Polymorphic Functions with Set-Theoretic Types

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Based on joint work with
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(This work was presented at **POPL '14** and **POPL '15**)

Outline

Motivations and goals

i.e., why unions, intersections, and negations of types are useful (and not just for XML)

Set-theoretic types for classic data structures

Red-black trees are balanced binary search trees that must satisfy 4 invariants:

1. the root of the tree is black
2. the leaves of the tree are black
3. no red node has a red child
4. every path from root to a leaf contains the same number of black nodes

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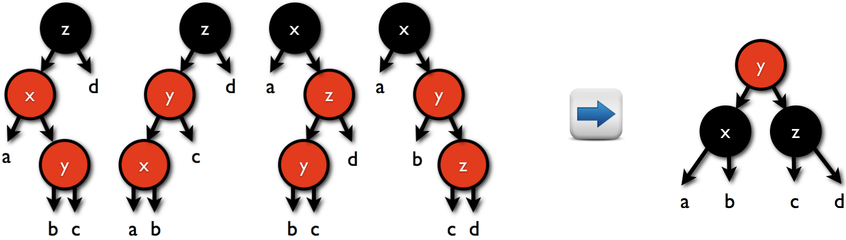
The key to implement `insert` is the function `balance` which transforms an *unbalanced tree*, into a *valid red tree* (as long as a, b, c, and d are valid):



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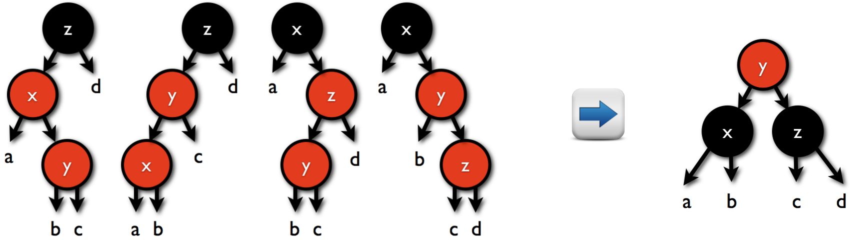

In ML-like languages this yields a simple pattern-matching implementation:
[due to Okasaki: *Purely Functional Data Structures*, Cambridge Univ Press, 1998]

The code as written in Okasaki's book

```
type α RBtree =
   | Leaf
   | Red( α , RBtree , RBtree)
   | Blk( α , RBtree , RBtree)

let balance =
 function
  | Blk( z , Red( x, a, Red(y,b,c) ) , d )
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  | Blk( x , a , Red( z, Red(y,b,c), d ) )
  | Blk( x , a , Red( y, b, Red(z,c,d) ) )
      -> Red ( y, Blk(x,a,b), Blk(z,c,d) )
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 function ( x , t ) ->
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     | c(y,a,b) as z ->
         if x < y  then balance c( y, (ins a), b ) else
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  in let _(y,a,b) = ins t in Blk(y,a,b)
```

Notice that ML types *do not* enforce the invariants of the previous slide

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ML needs extra auxiliary functions *and* GADTs to enforce these invariants.

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In set-theoretic types these functions are straightforwardly typed **as they are**

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(1) Write the correct type definitions

(2) Add type annotations to function definitions

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A simpler example of the same pattern

A motivating example in Haskell [no XML]

```
map :: (α → β) → [α] → [β]
map f l = case l of
            | [] -> []
            | (x : xs) -> (f x : map f xs)
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even :: (Int→Bool) ∧ ((α\Int) → (α\Int))
even x = case x of
           | Int -> (x `mod` 2) == 0
           |   _ -> x
```

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```

- Expression: if the argument is an integer then return the Boolean expression otherwise return the argument

A motivating example in Haskell (almost) [no XML]

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```

- Expression: if the argument is an integer then return the Boolean expression otherwise return the argument
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type variables

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> Common pattern for functional data structures: red-black trees balancing; ZDD operations; XML nodes modification

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The combination of type-case and intersections yields statically typed dynamic overloading.

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This example a… k. I want to define a language that:

Tough!

1. Can define … and even
2. Can *check* the … types specified in the signature
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$([\mathtt{Int}] \to [\mathtt{Bool}]) \wedge$ int lists are transformed into bool lists

$([\alpha\backslash\mathtt{Int}] \to [\alpha\backslash\mathtt{Int}]) \wedge$ lists w/o ints return the same type

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Difficult because of expansion: needs *a set of type substitutions* — rather than just one— to unify the domain and the argument types.

Formal framework

i.e., all the gory details you do not want the programmer to ever know

Formal calculus

Exprs	e	$::=$	$x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t\ ?\ e : e$
Types	t	$::=$	$B \mid t{\to}t \mid t{\vee}t \mid t{\wedge}t \mid \neg t \mid \mathbb{0} \mid \mathbb{1} \mid \alpha$

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Explicitly-typed functions:

- Needed by the type-case [e.g. $\mu f.\lambda x.f{\in}(\mathbb{1}{\to}\mathtt{Int})\,?\,\mathtt{true} : 42$]
- More expressive with the result type (parameter type not enough)

$\lambda^{\wedge_{i\in I} s_i \to t_i} x.e$: well typed if for all $i{\in}I$ from $x : s_i$ we can deduce $e : t_i$.

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Types may be **recursive** and have a **set-theoretic** interpretation:

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Constructors: $[\![\texttt{Int}]\!]{=}\{0,1,-1,...\}$. $[\![s \to t]\!]$ = all λ-abstractions that applied to arguments in $[\![s]\!]$ return only results in $[\![t]\!]$.

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Subtyping with type variables:

- it is *defined* as set-containment: $s\leq t \overset{def}{\iff} [\![s]\!]\subseteq[\![t]\!]$;
- it is such that forall type-substitutions σ: $s\leq t \ \Rightarrow\ s\sigma\leq t\sigma$;
- it is decidable. [ICFP2011].

Formal calculus: **new stuff**

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Meaning: types obtained by subsumption *and by instantiation*

- $\lambda^{\alpha\to\alpha}x.x : \mathbb{0} \to \mathbb{1}$ — subsumption
- $\lambda^{\alpha\to\alpha}x.x : \neg\texttt{Int}$ — subsumption
- $\lambda^{\alpha\to\alpha}x.x : \texttt{Int} \to \texttt{Int}$ — instantiation
- $\lambda^{\alpha\to\alpha}x.x : \texttt{Bool} \to \texttt{Bool}$ — instantiation

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Problem

Define an explicitly typed, polymorphic calculus
with intersection types and dynamic type-case

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Four simple points to show why dealing with this blend is quite problematic

1. Polymorphism needs instantiation:

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To apply $\lambda^{\alpha\to\alpha}x.x$ to 42 we must use the instance obtained by the type substitution $\{\mathtt{Int}/\alpha\}$:

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we *relabel* the function by instantiating its interface.

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To apply it to 42, relabeling the outer λ by $\{\mathtt{Int}/_{\alpha}\}$ does not suffice:

$$(\lambda^{\alpha\to\alpha}y.42)42$$

is not well typed. The body must be relabeled as well, by applying the $\{\mathtt{Int}/_{\alpha}\}$ yielding: $(\lambda^{\mathtt{Int}\to\mathtt{Int}}y.42)42$

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We can feed the identity $\lambda^{\alpha\to\alpha}x.x$ to a function which expects an argument of the type above. But *how do we relabel it?*

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So it has their intersection.
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Let us see why it is not well typed

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This "dependent relabeling" is the stumbling block for the definition of an explicitly-typed λ-calculus with intersection types.

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- The new decoration is statically used by the type system to ensure soundness.

Details follow, but remember we want to program in this language

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t\,?\,e : e$$

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Before proceeding we can already check our first yardstick:

$$\begin{array}{rcl} \texttt{even} & = & \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})} x\,.\, x{\in}\texttt{Int}\,?\,(x \bmod 2) = 0 : x \\ \texttt{map} & = & \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\ f\,. \\ & & \qquad \lambda^{[\alpha]\to[\beta]}\ell\,.\,\ell{\in}\texttt{nil}\,?\,\texttt{nil} : (f(\pi_1\ell), mf(\pi_2\ell)) \end{array}$$

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Some examples:

- ✗ $(\lambda^{\alpha\to\alpha}x.x)42$
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$$x@[\sigma_j]_{j\in J} \stackrel{\text{def}}{=} x$$

$$(\lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_k]_{k\in K}} x.e)@[\sigma_j]_{j\in J} \stackrel{\text{def}}{=} \lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_k]_{k\in K} \circ [\sigma_j]_{j\in J}} x.e$$

$$(e_1 e_2)@[\sigma_j]_{j\in J} \stackrel{\text{def}}{=} (e_1@[\sigma_j]_{j\in J})(e_2@[\sigma_j]_{j\in J})$$

$$(e{\in}t\,?\,e_1 : e_2)@[\sigma_j]_{j\in J} \stackrel{\text{def}}{=} e@[\sigma_j]_{j\in J}{\in}t\,?\,e_1@[\sigma_j]_{j\in J} : e_2@[\sigma_j]_{j\in J}$$

$$(e[\sigma_k]_{k\in K})@[\sigma_j]_{j\in J} \stackrel{\text{def}}{=} e@([\sigma_k]_{k\in K} \circ [\sigma_j]_{j\in J})$$

Reduction semantics

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

Relabeling operation $e@[\sigma_j]_{j\in J}$**:** [Pushes σ's down into λ's]

$$
\begin{array}{rcl}
x@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & x \\
(\lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_k]_{k\in K}} x.e)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & \lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_k]_{k\in K}\circ[\sigma_j]_{j\in J}} x.e \\
(e_1 e_2)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & (e_1@[\sigma_j]_{j\in J})(e_2@[\sigma_j]_{j\in J}) \\
(e{\in}t\,?\,e_1 : e_2)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & e@[\sigma_j]_{j\in J}{\in}t\,?\,e_1@[\sigma_j]_{j\in J} : e_2@[\sigma_j]_{j\in J} \\
(e[\sigma_k]_{k\in K})@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & e@([\sigma_k]_{k\in K} \circ [\sigma_j]_{j\in J})
\end{array}
$$

Notions of reduction:

$$
\begin{array}{rcl}
e[\sigma_j]_{j\in J} & \rightsquigarrow & e@[\sigma_j]_{j\in J} \\
(\lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_j]_{j\in J}} x.e)v & \rightsquigarrow & (e@[\sigma_j]_{j\in P})\{v/_x\} \qquad P = \{j{\in}J \mid \exists i{\in}I, \vdash v : t_i\sigma_j\} \\
v{\in}t\,?\,e_1 : e_2 & \rightsquigarrow & \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}
\end{array}
$$

Reduction semantics

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i\to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e\in t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

Relabeling operation $e@[\sigma_j]_{j\in J}$**:** [Pushes σ's down into λ's]

$$\begin{array}{rcl}
x@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & x \\
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(e_1 e_2)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & (e_1@[\sigma_j]_{j\in J})(e_2@[\sigma_j]_{j\in J}) \\
(e\in t\,?\,e_1 : e_2)@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & e@[\sigma_j]_{j\in J}\in t\,?\,e_1@[\sigma_j]_{j\in J} : e_2@[\sigma_j]_{j\in J} \\
(e[\sigma_k]_{k\in K})@[\sigma_j]_{j\in J} & \stackrel{\text{def}}{=} & e@([\sigma_k]_{k\in K}\circ[\sigma_j]_{j\in J})
\end{array}$$

Notions of reduction:

$$\begin{array}{rcll}
e[\sigma_j]_{j\in J} & \leadsto & e@[\sigma_j]_{j\in J} & \\
(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_j]_{j\in J}} x.e)v & \leadsto & (e@[\sigma_j]_{j\in P})\{v/x\} & P = \{j\in J \mid \exists i\in I, \vdash v : t_i\sigma_j\} \\
v\in t\,?\,e_1 : e_2 & \leadsto & \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases} &
\end{array}$$

Reduction semantics

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i\to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

Relabeling operation $e@[\sigma_j]_{j\in J}$**:** [Pushes σ's down into λ's]

$x@[\sigma_j]_{j\in J} \overset{\text{def}}{=} x$

$(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_k]_{k\in K}} x.e)@[\sigma_j \ldots \overset{\text{def}}{=} \lambda^{\wedge_{i\in I} t_i\to s_i} \ldots x.e$

$(e_1 e_2)@[\ldots$

$(e{\in}t\,?\,e_1 : e_2)@[\ldots : e_2@[\sigma_j]_{j\in J}$

$(e[\sigma_k]_{k\in K})@[\ldots$

Only keep the substitutions that make the type of the argument v match at least one input type of the interface

Notions of reduction:

$e[\sigma_j]_{j\in J} \rightsquigarrow e@[\sigma_j]_{j\in J}$

$(\lambda^{\wedge_{i\in I} t_i\to s_i}_{[\sigma_j]_{j\in J}} x.e)v \rightsquigarrow (e@[\sigma_j]_{j\in P})\{{}^{v}\!/\!{}_{x}\}$ $\qquad P = \{j{\in}J \mid \exists i{\in}I, \vdash v : t_i\sigma_j\}$

$v{\in}t\,?\,e_1 : e_2 \rightsquigarrow \begin{cases} e_1 & \text{if } \vdash v : t \\ e_2 & \text{otherwise} \end{cases}$

Example

$$(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)$$

Example

$$\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)z$$

Example

$$\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\texttt{Int}/_{\alpha}\},\{\texttt{Bool}/_{\alpha}\}]z$$

Example

$$(\lambda^{(\mathtt{Int}\to\mathtt{Int})\wedge(\mathtt{Bool}\to\mathtt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{{}^{\mathtt{Int}}\!/\!_{\alpha}\},\{{}^{\mathtt{Bool}}\!/\!_{\alpha}\}]z)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]z)42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]z)42$$

$$\rightsquigarrow\quad(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]42$$

$$\rightsquigarrow\quad(\lambda^{\alpha\to\alpha}_{[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]}x.(\lambda^{\alpha\to\alpha}y.x)x)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})}z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]z)42$$

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$$\rightsquigarrow\quad (\lambda^{\texttt{Int}\to\texttt{Int}}y.42)42$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]z)42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]42$$

$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}_{[\{\texttt{Int}/\alpha\},\{\texttt{Bool}/\alpha\}]} x.(\lambda^{\alpha\to\alpha} y.x)x)42$$

$$\rightsquigarrow \quad (\lambda^{\texttt{Int}\to\texttt{Int}} y.42)42$$

no Bool here

Example

$$(\lambda^{(\mathtt{Int}\to\mathtt{Int})\wedge(\mathtt{Bool}\to\mathtt{Bool})} z.(\lambda^{\alpha\to\alpha}x.(\lambda^{\alpha\to\alpha}y.x)x)[\{\mathtt{Int}/\alpha\},\{\mathtt{Bool}/\alpha\}]z)42$$

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Example

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$$\rightsquigarrow \quad (\lambda^{\alpha\to\alpha}_{[\{\texttt{Int}/_{\alpha}\},\{\texttt{Bool}/_{\alpha}\}]} x.(\lambda^{\alpha\to\alpha} y.x)x)42$$

REJECTED

$$\rightsquigarrow \quad (\lambda^{\texttt{Int}\to\texttt{Int}} y.42)42 \qquad \equiv (((\lambda^{\alpha\to\alpha} y.x)x)@[\{\texttt{Int}/_{\alpha}\}])\{42/_{x}\}$$

Example

$$(\lambda^{(\texttt{Int}\to\texttt{Int})\wedge(\texttt{Bool}\to\texttt{Bool})} z.(\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]z)42$$

$$\rightsquigarrow\quad (\lambda^{\alpha\to\alpha} x.(\lambda^{\alpha\to\alpha} y.x)x)[\{{}^{\texttt{Int}}/_{\alpha}\},\{{}^{\texttt{Bool}}/_{\alpha}\}]42$$

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$$\rightsquigarrow\quad 42$$

Type system

$$(\mathit{subsumption})\ \frac{\Gamma \vdash e : t_1 \quad t_1 \leq t_2}{\Gamma \vdash e : t_2}$$

$$(\mathit{appl})\ \frac{\Gamma \vdash e_1 : t_1 \to t_2 \qquad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$$

$$(\mathit{inst})\ \frac{\Gamma \vdash e : t}{\Gamma \vdash e[\sigma_j]_{j\in J} : \bigwedge_{j\in J} t\sigma_j}\quad \sigma_j \sharp\, \Gamma$$

$$(\mathit{abstr})\ \frac{\Gamma, x : t_i\sigma_j \vdash e[\sigma_j] : s_i\sigma_j}{\Gamma \vdash \lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_j]_{j\in J}} x.e : \bigwedge_{i\in I, j\in J} t_i\sigma_j \to s_i\sigma_j}\quad \begin{array}{l} i \in I \\ j \in J \end{array}$$

[plus the rules for type-case and variables]

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[plus the rules for type-case and variables]

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$$(\textit{abstr})\ \frac{\Gamma, x : t_i\sigma_j \vdash e[\sigma_j] : s_i\sigma_j}{\Gamma \vdash \lambda^{\wedge_{i\in I} t_i \to s_i}_{[\sigma_j]_{j\in J}} x.e : \bigwedge_{i\in I, j\in J} t_i\sigma_j \to s_i\sigma_j}\ \begin{array}{l} i \in I \\ j \in J \end{array}$$

[plus the rules for type-case and variables]

Properties

Theorem (Subject Reduction)

For every term e and type t, if $\Gamma \vdash e : t$ and $e \rightsquigarrow e'$, then $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e be a well-typed closed term. If e is not a value, then there exists a term e' such that $e \rightsquigarrow e'$.

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Let e be a well-typed closed term. If e is not a value, then there exists a term e' such that $e \rightsquigarrow e'$.

Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\lceil e \rceil$ the type erasure of e. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\lceil e \rceil = a$.

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Note that

$$e \quad ::= \quad x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

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For every term e *and type* t*, if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$*, then* $\Gamma \vdash e' : t$*.*

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Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\lceil e \rceil$ the type erasure of e. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\lceil e \rceil = a$.

Note that

$$e \quad ::= \quad x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid \not{e[\sigma_j]_{j\in J}}$$

satisfies the above theorem and is closed by reduction.

Properties

Theorem (Subject Reduction)

For every term e *and type* t*, if* $\Gamma \vdash e : t$ *and* $e \rightsquigarrow e'$*, then* $\Gamma \vdash e' : t$*.*

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Let e *be a well-typed closed term. If* e *is not a value, then there exists a term* e' *such that* $e \rightsquigarrow e'$*.*

Theorem

Let $\vdash_{BCD}$ *be Barendregt, Coppo, and Dezani, typing, and* $\lceil e \rceil$ *the type erasure of* e*. If* $\vdash_{BCD} a : t$*, then* $\exists e$ *s.t.* $\vdash e : t$ *and* $\lceil e \rceil = a$*.*

Note that

$$e \quad ::= \quad x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i}_{[\sigma_j]_{j\in J}} x.e \mid e{\in}t\,?\,e : e \mid e[\sigma_i]_{i\in I}$$

satisfies the above theorem and is closed by reduction, too.

Properties

Theorem (Subject Reduction)

For every term e and type t, if $\Gamma \vdash e : t$ and $e \rightsquigarrow e'$, then $\Gamma \vdash e' : t$.

Theorem (Progress)

Let e be a well-typed closed term. If e is not a value, then there exists a term e' such that $e \rightsquigarrow e'$.

Theorem

Let $\vdash_{BCD}$ be Barendregt, Coppo, and Dezani, typing, and $\lceil e \rceil$ the type erasure of e. If $\vdash_{BCD} a : t$, then $\exists e$ s.t. $\vdash e : t$ and $\lceil e \rceil = a$.

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The first n terms ($n = 3, 4, 5$) form an explicitly-typed λ-calculus with intersection types subsuming BCD.

Properties

The definitions we gave:

$$\begin{array}{rcl} \texttt{even} & = & \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})}x\,.\,x{\in}\texttt{Int}\,?\,(x \bmod 2) = 0 : x \\ \texttt{map} & = & \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\ f. \\ & & \qquad \lambda^{[\alpha]\to[\beta]}\ell\,.\,\ell{\in}\texttt{nil}\,?\,\texttt{nil} : (f(\pi_1\ell), mf(\pi_2\ell)) \end{array}$$

are well typed.

Properties

The definitions we gave:

$$\begin{aligned} \texttt{even} &= \lambda^{(\texttt{Int}\to\texttt{Bool})\wedge(\alpha\setminus\texttt{Int}\to\alpha\setminus\texttt{Int})}x\,.\,x{\in}\texttt{Int}\,?\,(x \bmod 2)=0 : x \\ \texttt{map} &= \mu m^{(\alpha\to\beta)\to[\alpha]\to[\beta]}\, f\,. \\ &\qquad \lambda^{[\alpha]\to[\beta]}\ell\,.\,\ell{\in}\texttt{nil}\,?\,\texttt{nil} : (f(\pi_1\ell), mf(\pi_2\ell)) \end{aligned}$$

are well typed.

A yardstick for the language

- ✔ Can define both `map` and `even`
- ✔ Can *check* the types specified in the signature
- ? Can *deduce* the type of the partial application `map even`

Inference of explicit type-substitutions

Two problems:

1. **Local type-substitution inference:** Given a term of

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t \,?\, e : e$$

find a sound & complete algorithm that, whenever possible, inserts sets of type-substitutions making it a well-typed term of

$$e ::= x \mid ee \mid \lambda_{[\,]}^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t \,?\, e : e \mid e[\sigma_j]_{j\in J}$$

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(and, yes, the type inferred for `map even` is as expected)

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(and, yes, the type inferred for `map even` is as expected)

2. **Type reconstruction:** Given a term

$$\lambda x.e$$

find, if possible, a set of type-substitutions $[\sigma_j]_{j\in J}$ such that

$$\lambda^{\alpha\to\beta}_{[\sigma_j]_{j\in J}} x.e$$

is well typed

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The rule for applications

1. In the type system: [with explicit type-subst.]

$$\textsc{(Appl)}\ \frac{\Gamma \vdash e_1 : s \to u \qquad \Gamma \vdash e_2 : s}{\Gamma \vdash e_1 e_2 : u}$$

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Tallying problem

The problem of inferring the type of an application is thus to find for s and t given, two sets $[\sigma_i]_{i\in I}, [\sigma'_j]_{j\in J}$ such that:

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This can be reduced to solving a suite of *tallying problems*:

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Generally: let $C = \{(s_1 \leq t_1), ..., (s_n \leq t_n)\}$ a *constraint set*. A type-substitution σ is a solution for the *tallying* of C iff $s\sigma \leq t\sigma$ for all $(s \leq t) \in C$.

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Type tallying is decidable and a sound and complete set of solutions for every tallying problem can be effectively found in **three** simple **steps**.

Step 1: Decompose constraints.

Use the set-theoretic decomposition rules to transform C into a set of constraint sets whose constraints are of the form $\alpha \leq t$ or $t \leq \alpha$.

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- if $\alpha \leq t_1$ and $\alpha \leq t_2$ are in C, then replace them by $\alpha \leq t_1 \wedge t_2$;
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Possibly decompose the new constraints generated by transitivity.

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1. select $s \leq \alpha \leq t$ and replace it by $\alpha = (s \vee \beta) \wedge t$ with β fresh.
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3. $\{(\mathtt{Int} \leq \alpha_1 \leq \mathsf{Real}), (\alpha_2 \leq \alpha_1{\wedge}\mathtt{Int})\}$

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At the end we have a sets of equations $\{\alpha_i = u_i \mid i \in [1..n]\}$ that (with some care) are *contractive*. By Courcelle there exists a solution, *ie*, a substitution for $\alpha_1, \ldots, \alpha_n$ into (possibly recursive regular) types $t_1, \ldots, t_n$ (in which the fresh β's are free variables).

Example: `map even`

Start with the following tallying problem:

$$(\alpha_1 \to \beta_1) \to [\alpha_1] \to [\beta_1] \leq s \to \gamma$$

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- The algorithm generates 9 constraint-sets: one is unsatisfiable ($s \leq \mathbb{0}$); four are implied by the others; remain
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- The last two are minimal and we take their intersection:
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Principality: This raises the problem of the existence of principal types: may an infinite sequence of increasingly general solutions exist?

Type reconstruction

- Solve *sets* of contraint-sets by the tallying algorithm:

$$\frac{}{\Gamma \vdash_{\mathcal{R}} x : \Gamma(x) \leadsto \{\varnothing\}} \qquad \frac{\Gamma, x : \alpha \vdash_{\mathcal{R}} e : t \leadsto \mathcal{S}}{\Gamma \vdash_{\mathcal{R}} \lambda x.e : \alpha \to \beta \leadsto \mathcal{S} \sqcap \{\{(t \leq \beta)\}\}}$$

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- It types more than ML

$$\lambda x.xx : \mu X.(\alpha \wedge (X \to \beta)) \to \beta \qquad (\leq \alpha \wedge (\alpha \to \beta)) \to \beta)$$

  and for functions typable in ML, it deduces a type at least as good:

$$map : ((\alpha \to \beta) \to [\alpha] \to [\beta]) \wedge ((\mathbb{0} \to \mathbb{1}) \to [] \to [])$$

Efficient evaluation

Monomorphic language

$$
\begin{array}{rcl}
e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e\in t\,?\,e : e \\
v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\end{array}
$$

Monomorphic language

$$
\begin{array}{rcl}
e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e \in t \, ? \, e : e \\
v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\end{array}
$$

$$
(\textsc{Closure}) \; \frac{}{\mathcal{E} \vdash_{\mathsf{m}} \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply}) \; \frac{\mathcal{E} \vdash_{\mathsf{m}} e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathsf{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathsf{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathsf{m}} e_1 e_2 \Downarrow v}
$$

Monomorphic language

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}x.e \mid ee \mid e \in t \,?\, e : e \\
v & ::= & c \mid \langle \lambda^{t}x.e, \mathcal{E} \rangle
\end{array}
$$

save the environment

$$
(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathrm{m}} \lambda^{t}x.e \Downarrow \langle \lambda^{t}x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply})\ \frac{\mathcal{E} \vdash_{\mathrm{m}} e_1 \Downarrow \langle \lambda^{t}x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathrm{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathrm{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathrm{m}} e_1 e_2 \Downarrow v}
$$

Monomorphic language

$$
\begin{array}{rcl}
e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e \in t\, ?\, e : e \\
v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\end{array}
$$

save the environment

$$(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathsf{m}} \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}$$

$$(\textsc{Apply})\ \frac{\mathcal{E} \vdash_{\mathsf{m}} e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \quad \mathcal{E} \vdash_{\mathsf{m}} e_2 \Downarrow v_0 \quad \mathcal{E}', x \mapsto v_0 \vdash_{\mathsf{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathsf{m}} e_1 e_2 \Downarrow v}$$

restore the environment

Monomorphic language

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^t x.e \mid ee \mid e \in t\, ?\, e : e \\
v & ::= & c \mid \langle \lambda^t x.e, \mathcal{E} \rangle
\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash_{\mathsf{m}} \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply})\ \frac{\mathcal{E} \vdash_{\mathsf{m}} e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash_{\mathsf{m}} e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash_{\mathsf{m}} e \Downarrow v}{\mathcal{E} \vdash_{\mathsf{m}} e_1 e_2 \Downarrow v}
$$

$$
(\textsc{Typecase True})\ \frac{\mathcal{E} \vdash_{\mathsf{m}} e_1 \Downarrow v_0 \qquad v_0 \in_{\mathsf{m}} t \qquad \mathcal{E} \vdash_{\mathsf{m}} e_2 \Downarrow v}{\mathcal{E} \vdash_{\mathsf{m}} e_1 \in t\, ?\, e_2 : e_3 \Downarrow v}
$$

$$
(\textsc{Typecase False})\ \frac{\mathcal{E} \vdash_{\mathsf{m}} e_1 \Downarrow v_0 \qquad v_0 \notin_{\mathsf{m}} t \qquad \mathcal{E} \vdash_{\mathsf{m}} e_3 \Downarrow v}{\mathcal{E} \vdash_{\mathsf{m}} e_1 \in t\, ?\, e_2 : e_3 \Downarrow v}
$$

$$
\begin{array}{rcl}
c \in_{\mathsf{m}} t & \stackrel{\text{def}}{=} & \{c\} \leq t \\
\langle \lambda^s x.e, \mathcal{E} \rangle \in_{\mathsf{m}} t & \stackrel{\text{def}}{=} & s \leq t
\end{array}
$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i \in I}$)

$$e \quad ::= \quad c \mid x \mid \lambda^{t}_{\sigma_I} x.e \mid ee \mid e \in t \,?\, e : e \mid e\sigma_I$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e \quad ::= \quad c \mid x \mid \lambda^{t}_{\sigma_I} x.e \mid ee \mid e \in t\ ?\ e : e \mid e\sigma_I$$
$$v \quad ::= \quad c \mid \langle \lambda^{t}_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e \quad ::= \quad c \mid x \mid \lambda^t_{\sigma_I} x.e \mid ee \mid e \in t \,?\, e : e \mid e\sigma_I$$
$$v \quad ::= \quad c \mid \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I ; \mathcal{E} \vdash_{\mathsf{p}} \lambda^t_{\sigma_J} x.e \Downarrow \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle}$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I} x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$
$$v ::= c \mid \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle$$

save the environment

$$(\textsc{Closure})\ \frac{}{\sigma_I; \mathcal{E} \vdash_{\mathsf{p}} \lambda^t_{\sigma_J} x.e \Downarrow \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle}$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$
$$v ::= c \mid \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle$$

save the environment

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathsf{p}}\lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}$$

save current type-substitutions

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I} x.e \mid ee \mid e \in t \,?\, e : e \mid e\sigma_I \\
v & ::= & c \mid \langle \lambda^{t}_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle
\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E} \vdash_{\mathsf{p}} \lambda^{t}_{\sigma_J} x.e \Downarrow \langle \lambda^{t}_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I \circ \sigma_J;\mathcal{E} \vdash_{\mathsf{p}} e \Downarrow v}{\sigma_I;\mathcal{E} \vdash_{\mathsf{p}} e\sigma_J \Downarrow v}
$$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I \\
v & ::= & c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle
\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} \lambda^{t}_{\sigma_J}x.e \Downarrow \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}
$$

(Apply)

$$
\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}
$$

where $\sigma_J = \sigma_H\circ\sigma_K$ and $P = \{j\in J \mid \exists \ell\in L : v_0\in_{\mathrm{p}} s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$
$$v ::= c \mid \langle\lambda^t_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\rm p}\lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\rm p} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\rm p} e\sigma_J\Downarrow v}$$

restore the environment

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\rm p} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\rm p} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\rm p} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\rm p} e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid \exists\ell\in L: v_0\in_{\rm p} s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e \;::=\; c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$
$$v \;::=\; c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm p}\lambda^{t}_{\sigma_J}x.e \Downarrow \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm p} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm p} e\sigma_J\Downarrow v}$$

restore the environment

$$(\textsc{Apply})\ \frac{\sigma_I;\mathcal{E}\vdash_{\mathrm p} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm p} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm p} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm p} e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid \exists \ell\in L: v_0\in_{\mathrm p} s_\ell\sigma_j\}$

restore the type substitutions

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I$$
$$v ::= c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} \lambda^{t}_{\sigma_J}x.e \Downarrow \langle\lambda^{t}_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}$$

$$(\textsc{Apply})\ \frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}$$

where $\sigma_J = \sigma_H\circ\sigma_K$ and $P = \{j\in J \mid \exists\ell\in L : v_0\in_{\mathrm{p}} s_\ell\sigma_j\}$

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I$$
$$v ::= c \mid \langle\lambda^t_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}}\lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}$$

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}$$

where $\sigma_J = \sigma_H\circ\sigma_K$ and $P = \{j\in J \mid \exists\ell\in L : v_0 \in_{\mathrm{p}} s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t}_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I \\
v & ::= & c \mid \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle
\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} \lambda^{t}_{\sigma_J}x.e \Downarrow \langle \lambda^{t}_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle}
\qquad
(\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e\sigma_J\Downarrow v}
$$

(Apply)

$$
\frac{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1\Downarrow \langle \lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e, \mathcal{E}', \sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathrm{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathrm{p}} e_1e_2\Downarrow v}
$$

where $\sigma_J = \sigma_H\circ\sigma_K$ and $P = \{j\in J \mid \exists \ell\in L : v_0 \in_{\mathrm{p}} s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

1. **compose** of two sets of type-substitution

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I} x.e \mid ee \mid e \in t\ ?\ e : e \mid e\sigma_I$$
$$v ::= c \mid \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle$$

$$(\text{Closure}) \ \frac{}{\sigma_I; \mathcal{E} \vdash_p \lambda^t_{\sigma_J} x.e \Downarrow \langle \lambda^t_{\sigma_J} x.e, \mathcal{E}, \sigma_I \rangle} \qquad (\text{Instance}) \ \frac{\sigma_I \circ \sigma_J; \mathcal{E} \vdash_p e \Downarrow v}{\sigma_I; \mathcal{E} \vdash_p e\sigma_J \Downarrow v}$$

(Apply)

$$\frac{\sigma_I; \mathcal{E} \vdash_p e_1 \Downarrow \langle \lambda^{\wedge_{\ell\in L} s_\ell \to t_\ell}_{\sigma_K} x.e, \mathcal{E}', \sigma_H \rangle \quad \sigma_I; \mathcal{E} \vdash_p e_2 \Downarrow v_0 \quad \sigma_P; \mathcal{E}', x \mapsto v_0 \vdash_p e \Downarrow v}{\sigma_I; \mathcal{E} \vdash_p e_1 e_2 \Downarrow v}$$

where $\sigma_J = \sigma_H \circ \sigma_K$ and $P = \{j \in J \mid \exists \ell \in L : v_0 \in_p s_\ell \sigma_j\}$

Problem:

At every application compute σ_P:

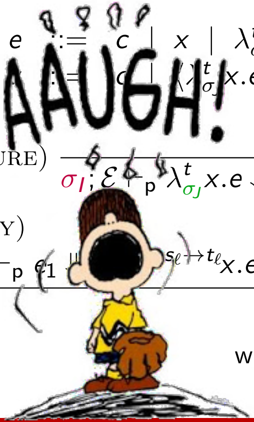
1. **compose** of two sets of type-substitution
2. **select** the substitutions compatible with the argument v_0

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e:e \mid e\sigma_I$$

$$v ::= c \mid \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle$$



(Closure) $$\frac{}{\sigma_I;\mathcal{E}\vdash_p \lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle}$$

(Instance) $$\frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_p e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_p e\sigma_J\Downarrow v}$$

(Apply)
$$\frac{\sigma_I;\mathcal{E}\vdash_p e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_J}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_p e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_p e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_p e_1e_2\Downarrow v}$$

where $\sigma_J=\sigma_H\circ\sigma_K$ and $P=\{j\in J\mid \exists\ell\in L: v_0\in_p s_\ell\sigma_j\}$

Problem:

At every application compute σ_P:

1. **compose** of two sets of type-substitution
2. **select** the substitutions compatible with the argument v_0

Polymorphic language: naive implementation

(σ_I short for $[\sigma_i]_{i\in I}$)

$$e ::= c \mid x \mid \lambda^t_{\sigma_I}x.e \mid ee \mid e\in t\,?\,e : e \mid e\sigma_I$$
$$v ::= c \mid \langle\lambda^t_{\sigma_J}x.e, \mathcal{E}, \sigma_I\rangle$$

$$(\textsc{Closure})\ \frac{}{\sigma_I;\mathcal{E}\vdash_{\mathsf{p}}\lambda^t_{\sigma_J}x.e \Downarrow \langle\lambda^t_{\sigma_J}x.e,\mathcal{E},\sigma_I\rangle} \qquad (\textsc{Instance})\ \frac{\sigma_I\circ\sigma_J;\mathcal{E}\vdash_{\mathsf{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathsf{p}} e\sigma_J\Downarrow v}$$

(Apply)

$$\frac{\sigma_I;\mathcal{E}\vdash_{\mathsf{p}} e_1\Downarrow\langle\lambda^{\wedge_{\ell\in L}s_\ell\to t_\ell}_{\sigma_K}x.e,\mathcal{E}',\sigma_H\rangle \quad \sigma_I;\mathcal{E}\vdash_{\mathsf{p}} e_2\Downarrow v_0 \quad \sigma_P;\mathcal{E}',x\mapsto v_0\vdash_{\mathsf{p}} e\Downarrow v}{\sigma_I;\mathcal{E}\vdash_{\mathsf{p}} e_1e_2\Downarrow v}$$

where $\sigma_J = \sigma_H\circ\sigma_K$ and $P = \{j\in J \mid \exists\ell\in L : v_0\in_{\mathsf{p}} s_\ell\sigma_j\}$

Solution:

Compute compositions and selections lazily.

Intermediate language as compilation target

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^{t} x.e \mid ee \mid e \in t \,?\, e : e \\
v & ::= & c \mid \langle \lambda^{t} x.e, \mathcal{E} \rangle
\end{array}
$$

$$
(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash \lambda^{t} x.e \Downarrow \langle \lambda^{t} x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply})\ \frac{\mathcal{E} \vdash e_1 \Downarrow \langle \lambda^{t} x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash e \Downarrow v}{\mathcal{E} \vdash e_1 e_2 \Downarrow v}
$$

$$
(\textsc{Typecase True})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \in t \qquad \mathcal{E} \vdash e_2 \Downarrow v}{\mathcal{E} \vdash e_1 \in t \,?\, e_2 : e_3 \Downarrow v}
\qquad
(\textsc{Typecase False})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \notin t \qquad \mathcal{E} \vdash e_3 \Downarrow v}{\mathcal{E} \vdash e_1 \in t \,?\, e_2 : e_3 \Downarrow v}
$$

$$
\begin{array}{rcl}
c \in t & \stackrel{\text{def}}{=} & \{c\} \leq t \\
\langle \lambda^{s} x.e, \mathcal{E} \rangle \in t & \stackrel{\text{def}}{=} & s \leq t
\end{array}
$$

Intermediate language as compilation target

$$
\begin{array}{lcl}
e & ::= & c \mid x \mid \lambda^t_{\Sigma} x.e \mid ee \mid e \in t \,?\, e : e \\
v & ::= & c \mid \langle \lambda^t_{\Sigma} x.e, \mathcal{E} \rangle \\
\Sigma & ::= & \sigma_I \mid \mathtt{comp}(\Sigma, \Sigma') \mid \mathtt{sel}(x, t, \Sigma)
\end{array}
$$

symbolic substitutions

$$
(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash \lambda^t x.e \Downarrow \langle \lambda^t x.e, \mathcal{E} \rangle}
$$

$$
(\textsc{Apply})\ \frac{\mathcal{E} \vdash e_1 \Downarrow \langle \lambda^t x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash e \Downarrow v}{\mathcal{E} \vdash e_1 e_2 \Downarrow v}
$$

$$
(\textsc{Typecase True})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \in t \qquad \mathcal{E} \vdash e_2 \Downarrow v}{\mathcal{E} \vdash e_1 \in t \,?\, e_2 : e_3 \Downarrow v}
\qquad
(\textsc{Typecase False})\ \frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \notin t \qquad \mathcal{E} \vdash e_3 \Downarrow v}{\mathcal{E} \vdash e_1 \in t \,?\, e_2 : e_3 \Downarrow v}
$$

$$
\begin{array}{rcl}
c \in t & \stackrel{\text{def}}{=} & \{c\} \leq t \\
\langle \lambda^s x.e, \mathcal{E} \rangle \in t & \stackrel{\text{def}}{=} & s \leq t
\end{array}
$$

Intermediate language as compilation target

$$e ::= c \mid x \mid \lambda^t_\Sigma x.e \mid ee \mid e \in t\,?\,e : e$$
$$v ::= c \mid \langle \lambda^t_\Sigma x.e, \mathcal{E} \rangle$$
$$\Sigma ::= \sigma_I \mid \mathtt{comp}(\Sigma, \Sigma') \mid \mathtt{sel}(x, t, \Sigma)$$

symbolic substitutions

$$(\textsc{Closure})\ \frac{}{\mathcal{E} \vdash \lambda^t_\Sigma x.e \Downarrow \langle \lambda^t_\Sigma x.e, \mathcal{E} \rangle}$$

$$(\textsc{Apply})\ \frac{\mathcal{E} \vdash e_1 \Downarrow \langle \lambda^t_\Sigma x.e, \mathcal{E}' \rangle \qquad \mathcal{E} \vdash e_2 \Downarrow v_0 \qquad \mathcal{E}', x \mapsto v_0 \vdash e \Downarrow v}{\mathcal{E} \vdash e_1 e_2 \Downarrow v}$$

(Typecase True)
$$\frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \in t \qquad \mathcal{E} \vdash e_2 \Downarrow v}{\mathcal{E} \vdash e_1 \in t\,?\,e_2 : e_3 \Downarrow v}$$

(Typecase False)
$$\frac{\mathcal{E} \vdash e_1 \Downarrow v_0 \qquad v_0 \notin t \qquad \mathcal{E} \vdash e_3 \Downarrow v}{\mathcal{E} \vdash e_1 \in t\,?\,e_2 : e_3 \Downarrow v}$$

$$c \in t \stackrel{\text{def}}{=} \{c\} \leq t$$
$$\langle \lambda^s x.e, \mathcal{E} \rangle \in t \stackrel{\text{def}}{=} s \leq t$$

Intermediate language as compilation target

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watch here !

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c \in t & \stackrel{\text{def}}{=} & \{c\} \leq t \\
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The only difference!

$$c \in t \overset{\text{def}}{=} \{c\} \leq t$$

$$\langle \lambda^{s}_{\Sigma}x.e, \mathcal{E}\rangle \in t \overset{\text{def}}{=} s(\mathsf{eval}(\mathcal{E},\Sigma)) \leq t$$

Compilation

1. Compile into the intermediate language

$$
\begin{array}{rcl}
[\![x]\!]_\Sigma & = & x \\
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[\![e_1 e_2]\!]_\Sigma & = & [\![e_1]\!]_\Sigma [\![e_2]\!]_\Sigma \\
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2. For $\langle \lambda^s_\Sigma x.e, \mathcal{E}\rangle \in t \stackrel{\text{def}}{=} s(\text{eval}(\mathcal{E},\Sigma)) \leq t$ we have $s(\text{eval}(\mathcal{E},\Sigma)) \neq s$ *only if* $\lambda^s_\Sigma x.e$ results from the partial application of a polymorphic function (*ie*, in s there occur free variables bound in the context).

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> Execution *may* be slowed *only* when testing the type of the result of a partial application of a polymorphic function.

3. Compilation can flag the functions that may require to compute eval:

$$[\![\lambda^{t}_{[\!]}x.e]\!]_{\Sigma} \quad = \quad \begin{cases} \lambda^{t}_{\Sigma}x.[\![e]\!]_{\mathtt{sel}(x,t,\Sigma)} & \text{if } \mathsf{var}(t) \cap \mathsf{dom}(\Sigma) = \varnothing \\ \hat{\boldsymbol{\lambda}}^{t}_{\Sigma}x.[\![e]\!]_{\mathtt{sel}(x,t,\Sigma)} & \text{otherwise} \end{cases}$$

and then we evaluate the symbolic substitutions only for marked functions:

$$\begin{array}{lcl} \langle\lambda^{s}_{\Sigma}x.e,\mathcal{E}\rangle \in t & \stackrel{\text{def}}{\Longleftrightarrow} & s \leq t \\ \langle\hat{\boldsymbol{\lambda}}^{s}_{\Sigma}x.e,\mathcal{E}\rangle \in t & \stackrel{\text{def}}{\Longleftrightarrow} & s(\mathsf{eval}(\mathcal{E},\Sigma)) \leq t \end{array}$$

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Bottom Line

The execution is as efficient as in the monorphic case, apart from a single well identified exception

Conclusion

Theory: All the theoretical machinery necessary to design and implement a programming language is there. The practical relevance of the open theoretical issues is negligible.

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Type reconstruction: Full usage needs more research, expecially about the production of human readable types and helpful error messages, but it is mature enough to use it to type local functions.

Extra slides

References

1. *Subtyping:* Set-theoretic Foundation of Parametric Polymorphism and Subtyping. **ICFP '11**
2. *Language:* Polymorphic Functions with Set-Theoretic Types. Part 1: Syntax, Semantics, and Evaluation. **POPL '14**
3. *Language:* Polymorphic Functions with Set-Theoretic Types. Part 2: Local Type Inference and Type Reconstruction. **POPL '15**.

Tallying problem

The problem of inferring the type of an application is thus to find for s and t given, $[\sigma_i]_{i\in I}, [\sigma'_j]_{j\in J}$ such that:

$$t[\sigma'_j]_{j\in J} \leq \mathbb{0} \to \mathbb{1} \quad \text{and} \quad s[\sigma_i]_{i\in I} \leq \text{dom}(t[\sigma'_j]_{j\in J})$$

This can be reduced to solving a suite of *tallying problems*:

Definition (Type tallying)

Let $C = \{(s_1, t_1), ..., (s_n, t_n)\}$ a *constraint set*. A type-substitution σ is a solution for the *tallying* of C iff $s\sigma \leq t\sigma$ for all $(s, t) \in C$.

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Type tallying is decidable and a sound and complete set of solutions for every tallying problem can be effectively found in three simple steps.

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Use the set-theoretic decomposition rules to transform C into a set of constraint sets whose constraints are of the form (α, t) or (t, α).

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Step 3: Transform into a set of equations.

After Step 2 we have constraint-sets of the form
$\{s_i \leq \alpha_i \leq t_i \mid i \in [1..n]\}$ where α_i are pairwise distinct.

1. select $s \leq \alpha \leq t$ and replace it by $\alpha = (s \vee \beta) \wedge t$ with β fresh.
2. in all other constraints in replace every α by $(s \vee \beta) \wedge t$
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At the end we have a sets of equations $\{\alpha_i = u_i \mid i \in [1..n]\}$ that (with some care) are *contractive*. By Courcelle there exists a solution, *ie*, a substitution for $\alpha_1, ..., \alpha_n$ into (possibly recursive regular) types $t_1, ..., t_n$ (in which the fresh β's are free variables).

The application problem

Definition (Inference application problem)

Given s and t types, find $[\sigma_i]_{i\in I}$ and $[\sigma'_j]_{j\in J}$ such that:

$$\bigwedge_{i\in I} t\sigma_i \leq \mathbb{0}\to\mathbb{1} \quad \text{and} \quad \bigwedge_{j\in J} s\sigma_j \leq \mathrm{dom}(\bigwedge_{i\in I} t\sigma_i)$$

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2. Split each substitution σ_k (for $k{\in}I{\cup}J$) in two: $\sigma_k = \rho_k \circ \sigma'_k$ where ρ_k is a renaming substitution mapping each variable of the domain of σ_k into a fresh variable:

$$\bigwedge_{i\in I}(t\rho_i)\sigma'_i \leq \mathbb{0}\to\mathbb{1} \quad \text{and} \quad \bigwedge_{j\in J}(s\rho_j)\sigma'_j \leq \mathrm{dom}(\ \bigwedge_{i\in I}(t\rho_i)\sigma'_i);$$

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3. Solve the tallying problem for
   $$\{(t_1, \mathbb{0}\to\mathbb{1}), (t_1, t_2\to\gamma)\}$$
   with $t_1 = \bigwedge_{i\in I} t\rho_i$, $t_2 = \bigwedge_{j\in J} s\rho_j$, and γ fresh

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2. Split each substitution σ_k (for $k\in I\cup J$) in two: $\sigma_k = \rho_k \circ \sigma'_k$ where ρ_k is a renaming substitution mapping each variable of the domain of σ_k into a fresh variable: $(\bigwedge_{i\in I} t\rho_i)\sigma \;\leq \mathbb{0}\to\mathbb{1}$ and $(\bigwedge_{j\in J} s\rho_j)\sigma \;\leq \mathrm{dom}((\bigwedge_{i\in I} t\rho_i)\sigma\;)$;
3. Solve the tallying problem for
$$\{(t_1, \mathbb{0}\to\mathbb{1}), (t_1, t_2\to\gamma)\}$$
with $t_1 = \bigwedge_{i\in I} t\rho_i$, $t_2 = \bigwedge_{j\in J} s\rho_j$, and γ fresh
   - if it fails at Step 1, then fail.
   - if it fails at Step 2, then change cardinalities (dove-tail)

The application problem

Definition (Inference application problem)

Given s and t types, find $[\sigma_i]_{i\in I}$ and $[\sigma'_j]_{j\in J}$ such that:

$$\bigwedge_{i\in I} t\sigma_i \leq \mathbb{0}\to\mathbb{1} \quad \text{and} \quad \bigwedge_{j\in J} s\sigma_j \leq \mathrm{dom}(\bigwedge_{i\in I} t\sigma_i)$$

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➡ **Every solution for γ is a solution for the application.**

Example: map even

Start with the following tallying problem:

$$\{(\alpha_1 \to \beta_1) \to [\alpha_1] \to [\beta_1] \leq t \to \gamma\}$$

where $t = (\texttt{Int} \to \texttt{Bool}) \wedge (\alpha \setminus \texttt{Int} \to \alpha \setminus \texttt{Int})$ is the type of even

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- At step 2 the algorithm generates 9 constraint-sets: one is unsatisfiable ($t \leq \mathbb{0}$); four are implied by the others; remain
  $\{\gamma \geq [\alpha_1] \to [\beta_1]\ ,\ \alpha_1 \leq \mathbb{0}\}$,
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  $\{\gamma = [\alpha \vee \mathtt{Int}] \to [(\alpha \setminus \mathtt{Int}) \vee \mathtt{Bool}]\}$.
- The last two are minimal and we take their intersection:
  $\{\gamma = ([\alpha \setminus \mathtt{Int}] \to [\alpha \setminus \mathtt{Int}]) \wedge ([\alpha \vee \mathtt{Int}] \to [(\alpha \setminus \mathtt{Int}) \vee \mathtt{Bool}])\}$

Type Reconstruction Algorithm

$$\frac{}{\Gamma \vdash_{\mathcal{R}} c : b_c \rightsquigarrow \{\varnothing\}}\text{(R-CONST)} \qquad \frac{}{\Gamma \vdash_{\mathcal{R}} x : \Gamma(x) \rightsquigarrow \{\varnothing\}}\text{(R-VAR)}$$

$$\frac{\Gamma \vdash_{\mathcal{R}} m_1 : t_1 \rightsquigarrow \mathcal{S}_1 \qquad \Gamma \vdash_{\mathcal{R}} m_2 : t_2 \rightsquigarrow \mathcal{S}_2}{\Gamma \vdash_{\mathcal{R}} m_1 m_2 : \alpha \rightsquigarrow \mathcal{S}_1 \sqcap \mathcal{S}_2 \sqcap \{\{(t_1 \leq t_2 \to \alpha)\}\}}\text{(R-APPL)}$$

$$\frac{\Gamma, x : \alpha \vdash_{\mathcal{R}} m : t \rightsquigarrow \mathcal{S}}{\Gamma \vdash_{\mathcal{R}} \lambda x.m : \alpha \to \beta \rightsquigarrow \mathcal{S} \sqcap \{\{(t \leq \beta)\}\}}\text{(R-ABSTR)}$$

(R-CASE)

$$\frac{\begin{array}{rl} \mathcal{S} = & (\mathcal{S}_0 \sqcap \{\{(t_0 \leq \mathbb{0})\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_1 \sqcap \{\{(t_0 \leq t), (t_1 \leq \alpha)\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_2 \sqcap \{\{(t_0 \leq \neg t), (t_2 \leq \alpha)\}\}) \\ \sqcup & (\mathcal{S}_0 \sqcap \mathcal{S}_1 \sqcap \mathcal{S}_2 \sqcap \{\{(t_1 \vee t_2 \leq \alpha)\}\}) \end{array} \\ \Gamma \vdash_{\mathcal{R}} m_0 : t_0 \rightsquigarrow \mathcal{S}_0 \qquad \Gamma \vdash_{\mathcal{R}} m_1 : t_1 \rightsquigarrow \mathcal{S}_1 \qquad \Gamma \vdash_{\mathcal{R}} m_2 : t_2 \rightsquigarrow \mathcal{S}_2}{\Gamma \vdash_{\mathcal{R}} (m_0 \in t \,?\, m_1 : m_2) : \alpha \rightsquigarrow \mathcal{S}}$$

where α, α_i and β in each rule are fresh type variables.

Semantic subtyping with type variables

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and that it is itself contained in the union of the two, that is:

$$[\alpha] \sim (\mu z.(\alpha \times (\alpha \times z)) \vee \text{nil}) \vee (\mu z.(\alpha \times (\alpha \times z)) \vee (\alpha \times \text{nil}))$$

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Most importantly we can use standard set-theoretic laws to show:

- that every type is equivalent to a type in disjunctive normal form
- to *deduce* decomposition rules used in algorithms such as

$$s_1 \times s_2 \leq t_1 \times t_2 \iff \big(s_1 \leq \mathbb{0} \text{ or } s_2 \leq \mathbb{0} \text{ or } (s_1 \leq t_1 \text{ and } s_2 \leq t_2) \big)$$

Expressions

$$e ::= x \mid ee \mid \lambda^{\wedge_{i\in I} s_i \to t_i} x.e \mid e{\in}t\,?\,e : e \mid (e, e) \mid \pi_i e$$

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Why explicitly-typed functions:

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- Intersection types with "real" overloading vs. coherent one
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- Explicitly assign the type $\mathbb{1} \to \texttt{Int}{\vee}\texttt{Bool}$ to it.
- More expressive with the result type (type of x not enough)

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Syntax for λ-abstractions

Add to expressions

$$\lambda^{\wedge_{i\in I} s_i\to t_i}x.e$$

Well typed if from $x : s_i$ we can deduce $e : t_i$, for all $i\in I$.