  [e.g., non diverging functions in \((\text{Int} \rightarrow \text{Bool}) \land (\text{Bool} \rightarrow \text{Int})\)]

- The following containment is strict:
  \[ s_1 \lor s_2 \rightarrow t_1 \land t_2 \leq (s_1 \rightarrow t_1) \land (s_2 \rightarrow t_2) \]

Why explicitly-typed functions:
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\[ e ::= x \mid ee \mid \lambda^{i \in I} s_i \to t_i \ x.e \mid e \in t \ ? e : e \mid (e, e) \mid \pi_i e \]

Why a type-case:
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  \[ s_1 \lor s_2 \to t_1 \land t_2 \not\subset (s_1 \to t_1) \land (s_2 \to t_2) \]

Why explicitly-typed functions:
- [a consequence of the type-case]
- Avoid paradoxes:
  \[ \mu f. \lambda x. f \in (\mathbb{1} \to \text{Int}) ? \text{true} : 42 \]
  It has type \( \mathbb{1} \to \text{Int} \) iff it *does not* have type \( \mathbb{1} \to \text{Int} \).
### Expressions

\[ e ::= x \mid ee \mid \lambda i \in I \to s_i \to t_i \cdot x.\ e \mid e \in t \ ? \ e : e \mid (e, e) \mid \pi \cdot e \]

**Why a type-case:**
- Intersection types with “real” overloading vs. coherent one
  
  \[ \text{eg, non diverging functions in } (\text{Int} \to \text{Bool}) \land (\text{Bool} \to \text{Int}) \]

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  \[ s_1 \lor s_2 \to t_1 \land t_2 \preccurlyeq (s_1 \to t_1) \land (s_2 \to t_2) \]

**Why explicitly-typed functions:**  
[a consequence of the type-case]

Avoid paradoxes:

\[ \mu f . \lambda x . f \in (1 \to \text{Int}) \ ? \ true : 42 \]

It has type \( 1 \to \text{Int} \) iff it *does not* have type \( 1 \to \text{Int} \).

- Explicitly assign the type \( 1 \to \text{Int} \lor \text{Bool} \) to it.
- More expressive with the result type (type of \( x \) not enough)
How to type-annotate functions?

\[ \lambda x. (x \in \text{Int} \, ? \, \text{true} : 42) \]
How to type-annotate functions?

\[ \lambda x. (x \in \text{Int} \ ? \ \text{true} \ : \ 42) \]

It has type \((\text{Int} \rightarrow \text{Bool}) \land (\neg \text{Int} \rightarrow \text{Int})\) but we will be content with 
\((\text{Int} \rightarrow \text{Bool}) \land (\text{Bool} \rightarrow \text{Int})\)
How to type-annotate functions?

\[ \lambda x^{t?}.(x\in\text{Int} \? \text{true} : 42) \]

It has type \((\text{Int} \to \text{Bool}) \wedge \neg(\text{Int} \to \text{Int})\) but we will be content with \((\text{Int} \to \text{Bool}) \wedge (\text{Bool} \to \text{Int})\)

- Church style?
How to type-annotate functions?

\[ \lambda x^{\text{Int} \lor \text{Bool}}. (x \in \text{Int} \, ? \, \text{true} \, : \, 42) \]

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  If we assign \(\text{Int} \lor \text{Bool}\) to \(x\) the type, we can only deduce
  \(\text{Int} \lor \text{Bool} \to \text{Int} \lor \text{Bool}\)
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- **CDuce solution:** annotate \(\lambda\)'s with their intersection type
How to type-annotate functions?

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- **CDuce solution:** annotate \(\lambda\)'s with their intersection type

**Syntax for \(\lambda\)-abstractions**

Add to expressions

\[ \lambda^{i \in I \, s_i \to t_i} \, x. \, e \]

Well typed if from \(x : s_i\) we can deduce \(e : t_i\), for all \(i \in I\).