Computing connected proof(-structure)s from their Taylor expansion

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Abstract

We show that every connected Multiplicative Exponential Linear Logic (MELL) proof-structure (with or without cuts) is uniquely determined by a (well-chosen) element of its Taylor expansion. As a consequence, we show that the relational model is injective with respect to connected MELL proof-structures.

Keywords: Linear logic, differential linear logic, proof-nets, relational model

1. Introduction

Starting from investigations on denotational semantics of System F (second order typed λ-calculus), in 1987 Girard [11] introduced linear logic (LL), a refinement of intuitionistic logic. In LL two new modalities, ! and ?, give a logical status to structural rules and allow to distinguish between linear resources (i.e. usable exactly once during the evaluation/cut-elimination process) and resources available at will. One of the main features of LL is the possibility of representing proofs (and λ-terms) geometrically by means of particular graphs: *proof-structures*.

Ehrhard [5] introduced finiteness spaces, a denotational model of LL (and λ-calculus) which interprets formulas by topological vector spaces and proofs by analytical functions: in this model the operations of differentiation and Taylor expansion make sense. Ehrhard and Regnier [7, 8, 9] internalized these operations in the syntax and thus introduced differential linear logic DiLL (and differential

λ-calculus), in addition to promotion rule (the only one in LL which is responsible for introducing the !-modality and hence creating resources available at will, marked in proof-structures by *boxes*) there are three "finitary" *co-structural* rules handling !-modality (co-weakening, co-contraction and co-dereliction) which are perfectly symmetric to the *structural* rules (weakening, contraction and dereliction) for the ?-modality: this allows a more subtle analysis of the resources consumption during the cut-elimination process. At the syntactic level, Taylor expansion decomposes a LL proof-structure in a (infinite in general) formal sum of *diffnets* (or DiLL_0 proof-structures, where DiLL_0 is the promotion-free fragment of DiLL), each of which contains resources usable only a fixed number of times because boxes are replaced by the co-contraction of $n \in \mathbb{N}$ copies of their content.

The main result of the present paper (Theorem 45) shows that a (*box-*)*connected* (Definition 39) proof-structure of the Multiplicative and Exponential fragment of Linear Logic (MELL, sufficiently expressive to encode λ-calculus) can be computed by a well-chosen single diffnet of *its* Taylor expansion: provided we know that a certain set of diffnets is the Taylor expansion of some (box-)connected proof-structure, we show constructively that there is only one way to build this proof-structure. Actually, this can be understood as the difficult part of the proof of injectivity for connected proof-structures w.r.t. the relational model (the category **Rel** of sets and relations), a result already proven in [4]: when two (cut-free with atomic axioms) connected proof-structures have the same interpretation in the relational model, then they are the same. The reader acquainted with this question will easily guess how one can deduce injectivity from Theorem 45. It is already sketched in [19], and for lack of space we leave the details to future work. In any case, the proof that a (box)-connected proof-structure can be recovered by one single diffnet of its Taylor expansion is obviously already interesting in itself, while the relation between such a result and injectivity in the relational model corroborates the intuition that the Taylor expansion of proof-structures is a bridge between syntax and semantics.

There are several evidences that our work is inspired by the question of injectivity, one of them is that we can find here tools similar to the ones used in [4], as we now try to explain. But let us first stress one difference: while working with the interpretation of a proof-structure in the relational model one can extract informations on the *cut-free* proof-structure, our analysis suits perfectly well also the case of proof-structures *with cuts* (our main theorem holds in presence of cuts too). Diffnets have both a semantic nature (again, see [19] for a technical version of this -rather obvious- statement) and a syntactic one (cut-elimination is defined on diffnets). Like in [4], there are two main ingredients in our proof:

1. to establish a precise relation between cells and ports of a proof-structure Φ and cells and ports of (some) elements γ of the Taylor expansion $\mathcal{T}_\Phi$ of Φ. There is absolutely nothing deep here; anyone immediately understands by a little drawing on the blackboard what one means. But this is awfully complicated to formalize. In Section 2, one can find a detailed description

of the syntax, inspired by previous works: the point here is to have “the language” necessary to express our result. In Section 3, we define the Taylor expansion $\mathcal{T}_\Phi$ of a proof-structure Φ, and Lemma 28 relates cells and ports of Φ to cells and ports of $\gamma \in \mathcal{T}_\Phi$, by means of “names”, which remind in some sense the “experiments of $PLPS$” of [4]. As expected, Lemma 28 is difficult to state but completely straightforward to prove.

2. to recover, based on purely geometric grounds, the boxes and the contractions of a proof-structure Φ from a (well-chosen) $\gamma \in \mathcal{T}_\Phi$. Here is the mathematical content of our contribution and the point where box-connectedness comes into the picture. In Section 4, through the notion of accessibility (Definition 36), we show how to recover the frontier of exponential boxes: the crucial Lemma 41 essentially proves that a port p is accessible from the unique premise of a !-cell which is a box if and only if p is inside the box. Notice that this equivalence does not hold in general when the proof-structure is not box- connected. In the case of a cut-free proof-structure Φ, the notion of accessibility for $\gamma \in \mathcal{T}_\Phi$ is closely related to the notion of “bridge” of [4]. However, in [4] we did not notice that this is enough in the connected case: once we have “split” in γ the copies of the content of boxes (by means of the notion of accessibility in the language of diffnets and by means of bridges in the language of [4]), we can recover Φ. In particular, we do not need elements of $\mathcal{T}_\Phi$ with size k (or k-points with the language of [4]), where k depends on $\mathcal{T}_\Phi$: any element of $\mathcal{T}_\Phi$ will do, provided it is “strongly fat” (Definition 2), and thus taking (for example) $\gamma \in \mathcal{T}_\Phi$ which contains 2 copies of every box is enough (a 2-point is enough, with the language of [4]) for any connected proof-structure Φ.

Preliminaries and notations

We set $\mathcal{L}_{\mathsf{MELL}} = \{1, \bot, \otimes, \parr, !, ?, ax, cut\}$. The MELL *connectives* are $1, \bot, \otimes, \parr, !, ?$. We say that 1, $\bot$, $\otimes$, $\parr$ (resp. $!$, $?$) are the *multiplicative* (resp. *exponential*) connectives, and 1, $\bot$ are the *units*.

Let $\mathcal{V}_{\mathsf{MELL}}$ be a countably infinite set whose elements, denoted by $X, Y, Z, \dots$, are called *propositional variables*. The set $\mathcal{F}_{\mathsf{MELL}}$ of MELL *formulas*, denoted by $A, B, C, \dots$, is generated by the grammar:

$$A, B, C ::= X \mid X^\bot \mid 1 \mid \bot \mid A \otimes B \mid A \parr B \mid !A \mid ?A\,.$$

If $\Gamma = (A_1, \dots, A_n)$ is a finite sequence of MELL formulas (with $n \in \mathbb{N}$), then $\parr\Gamma = A_1 \parr \cdots \parr A_n$; in particular, if $n = 0$ then $\parr\Gamma = \bot$.

For every $A \in \mathcal{F}_{\mathsf{MELL}}$, the *dual*, or *negation*, *of* A, denoted by $(A)^\bot$ or $A^\bot$, is defined by induction as follows: $(X)^\bot = X^\bot$, $(X^\bot)^\bot = X$, $(1)^\bot = \bot$, $(\bot)^\bot = 1$, $(A \otimes B)^\bot = (A)^\bot \parr (B)^\bot$, $(A \parr B)^\bot = (A)^\bot \otimes (B)^\bot$, $(!A)^\bot = ?(A)^\bot$ and $(?A)^\bot = !(A)^\bot$. Therefore, $A^{\bot\bot} = A$ for any $A \in \mathcal{F}_{\mathsf{MELL}}$.

Notation. Let $\mathcal{A}$ be a set: $\mathsf{card}(\mathcal{A})$ is the cardinality of $\mathcal{A}$, $\mathfrak{P}(\mathcal{A})$ is the power set of $\mathcal{A}$, $\mathcal{A}^*$ is the set of finite sequences over a set $\mathcal{A}$.

Elements of $\mathcal{A}^*$ are denoted by $(a_1, \ldots, a_n)$ or $\langle a_1, \ldots, a_n \rangle$, where $n \in \mathbb{N}$ and $a_i \in \mathcal{A}$ for any $1 \leq i \leq n$; in particular, the empty sequence is denoted by $()$ or $\langle\rangle$; often $(a_1) \in \mathcal{A}^*$ is denoted by a_1. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ for some $n, m \in \mathbb{N}$: we set $\mathsf{length}(a) = n$, $\mathsf{supp}(a) = \{a_1, \ldots, a_n\}$ and $a \cdot b = (a_1, \ldots, a_n, b_1, \ldots, b_m)$; if moreover $n > 0$, we set $a^- = (a_1, \ldots, a_{n-1})$.

Let $\mathcal{A}, \mathcal{B}$ be sets and $\mathsf{f} \colon \mathcal{A} \to \mathcal{B}$ be a a function (resp. partial function): we set $\mathsf{dom}(\mathsf{f}) = \mathcal{A}$ (resp. $\mathsf{dom}(\mathsf{f}) = \{a \in \mathcal{A} \mid f(a) \text{ is defined}\}$) i.e. the domain of f, and $\mathsf{im}(\mathsf{f}) = \{\mathsf{f}(a) \mid a \in \mathsf{dom}(f)\}$ i.e. the image of f; the function (resp. partial function) $\bar{\mathsf{f}} \colon \mathfrak{P}(\mathcal{A}) \to \mathfrak{P}(\mathcal{B})$ is defined by $\bar{\mathsf{f}}(\mathcal{A}') = \{\mathsf{f}(a) \mid a \in \mathcal{A}' \cap \mathsf{dom}(\mathsf{f})\}$ for any $\mathcal{A}' \subseteq \mathcal{A}$; the function (resp. partial function) $\widehat{\mathsf{f}} \colon \mathcal{A}^* \to \mathcal{B}^*$ is defined by $\widehat{\mathsf{f}}((a_1, \ldots, a_n)) = (\mathsf{f}(a_1), \ldots, \mathsf{f}(a_n))$ for any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathcal{A} \cap \mathsf{dom}(\mathsf{f})$, otherwise $\widehat{\mathsf{f}}((a_1, \ldots, a_n))$ is not defined; given $\mathcal{A}' \subseteq \mathcal{A}$, the function (resp. partial function) $\mathsf{f}\restriction_{\mathcal{A}'} \colon \mathcal{A}' \to \mathcal{B}$ is defined by $\mathsf{dom}(\mathsf{f}\restriction_{\mathcal{A}'}) = \mathsf{dom}(\mathsf{f}) \cap \mathcal{A}'$ and $\mathsf{f}\restriction_{\mathcal{A}'}(a) = \mathsf{f}(a)$ for any $a \in \mathsf{dom}(\mathsf{f}\restriction_{\mathcal{A}'})$. If $\mathsf{f}_0 \colon \mathcal{A}_0 \to \mathcal{B}_0$ and $f_1 \colon \mathcal{A}_1 \to \mathcal{B}_1$ are two partial functions such that $\mathsf{f}_0(a) = \mathsf{f}_1(a)$ for any $a \in \mathsf{dom}(\mathsf{f}_0) \cap \mathsf{dom}(\mathsf{f}_1)$, the partial function $\mathsf{f}_0 \cup \mathsf{f}_1 \colon \mathcal{A}_0 \cup \mathcal{A}_1 \to \mathcal{B}_0 \cup \mathcal{B}_1$ is defined by: $(\mathsf{f}_0 \cup \mathsf{f}_1)(a) = \mathsf{f}_i(a)$ if $a \in \mathcal{A}_i$ for some $i \in \{0, 1\}$, otherwise $(\mathsf{f}_0 \cup \mathsf{f}_1)(a)$ is not defined. The identity function on a set $\mathcal{A}$ is denoted by $\mathsf{id}_{\mathcal{A}}$.

If f is an enumeration of a finite set $\mathcal{A}$, i.e. a bijection from $\{1, \ldots, \mathsf{card}(\mathcal{A})\}$ into $\mathcal{A}$, then f is also denoted like a finite sequence $(\mathsf{f}(1), \ldots, \mathsf{f}(\mathsf{card}(\mathcal{A})))$.

2. A non-inductive syntax for proof structures

The syntax that we present here is inspired by the ones used in [15, 17, 18, 22, 3, 4]. We use terminology of interactions nets [13, 8], even if properly speaking our objects are not interaction nets. The main novelties in our syntax are that there are no wires (the same port may be auxiliary for some cell and principal for another cell) and boxes have not an explicit constructor, they are recovered in a non-inductive way.

As in [15, 17, 18] and unlike [3, 4], our syntactic objects are typed by MELL formulas, but they can easily be defined in the untyped case too: we have opted for the typed version only to keep out immediately the possibility of "vicious cycles" (see Fact 10). We use !- and ?-cells with $n \in \mathbb{N}$ premises, as in [1, 15], for generalized (co-)contractions/(co-)derelictions/(co-)weakenings: the type of their premises is different from the type of their conclusion. All the results in this paper can be achieved also in the untyped case.

2.1. Pre-proof-structures and DiLL_0*-proof structures*

We define here our basic syntactical object: *pre-proof structure* (*pps*, for short). All other syntactical objects, in particular the ones corresponding to some framework or extension of LL (DiLL-, MELL- and DiLL_0-proof structures), are some special cases of pps.

Unlike [18, 4], our syntactical objects are not necessarily cut-free (nor η-expanded). It is possible to define the cut-elimination and η-expansion[1] for DiLL-proof structures (and in particular for DiLL_0- and MELL-proof structures), but their rewriting rules are not given here since we will not use them.

Definition 1 (Pre-proof-structure, ports, cells)**.** *A* pre-proof-structure *(*pps *for short) is a* 9*-tuple* $\Phi = (\mathcal{P}_\Phi, \mathcal{C}_\Phi, \mathsf{tc}_\Phi, \mathsf{P}^{\mathsf{pri}}_\Phi, \mathsf{P}^{\mathsf{aux}}_\Phi, \mathsf{P}^{\mathsf{left}}_\Phi, \mathsf{tp}_\Phi, \mathsf{auxd}_\Phi, \mathsf{cutd}_\Phi)$ *such that:*

- $\mathcal{P}_\Phi$ *(resp.* $\mathcal{C}_\Phi$*) is a finite set, whose elements are the* ports *(resp.* cells *or* links*)* of Φ*;*
- tc_Φ *is a function from* $\mathcal{C}_\Phi$ *to* $\mathcal{L}_{\mathsf{MELL}}$*; for every* $l \in \mathcal{C}_\Phi$*,* $\mathsf{tc}_\Phi(l)$ *is the* label*, or* type*, of* l*; for every* $t, t' \in \mathcal{L}_{\mathsf{MELL}}$*, we set* $\mathcal{C}^t_\Phi = \{l \in \mathcal{C}_\Phi \mid \mathsf{tc}_\Phi(l) = t\}$ *(whose elements are the* t-cells*, or* t-links*,* of Φ*) and* $\mathcal{C}^{t,t'}_\Phi = \mathcal{C}^t_\Phi \cup \mathcal{C}^{t'}_\Phi$*;*
- $\mathsf{P}^{\mathsf{pri}}_\Phi$ *is a function from* $\mathcal{C}_\Phi$ *to* $\mathfrak{P}(\mathcal{P}_\Phi)$ *such that* $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\Phi) = \mathcal{P}_\Phi$*, and moreover, for all* $l, l' \in \mathcal{C}_\Phi$*,*
  - *if* $l \neq l'$ *then* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{pri}}_\Phi(l') = \emptyset$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 1$ *if* $\mathsf{tc}_\Phi(l) \in \{1, \bot, \otimes, \parr, !, ?\}$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 2$ *(resp.* $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 0$*) if* $\mathsf{tc}_\Phi(l) = ax$ *(resp.* $\mathsf{tc}_\Phi(l) = cut$*);*

  for any $l \in \mathcal{C}_\Phi$*, the elements of* $\mathsf{P}^{\mathsf{pri}}_\Phi(l)$ *are the* principal ports*, or* conclusions*,* of l in Φ*;*
- $\mathsf{P}^{\mathsf{aux}}_\Phi$ *is a function from* $\mathcal{C}_\Phi$ *to* $\mathfrak{P}(\mathcal{P}_\Phi)$ *such that, for all* $l, l' \in \mathcal{C}_\Phi$*,*
  - $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l') = \emptyset$ *if* $l \neq l'$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 0$ *(resp.* $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 2$*) if* $\mathsf{tc}_\Phi(l) \in \{1, \bot, ax\}$ *(resp.* $\mathsf{tc}_\Phi(l) \in \{\otimes, \parr, cut\}$*);*

  for any $l \in \mathcal{C}_\Phi$*, the elements of* $\mathsf{P}^{\mathsf{aux}}_\Phi(l)$ *are the* auxiliary ports*, or* premises*,* of l in Φ*; we set* $\mathcal{P}^{\mathsf{aux}}_\Phi = \bigcup \mathsf{im}(\mathsf{P}^{\mathsf{aux}}_\Phi)$ *whose elements are the* auxiliary ports of Φ*,* $\mathcal{P}^{\mathsf{free}}_\Phi = \mathcal{P}_\Phi \smallsetminus \mathcal{P}^{\mathsf{aux}}_\Phi$ *whose elements are the* free ports*, or* conclusions*,* of Φ*.*
- $\mathsf{P}^{\mathsf{left}}_\Phi$ *is a function from* $\mathcal{C}^{\otimes,\parr}_\Phi$ *to* $\mathcal{P}^{\mathsf{aux}}_\Phi$ *such that* $\mathsf{P}^{\mathsf{left}}_\Phi(l) \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ *for any* $l \in \mathcal{C}^{\otimes,\parr}_\Phi$*;*
- tp_Φ *is a function from* $\mathcal{P}_\Phi$ *to* $\mathcal{F}_{\mathsf{MELL}}$ *such that, for any* $l \in \mathcal{C}_\Phi$*, one has*
  - $\mathsf{tp}_\Phi(p_1) = A$ *and* $\mathsf{tp}_\Phi(p_2) = A^\bot$*, if* $\mathsf{tc}_\Phi(l) = ax$ *(resp.* $\mathsf{tc}_\Phi(l) = cut$*),* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p_1, p_2\}$ *(resp.* $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$*) and* $A \in \mathcal{F}_{\mathsf{MELL}}$*,*

[1] The η-expansion of a pps is the substitution of every ax-cell whose conclusions are typed by $A, A^\bot$ with the "standard" pps with conclusions typed by $A, A^\bot$ and where the conclusions of every ax-cell are now typed by dual atomic formulas.

– $\mathsf{tp}_\Phi(p) = A$, *if* $\mathsf{tc}_\Phi(l) = A \in \{1, \bot\}$ *and* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$,

– $\mathsf{tp}_\Phi(p) = \mathsf{tp}_\Phi(p_1) \odot \mathsf{tp}_\Phi(p_2)$ *if* $\mathsf{tc}_\Phi(l) = \odot \in \{\otimes, ⅋\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$, $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$ *and* $\mathsf{P}^{\mathsf{left}}_\Phi(l) = p_1$,

– $\mathsf{tp}_\Phi(p) = \Diamond A$ *and* $\mathsf{tp}_\Phi(p_i) = A$ *for any* $1 \leq i \leq n$, *if* $\mathsf{tc}_\Phi(l) = \Diamond \in \{!, ?\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$, $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, \ldots, p_n\}$ *for some* $n \in \mathbb{N}$, *and* $A \in \mathcal{F}_{\mathsf{MELL}}$*;*

for every $p \in \mathcal{P}_\Phi$, $\mathsf{tp}_\Phi(p)$ *is the* label*, or* type*, of* p*; if* $\mathsf{tp}_\Phi(p) = A$ *we write* $p : A$*;*

- auxd_Φ *is a partial function from* $\mathcal{C}^!_\Phi$ *to* $\mathfrak{P}(\mathcal{P}^{\mathsf{aux}}_\Phi)$ *such that for every* $l \in \mathsf{dom}(\mathsf{auxd}_\Phi)$ *one has*

  – $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 1$,

  – *if* $q \in \mathsf{auxd}_\Phi(l)$*, then* $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l')$ *for some* $l' \in \mathcal{C}^?_\Phi$*;*

  we set $\mathcal{C}^{\mathsf{prom}}_\Phi = \mathsf{dom}(\mathsf{auxd}_\Phi)$ *(resp.* $\mathcal{A}uxdoors_\Phi = \bigcup \mathsf{im}(\mathsf{auxd}_\Phi)$*;* $\mathcal{P}ridoors_\Phi = \overline{\mathsf{P}^{\mathsf{aux}}_\Phi}(\mathcal{C}^{\mathsf{prom}}_\Phi)$*) whose elements are the* promotion cells *or* links *(resp. aux-*doors*; pri-*doors*)* of Φ*; if* $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$ *and* $\mathcal{P}^{\mathsf{aux}}_\Phi(l) = \{p\}$*, the pri-*door of l in Φ *is* $\mathsf{prid}_\Phi(l) = p$*, and the aux-*doors of l in Φ *are the elements of* $\mathsf{auxd}_\Phi(l)$*;*

- cutd_Φ *is a function from* $\mathcal{C}^{\mathsf{prom}}_\Phi$ *to* $\mathfrak{P}(\mathcal{P}^{\mathsf{aux}}_\Phi)$ *such that, for every* $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$,

  – *there exists* $n \in \mathbb{N}$ *and* $l_1, \ldots, l_n \in \mathcal{C}^{cut}_\Phi$ *such that* $\mathsf{cutd}_\Phi(l) = \bigcup_{i=1}^n \mathsf{P}^{\mathsf{aux}}_\Phi(l_i)$,

  – $\mathsf{cutd}_\Phi(l) \cap \mathsf{cutd}_\Phi(l') = \emptyset$ *for any* $l' \in \mathcal{C}^{\mathsf{prom}}_\Phi$ *such that* $l \neq l'$*;*

  for any $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$ *the elements of* $\mathsf{cutd}_\Phi(l)$ *are the cut-*doors of l in Φ*; we set* $\mathcal{C}utdoors_\Phi = \bigcup \mathsf{im}(\mathsf{cutd}_\Phi)$*, whose elements are the cut-*doors of Φ*.*

If $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$*, we set* $\mathsf{doors}_\Phi(l) = \mathsf{P}^{\mathsf{aux}}_\Phi(l) \cup \mathsf{auxd}_\Phi(l) \cup \mathsf{cutd}_\Phi(l)$*, whose elements are the* doors of l in Φ*. We set* $\mathcal{C}^{\mathsf{free}}_\Phi = \{l \in \mathcal{C}_\Phi \mid \emptyset \neq \mathsf{P}^{\mathsf{pri}}_\Phi(l) \subseteq \mathcal{P}^{\mathsf{free}}_\Phi\} \cup \{l \in \mathcal{C}^{cut}_\Phi \mid \mathcal{C}utdoors_\Phi \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l) = \emptyset\}$*, whose elements are the* free*, or* terminal*,* cells of Φ*.*[2]

The empty pps *is the pps* Φ *such that* $\mathcal{P}_\Phi = \emptyset = \mathcal{C}_\Phi$ *and* tc_Φ, $\mathsf{P}^{\mathsf{pri}}_\Phi$, $\mathsf{P}^{\mathsf{aux}}_\Phi$, $\mathsf{P}^{\mathsf{left}}_\Phi$, tp_Φ, auxd_Φ, cutd_Φ *are the empty function.*

In a pps Φ, the function $\mathsf{P}^{\mathsf{left}}_\Phi$ fixes an order on the two premises of any $\otimes$- and ⅋-cell of Φ; the premises of the other types of cells are unordered. The conditions $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\Phi) = \mathcal{P}_\Phi$ and, for all $l, l' \in \mathcal{C}_\Phi$, if $l \neq l'$ then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{pri}}_\Phi(l') = \emptyset = \mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l')$, mean that every port is conclusion of exactly one cell and premise of at most one cell; the ports that are not premises of any cell are the conclusions of Φ. Notice that $\mathcal{C}^{\mathsf{free}}_\Phi \cap \mathcal{C}^{cut}_\Phi = \emptyset$, and no condition is required for $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l))$ when $l \in \mathcal{C}^{!,?}_\Phi \smallsetminus \mathcal{C}^{\mathsf{prom}}_\Phi$ (l can have $n \in \mathbb{N}$ premises): we use generalized ?- and !-cells for (co-)contraction, (co-)weakening and (co-)dereliction. If $l \in \mathcal{C}^!_\Phi$, note the difference between "$\mathsf{auxd}_\Phi(l) = \emptyset$" and "$\mathsf{auxd}_\Phi(l)$ is not defined": in

[2]Therefore, a cell l of a pps Φ is terminal iff either l is a ax-cell and both its conclusions are conclusions of Φ, or l is a cut-cell and both its premises are not in $\mathsf{cutd}_\Phi(l')$ for any promotion cell l', or l is neither an ax- nor a cut-cell and its unique conclusion is a conclusion of Φ.

the first case $l \in \mathcal{C}_{\Phi}^{\mathsf{prom}}$ (and l has no *aux*-doors), in the second one $l \notin \mathcal{C}_{\Phi}^{\mathsf{prom}}$; similarly for cutd_{Φ}. Note that a premise of a *cut*-cell may be *cut*-door of at most one promotion cell, while a premise of an *aux*-cell may be *aux*-door of several promotion cells (this is due to the fact that we use generalized ?-cells).

With every pps Φ are associated two directed labelled hypergraphs $\mathfrak{G}^{\leq}(\Phi)$ and $\mathfrak{G}^{\preccurlyeq}(\Phi)$ such that:

- the nodes of $\mathfrak{G}^{\leq}(\Phi)$ are the ports of Φ, labelled by their type; the hyperedges of $\mathfrak{G}^{\leq}(\Phi)$ are the cells of Φ, labelled by their type and oriented from their premises to their conclusions;
- the labelled nodes and the labelled oriented hyperedges of $\mathfrak{G}^{\preccurlyeq}(\Phi)$ are exactly the labelled nodes and the labelled oriented hyperedges of $\mathfrak{G}^{\leq}(\Phi)$; moreover, for every promotion cell l of Φ, there is in $\mathfrak{G}^{\preccurlyeq}(\Phi)$ an unlabelled hyperedge oriented from the *aux*- and *cut*-doors of l to the *pri*-door of l.

We denote by $\mathfrak{G}^{\leq}_{\mathrm{undir}}(\Phi)$ (resp. $\mathfrak{G}^{\preccurlyeq}_{\mathrm{undir}}(\Phi)$) the undirected version of $\mathfrak{G}^{\leq}(\Phi)$ (resp. $\mathfrak{G}^{\preccurlyeq}(\Phi)$).

A cell is graphically depicted as a trapezoid with its label inside it (the label of a promotion cell is depicted as $\mathsf{!p}$), its principal port being on the shorter base and its auxiliary ports on the longer base (in such a way that when the principal port is downwards the left auxiliary ports of a $\otimes$- or $⅋$-cell is placed on the left). A port which is principal for one cell l and auxiliary for another cell l' is depicted as an oriented wire from l to l'. The fact that a premise q of a ?- or *cut*-cell is an *aux*- or *cut*-door of a promotion cell l is represented graphically by a dotted arrow from q to the auxiliary port of l. In such graphical representations the names of ports and cells are omitted, unless indicated to the contrary.

Observe that any non-empty pps has at least one terminal cell.

Definition 2 (DiLL_0-proof structure, fatness, wideness, well-naming, interface)**.** *Let Φ and Ψ be pps. We say that:*

- *Φ is a* DiLL_0-proof structure *(*DiLL_0-ps *or* diffnet *for short) if $\mathcal{C}_{\Phi}^{\mathsf{prom}} = \emptyset$;*
- *Φ is* k-wide *if Φ is a DiLL_0-ps and there exists $k \in \mathbb{N}$ such that $\mathsf{card}(\mathsf{P}_{\Phi}^{\mathsf{aux}}(l)) = k$ for any $l \in \mathcal{C}_{\Phi}^{!}$;*
- *Φ is* fat *(resp.* strongly fat*) if $\mathsf{card}(\mathsf{P}_{\Phi}^{\mathsf{aux}}(l)) \geq 1$ (resp. $\mathsf{card}(\mathsf{P}_{\Phi}^{\mathsf{aux}}(l)) \geq 2$) for any $l \in \mathcal{C}_{\Phi}^{!}$;*
- *Φ is* well-named *if for every $d \in \mathcal{P}_{\Phi} \cup \mathcal{C}_{\Phi}$ there exists a finite sequence a of ordered pairs such that $d = (d', a)$ for some d', and every $c \in \mathcal{P}_{\Phi}^{\mathsf{free}} \cup \mathcal{C}_{\Phi}^{\mathsf{free}}$ is such that $c = (c', ())$ for some c'; Φ is* empty-named *if every $c \in \mathcal{P}_{\Phi} \cup \mathcal{C}_{\Phi}$ is such that $c = (c', ())$ for some c';*
- *Φ and Ψ are* weakly interfaced *if $\mathcal{P}_{\Phi}^{\mathsf{free}} = \mathcal{P}_{\Psi}^{\mathsf{free}}$ and $\mathsf{tp}_{\Phi}\restriction_{\mathcal{P}_{\Phi}^{\mathsf{free}}} = \mathsf{tp}_{\Psi}\restriction_{\mathcal{P}_{\Psi}^{\mathsf{free}}}$; Φ and Ψ are* interfaced *if they are weakly interfaced and $\mathcal{C}_{\Phi}^{\mathsf{free}} = \mathcal{C}_{\Psi}^{\mathsf{free}}$, $\mathsf{P}_{\Phi}^{\mathsf{pri}}\restriction_{\mathcal{C}_{\Phi}^{\mathsf{free}}} = \mathsf{P}_{\Psi}^{\mathsf{pri}}\restriction_{\mathcal{C}_{\Psi}^{\mathsf{free}}}$ and $\mathsf{tc}_{\Phi}\restriction_{\mathcal{C}_{\Phi}^{\mathsf{free}}} = \mathsf{tc}_{\Psi}\restriction_{\mathcal{C}_{\Psi}^{\mathsf{free}}}$; Φ and Ψ are* strongly interfaced *if they are interfaced and $\mathcal{P}_{\Phi} \cap \mathcal{P}_{\Psi} = \mathcal{P}_{\Phi}^{\mathsf{free}}$ and $\mathcal{C}_{\Phi} \cap \mathcal{C}_{\Psi} = \mathcal{C}_{\Phi}^{\mathsf{free}}$.*

The set of DiLL_0*-ps is denoted by* $\mathbf{PS}_{\mathsf{DiLL}_0}$.

We will use well- and empty-named pps in order to define the Taylor expansion (see Definition 22). We will require that some pps are strongly interfaced in order to define their product (see Definition 5).

Definition 3 (Isomorphism on pre-proof structures)**.** *Let* Φ *and* Ψ *be some pps.*

An isomorphism from Φ to Ψ *is a pair* $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}})$ *of bijections* $\varphi_{\mathcal{P}} \colon \mathcal{P}_\Phi \to \mathcal{P}_\Psi$ *and* $\varphi_{\mathcal{C}} \colon \mathcal{C}_\Phi \to \mathcal{C}_\Psi$ *such that* $\mathsf{im}(\varphi_{\mathcal{C}}\restriction_{\mathcal{C}_\Phi^{\mathsf{prom}}}) = \mathcal{C}_\Psi^{\mathsf{prom}}$ *and all diagrams* (1)-(2) *commute. We write then* $\varphi : \Phi \simeq \Psi$*.*

$$
\begin{array}{ccccc}
\mathfrak{P}(\mathcal{P}_\Phi) & \xleftarrow{\mathsf{P}_\Phi^{\mathsf{aux}}} & \mathcal{C}_\Phi & \xrightarrow{\mathsf{P}_\Phi^{\mathsf{pri}}} & \mathfrak{P}(\mathcal{P}_\Phi) \\
\downarrow{\scriptstyle \overline{\varphi_{\mathcal{P}}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{C}}} & & \downarrow{\scriptstyle \overline{\varphi_{\mathcal{P}}}} \\
\mathfrak{P}(\mathcal{P}_\Psi) & \xleftarrow{\mathsf{P}_\Psi^{\mathsf{aux}}} & \mathcal{C}_\Psi & \xrightarrow{\mathsf{P}_\Psi^{\mathsf{pri}}} & \mathfrak{P}(\mathcal{P}_\Psi)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_\Phi & \xrightarrow{\mathsf{tc}_\Phi} & \mathcal{L}_{\mathsf{MELL}} \\
\downarrow{\scriptstyle \varphi_{\mathcal{C}}} & \nearrow{\scriptstyle \mathsf{tc}_\Psi} & \\
\mathcal{C}_\Psi & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}_\Phi & \xrightarrow{\mathsf{tp}_\Phi} & \mathcal{F}_{\mathsf{MELL}} \\
\downarrow{\scriptstyle \varphi_{\mathcal{P}}} & \nearrow{\scriptstyle \mathsf{tp}_\Psi} & \\
\mathcal{P}_\Psi & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_\Phi^{\otimes,⅋} & \xrightarrow{\mathsf{P}_\Phi^{\mathsf{left}}} & \mathcal{P}_\Phi \\
\downarrow{\scriptstyle \varphi_{\mathcal{C}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{P}}} \\
\mathcal{C}_\Psi^{\otimes,⅋} & \xrightarrow{\mathsf{P}_\Psi^{\mathsf{left}}} & \mathcal{P}_\Psi
\end{array}
\tag{1}
$$

$$
\begin{array}{ccccc}
\mathfrak{P}(\mathcal{C}utdoors_\Phi) & \xleftarrow{\mathsf{cutd}_\Phi} & \mathcal{C}_\Phi^{\mathsf{prom}} & \xrightarrow{\mathsf{auxd}_\Phi} & \mathfrak{P}(\mathcal{A}uxdoors_\Phi) \\
\downarrow{\scriptstyle \overline{\varphi_{\mathcal{P}}}} & & \downarrow{\scriptstyle \varphi_{\mathcal{C}}} & & \downarrow{\scriptstyle \overline{\varphi_{\mathcal{P}}}} \\
\mathfrak{P}(\mathcal{C}utdoors_\Psi) & \xleftarrow{\mathsf{cutd}_\Psi} & \mathcal{C}_\Psi^{\mathsf{prom}} & \xrightarrow{\mathsf{auxd}_\Psi} & \mathfrak{P}(\mathcal{A}uxdoors_\Psi)
\end{array}
\tag{2}
$$

Given $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) = \Phi \simeq \Psi$*, we set* $\varphi^{-1} = (\varphi_{\mathcal{P}}^{-1}, \varphi_{\mathcal{C}}^{-1})$*.*

If there is an isomorphism from Φ *to* Ψ*, we say that* Φ and Ψ are isomorphic *and we write* $\Phi \simeq \Psi$*.*

The idea is that two pps are isomorphic iff they are identical up to the names of their cells and ports: two isomorphic pps are "morally" the same object (bijections $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ of an isomorphism φ are a renaming of ports and cells, ports and cells being nothing but their names) and indeed they have the same graphical representation up to the order of the premises of their !- and ?-cells. The relation $\simeq$ is an equivalence on the set of pps and a somehow "canonical" representative of an equivalence class with respect to $\simeq$ on the set of pps is the graphical representation of any pps in this class up to the order of the premises of its !- and ?-cells.

Obviously, given two pps Φ and Ψ, $\varphi \colon \Phi \simeq \Psi$ if and only if $\varphi^{-1} \colon \Psi \simeq \Phi$.

Remark 4. Let Φ and Ψ be pps. If $\varphi_{\mathcal{P}} \colon \mathcal{P}_\Phi \to \mathcal{P}_\Psi$ and $\varphi_{\mathcal{C}} \colon \mathcal{C}_\Phi \to \mathcal{P}_\Psi$ are two bijections such that all diagrams (1) of Definition 3 commute, then $\mathcal{P}_\Psi^{\mathsf{free}} = \overline{\varphi_{\mathcal{P}}}(\mathcal{P}_\Phi^{\mathsf{free}})$, $\mathcal{C}_\Psi^{\mathsf{free}} = \overline{\varphi_{\mathcal{C}}}(\mathcal{C}_\Phi^{\mathsf{free}})$ and $\mathsf{card}(\mathsf{P}_\Phi^{\mathsf{aux}}(l)) = \mathsf{card}(\mathsf{P}_\Psi^{\mathsf{aux}}(\varphi_{\mathcal{C}}(l)))$ for any $l \in \mathcal{C}_\Phi$; hence, Φ is (strongly) fat iff Ψ is (strongly) fat; Φ is k-wide iff Ψ is k-wide.

The following Definitions 5–8 will be used to define the Taylor expansion (Definitions 22 and 24). We leave it to the reader to check that the objects defined in Definitions 5 and 7 are indeed pps.

Definition 5 (Product of pre-proof structures)**.** *Let* $\mathcal{A} = \{\Phi_1, \dots, \Phi_n\}$ *(with* $n \in \mathbb{N}$*) be a finite set of pairwise strongly interfaced pps such that* $\mathcal{C}^{\mathsf{free}}_{\Phi_i} \subseteq \mathcal{C}^{!,?}_{\Phi_i} \smallsetminus \mathcal{C}^{\mathsf{prom}}_{\Phi_i}$ *for any* $1 \leq i \leq n$*. The* product of $\mathcal{A}$ *is the pps* $\Psi = \prod_{i=1}^{n} \Phi_i = \prod \mathcal{A}$ *defined by:*

- $\mathcal{P}_\Psi = \bigcup_{i=1}^{n} \mathcal{P}_{\Phi_i}$ *and* $\mathcal{C}_\Psi = \bigcup_{i=1}^{n} \mathcal{C}_{\Phi_i}$*;*
- $\mathsf{tc}_\Psi = \bigcup_{i=1}^{n} \mathsf{tc}_{\Phi_i}$, $\mathsf{P}^{\mathsf{pri}}_\Psi = \bigcup_{i=1}^{n} \mathsf{P}^{\mathsf{pri}}_{\Phi_i}$ *and* $\mathsf{P}^{\mathsf{aux}}_\Psi$ *is such that* $\mathsf{P}^{\mathsf{aux}}_\Psi \restriction_{\mathcal{C}_\Psi \smallsetminus \mathcal{C}^{\mathsf{free}}_{\mathcal{A}}} = \bigcup_{i=1}^{n} \mathsf{P}^{\mathsf{aux}}_{\Phi_i} \restriction_{\mathcal{C}_{\Phi_i} \smallsetminus \mathcal{C}^{\mathsf{free}}_{\Phi_i}}$ *and* $\mathsf{P}^{\mathsf{aux}}_\Psi(l) = \bigcup_{i=1}^{n} \mathsf{P}^{\mathsf{aux}}_{\Phi_i}(l)$ *for any* $l \in \mathcal{C}^{\mathsf{free}}_{\mathcal{A}}$*, where* $\mathcal{C}^{\mathsf{free}}_{\mathcal{A}} = \bigcup_{i=1}^{n} \mathcal{C}^{\mathsf{free}}_{\Phi_i}$*;*[3]
- $\mathsf{P}^{\mathsf{left}}_\Psi = \bigcup_{i=1}^{n} \mathsf{P}^{\mathsf{left}}_{\Phi_i}$ *and* $\mathsf{tp}_\Psi = \bigcup_{i=1}^{n} \mathsf{tp}_{\Phi_i}$, $\mathsf{auxd}_\Psi = \bigcup_{i=1}^{n} \mathsf{auxd}_{\Phi_i}$ *and* $\mathsf{cutd}_\Psi = \bigcup_{i=1}^{n} \mathsf{cutd}_{\Phi_i}$*.*

Notice that if $\mathcal{A}$ is a finite set of pairwise strongly interfaced pps such that $\mathcal{C}^{\mathsf{free}}_{\Phi} \subseteq \mathcal{C}^{!,?}_{\Phi} \smallsetminus \mathcal{C}^{\mathsf{prom}}_{\Phi}$ for any $\Phi \in \mathcal{A}$, then $\mathcal{C}^{\mathsf{free}}_{\prod \mathcal{A}} \subseteq \mathcal{C}^{!,?}_{\prod \mathcal{A}} \smallsetminus \mathcal{C}^{\mathsf{prom}}_{\prod \mathcal{A}}$, and Φ and $\prod \mathcal{A}$ are interfaced for any $\Phi \in \mathcal{A}$. If $\mathcal{A}$ is the empty set, then $\prod \mathcal{A}$ is the empty pps, i.e. $\mathcal{P}_{\prod \mathcal{A}} = \emptyset = \mathcal{C}_{\prod \mathcal{A}}$ and $\mathsf{tc}_{\prod \mathcal{A}}$, $\mathsf{P}^{\mathsf{pri}}_{\prod \mathcal{A}}$, $\mathsf{P}^{\mathsf{aux}}_{\prod \mathcal{A}}$, $\mathsf{P}^{\mathsf{left}}_{\prod \mathcal{A}}$, $\mathsf{tp}_{\prod \mathcal{A}}$, $\mathsf{auxd}_{\prod \mathcal{A}}$ and $\mathsf{cutd}_{\prod \mathcal{A}}$ are the empty function. Since the premises of !- and ?-cells are unordered, it is not necessary to fix an order on the elements of $\mathcal{A}$ to compute $\prod \mathcal{A}$.

Definition 6 (Sub-pre-proof structure, restriction of an isomorphism to a sub-pps)**.** *Let* Φ *and* Ψ *be pps.*

A sub-pre-proof structure *(*sub-pps *for short)* of Φ *is a pps* Φ_0 *such that:*

- $\mathcal{P}_{\Phi_0} \subseteq \mathcal{P}_\Phi$, $\mathcal{C}_{\Phi_0} \subseteq \mathcal{C}_\Phi$ *and* $\mathsf{tc}_{\Phi_0} = \mathsf{tc}_\Phi \restriction_{\mathcal{C}_{\Phi_0}}$*;*
- $\mathsf{P}^{\mathsf{pri}}_{\Phi_0} = \mathsf{P}^{\mathsf{pri}}_\Phi \restriction_{\mathcal{C}_{\Phi_0}}$, $\mathsf{P}^{\mathsf{aux}}_{\Phi_0}(l) = \mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathcal{P}_{\Phi_0}$ *for any* $l \in \mathcal{C}_{\Phi_0}$, $\mathsf{P}^{\mathsf{left}}_{\Phi_0} = \mathsf{P}^{\mathsf{left}}_\Phi \restriction_{\mathcal{C}^{\otimes,\parr}_{\Phi_0}}$ *and* $\mathsf{tp}_{\Phi_0} = \mathsf{tp}_\Phi \restriction_{\mathcal{P}_{\Phi_0}}$*;*
- $\mathsf{auxd}_{\Phi_0} = \mathsf{auxd}_\Phi \restriction_{\mathcal{C}_{\Phi_0}}$ *and* $\mathsf{cutd}_{\Phi_0} = \mathsf{cutd}_\Phi \restriction_{\mathcal{C}_{\Phi_0}}$.[4]

Let $\varphi \colon \Phi \simeq \Psi$ *and* Φ_0 *be a sub-pps of* Φ*. We denote by* $\varphi(\Phi_0)$ *the sub-pps of* Ψ *defined by:* $\mathcal{P}_{\varphi(\Phi_0)} = \overline{\varphi_{\mathcal{P}}}(\mathcal{P}_{\Phi_0})$ *and* $\mathcal{C}_{\varphi(\Phi_0)} = \overline{\varphi_{\mathcal{C}}}(\mathcal{C}_{\Phi_0})$*. The* restriction of φ to Φ_0 *is the isomorphism* $\varphi\restriction_{\Phi_0} = (\varphi_{\mathcal{P}}\restriction_{\mathcal{P}_{\Phi_0}}, \varphi_{\mathcal{C}}\restriction_{\mathcal{C}_{\Phi_0}}) \colon \Phi_0 \simeq \varphi(\Phi_0)$.

The notion of sub-pps generalizes that one of sub-hypergraph. In Definition 6, the request that a sub-pps Φ_0 of a pps Φ is itself a pps has a lot of implications: for example, if $l \notin \mathcal{C}^{!,?}_{\Phi_0}$ then $\mathsf{P}^{\mathsf{aux}}_{\Phi_0}(l) = \mathsf{P}^{\mathsf{aux}}_\Phi(l)$; if $l \in \mathcal{C}^{\mathsf{prom}}_\Phi \cap \mathcal{C}_{\Phi_0}$ then $l \in \mathcal{C}^{\mathsf{prom}}_{\Phi_0}$ (so $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_{\Phi_0}(l)) = 1$ and $\mathsf{tc}_{\Phi_0}(l) = !$), $\mathsf{auxd}_\Phi(l) = \mathsf{auxd}_{\Phi_0}(l) \subseteq \mathcal{P}_{\Phi_0} \cap \{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi_0}(l') \mid l' \in \mathcal{C}^{?}_{\Phi_0}\}$ and $\mathsf{cutd}_\Phi(l) = \mathsf{cutd}_{\Phi_0}(l) \subseteq \mathcal{P}_{\Phi_0} \cap \{p \in \mathsf{P}^{\mathsf{aux}}_{\Phi_0}(l') \mid l' \in \mathcal{C}^{cut}_{\Phi_0}\}$.

---

[3] If $n = 0$ then $\mathcal{C}^{\mathsf{free}}_{\mathcal{A}} = \emptyset$, otherwise $\mathcal{C}^{\mathsf{free}}_{\mathcal{A}} = \mathcal{C}^{\mathsf{free}}_{\Phi_i}$ for any $1 \leq i \leq n$, because $\mathcal{A}$ is a set of pairwise interfaced pps.

[4] Observe that a sub-pps Φ_0 of a pps Φ is univocally determined by $\mathcal{P}_{\Phi_0}$ and $\mathcal{C}_{\Phi_0}$. Otherwise stated: given $\mathcal{P}_0 \subseteq \mathcal{P}_\Phi$ and $\mathcal{C}_0 \subseteq \mathcal{C}_\Phi$ there exists at most one sub-pps Φ_0 of Φ such that $\mathcal{P}_{\Phi_0} = \mathcal{P}_0$ and $\mathcal{C}_{\Phi_0} = \mathcal{C}_0$.

Definition 7 (Substitution of sub-pre-proof structures)**.** *Let* $n \in \mathbb{N}$, *let* $\mathit{\Phi}, \mathit{\Psi}_1, \ldots, \mathit{\Psi}_n$ *be some pps and, for every* $1 \leq i \leq n$, *let* $\mathit{\Phi}_i$ *be a sub-pps of* $\mathit{\Phi}$ *such that* $\mathit{\Phi}_i$ *and* $\mathit{\Psi}_i$ *are weakly interfaced. If for every* $1 \leq i \neq j \leq n$ *one has* $\mathcal{P}_{\mathit{\Phi}_i} \cap \mathcal{P}_{\mathit{\Phi}_j} = \emptyset = \mathcal{C}_{\mathit{\Phi}_i} \cap \mathcal{C}_{\mathit{\Phi}_j}$, $\mathcal{P}_{\mathit{\Psi}_i} \cap \mathcal{P}_{\mathit{\Psi}_j} = \emptyset = \mathcal{C}_{\mathit{\Psi}_i} \cap \mathcal{C}_{\mathit{\Psi}_j}$, $\mathcal{P}_{\mathit{\Psi}_i} \cap (\mathcal{P}_{\mathit{\Phi}} \smallsetminus \bigcup_{k=1}^{n} \mathcal{P}_{\mathit{\Phi}_k}) = \emptyset = \mathcal{C}_{\mathit{\Psi}_i} \cap (\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{k=1}^{n} \mathcal{C}_{\mathit{\Phi}_k})$ *and* $\mathcal{A}uxdoors_{\mathit{\Phi}_i} \cap \mathsf{auxd}_{\mathit{\Phi}}(l) = \emptyset$ *for any* $l \in \mathcal{C}_{\mathit{\Phi}}^{\mathsf{prom}} \smallsetminus \mathcal{C}_{\mathit{\Phi}_i}^{\mathsf{prom}}$, *then the* substitution of $\mathit{\Psi}_1, \ldots, \mathit{\Psi}_n$ for $\mathit{\Phi}_1, \ldots, \mathit{\Phi}_n$ in $\mathit{\Phi}$ *is the pps* $\mathit{\Psi} = \mathit{\Phi}[\mathit{\Psi}_1/\mathit{\Phi}_1, \ldots, \mathit{\Psi}_n/\mathit{\Phi}_n]$ *defined by:*

- $\mathcal{P}_{\mathit{\Psi}} = (\mathcal{P}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{P}_{\mathit{\Phi}_i}) \cup \bigcup_{i=1}^{n} \mathcal{P}_{\mathit{\Psi}_i}$, $\mathcal{C}_{\mathit{\Psi}} = (\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}) \cup \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Psi}_i}$ *and* $\mathsf{tc}_{\mathit{\Psi}} = \mathsf{tc}_{\mathit{\Phi}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{tc}_{\mathit{\Psi}_i}$;

- $\mathsf{P}_{\mathit{\Psi}}^{\mathsf{pri}} = \mathsf{P}_{\mathit{\Phi}}^{\mathsf{pri}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{P}_{\mathit{\Psi}_i}^{\mathsf{pri}}$, $\mathsf{P}_{\mathit{\Psi}}^{\mathsf{aux}} = \mathsf{P}_{\mathit{\Phi}}^{\mathsf{aux}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{P}_{\mathit{\Psi}_i}^{\mathsf{aux}}$, $\mathsf{P}_{\mathit{\Psi}}^{\mathsf{left}} = \mathsf{P}_{\mathit{\Phi}}^{\mathsf{left}} \restriction_{\mathcal{C}_{\mathit{\Phi}}^{\otimes,\parr} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}^{\otimes,\parr}} \cup \bigcup_{i=1}^{n} \mathsf{P}_{\mathit{\Psi}_i}^{\mathsf{left}}$ *and* $\mathsf{tp}_{\mathit{\Psi}} = \mathsf{tp}_{\mathit{\Phi}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{tp}_{\mathit{\Psi}_i}$;

- $\mathsf{auxd}_{\mathit{\Psi}} = \mathsf{auxd}_{\mathit{\Phi}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{auxd}_{\mathit{\Psi}_i}$ *and* $\mathsf{cutd}_{\mathit{\Psi}} = \mathsf{cutd}_{\mathit{\Phi}} \restriction_{\mathcal{C}_{\mathit{\Phi}} \smallsetminus \bigcup_{i=1}^{n} \mathcal{C}_{\mathit{\Phi}_i}} \cup \bigcup_{i=1}^{n} \mathsf{cutd}_{\mathit{\Psi}_i}$.

Given some pps $\mathit{\Phi}, \mathit{\Psi}_1, \ldots, \mathit{\Psi}_n$ and some sub-pps $\mathit{\Phi}_1, \ldots, \mathit{\Phi}_n$ of $\mathit{\Phi}$, $\mathit{\Phi}[\mathit{\Psi}_1/\mathit{\Phi}_1, \ldots, \mathit{\Psi}_1/\mathit{\Phi}_1]$ is the pps obtained from $\mathit{\Phi}$ by simultaneous substitution of $\mathit{\Psi}_i$ for $\mathit{\Phi}_i$, for all $1 \leq i \leq n$. According to Definition 7, for this operation to make sense and yield a pps, some conditions have to be fulfilled: for example, for any $1 \leq i \leq n$, $\mathit{\Phi}_i$ and $\mathit{\Psi}_i$ have to be weakly interfaced, ports and cells of $\mathit{\Psi}_i$ cannot have the same name of ports and cells of $\mathit{\Phi}$ that are not substituted, $\mathit{\Phi}_i$ and $\mathit{\Phi}_j$ does not overlap, and if a promotion cell of $\mathit{\Phi}$ is not substituted then all its *aux*-doors are not substituted.

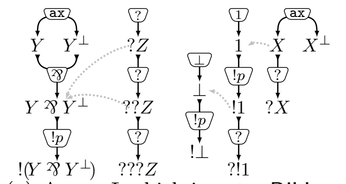
Definition 8 (Empty-renaming, n-th copy of a pre-proof structure)**.** *Let* $\mathit{\Phi}$ *be a pps. The* empty-renaming of $\mathit{\Phi}$ *is the empty-named pps* $\mathit{\Phi}_\varepsilon$ *such that* $\mathcal{P}_{\mathit{\Phi}_\varepsilon} = \{(p, ()) \mid p \in \mathcal{P}_{\mathit{\Phi}}\}$, $\mathcal{C}_{\mathit{\Phi}_\varepsilon} = \{(l, ()) \mid l \in \mathcal{C}_{\mathit{\Phi}}\}$ *and* $\varphi\colon \mathit{\Phi} \simeq \mathit{\Phi}_\varepsilon$ *where* φ *is such that* $\varphi_{\mathcal{P}}(p) = (p, ())$ *and* $\varphi_{\mathcal{C}}(l) = (l, ())$ *for any* $p \in \mathcal{P}_{\mathit{\Phi}}$ *and* $l \in \mathcal{C}_{\mathit{\Phi}}$.

Let $n \in \mathbb{N}^+$, $\mathit{\Phi}$ *be a well-named pps and* $(l, ()) \in \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}$. *The* n-th l-copy of $\mathit{\Phi}$ *is the pps* $\mathit{\Phi}_n$ *such that* $\mathcal{P}_{\mathit{\Phi}_n} = \mathcal{P}_{\mathit{\Phi}}^{\mathsf{free}} \cup \{(p, (l, n) \cdot a) \mid (p, a) \in \mathcal{P}_{\mathit{\Phi}} \smallsetminus \mathcal{P}_{\mathit{\Phi}}^{\mathsf{free}}\}$, $\mathcal{C}_{\mathit{\Phi}_n} = \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}} \cup \{(l', (l, n) \cdot a) \mid (l', a) \in \mathcal{C}_{\mathit{\Phi}} \smallsetminus \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}\}$ *and* $\varphi\colon \mathit{\Phi} \simeq \mathit{\Phi}_n$ *where* φ *is such that* $\varphi_{\mathcal{P}}((p, a)) = (p, (l, n) \cdot a)$ *and* $\varphi_{\mathcal{C}}((l', a)) = (l', (l, n) \cdot a)$ *for any* $(p, a) \in \mathcal{P}_{\mathit{\Phi}} \smallsetminus \mathcal{P}_{\mathit{\Phi}}^{\mathsf{free}}$ *and* $(l', a) \in \mathcal{C}_{\mathit{\Phi}} \smallsetminus \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}$, *and* $\varphi_{\mathcal{P}} \restriction_{\mathcal{P}_{\mathit{\Phi}}^{\mathsf{free}}} = \mathsf{id}_{\mathcal{P}_{\mathit{\Phi}}^{\mathsf{free}}}$ *and* $\varphi_{\mathcal{C}} \restriction_{\mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}} = \mathsf{id}_{\mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}}$.

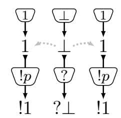
The intuition is: if $\mathit{\Phi}_n$ is the n-th l-copy of a well-named pps $\mathit{\Phi}$ with $(l, ()) \in \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}$ and $c = (p, (l, n) \cdot a) \in \mathcal{P}_{\mathit{\Phi}_n}$ (resp. $c = (l', (l, n) \cdot a) \in \mathcal{C}_{\mathit{\Phi}_n}$) then c is the n-th copy of $(p, a) \in \mathcal{P}_{\mathit{\Phi}}$ (resp. $(l', a) \in \mathcal{C}_{\mathit{\Phi}}$) associated with l. In this way, for any pps $\mathit{\Phi}$, taking its empty-renaming $\mathit{\Phi}_\varepsilon$ it is possible to build n different copies of $\mathit{\Phi}$ (associated with some $l \in \mathcal{C}_{\mathit{\Phi}}^{\mathsf{free}}$) and make their product, as in Definition 22.

2.2. Boxes, DiLL- *and* MELL*-proof structures*

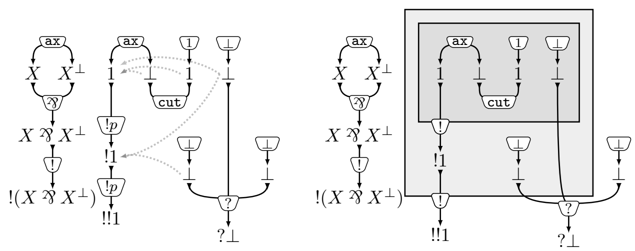
We present here if and how a box can be associated with a promotion cell of a pps. We define then DiLL-proof structures (DiLL-ps for short). Full differential linear logic (DiLL) is an extension of MELL (with the same language as MELL) provided with both promotion rule (i.e. boxes) and co-structural rules (i.e. the duals of the structural rules handling ? modality) for ! modality: DiLL_0 and MELL



(a) A pps Φ which is not a DiLL-ps.



(b) A pps Ψ which is not a DiLL-ps.



(c) A DiLL-ps in its two possibles presentations: the "arrow-like" one (on the left) and the "inductive-like" one (on the right).

Figure 1: Some examples of pps. In Φ (Figure 1(a)) the three promotion cells are not box cells (see Definition 11). In Ψ (Figure 1(b)) the promotion cell is a box cell but the nesting condition does not hold (see Definition 16).

are particular subsystems of DiLL, respectively corresponding to the promotion-free (i.e. without boxes) fragment of DiLL and the fragment of DiLL without co-structural rules. The reader have to pay attention that the set of our DiLL-ps is a proper subset of all proof-structures that can be generated in DiLL: indeed, in [17, 21, 22] DiLL-ps are defined (inductively on their depth) in such a way that inside a box there is a sum (i.e. a set) of DiLL-ps, not necessarily exactly one DiLL-ps as in our syntax (see Definition 14). However, all proof structures that can be generated in DiLL_0 and MELL are particular cases of our DiLL-ps. Our interest for DiLL-ps is just to have an unitary syntax subsuming both MELL-proof structures (MELL-ps for short) and DiLL_0-proof structures (DiLL_0-ps for short): for technical reasons, this approach allows for a more simple definition of Taylor expansion of a MELL-ps, especially when making the product of several copies of the content of a box (see Definition 22).

Unlike the usual syntaxes of LL- or DiLL-proof structures (see for example [11, 14, 17, 3, 22]), in our syntax there is no explicit (inductive) constructor for boxes: a box in a pps Φ is defined as sub-graph of Φ satisfying some conditions. This more "geometrical" approach was followed for example in [1, 20, 15, 4]. In our syntax we have to reconstruct the boxes in a pps Φ by using some "geometrical" informations coming from Φ, in particular the partial functions auxd_{Φ} (and cutd_{Φ}) play a crucial role to determine the frontier of boxes. The idea is that a promotion cell l in a pps Φ is a "candidate for a box": if the sets $\mathsf{auxd}_\Phi(l)$ and $\mathsf{cutd}_\Phi(l)$ fulfill some conditions (see Definition 11) then one can compute the box associated with l in Φ (see Definition 14).

Definition 9 ((Pre-)order relations on the ports of a pre-proof structure)**.** *Let Φ be a pps.*

The binary relations $<^1_\Phi$ and $\prec^1_\Phi$ on $\mathcal{P}_\Phi$ are defined by:

- *$p <^1_\Phi q$ if there exists $l \in \mathcal{C}_\Phi$ such that $p \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ and $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$;*
- *$p \prec^1_\Phi q$ if either $p <^1_\Phi q$ or there exists $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$ such that $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ and $q \in \mathsf{auxd}_\Phi(l) \cup \mathsf{cutd}_\Phi(l)$.*

The preorder relation $\leq_\Phi$ (resp. $\preccurlyeq_\Phi$) on $\mathcal{P}_\Phi$ is the reflexive-transitive closure of $<^1_\Phi$ (resp. $\prec^1_\Phi$). For all $p, q \in \mathcal{P}_\Phi$, when $p \leq_\Phi q$ (resp. $p \preccurlyeq_\Phi q$) we say that q is straightly above *(resp.* above*) p.*

For all $p, q \in \mathcal{P}_\Phi$, we set: $p <_\Phi q$ (resp. $\prec_\Phi$) iff $p \leq_\Phi q$ (resp. $p \preccurlyeq_\Phi q$) and $p \neq q$.

In a pps Φ, the relations $\leq_\Phi$ and $\preccurlyeq_\Phi$ have a geometrical meaning: for any $p, q \in \mathcal{P}_\Phi$, if $p \leq_\Phi q$ (resp. $p \preccurlyeq_\Phi q$) then in the directed hypergraph $\mathfrak{G}^{\leq}(\Phi)$ (resp. $\mathfrak{G}^{\preccurlyeq}(\Phi)$) defined at page 7 after Definition 1 there is a directed path from q to p not crossing any *ax*- or *cut*-cell.

With respect to $\leq_\Phi$, the conclusions of Φ and the premises of the *cut*-cells of Φ are the minimal elements of $\mathcal{P}_\Phi$. With respect to $\leq_\Phi$ and $\preccurlyeq_\Phi$, the conclusions of the cells of Φ with no premises (i.e. *ax*-, 1- and $\bot$-cells, and !- and ?-cells with no premises) are the maximal elements of $\mathcal{P}_\Phi$.

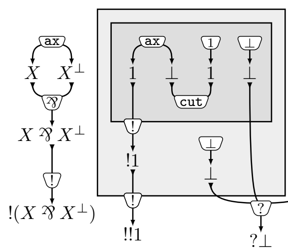
It is immediate to check that $\leq_\Phi \subseteq \preccurlyeq_\Phi$ (and $\leq_\Phi = \preccurlyeq_\Phi$ if Φ is a DiLL_0-ps).

Fact 10. *Let Φ be a pps: $\leq_\Phi$ is an order relation on $\mathcal{P}_\Phi$ and, for any $p, q \in \mathcal{P}_\Phi$, $p <^1_\Phi q$ implies $p \neq q$.*

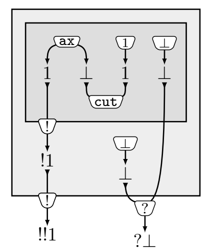
Proof. Let $p, q \in \mathcal{P}_\Phi$. If $p <^1_\Phi q$ then $\mathsf{tp}_\Phi(q)$ is a proper subformula of $\mathsf{tp}_\Phi(p)$, thus $p \neq q$. Hence, if $p \leq_\Phi q$ and $q \leq_\Phi p$ then $p = q$. So, $\leq_\Phi$ is antisymmetric and thus an order, according to Definition 9. □

According to Fact 10, in a pps Φ the (natural top-down) orientation of the hyperedges of $\mathfrak{G}^{\leq}(\Phi)$ induces the order relation $\leq_\Phi$ on the ports of Φ. This means that Φ cannot have "vicious cycles" like for example a cell l such that $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\} \subseteq \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ (i.e., p is both a premise and the conclusion of l), or more generally a sequence $(l_1, \ldots, l_n)$ of cells, for some $n \in \mathbb{N}^+$, such that $\mathsf{P}^{\mathsf{pri}}_\Phi(l_i) = \{p_i\} \subseteq \mathsf{P}^{\mathsf{aux}}_\Phi(l_{i+1})$ for any $1 \leq i \leq n-1$ and $\mathsf{P}^{\mathsf{pri}}_\Phi(l_n) = \{p_n\} \subseteq \mathsf{P}^{\mathsf{aux}}_\Phi(l_1)$. Such a cycle is called "vicious" to distinguish it from the cycles in the so-called correctness graphs (see [20]).

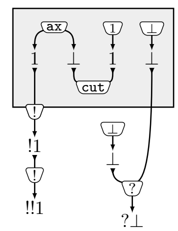
A pps is a very "light" structure and in order to associate with some $l \in \mathcal{C}^{\mathsf{prom}}_\Phi$ where $\mathsf{P}^{\mathsf{aux}}_\phi(l) = \{p_l\}$ a sub-pps, the box of l, some conditions need be satisfied: for example, when Φ is cut-free, if the port q is (not straightly) above p_l, then there exists a unique *aux*-door r of l such that q is straightly above r. In presence of cuts, such an r can be either an *aux*-door of l or the *cut*-door of some $l' \in \mathcal{C}^{\mathsf{prom}}_\Phi$ whose principal port is itself above p_l: in any case, r is above p_l.



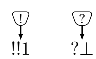
(a) The DiLL-ps S with $\mathcal{C}_S^{\mathsf{box}_0} = \{l\}$.



(b) The box of l.



(c) The content of l.



(d) The border of l.

Figure 2: A DiLL-ps S (where $\mathcal{C}_S^{\mathsf{box}_0} = \{l\}$), with the box of l, the content of l, the border of l.

Definition 11 (Definability of a box)**.** *Let Φ be a pps.*

Let $l \in \mathcal{C}_\Phi^{\mathsf{prom}}$.[5] *We say that* "the box of l is defined in Φ" *or* "l has a box in Φ" *when, for every $q \in \mathcal{P}_\Phi$, if at least one of the followings holds:*

- $\mathsf{prid}_\Phi(l) \preccurlyeq_\Phi q$
- $\mathsf{prid}_\Phi(l) \preccurlyeq_\Phi q'$ *where* $\mathsf{P}_\Phi^{\mathsf{pri}}(l') = \{q, q'\}$ *for some* $l' \in \mathcal{C}_\Phi^{ax}$,

then there exists a unique $r \in \mathcal{P}_\Phi$ such that $r \leq_\Phi q$ and (exactly) one of the following holds:

- $r \in \mathsf{auxd}_\Phi(l) \cup \{\mathsf{prid}_\Phi(l)\}$
- $\mathsf{prid}_\Phi(l) \preccurlyeq_\Phi r$ *and* $r \in \mathsf{cutd}_\Phi(l'')$ *for some* $l'' \in \mathcal{C}_\Phi^{\mathsf{prom}}$.

We set $\mathcal{C}_\Phi^{\mathsf{box}} = \{l \in \mathcal{C}_\Phi^{\mathsf{prom}} \mid l$ has a box in $\Phi\}$, whose elements are the box-cells *of Φ. For every $l \in \mathcal{C}_\Phi^{\mathsf{box}}$, we set* $\mathsf{inbox}_\Phi(l) = \{q \in \mathcal{P}_\Phi \mid \mathsf{prid}_\Phi(l) \preccurlyeq_\Phi q\}$.

Remark 12. If Φ and Ψ are pps and $\varphi \colon \Phi \simeq \Psi$ then, for any $p, q \in \mathcal{P}_\Phi$, one has: $p <_\Phi^1 q$ (resp. $p \prec_\Phi^1 q$) iff $\varphi_\mathcal{P}(p) <_\Phi^1 \varphi_\mathcal{P}(q)$ (resp. $\varphi_\mathcal{P}(p) \prec_\Phi^1 \varphi_\mathcal{P}(q)$); therefore, $p \leq_\Phi q$ (resp. $p \preccurlyeq_\Phi q$) iff $\varphi_\mathcal{P}(p) \leq_\Phi \varphi_\mathcal{P}(q)$ (resp. $\varphi_\mathcal{P}(p) \preccurlyeq_\Phi \varphi_\mathcal{P}(q)$). As a consequence, $\mathsf{im}(\varphi_\mathcal{C} \restriction_{\mathcal{C}_\Phi^{\mathsf{box}}}) = \mathcal{C}_\Psi^{\mathsf{box}}$ (since $\mathsf{im}(\varphi_\mathcal{C} \restriction_{\mathcal{C}_\Phi^{\mathsf{prom}}}) = \mathcal{C}_\Psi^{\mathsf{prom}}$ by Definition 3) and $\overline{\varphi_\mathcal{P}}(\mathsf{inbox}_\Phi(l)) = \mathsf{inbox}_\Psi(\varphi_\mathcal{C}(l))$ for any $l \in \mathcal{C}_\Phi^{\mathsf{box}}$.

Given a pps Φ, in general $\preccurlyeq_\Phi$ is not an order on $\mathcal{P}_\Phi$ because it is not antisymmetric. For instance, take a pps Ψ consisting of a 1-cell whose principal port q is the auxiliary port of a promotion cell l whose principal port is the only auxiliary port p of a ?-cell, with $\mathsf{auxd}_\Psi(l) = \{p\}$: then $p \prec_\Psi q$ and $q \prec_\Psi p$ (and

[5] Remind that $\mathsf{card}(\mathsf{P}_\Phi^{\mathsf{aux}}(l)) = 1$ and $\mathsf{tc}_\Phi(l) = !$, since $l \in \mathcal{C}_\Phi^{\mathsf{prom}}$.

$l \notin \mathcal{C}_{\Psi}^{\mathsf{box}}$). When Φ is a pps such that all the promotion cells are *box*-cells, then $\preccurlyeq_{\Phi}$ is an order relation on $\mathcal{P}_{\Phi}$, according to next Fact 13.2. Nevertheless, the converse does not hold: in the pps Φ in Figure 1(a), the three promotion cells are not *box*-cells but $\preccurlyeq_{\Phi}$ is an order relation on $\mathcal{P}_{\Phi}$.

Fact 13. *Let Φ be a pps.*

1. *For every $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ and every $p, q \in \mathcal{P}_{\Phi}$, if $p <_{\Phi} q$ and $q \in \mathsf{doors}_{\Phi}(l)$, then $p \notin \mathsf{inbox}_{\Phi}(l)$.*

2. *If $\mathcal{C}_{\Phi}^{\mathsf{prom}} = \mathcal{C}_{\Phi}^{\mathsf{box}}$ then $\preccurlyeq_{\Phi}$ is an order relation on $\mathcal{P}_{\Phi}$; in particular, for every $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$, $\mathsf{prid}_{\Phi}(l)$ is the least element of $\mathsf{inbox}_{\Phi}(l)$ with respect to $\preccurlyeq_{\Phi}$.*

3. *If $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ and $l' \in \mathcal{C}_{\Phi}^{ax}$ (resp. $l' \in \mathcal{C}_{\Phi}^{cut}$) then either $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') \subseteq \mathsf{inbox}_{\Phi}(l)$ or $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') \cap \mathsf{inbox}_{\Phi}(l) = \emptyset$ (resp. $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \subseteq \mathsf{inbox}_{\Phi}(l)$ or $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \cap \mathsf{inbox}_{\Phi}(l) = \emptyset$).*

Proof. 1. Suppose $p \in \mathsf{inbox}_{\Phi}(l)$ and $q \in \mathsf{doors}_{\Phi}(l)$: then $p_l \preccurlyeq_{\Phi} p$ where $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$, so (by Definition 11) there exists a unique $r \in \mathcal{P}_{\Phi}$ such that $r \leq_{\Phi} p$ and

$$\text{either } r \in \mathsf{auxd}_{\Phi}(l) \cup \{p_l\}, \text{ or } p_l \preccurlyeq_{\Phi} r \text{ and } r \in \mathsf{cutd}_{\Phi}(l') \text{ for some } l' \in \mathcal{C}_{\Phi}^{\mathsf{prom}}. \tag{3}$$

From the uniqueness of r it follows that $p \not<_{\Phi} q$, otherwise $r <_{\Phi} q$ where $p_l \preccurlyeq_{\Phi} q$ and q also fulfils condition (3).

2. The second part of the statement is an immediate consequence of the first part. In order to prove that $\preccurlyeq_{\Phi}$ is an order relation on $\mathcal{P}_{\Phi}$, we have just to show that $\preccurlyeq_{\Phi}$ is antisymmetric. Suppose not: then there would be $p, q \in \mathcal{P}_{\Phi}$ such that $p \prec_{\Phi} q$ and $q \prec_{\Phi} p$, thus (since it is impossible that $p <_{\Phi} q$ and $q <_{\Phi} p$, see Fact 10) there would be $l \in \mathcal{C}_{\Phi}^{\mathsf{prom}} = \mathcal{C}_{\Phi}^{\mathsf{box}}$ and $p_l, q_l \in \mathcal{P}_{\Phi}$ such that $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$, $q_l \in \mathsf{auxd}_{\Phi}(l) \cup \mathsf{cutd}_{\Phi}(l)$ and either $p \preccurlyeq_{\Phi} p_l$ and $q_l \preccurlyeq_{\Phi} q$, or $q \preccurlyeq_{\Phi} p_l$ and $q_l \preccurlyeq_{\Phi} p$.

   In the case where $p \preccurlyeq_{\Phi} p_l$ and $q_l \preccurlyeq_{\Phi} q$, let $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{p'\}$: then, $p' <_{\Phi}^{1} p_l$ and, from $q \prec_{\Phi} p$ and the facts that l has a unique principal port and $q \prec_{\Phi} p_l \notin \mathcal{A}uxdoors_{\Phi} \cup \mathcal{C}utdoors_{\Phi}$, it follows that $q \preccurlyeq_{\Phi} p'$, hence $p' \in \mathsf{inbox}_{\Phi}(l)$ because $p_l \prec_{\Phi} q_l \preccurlyeq_{\Phi} q$, but $p' <_{\Phi} p_l$, that is impossible by Fact 13.1 since $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$.

   The case where $q \preccurlyeq_{\Phi} p_l$ and $q_l \preccurlyeq_{\Phi} p$ is analogous.

3. For $l' \in \mathcal{C}_{\Phi}^{ax}$, let $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') = \{q, q'\}$ and $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$. If $q' \in \mathsf{inbox}_{\Phi}(l)$ then (by Definition 11) there exists $r \in \mathcal{P}_{\Phi}$ such that $r \leq_{\Phi} q$ and $p_l \preccurlyeq_{\Phi} r$, thus $p_l \preccurlyeq_{\Phi} q$ and hence $q \in \mathsf{inbox}_{\Phi}(l)$.

   For $l' \in \mathcal{C}_{\Phi}^{cut}$, let $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l') = \{q, q'\}$ and $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$. If $q' \in \mathsf{inbox}_{\Phi}(l)$ then (since there is no $q'' \in \mathcal{P}_{\Phi}$ such that $q'' \leq_{\Phi} q, q'$) there exists $l'' \in \mathcal{C}_{\Phi}^{\mathsf{prom}}$ with $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l'') = \{p''\}$ such that $\mathsf{cutd}_{\Phi}(l'') = \{q, q'\}$ and $p_l \preccurlyeq_{\Phi} p''$, thus $p_l \preccurlyeq_{\Phi} q$ and hence $q \in \mathsf{inbox}_{\Phi}(l)$. □

Facts 13.1-2 say that in a pps Φ such that $\mathcal{C}_{\Phi}^{\mathsf{box}} = \mathcal{C}_{\Phi}^{\mathsf{prom}}$, for every $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$, the doors of l in Φ (i.e. the premise p_l of l and the ports in $\mathsf{auxd}_{\Phi}(l)$ and $\mathsf{cutd}_{\Phi}(l)$) determine the "boundary" of the box of l in Φ, and the content of the box of l in Φ is "all that is above" p_l in the sense of $\preccurlyeq_{\Phi}$ (for technical reasons, the cells whose premises are among the doors of l are also considered to be in the content of the box of l).

Fact 13.3 means that in a pps Φ the *cut*- and *ax*-cells cannot cross the "boundary" of a box in Φ.

Definition 14 (Box, content and border of a box). *Let Φ be pps and $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ with $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$.*

The box of l in Φ *(resp.* the content of the box of l in Φ*) is the pps Φ_l defined by:*

- $\mathcal{P}_{\Phi_l} = \mathsf{inbox}_{\Phi}(l) \cup \bigcup \{\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') \mid l' \in \mathcal{C}_{\Phi}$ *and* $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \cap (\{p_l\} \cup \mathsf{auxd}_{\Phi}(l)) \neq \emptyset\}$*;*
- $\mathcal{C}_{\Phi_l} = \{l' \in \mathcal{C}_{\Phi} \mid (\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') \cup \mathsf{P}_{\Phi}^{\mathsf{aux}}(l')) \cap \mathsf{inbox}_{\Phi}(l) \neq \emptyset\}$ *and* $\mathsf{tc}_{\Phi_l} = \mathsf{tc}_{\Phi}\restriction_{\mathcal{C}_{\Phi_l}}$*;*
- $\mathsf{P}_{\Phi_l}^{\mathsf{pri}} = \mathsf{P}_{\Phi}^{\mathsf{pri}}\restriction_{\mathcal{C}_{\Phi_l}}$, $\mathsf{P}_{\Phi_l}^{\mathsf{aux}}(l') = \mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \cap \mathcal{P}_{\Phi_l}$ *for any* $l' \in \mathcal{C}_{\Phi_l}$, $\mathsf{P}_{\Phi_l}^{\mathsf{left}} = \mathsf{P}_{\Phi}^{\mathsf{left}}\restriction_{\mathcal{C}_{\Phi_l}^{\otimes,\parr}}$ *and* $\mathsf{tp}_{\Phi_l} = \mathsf{tp}_{\Phi}\restriction_{\mathcal{P}_{\Phi_l}}$*;*
- $\mathsf{auxd}_{\Phi_l} = \mathsf{auxd}_{\Phi}\restriction_{\mathcal{C}_{\Phi_l}}$ *(resp.* $\mathsf{auxd}_{\Phi_l} = \mathsf{auxd}_{\Phi}\restriction_{\mathcal{C}_{\Phi_l} \smallsetminus \{l\}}$*) and* $\mathsf{cutd}_{\Phi_l} = \mathsf{cutd}_{\Phi}\restriction_{\mathcal{C}_{\Phi_l}}$ *(resp.* $\mathsf{cutd}_{\Phi_l} = \mathsf{cutd}_{\Phi}\restriction_{\mathcal{C}_{\Phi_l} \smallsetminus \{l\}}$*).*

If Φ_l is the box of l in Φ, we say that p_l is the pri-door of Φ_l*, the elements of* $\mathsf{auxd}_{\Phi}(l)$ *are the* aux-doors of Φ_l*, and the elements of* $\mathsf{cutd}_{\Phi}(l)$ *are the* cut-doors of Φ_l*.*

The border of the box of l in Φ *is the* DiLL_0*-ps Φ_0 defined by:*

- $\mathcal{C}_{\Phi_0} = \{l' \in \mathcal{C}_{\Phi} \mid \mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \cap (\{p_l\} \cup \mathsf{auxd}_{\Phi}(l)) \neq \emptyset\}$ *and* $\mathcal{P}_{\Phi_0} = \bigcup\{\mathsf{P}_{\Phi}^{\mathsf{pri}}(l') \mid l' \in \mathcal{C}_{\Phi_0}\}$*;*
- $\mathsf{tc}_{\Phi_0} = \mathsf{tc}_{\Phi}\restriction_{\mathcal{C}_{\Phi_0}}$, $\mathsf{P}_{\Phi_0}^{\mathsf{pri}} = \mathsf{P}_{\Phi}^{\mathsf{pri}}\restriction_{\mathcal{C}_{\Phi_0}}$, $\mathsf{P}_{\Phi_0}^{\mathsf{aux}}(l) = \emptyset$ *for any* $l \in \mathcal{C}_{\Phi_0}$*, and* $\mathsf{tp}_{\Phi_0} = \mathsf{tp}_{\Phi}\restriction_{\mathcal{P}_{\Phi_0}}$*;*
- auxd_{Φ_0}, cutd_{Φ_0} *and* $\mathsf{P}_{\Phi_0}^{\mathsf{left}}$ *are the empty function.*

We set $\mathcal{B}ox_{\Phi} = \bigcup_{l \in \mathcal{C}_{\Phi}^{\mathsf{box}}}\{\Phi_l \mid \Phi_l$ is the box of l in $\Phi\}$, whose elements are the boxes in Φ*.*

Remark 15. Let Φ be a pps, let $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ with $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$, and let Φ_l be the box or the content of the box of l in Φ. The terminal cells of Φ_l are the cells of Φ having at least one auxiliary port among the doors of l, i.e., $\mathcal{C}_{\Phi_l}^{\mathsf{free}} = \{l' \in \mathcal{C}_{\Phi} \mid \mathsf{P}_{\Phi}^{\mathsf{aux}}(l') \cap (\{p_l\} \cup \mathsf{auxd}_{\Phi}(l)) \neq \emptyset\}$; thus a terminal cell of Φ_l is either l (which is a !-cell) or a ?-cell with at least one premise. More precisely, in case Φ_l is the box of l one has $l \in \mathcal{C}_{\Phi_l}^{\mathsf{free}} \cap \mathcal{C}_{\Phi_l}^{\mathsf{box}}$, $\mathcal{C}_{\Phi_l}^{\mathsf{free}} \smallsetminus \{l\} \subseteq \mathcal{C}_{\Phi_l}^{?}$, and Φ_l is a sub-pps of Φ; whereas in case Φ_l is the content of the box of l one has $l \in \mathcal{C}_{\Phi_l}^{\mathsf{free}} \cap (\mathcal{C}_{\Phi_l}^{!} \smallsetminus \mathcal{C}_{\Phi_l}^{\mathsf{prom}})$, $\mathcal{C}_{\Phi_l}^{\mathsf{free}} \smallsetminus \{l\} \subseteq \mathcal{C}_{\Phi_l}^{?}$, and Φ_l is not a sub-pps of Φ. Notice also that $\mathsf{P}_{\Phi_l}^{\mathsf{aux}}(l') = \mathsf{P}_{\Phi}^{\mathsf{aux}}(l')$ for any $l' \in (\mathcal{C}_{\Phi_l} \smallsetminus \mathcal{C}_{\Phi_l}^{\mathsf{free}}) \cup \{l\}$ and $\mathcal{C}_{\Phi_l}^{t} \subseteq \mathcal{C}_{\Phi}^{t}$ for any $t \in \mathcal{L}_{\mathsf{MELL}}$.

Note that if Φ is a pps and $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ then the box, the content of the box and the border of the box of l in Φ are interfaced. The only difference between the box and the content of the box of l in Φ is that l is a promotion cell in the box of l in Φ, whereas it is a !-cell and not a promotion one in the content of the box of l in Φ. The border of the box of l in Φ consists of the terminal cells of the (content of the) box of l in Φ with their principal ports and their labels but with no auxiliary port.

Definition 16 (Proof structure, depth)**.** *A pps R is a* DiLL-proof structure *(*DiLL-ps *for short) if:*

- $\mathcal{C}_R^{\mathsf{prom}} = \mathcal{C}_R^{\mathsf{box}}$ *and*
- *(*nesting condition*) for all $l, l' \in \mathcal{C}_R^{\mathsf{box}}$, if $l \neq l'$ then either* $\mathsf{inbox}_R(l) \subsetneq \mathsf{inbox}_R(l')$ *or* $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$ *or* $\mathsf{inbox}_R(l) \cap \mathsf{inbox}_R(l') = \emptyset$*.*

We say that R is a MELL-proof structure *(*MELL-ps *for short) if R is a* DiLL*-ps and $\mathcal{C}_R^{!} = \mathcal{C}_R^{\mathsf{box}}$.*

The set of DiLL*-ps (resp.* MELL*-ps) is denoted by $\mathbf{PS}_{\mathsf{DiLL}}$ (resp. $\mathbf{PS}_{\mathsf{MELL}}$) and its elements are denoted by R, S, etc.*

Let R be a DiLL*-ps. The depth of ports and cells in R is defined as follows:*

- *for every $p \in \mathcal{P}_R$, the* depth of p in R *is* $\mathsf{depth}_R(p) = \mathsf{card}(\{l \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l)\})$*;*
- *for every $l \in \mathcal{C}_R \smallsetminus \mathcal{C}_R^{ax,cut}$, the* depth of l in R *is* $\mathsf{depth}_R(l) = \mathsf{depth}_R(q)$ *where* $\mathsf{P}_R^{\mathsf{pri}}(l) = \{q\}$*;*
- *for every $l \in \mathcal{C}_R^{ax}$ (resp. $l \in \mathcal{C}_R^{cut}$), the* depth of l in R *is* $\mathsf{depth}_R(l) = \mathsf{depth}_R(q) = \mathsf{depth}_R(q')$ *where* $\mathsf{P}_R^{\mathsf{pri}}(l) = \{q, q'\}$ *(resp.* $\mathsf{P}_R^{\mathsf{aux}}(l) = \{q, q'\}$*).*[6]

We set $\mathcal{C}_R^{\mathsf{box}_0} = \{l \in \mathcal{C}_R^{\mathsf{box}} \mid \mathsf{depth}_R(l) = 0\}$*. The* depth of R *is* $\mathsf{depth}(R) = \sup\{\mathsf{depth}_R(p) \in \mathbb{N} \mid p \in \mathcal{P}_R\}$*.*[7]

For any $p \in \mathcal{P}_R$, we set $\mathsf{boxesof}_R(p) = (l_1, \ldots, l_m)$ *where* $\{l_1, \ldots, l_m\} = \{l \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l)\}$ *and* $\mathsf{inbox}_R(l_j) \subsetneq \mathsf{inbox}_R(l_i)$ *for any* $1 \leq i < j \leq m$*. Similarly, for any $l \in \mathcal{C}_R \smallsetminus \mathcal{C}_R^{cut}$ (resp. $l \in \mathcal{C}_R^{cut}$), we set* $\mathsf{boxesof}_R(l) = (l_1, \ldots, l_m)$ *where* $\{l_1, \ldots, l_m\} = \{l' \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l')$ *for any* $p \in \mathsf{P}_R^{\mathsf{pri}}(l)\}$ *(resp.* $\{l_1, \ldots, l_m\} = \{l' \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l')$ *for any* $p \in \mathsf{P}_R^{\mathsf{aux}}(l)\}$*) and* $\mathsf{inbox}_R(l_j) \subsetneq \mathsf{inbox}_R(l_i)$ *for any* $1 \leq i < j \leq m$*.*[8]

For all $l, l' \in \mathcal{C}_R^{\mathsf{box}}$ with $\mathsf{prid}_R(l) = p$ *and* $\mathsf{prid}_R(l') = p'$*, we set* $l \preccurlyeq_{\mathcal{C}_R^{\mathsf{box}}} l'$ *(resp.* $l \prec_{\mathcal{C}_R^{\mathsf{box}}} l'$*) iff* $p \preccurlyeq_R p'$ *(resp.* $p \prec_R p'$*).*

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[6] For any $l \in \mathcal{C}_R$, $\mathsf{depth}_R(l)$ is well-defined by Fact 13.3 and because all cells but *ax*- and *cut*-cells have exactly one conclusion.

[7] Recall that for every $\mathcal{A} \subseteq \mathbb{N}$, if $\mathcal{A} = \emptyset$ then $\sup(\mathcal{A}) = 0$. Hence, the depth of the empty-pps (which is a DiLL-ps) is 0.

[8] For every $p \in \mathcal{P}_R$ and $l \in \mathcal{C}_R$, $\mathsf{boxesof}_R(p)$ and $\mathsf{boxesof}_R(l)$ are well-defined by the nesting condition and Fact 13.3.

Observe that every DiLL_0-ps S is a DiLL-ps with depth 0, because $\mathcal{C}_S^{\mathsf{box}} = \mathcal{C}_S^{\mathsf{prom}} = \emptyset$. The DiLL_0-ps (resp. MELL-ps) R consisting of a 1-cell whose principal port is the auxiliary port of a !-cell l and such that auxd_R is the empty function (resp. $\mathsf{auxd}_R(l) = \emptyset$), is not a MELL-ps (resp. not a DiLL_0-ps). Every pps Ψ such that $\mathcal{C}_\Psi^! = \emptyset$ is a MELL-ps and a DiLL_0-ps.

Let R be a DiLL-ps and let $l, l' \in \mathcal{C}_R^{\mathsf{box}}$ be such that $l \neq l'$: if $\mathsf{auxd}_R(l) \cap \mathsf{auxd}_R(l') \neq \emptyset$, then either $\mathsf{inbox}_R(l) \subsetneq \mathsf{inbox}_R(l')$ or $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$, by the nesting condition (i.e. an *aux*-door of some box B in Φ might be an *aux*-door of another box B' in R but then one between B and B' is nested in the other one); if $p \in \mathsf{cutd}_R(l)$ and R_l is the content of the box of l in R, then $p \in \mathcal{P}_{R_l}$ and $\mathsf{depth}_{R_l}(p) = 0$.

According to Fact 13.2, given a DiLL-ps R, the conclusions of R and the premises of the *cut*-cells with depth 0 in R are the minimal elements of $\mathcal{P}_R$ with respect to $\preccurlyeq_R$.

Given a DiLL-ps R and $p \in \mathcal{P}_R$ and $l \in \mathcal{C}_R$, $\mathsf{boxesof}_R(p)$ and $\mathsf{boxesof}_R(l)$ give the sequence of nested boxes in R containing p and l, respectively, thus one has $\mathsf{length}(\mathsf{boxesof}_R(p)) = \mathsf{depth}_R(p)$ and $\mathsf{length}(\mathsf{boxesof}_R(l)) = \mathsf{depth}_R(l)$.

As usual in the literature, given DiLL-ps R and $l \in \mathcal{C}_R^{\mathsf{box}}$, the box R_l of l in R is often graphically depicted (instead of using dotted arrows to pick out its *aux*- and *cut*-doors) by a rectangular frame containing all ports in $\mathsf{inbox}_R(l)$, whereas the free ports and free cells of R_l are outside this frame.

Remark 17. Let R be a MELL-ps (resp. DiLL-ps) and $l \in \mathcal{C}_R^{\mathsf{box}}$: the box B of l in R is a MELL-ps (resp. DiLL-ps) while the content K_B of B is a DiLL-ps and not a MELL-ps, moreover $\mathsf{depth}(B) = \mathsf{depth}(K_B) + 1$.

Fact 18.

1. *If R is a* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps) and Ψ is a sub-pps of R, then Ψ is a* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps).*

2. *Let $R, S_1, \ldots, S_n$ be some* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps) and, for every $1 \leq i \leq n$, let R_i be a sub-pps of R such that the conditions of Definition 7 are fulfilled: then, $R[S_1/R_1, \ldots, S_n/R_n]$ is a* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps).*

3. *If R is a* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps) and Ψ is a pps such that $R \simeq \Psi$, then Ψ is a* DiLL*-ps (resp.* DiLL_0*-ps;* MELL*-ps).*

4. *If R is a* DiLL*-ps, then $\preccurlyeq_{\mathcal{C}_R^{\mathsf{box}}}$ is a tree-order on $\mathcal{C}_R^{\mathsf{box}}$ and, for every $l, l' \in \mathcal{C}_R^{\mathsf{box}}$, $l \prec_{\mathcal{C}_R^{\mathsf{box}}} l'$ if and only if $\mathsf{inbox}_R(l') \subsetneq \mathsf{inbox}_R(l)$.*

Proof. The proof of Fact 18.1 is straightforward and left to the reader. Fact 18.2 is due to Fact 18.1. Fact 18.3 is a consequence of Remark 12. Fact 18.4 follows from Fact 13.2 and the nesting condition. □

Facts 18.1-2 mean that the set of DiLL-ps (resp. DiLL_0-ps; MELL-ps) is closed under sub-pps and substitution. In particular, Fact 18.1 allows us to define a *sub-*DiLL*-ps* (resp. *sub-*DiLL_0*-ps*; *sub-*MELL*-ps*) of a DiLL-ps (resp. DiLL_0-ps;

MELL-ps) R as a sub-pps of R. Remark 12 and Fact 18.3 say that, given two MELL-ps or DiLL-ps, in order to prove that they represent the "same object" we have to prove only that they are isomorphic according to Definition 3, because boxes are preserved by such an isomorphism. According to Fact 18.4, in a DiLL-ps R, $\preccurlyeq_{\mathcal{C}_R^{\mathsf{box}}}$ represents the tree-order of the boxes in R induced by the nesting condition: whenever two *box*-cells have a sup with respect to $\preccurlyeq_{\mathcal{C}_R^{\mathsf{box}}}$, then they are $\preccurlyeq_{\mathcal{C}_R^{\mathsf{box}}}$-comparable.

3. Computing the Taylor expansion of a DiLL-proof structure

The Taylor expansion of a MELL-ps or more generally a DiLL-ps R, is a set $\mathcal{T}_R$ of DiLL_0-ps: roughly speaking, each element of $\mathcal{T}_R$ is obtained from R by replacing each box B in R with n_B copies of its content (for any $n_B \in \mathbb{N}$), recursively on the depth of R. In [12] we introduced proto-nets, an abstract object saying recursively how many copies to take for each box, thus we can build any element of the labelled Taylor expansion $\mathcal{T}_R$ of R from one element of the proto-Taylor expansion $\mathcal{T}_R^{\mathsf{proto}}$ of R, $\mathcal{T}_R^{\mathsf{proto}}$ being the set of proto-nets associated with R: one of the advantages of our pointwise approach with respect to the other ones such as [15, 18] is that it makes easier to define the correspondence between ports and cells of any $\rho \in \mathcal{T}_R$ and ports and cells of R, to recognize in which box (if any) the correspondents in R of ports and cells of ρ are, and to differentiate the various copies in ρ of the content of a same box in R (see Lemma 28). For this purpose, $\rho \in \mathcal{T}_R$ is a well-named DiLL_0-ps and a port or cell of ρ is of the shape (c, a), where c is the corresponding port of R and the finite sequence a has to be intended as a list of indexes which keeps track of all boxes containing c and says in which copy of the content of each box (c, a) is. These indexes are a syntactic counterpart of the one used in the definition of k-experiment of *PLPS* in [4, Def. 35].

In the original formulation [9] the Taylor expansion of a λ-term is a (usually infinite) linear combination of resource λ-terms with scalars in some semiring. With respect to the results achieved in our work, scalars play no role, hence we do not tackle coefficients issue, and we will define the Taylor expansion of a DiLL-ps as a set (and not a linear combination) of DiLL_0-ps, as in [15, 18].

Definition 19 (Proto-net)**.** Proto-nets *are defined by induction as follows: if $n, k_1, \ldots, k_n \in \mathbb{N}$ and $\alpha_1^i, \ldots, \alpha_{k_i}^i$ are proto-nets for any $1 \leq i \leq n$, then $(\langle \alpha_1^1, \ldots, \alpha_{k_1}^1 \rangle, \ldots, \langle \alpha_1^n, \ldots, \alpha_{k_n}^n \rangle)$ is a proto-net.*[9]

Definition 20 (Representative of a DiLL-proof structure)**.** *A* representative of a DiLL-ps *R is a pair $R' = (\mathsf{box}_{R'}^0, \mathsf{rep}_{R'}^0)$ defined by induction on $\mathsf{depth}(R) \in \mathbb{N}$, where $\mathsf{box}_{R'}^0$ is an enumeration of $\mathcal{C}_R^{\mathsf{box}_0}$ and $\mathsf{rep}_{R'}^0$ is a map associating with any $l \in \mathcal{C}_R^{\mathsf{box}_0}$ a representative of the content of the box of l in R.*

[9] In particular, $()$ is a proto-net (take $n = 0$). More generally, $(\langle\rangle, \ldots, \langle\rangle)$ with $n \in \mathbb{N}$ times $\langle\rangle$ is a proto-net (take $k_1 = \ldots = k_n = 0$): these are the base cases of the inductive definition.

Intuitively, to take a representative of a DiLL-ps R amounts to fix an order on the promotion cells of R, recursively on the depth of R. This order has nothing to do with the left-right order of the promotion cells of R when R is depicted or that might be inherited by an order in the conclusions of R. In general, a DiLL-ps R has several representatives: suppose for example that R is the DiLL-ps consisting of two promotion cells whose principal ports are auxiliary ports of a terminal ?-cell; the choice of the first promotion cell is then arbitrary (since the premises of a ?-cell are unordered) and two distinct representatives of R exist.

Definition 21 (Proto-Taylor expansion of a representative of a DiLL-ps)**.** *Let $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ be a representative of a DiLL-ps R, where $\mathsf{box}^0_{R'} = (l_1, \ldots, l_n)$, with $n \in \mathbb{N}$. The* proto-Taylor expansion $\mathcal{T}^{\mathsf{proto}}_{R'}$ of R' *is the following set of proto-nets, defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*:*

$$\mathcal{T}^{\mathsf{proto}}_{R'} = \big\{(\langle\alpha^1_1, \ldots, \alpha^1_{k_1}\rangle, \ldots, \langle\alpha^n_1, \ldots, \alpha^n_{k_n}\rangle) \mid k_i \in \mathbb{N} \textit{ and } \alpha^i_1, \ldots, \alpha^i_{k_i} \in \mathcal{T}^{\mathsf{proto}}_{\mathsf{rep}^0_{R'}(l_i)}, \textit{ for all } 1 \le i \le n\big\}.$$

Notice that while the proto-Taylor expansion of some representative of a DiLL-ps R depends on the representative of R chosen, this is not the case for the labelled Taylor expansion of R: this will be fixed by Proposition 23 and Definition 24.

Given a (empty-named) DiLL-ps R, an element of the proto-Taylor expansion of a representative $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ of R allows one to build an element of the labelled Taylor expansion of R', as stated precisely in next Definition 22, in such a way that sequences of the shape $(\cdot)$ and $\langle\cdot\rangle$ play a completely different role: if $\alpha = (a_1, \ldots, a_n) \in \mathcal{T}^{\mathsf{proto}}_{R'}$ with $n \in \mathbb{N}$ and, for any $1 \le i \le n$, $a_i = \langle\alpha^i_1, \ldots, \alpha^i_{k_i}\rangle$ for some $k_i \in \mathbb{N}$, then R has exactly n *box*-cells with depth 0 and for the i-th *box*-cell l_i with depth 0, according to the order fixed by $\mathsf{box}^0_{R'}$, a_i asks for taking k_i copies of (the representative according to $\mathsf{rep}^0_{R'}$ of) the content of the box of l_i in R.

Definition 22 (Labelled Taylor expansion of a representative of a empty–named DiLL-proof structure)**.** *Let R be an empty-named DiLL-ps where $\mathcal{C}^{\mathsf{box}_0}_R = \{(l_1, ()), \ldots, (l_n, ())\}$ for some $n \in \mathbb{N}$ and some pairwise distinct $l_1, \ldots, l_n$; for any $1 \le i \le n$, let R_i (resp. R^0_i) be the box (resp. the border of the box) of l_i in R.*[10] *Let $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ be a representative of R, where $\mathsf{box}^0_{R'} = \big((l_1, ()), \ldots, (l_n, ())\big)$.*

We define, by induction on $\mathsf{depth}(R)$, the function $\tau_{R'} \colon \mathcal{T}^{\mathsf{proto}}_{R'} \to \mathbf{PS}_{\mathsf{DiLL}_0}$ such that with $\alpha = (\langle\alpha^1_1, \ldots, \alpha^1_{k_1}\rangle, \ldots, \langle\alpha^n_1, \ldots, \alpha^n_{k_n}\rangle)$, where $k_1, \ldots, k_n \in \mathbb{N}$, is associated the well-named DiLL_0-ps $\tau_{R'}(\alpha)$ which is interfaced with R and such that for all $\alpha, \beta \in \mathcal{T}^{\mathsf{proto}}_{R'}$ one has that $\tau_{R'}(\alpha)$ and $\tau_{R'}(\beta)$ are strongly interfaced; for all $1 \le i \le n$ and all $1 \le j \le k_i$, let ρ^j_i be the j-th l_i-copy of $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$

[10] By definition of $\mathcal{C}^{\mathsf{box}_0}_R$ and the nesting condition (see Definition 16), $l_i \neq l_{i'}$ entails $\mathcal{P}_{R_i} \cap \mathcal{P}_{R_{i'}} = \emptyset = \mathcal{C}_{R_i} \cap \mathcal{C}_{R_{i'}}$ for any $1 \le i \neq i' \le n$.

and let[11]

$$\rho_i = \begin{cases} \prod_{j=1}^{k_i} \rho_i^j & \textit{if } k_i > 0 \\ R_i^0 & \textit{otherwise.} \end{cases} \qquad \textit{We then set: } \tau_{R'}(\alpha) = R[\rho_1/R_1, \dots, \rho_n/R_n].$$

The labelled Taylor expansion of R' *is* $\mathcal{T}_{R'} = \{\tau_{R'}(\alpha) \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \alpha \in \mathcal{T}_{R'}^{\mathsf{proto}}\}$.

The next proposition (whose proof can be bound in [12, Fact 25 and Prop. 26]) allows us to define the labelled Taylor expansion of a DiLL-ps R *independently of the representative* of R chosen (see Definition 24).

Proposition 23. *Let R' and R'' be some representatives of a empty-named* DiLL*-ps R. Then $\mathcal{T}_{R'} = \mathcal{T}_{R''}$.*

Proof. (Sketch) We can define a "recursive permutation" p inducing two functions (both denoted by p) such that: $\mathsf{p}(R') = R''$ and, for any $\alpha \in \mathcal{T}_{R'}^{\mathsf{proto}}$, $\mathsf{p}(\alpha) \in \mathcal{T}_{R''}^{\mathsf{proto}}$ and $\tau_{R'}(\alpha) = \tau_{R''}(\mathsf{p}(\alpha))$. □

Definition 24 (Labelled Taylor expansion of a DiLL-proof structure)**.** *Let R be a* DiLL*-ps and R_ε be the empty-renaming of R. The* labelled Taylor expansion *of R is $\mathcal{T}_R = \mathcal{T}_{R'_\varepsilon}$ where R'_ε is any representative of R_ε.*

For every DiLL-ps R, its labelled Taylor expansion $\mathcal{T}_R$ is well defined thanks to Proposition 23.

Remark 25. If R is a DiLL-ps and $\rho \in \mathcal{T}_R$ then R and ρ are not (weakly) interfaced strictly speaking, anyway there is a one-to-one correspondence between $\mathcal{P}_R^{\mathsf{free}}$ and $\mathcal{P}_\rho^{\mathsf{free}}$ (resp. $\mathcal{C}_R^{\mathsf{free}}$ and $\mathcal{C}_\rho^{\mathsf{free}}$), since $p \in \mathcal{P}_R^{\mathsf{free}}$ iff $(p, ()) \in \mathcal{P}_\rho^{\mathsf{free}}$ (resp. $l \in \mathcal{C}_R^{\mathsf{free}}$ iff $(l, ()) \in \mathcal{C}_\rho^{\mathsf{free}}$); moreover $\mathsf{tp}_R(p) = \mathsf{tp}_\rho((p, ()))$ for any $p \in \mathcal{P}_R$ (resp. $\mathsf{tc}_R(l) = \mathsf{tc}_\rho((l, ()))$ for any $l \in \mathcal{C}_R$), by Lemma 28.3. One can say that R and ρ are "morally interfaced".

Example 26. Let R be the MELL-ps as in Figure 3(a), R_ε be its empty-renaming (Figure 3(b)), $\alpha = (\langle(\langle()\rangle), (\langle(), ()\rangle)\rangle)$ and $\beta = (\langle(\langle(), ()\rangle), (\langle()\rangle)\rangle)$: obviously, there is only one representative R'_ε of R_ε, and $\alpha, \beta \in \mathcal{T}_{R'_\varepsilon}^{\mathsf{proto}}$ with $\alpha \neq \beta$, $\tau_{R'_\varepsilon}(\alpha) \neq \tau_{R'_\varepsilon}(\beta)$ (indeed, $(p, (l, 2) \cdot (k, 2)) \in \mathcal{P}_{\tau_{R'_\varepsilon}(\alpha)} \smallsetminus \mathcal{P}_{\tau_{R'_\varepsilon}(\beta)}$, see Figures 3(c)-3(d)) but $\tau_{R'_\varepsilon}(\alpha) \simeq \tau_{R'_\varepsilon}(\beta)$ (and $\tau_{R'_\varepsilon}(\alpha), \tau_{R'_\varepsilon}(\beta) \in \mathcal{T}_R$).

Example 26 shows that the labelled Taylor expansion of a DiLL-ps may contain several elements which are "morally identical" and differ each other only for the name of their ports and cells. Furthermore, from $\rho \in \mathcal{T}_R$ for some DiLL-ps

[11] See Definitions 2 (interface), 5 (product), 7 (substitution) and 8 (l_i-copy). Note that, for any $1 \leq i \leq n$ and $1 \leq j \neq h \leq k_i$, one has that: $\mathsf{rep}^0_{R'}((l_i, ()))$ is a representative of the content R_{l_i} of the box of l_i in R, with $\mathsf{depth}(R_{l_i}) < \mathsf{depth}(R)$; $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$ is well-named by induction hypothesis; ρ_i^j and ρ_i^h are strongly interfaced; R_i, R_i^0, $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$ and $\prod_{j=1}^{k_i} \rho_i^j$ are interfaced.

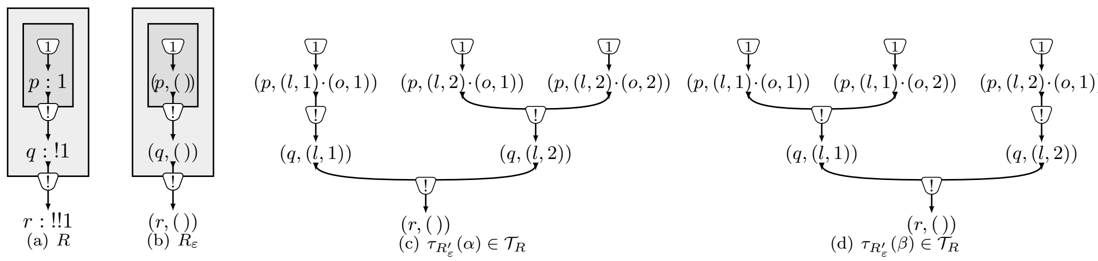


Figure 3: A MELL-ps R (Figure 3(a), where o and l are the internal and external *box*-cells, respectively), its empty-renaming R_ε (Figure 3(b)) having only one representative R'_ε, and two different but isomorphic elements $\tau_{R'_\varepsilon}(\alpha)$ (Figure 3(c)) and $\tau_{R'_\varepsilon}(\beta)$ (Figure 3(d)) of $\mathcal{T}_R$, where $\alpha = \big(\langle\langle(\langle\langle()\rangle\rangle, (\langle\langle(), ()\rangle\rangle)\rangle\rangle\big)$ and $\beta = \big(\langle\langle(\langle\langle(), ()\rangle\rangle, (\langle\langle()\rangle\rangle)\rangle\rangle\big)$. In Figures 3(b),3(c),3(d) the type of the ports are omitted.

R and $\sigma \simeq \rho$, it does not follow that $\sigma \in \mathcal{T}_R$ (and there may be a DiLL-ps $S \not\simeq R$ with $\sigma \in \mathcal{T}_S$). This means that all the informations about R available in ρ thanks to the names of its ports and cells (see next Lemmas 28-29) are lost in σ, though ρ and σ "morally" represent the same object.

The definitions of Taylor expansion of a MELL-ps in [15, Def. 9] and [18, Def. 5] forget all the information encoded in our labelled Taylor expansion, so they correspond to the following definition.

Definition 27 (Taylor expansion)**.** *Let R be a* DiLL*-ps. The* Taylor expansion of R *is* $\mathcal{T}_R^{\simeq} = \big\{\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\} \mid \rho \in \mathcal{T}_R\big\}$.

For every DiLL-ps R, the binary relation $\approx_R$ on $\mathbf{PS}_{\mathsf{DiLL}_0}$defined by "$\tau \approx_R \tau'$ iff there is $\rho \in \mathcal{T}_R$ such that $\tau \simeq \rho \simeq \tau'$" is a partial equivalence, and, for any $\rho \in \mathcal{T}_R$, $\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\}$ is a partial equivalence class on $\mathbf{PS}_{\mathsf{DiLL}_0}$ modulo $\approx_R$.

Next Lemmas 28-29 show how informative our definition of labelled Taylor expansion of a DiLL-ps is. If R is a DiLL-ps and $\rho \in \mathcal{T}_R$ then, for any port or cell c of ρ we can individuate the corresponding port or cell c_R in R (this correspondence $(\cdot)_R$ is not necessarily injective nor surjective). Conversely, for any port or cell c of R, there is at least one corresponding port or cell c_ρ of ρ (in such a way that $(c_\rho)_R = c$ for any port or cell c of R), provided that ρ takes at least one copy of the content of any box in R (see Definition 31). Besides, given a port or cell c of R and a port or cell c' of ρ which correspond each other, looking at c' we can say in how many and which boxes of R c is, and we can differentiate the various copies in ρ of a same box in R. The intuition is: if $(p, a\cdot(l, n)) \in \mathcal{P}_\rho$ then l is a *box*-cell of R, p is inside the box of l in R (and inside all the boxes containing l, encoded by the sequence a) and $(p, a\cdot(l, n))$ is a port (corresponding to p) of the n-th copy of the content of the box of l in ρ.

Lemma 28. *Let R be a* DiLL*-ps and let $\rho \in \mathcal{T}_R$.*

1. *If* $(p,a) \in \mathcal{P}_\rho$ *(resp.* $(l,b) \in \mathcal{C}_\rho$*), then* $p \in \mathcal{P}_R$ *(resp.* $l \in \mathcal{C}_R$*). If furthermore* $(p,a) \in \mathsf{P}^{\mathsf{pri}}_\rho((l,b))$*, then* $a = b$*. Similarly, if* $(p,a) \in \mathsf{P}^{\mathsf{aux}}_\rho((l,b))$ *and* $(l,b) \in \mathcal{C}^{cut}_\rho \cup \mathcal{C}^{\otimes,\parr}_\rho$*, then* $a = b$*.*

2. *If* ρ *is* R*-fat and* $p \in \mathcal{P}_R$ *(resp.* $l \in \mathcal{C}_R$*), then* $(p,a) \in \mathcal{P}_\rho$ *(resp.* $(l,a) \in \mathcal{C}_\rho$*) for some finite sequence* a*.*

3. *If* $(p,a) \in \mathcal{P}_\rho$ *and* $p \in \mathcal{P}_R$*, then:* $\mathsf{tp}_\rho((p,a)) = \mathsf{tp}_R(p)$*,* $a = ((l_1,n_1),\ldots,(l_m,n_m))$ *with* $m \in \mathbb{N}$ *and* $n_1,\ldots,n_m \in \mathbb{N}^+$*,* $\mathsf{depth}_R(p) = m$ *and* $\mathsf{boxesof}_R(p) = (l_1,\ldots,l_m)$*.*

   Similarly, if $(l,b) \in \mathcal{C}_\rho$ *and* $l \in \mathcal{C}_R$*, then:* $\mathsf{tc}_\rho((l,b)) = \mathsf{tc}_R(l)$*,* $b = ((l_1,n_1),\ldots,(l_m,n_m))$ *with* $m \in \mathbb{N}$ *and* $n_1,\ldots,n_m \in \mathbb{N}^+$*,* $\mathsf{depth}_R(l) = m$ *and* $\mathsf{boxesof}_R(l) = (l_1,\ldots,l_m)$*. In particular,* $(l_i,(l_1,n_1)\cdot\ldots\cdot(l_{i-1},n_{i-1})) \in \mathcal{C}^!_\rho$ *for any* $1 \le i \le m$*.*

   For every $(p,a) \in \mathcal{P}_\rho$ *and* $(l,b) \in \mathcal{C}_\rho$*:* $p \in \mathsf{P}^{\mathsf{pri}}_R(l)$ *iff* $(p,a) \in \mathsf{P}^{\mathsf{pri}}_\rho((l,b))$*;* $p \in \mathsf{P}^{\mathsf{aux}}_R(l)$ *iff* $(p,a) \in \mathsf{P}^{\mathsf{aux}}_\rho((l,b))$*; if moreover* $l \in \mathcal{C}^{\otimes,\parr}_R$ *and* $p \in \mathsf{P}^{\mathsf{aux}}_R(l)$*, then:* $p \in \mathsf{P}^{\mathsf{left}}_R(l)$ *iff* $(p,a) \in \mathsf{P}^{\mathsf{left}}_\rho((l,b))$*.*

4. *Let* $(l,b) \in \mathcal{C}_\rho$*. If* $a = b$ *for any* $(p,a) \in \mathsf{P}^{\mathsf{aux}}_\rho((l,b))$ *and either* ρ *is* R*-fat or* $\mathsf{P}^{\mathsf{aux}}_\rho((l,b)) \neq \emptyset$*, then* $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_R(l)) = \mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l,b)))$ *and* $l \notin \mathcal{C}^{\mathsf{box}}_R$ *and* $\mathsf{P}^{\mathsf{aux}}_R(l) \cap \mathcal{A}uxdoors_R = \emptyset$*.*

5. *If* $(p,a) \in \mathsf{P}^{\mathsf{aux}}_\rho((l,b))$ *for some* $(l,b) \in \mathcal{C}_\rho$ *with* $a \neq b$*, then there are* $k,n_1,\ldots,n_k \in \mathbb{N}^+$ *and* $l_1,\ldots,l_k \in \mathcal{C}^{\mathsf{box}}_R$ *such that* $a = b\cdot((l_1,n_1),\ldots,(l_k,n_k))$ *and either* $p \in \mathsf{auxd}_R(l_i)$ *for all* $1 \le i \le k$ *and* $l \in \mathcal{C}^?_R$*, or* $k = 1$ *and* $l = l_1$*. Moreover, for any* $1 \le i \le k$*, if* $\mathsf{card}\big(\mathsf{P}^{\mathsf{aux}}_\rho\big((l_i, b\cdot(l_1,n_1)\cdot\ldots\cdot(l_{i-1},n_{i-1}))\big)\big) \ge 2$ *then* $\{(p, b\cdot(l_1,n_1)\cdot\ldots\cdot(l_i,1)\cdot\ldots,(l_k,n_k)),\ (p, b\cdot(l_1,n_1)\cdot\ldots\cdot(l_i,2)\cdot\ldots\cdot(l_k,n_k))\} \subseteq \mathsf{P}^{\mathsf{aux}}_\rho((l,b))$ *and* $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_R(l)) \ge 1$ *and, if* ρ *is* R*-fat, then* $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_R(l)) < \mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l,b)))$*.*

6. *Let* $p,p' \in \mathcal{P}_R$ *(resp.* $l,l' \in \mathcal{C}_R$*) with* $\mathsf{boxesof}_R(p) = (l_1,\ldots,l_m)$ *and* $\mathsf{boxesof}_R(p') = (l_1,\ldots,l_{m'})$ *(resp.* $\mathsf{boxesof}_R(l) = (l_1,\ldots,l_m)$ *and* $\mathsf{boxesof}_R(l') = (l_1,\ldots,l_{m'})$*) for some* $m' \le m$*: if* $(p,(l_1,n_1)\cdot\ldots\cdot(l_m,n_m)) \in \mathcal{P}_\rho$ *(resp.* $(l,(l_1,n_1)\cdot\ldots\cdot(l_m,n_m)) \in \mathcal{C}_\rho$*) for some* $n_1,\ldots,n_m \in \mathbb{N}^+$*, then* $(p',(l_1,n_1)\cdot\ldots\cdot(l_{m'},n_{m'})) \in \mathcal{P}_\rho$ *(resp.* $(l,(l_1,n_1)\cdot\ldots\cdot(l_{m'},n_{m'})) \in \mathcal{C}_\rho$*).*

Proof. By induction on $\mathsf{depth}(R) \in \mathbb{N}$, following Definitions 22 and 24. □

Thanks to Lemmas 28.4-5, given a DiLL-ps R, $\rho \in \mathcal{T}_R$ and a (co-)contraction cell l of ρ (i.e. $l \in \mathcal{C}^{!,?}_\rho$ and $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho(l)) \ge 2$), it is possible to distinguish if l is a "real" (co-)contraction (i.e. the corresponding !- or ?-cell l' of R has at least 2 premises) or not (and then l' has only one premise which is a *pri*- or *aux*-door of some box in R): in the first case, and only in this one, there are two premises (p,a) and (q,b) of l with $p \neq q$.

The following lemma says that, given a DiLL-ps R and $\rho \in \mathcal{T}_R$ R-fat, the ports inside the box of some promotion cell of R can be identified by looking only at the name of the ports of ρ.

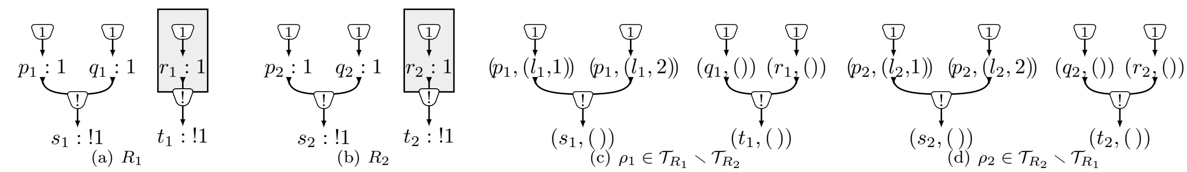


Figure 4: Two isomorphic MELL-ps R_1 (Figure 4(a), where $\mathcal{C}^{\mathsf{box}_0}_{R_1} = \{l_1\}$ and $\mathcal{C}^{\mathsf{box}}_{R_1} \setminus \mathcal{C}^{\mathsf{box}_0}_{R_1} = \{o_1\}$) and R_2 (Figure 4(b), where $\mathcal{C}^{\mathsf{box}_0}_{R_2} = \{l_2\}$ and $\mathcal{C}^{\mathsf{box}}_{R_2} \setminus \mathcal{C}^{\mathsf{box}_0}_{R_2} = \{o_2\}$), and their (isomorphic) 2-diffnets ρ_1 (Figure 4(c)) and ρ_2 (Figure 4(d)).

Lemma 29. *Let R be a* DiLL*-ps and $\rho \in \mathcal{T}_R$, let $(p, a) \in \mathcal{P}_\rho$ with $a = ((l_1, n_1), \dots, (l_m, n_m))$, $m \in \mathbb{N}^+$ and $\mathsf{P}^{\mathsf{aux}}_R(l_m) = \{p\}$. If ρ is R-fat then $\mathsf{inbox}_R(l_m) = \{q \in \mathcal{P}_R \mid (q, a \cdot b) \in \mathcal{P}_\rho$ for some finite sequence $b\}$.*

Proof. By Lemma 28.1, $p \in \mathcal{P}_R$. By Lemma 28.3, $n_1, \dots, n_m \in \mathbb{N}^+$ and $\mathsf{boxesof}_R(p) = (l_1, \dots, l_m)$. If $q \in \mathsf{inbox}_R(l_m)$ then $q \in \mathcal{P}_R$ and there exists a finite sequence b' such that $(q, b') \in \mathcal{P}_\rho$ by Lemma 28.2; according to Lemma 28.3, $b' = ((l_1, n_1), \dots, (l_m, n_m)) \cdot b$ for some finite sequence b, since $q \in \mathsf{inbox}_R(l_m)$. Conversely, if $(q, ((l_1, n_1), \dots, (l_m, n_m)) \cdot b) \in \mathcal{P}_\rho$ for some finite sequence b, then $q \in \mathcal{P}_R$ by Lemma 28.1, and $q \in \mathsf{inbox}_R(l_m)$ by Lemma 28.3 again. □

Remark 30. Let R and S be MELL-ps, let $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ such that $\varphi = (\varphi_\mathcal{P}, \varphi_\mathcal{C})\colon \rho \simeq \sigma$. Let $l \in \mathcal{C}^{\mathsf{box}}_R$ with $\mathsf{prid}_R(l) = p$. If $(p, a) \in \mathcal{P}_\rho$ and $(q, b) \in \mathcal{P}_\sigma$ are such that $\varphi_\mathcal{P}((p, a)) = (q, b)$, then there exists $o \in \mathcal{C}^{\mathsf{box}}_S$ such that $\mathsf{prid}_R(o) = q$ and $\varphi_\mathcal{C}((l, a^-)) = (o, b^-)$. This does not hold in general if R or S is a DiLL-ps, because only in a MELL-ps !-cells and *box*-cells coincide. For instance, let $R_1, R_2 \in \mathbf{PS}_{\mathsf{DiLL}} \setminus \mathbf{PS}_{\mathsf{MELL}}$ and $\rho_1 \in \mathcal{T}_{R_1} \setminus \mathcal{T}_{R_2}$ and $\rho_2 \in \mathcal{T}_{R_2} \setminus \mathcal{T}_{R_1}$ as in Figure 4: it easy to check that there exists $\varphi = (\varphi_\mathcal{P}, \varphi_\mathcal{C})\colon \rho_1 \simeq \rho_2$ such that

$$\varphi_\mathcal{P}\big((p_1, (l_1, 1))\big) = (q_2, ()) \quad \varphi_\mathcal{P}\big((q_1, ())\big) = (p_2, (l_2, 1)) \quad \varphi_\mathcal{C}\big((l_1, ())\big) = (o_2, ())$$
$$\varphi_\mathcal{P}\big((p_1, (l_1, 2))\big) = (r_2, ()) \quad \varphi_\mathcal{P}\big((r_1, ())\big) = (p_2, (l_2, 2) \quad \varphi_\mathcal{C}\big((o_1, ())\big) = (l_2, ()).$$

Let us consider some particular (and informative) elements of the labelled Taylor expansion of a DiLL-ps.

Definition 31 (R-fatness, k-diffnet of a DiLL-ps)**.** *Let R be a* DiLL*-ps and let $\rho \in \mathcal{T}_R$. We say that:*

- *ρ is R-*fat *(resp.* strongly R-fat*) if $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, b))) \geq 1$ (resp. $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, b))) \geq 2$) for any $(l, b) \in \mathcal{C}^!_\rho$ such that $l \in \mathcal{C}^{\mathsf{box}}_R$;*
- *ρ is a k-*diffnet of R *if there is $k \in \mathbb{N}$ such that $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l, b))) = k$ for any $(l, b) \in \mathcal{C}^!_\rho$ with $l \in \mathcal{C}^{\mathsf{box}}_R$.*

Remark 32. Let R be a DiLL-ps, let $\rho \in \mathcal{T}_R$ and $k \in \mathbb{N}$.

1. If R is a MELL-ps, one has: ρ is fat (resp. strongly fat) iff ρ is R-fat (resp. strongly R-fat); ρ is a k-diffnet of R iff ρ is k-wide (see Definition 2).

2. If ρ and σ are k-diffnets of R, then $\sigma = \rho$ (the k-diffnet of a DiLL-ps is unique). Indeed, there is only one $\alpha \in \mathcal{T}^{\mathsf{proto}}_{R'_\varepsilon}$ such that $\sigma = \tau_{R'_\varepsilon}(\alpha) = \rho$ for any representative R'_ε of the empty-renaming R_ε of R.

3. If $k > 0$ and ρ is the k-diffnet of R then: $(p, a) \in \mathcal{P}_\rho$ (resp. $(l, a) \in \mathcal{C}_\rho$) iff $p \in \mathcal{P}_R$ (resp. $l \in \mathcal{C}_R$) and $a = ((l_1, n_1), \ldots, (l_m, n_m))$ where $\mathsf{depth}_R(p) = m$ (resp. $\mathsf{depth}_R(l) = m$), $\mathsf{boxesof}_R(p) = (l_1, \ldots, l_m)$ (resp. $\mathsf{boxesof}_R(l) = (l_1, \ldots, l_m)$) and $n_1, \ldots, n_m \in \mathbb{N}^+$ are such that $1 \leq n_i \leq k$ for any $1 \leq i \leq m$.

4. The 1-(resp. 2-)diffnet of R is a sub-DiLL_0-ps of any R-fat (resp. strongly R-fat) element of $\mathcal{T}_R$.

Notation. Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$, $\rho \in \mathcal{T}_R$, $(p, a) \in \mathcal{P}_\rho$ and $k \in \mathbb{N}^+$. We write $|a| = k$ if $a = ((l_1, n_1), \ldots, (l_m, n_m))$ for some $m \in \mathbb{N}$ and some $l_1, \ldots, l_m \in \mathcal{C}^{\mathsf{box}}_R$, and $n_i = k$ for any $1 \leq i \leq m$.

Given a DiLL-ps R and $\rho \in \mathcal{T}_R$, when $(p, a) \in \mathcal{P}_\rho$ with $|a| = k$ for some $k \in \mathbb{N}^+$ it means that, in ρ, (p, a) is located in the k-th copy of the content of the box of l, for any $l \in \mathcal{C}^{\mathsf{box}}_R$ such that $p \in \mathsf{inbox}_R(l)$.

Fact 33. *Let R be a DiLL-ps and let $\rho \in \mathcal{T}_R$. If ρ is R-fat then, for every $p \in \mathcal{P}_R$ (resp. $l \in \mathcal{C}_R$), there exists a unique finite sequence a such that $|a| = 1$ and $(p, a) \in \mathcal{P}_\rho$ (resp. $(l, a) \in \mathcal{C}_\rho$).*

Proof. Let $p \in \mathcal{P}_R$ (resp. $l \in \mathcal{C}_R$). By Remark 32.4, the 1-diffnet of R is a sub-DiLL_0-ps of ρ and hence, by Remark 32.3, $(p, a) \in \mathcal{P}_\rho$ (resp. $(l, a) \in \mathcal{C}_\rho$) where $a = ((l_1, 1), \ldots (l_m, 1))$ and $\mathsf{boxesof}_R(p) = (l_1, \ldots, l_m)$ (resp. $\mathsf{boxesof}_R(l) = (l_1, \ldots, l_m)$), so $|a| = 1$. According to Lemma 28.3, if $(p, a') \in \mathcal{P}_\rho$ (resp.$(l, a') \in \mathcal{C}_R$) with $|a'| = 1$, then $a' = ((l_1, 1), \ldots (l_m, 1)) = a$. □

By Lemmas 28-29, a DiLL-ps R is completely characterized by any $\rho \in \mathcal{T}_R$ R-fat, i.e. if $\rho \in \mathcal{T}_R \cap \mathcal{T}_S$ for some DiLL-ps R, S and ρ is R-fat, then $R = S$: it is sufficient to look at the names of ports and cells of ρ. For example, if ρ is the 1-diffnet of R, then ρ is essentially R having forgotten the boundary of the boxes of R (see Fact 35.1), but we can recover auxd_R and cutd_R thanks to Lemmas 28-29. This is trivial and should not be confused with the (not all trivial) fact that when $\rho \in \mathcal{T}_R$, $\sigma \in \mathcal{T}_S$, σ and ρ are strongly fat and $\rho \simeq \sigma$, then $R \simeq S$; which is the whole point of the paper and will be proven (in the connected case) in Section 4, Theorem 45.

Definition 34 (Forgetful functions)**.** *Let R be a DiLL-ps and let $\rho \in \mathcal{T}_R$. We define the* forgetful functions $\mathsf{forget}^{\rho,R}_{\mathcal{P}} \colon \mathcal{P}_\rho \to \mathcal{P}_R$ *and* $\mathsf{forget}^{\rho,R}_{\mathcal{C}} \colon \mathcal{C}_\rho \to \mathcal{C}_R$ *as follows:* $\mathsf{forget}^{\rho,R}_{\mathcal{P}}((p, a)) = p$ *and* $\mathsf{forget}^{\rho,R}_{\mathcal{C}}((l, b)) = l$ *for any $(p, a) \in \mathcal{P}_\rho$ and $(l, b) \in \mathcal{C}_\rho$.*

Given a DiLL-ps R and $\rho \in \mathcal{T}_R$, by forgetting the indexes associated with the ports and cells of ρ, the forgetful functions $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ make explicit the correspondence between ports and cells of ρ and ports and cells of R, according to Lemmas 28.1-3. More precisely, Lemma 28.1 ensures that $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are functions, Lemma 28.2 says that $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are surjections when ρ is R-fat.

Fact 35. *Let R, S be* DiLL*-ps, and ρ (resp. σ) be the* 1*-diffnet of R (resp. S).*

1. *The forgetful functions $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are bijections and all diagrams* (1) *of Definition 3 commute (replacing φ with $\mathsf{forget}^{\rho,R}$).*

2. *Suppose $\varphi_1 \colon \rho \simeq \sigma$. We denote by $\varphi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ (resp. $\varphi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$) the function defined by:*

$$\varphi_{\mathcal{P}}(p) = \mathsf{forget}_{\mathcal{P}}^{\sigma,S}(\varphi_{1\mathcal{P}}((p,a))) \qquad \text{for every } p \in \mathcal{P}_R, \tag{4}$$

$$\varphi_{\mathcal{C}}(l) = \mathsf{forget}_{\mathcal{C}}^{\sigma,S}(\varphi_{1\mathcal{C}}((l,b))) \qquad \text{for every } l \in \mathcal{C}_R, \tag{5}$$

where, for $p \in \mathcal{P}_R$ and $l \in \mathcal{C}_R$, a and b are the unique[12] *finite sequences such that $(p,a) \in \mathcal{P}_\rho$ and $(l,b) \in \mathcal{C}_\rho$. Then $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{C}}$ are bijections and all diagrams* (1) *of Definition 3 commute.*

Proof.

1. The functions $\mathsf{forget}_{\mathcal{P}}^{\rho,R}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,R}$ are well-defined by Lemma 28.1 and are invertible by Remark 32.3 (where $k = 1$): since $k = 1$, for $(p,a) \in \mathcal{P}_\rho$, one has $a = ((l_1,1),\ldots,(l_n,1))$ where $\{l_1,\ldots,l_m\} = \{l' \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l')\}$. All diagrams (1) of Definition 3 commute by Lemma 28.3.

2. It follows immediately from Fact 35.1, since $\varphi_1 \colon \rho \simeq \sigma$.

$\square$

The fact that $\rho \in \mathcal{T}_R$ for some DiLL-ps R and $\sigma \simeq \rho$ do not imply that $\sigma \in \mathcal{T}_R$ (and there may exist a DiLL-ps $S \not\simeq R$ such that $\sigma \in \mathcal{T}_S$), so all the information about R available in ρ thanks to the names of its ports and cells (see Lemma 28) is lost in σ, though ρ and σ "morally" represent the same object. Even if ρ is a k-diffnet of R for a very large $k \in \mathbb{N}$ and $\sigma \simeq \rho$, in general looking at σ one is not able to recognize where the *aux*- and *cut*-doors of the boxes in R are. Fact 35.2 only says that if R, S are DiLL-ps and ρ (resp. σ) is the 1-diffnet of R (resp. S) with $\varphi_1 \colon \rho \simeq \sigma$, then φ_1 extends to a "quasi-isomorphism" φ from R to S, but in general φ does not make commute diagrams (2) of Definition 3.

[12] Uniqueness is due to Remark 32.3 where $k = 1$.

4. The connected case: computing a MELL-ps from its Taylor expansion

We show here our principal result (Theorem 45): a connected (in the sense of Definition 39) MELL-ps R is completely characterized by any $\gamma \in \mathcal{T}_R^{\simeq}$ strongly fat[13]. The idea is that, by means of the "geometry" of γ, we can recover the informations about R encoded in the names of ports and cells of some *suitable* $\rho \in \mathcal{T}_R \cap \gamma$: in particular, we can identify the "real" contraction cells from the "fake" ones. A key-tool for this approach is the notion of $\mathcal{L}$-accessibility: it allows to separate the different copies of the content of a box, so it plays at a more concrete syntactic level the same role which is played by bridges in [4, Def. 73].

Definition 36 ($\mathcal{L}$-accessibility)**.** *Let Φ be a pps and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$. The* $\mathcal{L}$-accessibility function $\mathcal{L}\text{-}\mathsf{acces}_\Phi \colon \mathcal{P}_\Phi \to \mathfrak{P}(\mathcal{P}_\Phi)$ *associates with any $p \in \mathcal{P}_\Phi$ the set $\mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$, whose elements are the* $\mathcal{L}$-accessible ports from p in Φ*, which is the smallest subset of $\mathcal{P}_\Phi$ obtained by applying the following rules:*

- *$p \in \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$;*
- *if $l \in \mathcal{C}_\Phi$ and $q \in \mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$, then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$;*
- *if $l \in \mathcal{C}_\Phi$ with $\mathsf{tc}_\Phi(l) \notin \mathcal{L}$ and $q \in (\mathsf{P}^{\mathsf{aux}}_\Phi(l) \smallsetminus \{p\}) \cap \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$, then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$;*
- *if $l \in \mathcal{C}_\Phi$, $\mathsf{tc}_\Phi(l) \in \mathcal{L}$, $\emptyset \neq \mathsf{P}^{\mathsf{aux}}_\Phi(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$ and $p \notin \mathsf{P}^{\mathsf{aux}}_\Phi(l)$, then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cup \mathsf{P}^{\mathsf{aux}}_\Phi(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$.*

Let Φ be a pps, $p \in \mathcal{P}_\Phi$ and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$. The set of $\mathcal{L}$-accessible ports from p is upward-closed: if $q \in \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$ and $q \leq_R q'$ then $q' \in \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$; if $l \in \mathcal{C}^{ax}_\Phi$ then either $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$ or $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathcal{L}\text{-}\mathsf{acces}_\Phi(p) = \emptyset$. The set of $\mathcal{L}$-accessible ports from p is "often" downward-closed: if $q \in \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$ and $q' \notin \mathcal{L}\text{-}\mathsf{acces}_\Phi(p)$ with $q \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ and $q' \in \mathsf{P}^{\mathsf{pri}}_\Phi(l)$ for some $l \in \mathcal{C}_\Phi$, then $p \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$, or $\mathsf{tc}_\Phi(l) \in \mathcal{L}$ and $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \not\subseteq \mathcal{L}\text{-}\mathsf{acces}_\Phi(l)$. Functions auxd_Φ and cutd_Φ (which mark the boundary of boxes in Φ) play no role in defining $\mathcal{L}$-accessibility.

Intuitively, q is a $\mathcal{L}$-accessible port from p if there is a path in $\mathfrak{G}^{\leq}_{\mathrm{undir}}(\Phi)$ (see page 7) starting upward from p and ending in q, paying attention that this kind of path may end when crossing downward a cell with type in $\mathcal{L}$ (here "upward" and "downward" are in the sense of the order relation $\leq_\Phi$ of Definition 9).

Remark 37. Recalling Remark 12, one can easily see that, if $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$ and Φ and Ψ are pps such that $\varphi \colon \Phi \simeq \Psi$, then $\overline{\varphi_{\mathcal{P}}}(\mathcal{L}\text{-}\mathsf{acces}_\Phi(p)) = \mathcal{L}\text{-}\mathsf{acces}_\Psi(\varphi_{\mathcal{P}}(p))$ for any $p \in \mathcal{P}_\Phi$.

[13] According to Definition 2, (strong) fatness is not defined for a set of pps, but the definition can be extended to a set of isomorphic pps thanks to Remark 4. More generally, all elements of γ are isomorphic and thus have the same "geometry".

Our definition of $\mathcal{L}$-accessibility is deeply related to the notion of empire, a well-known tool introduced by Girard in [11] in order to prove the sequentialization theorem for ACC proof-structures of the multiplicative fragment of LL (see [20, Def. A.6 and Rmk. A.7] for a definition of ACC for MELL-ps). Roughly speaking, a MELL-ps R is ACC if certain undirected graphs obtained from R are acyclic and connected; if a MELL-ps R is ACC, the empire of a port p of R is the biggest ACC sub-MELL-ps such that p is among the conclusions of R. The notions of ACC and empire can be adapted to DiLL-ps, by considering ?-cells (resp. !-cells which are not promotions) as generalized $⅋$-cells (resp. $\otimes$-cells). It can be shown that if R is a ACC DiLL_0-ps then, for any $p \in \mathcal{P}_R$, $\{?, ⅋\}\text{-}\mathsf{acces}_R(p)$ is the empire of p in R.

The next lemma shows how, given a box B in a DiLL-ps R, the $\mathcal{L}$-accessible ports in B from the *pri*-door p of B are related to the $\mathcal{L}$-accessible ports from a port corresponding to p in a R-fat $\rho \in \mathcal{T}_R$.

Lemma 38. *Let R be a DiLL-ps and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$, let $\rho \in \mathcal{T}_R$ be R-fat and $(p, a) \in \mathcal{P}_\rho$ where $a = ((l_1, n_1), \dots, (l_m, n_m))$ and $m \in \mathbb{N}^+$. If $\mathsf{prid}_R(l_m) = p$ and R_{l_m} is the content of the box of l_m in R, then $\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)) \supseteq \{(q, a \cdot b) \in \mathcal{P}_\rho \mid q \in \mathcal{L}\text{-}\mathsf{acces}_{R_{l_m}}(p)$ and b is some finite sequence$\}$.*

Proof. By induction on the definition of $q \in \mathcal{L}\text{-}\mathsf{acces}_{R_{l_m}}(p)$, using Lemmas 28.1-3. □

Given a DiLL-ps R and $\rho \in \mathcal{T}_R$, $\mathcal{L}$-accessibility allows one to recover some informations about R from ρ in a purely "geometric" way, without looking at the names of ports and cells of ρ, provided that ρ and R fulfil some conditions: essentially, ρ has to be strongly R-fat and R has to be ?-box-connected (see Lemma 41).

Definition 39 ($\mathcal{L}$-box-connectedness)**.** *Let R be a DiLL-ps and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$: R is* $\mathcal{L}$-box-connected *if, for any $l \in \mathcal{C}_R^{\mathsf{box}}$ where R_l is the content of the box of l in R, $\mathcal{L}\text{-}\mathsf{acces}_{R_l}(\mathsf{prid}_R(l)) = \mathcal{P}_{R_l} \smallsetminus \mathsf{P}_R^{\mathsf{pri}}(l)$.*

Intuitively, saying that a DiLL-ps R is $\mathcal{L}$-box-connected means that, for every *box*-cell l in R, if R_l is the content of the box of l in R then all the ports of R_l are connected to the auxiliary port of l by means of a path *inside* R_l, where paths are intended in the sense outlined after Definition 36. For example, the DiLL-ps S in Figure 2(a) in not $\mathcal{L}$-box-connected for any $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$.

Obviously, all DiLL_0-ps is $\mathcal{L}$-box-connected, for any $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$. Notice also that any DiLL-ps whose boxes have no *aux*-doors nor *cut*-doors are $\mathcal{L}$-box-connected, for any $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$.

We would like to stress that the box-connectedness condition (which is a crucial hypothesis in our main results) is quite general and not *ad hoc*. Indeed, it can be proven that: every ACC DiLL-ps (in particular, every ACC MELL-ps) without $\bot$-cells and weakenings (i.e. ?-cells with no premises) is $\{?, ⅋\}$-box-connected; every MELL-ps which is the translation of either a λ-term or a derivation in MELL sequent calculus (without mix rule) is $\{?, ⅋\}$-box-connected; $\mathcal{L}$-box-connectedness is preserved under cut-elimination, for any $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$.

Remark 40. Let R be a pps and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$.

1. If $p \in \mathcal{P}_R$ and $\mathcal{L}' \subseteq \mathcal{L}$ then $\mathcal{L}\text{-}\mathsf{acces}_R(p) \subseteq \mathcal{L}'\text{-}\mathsf{acces}_R(p)$: this follows immediately from the "closure" properties of $\mathcal{L}$-accessibility. As a consequence, if moreover R is a $\mathcal{L}$-box-connected DiLL-ps then it is $\mathcal{L}'$-box-connected too.

2. If R is a $\mathcal{L}$-box-connected DiLL-ps and $l \in \mathcal{C}_R^{\mathsf{box}}$, where $\mathsf{P}_R^{\mathsf{aux}}(l) = \{p\}$ and R_l is the content of the box of l in R, then $\mathsf{inbox}_R(l) \subseteq \mathcal{L}\text{-}\mathsf{acces}_{R_l}(p)$.

Lemma 41 (Geometric characterization of boxes in an element of the Taylor expansion)**.** *Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$, $\rho \in \mathcal{T}_R$, let $(p, a) \in \mathcal{P}_\rho$ with $a = ((l_1, n_1), \ldots, (l_m, n_m))$ and $m, n_1, \ldots n_m \in \mathbb{N}^+$. Let $\mathcal{P}_\rho{}^a = \{(q, a{\cdot}b) \in \mathcal{P}_\rho \mid b$ is some finite sequence$\}$ and $\mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}}$.[14] Then, $(l_m, a^-) \in \mathcal{C}_\rho^!$ and:*

1. *If ρ is R-fat and $\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)) = \mathcal{P}_\rho{}^a$ with $! \notin \mathcal{L}$, then $\mathsf{P}_R^{\mathsf{aux}}(l_m) = \{p\}$.*

2. *If $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}((l_m, a^-))) \geq 2$, $? \in \mathcal{L}$ and $\mathsf{P}_R^{\mathsf{aux}}(l_m) = \{p\}$, then $\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)) \subseteq \mathcal{P}_\rho{}^a$.*

3. *If ρ is R-fat and R is $\mathcal{L}$-box-connected with $\mathsf{P}_R^{\mathsf{aux}}(l_m) = \{p\}$, then $\mathcal{P}_\rho{}^a \subseteq \mathcal{L}\text{-}\mathsf{acces}_\rho((p, a))$.*

4. *If R is $\mathcal{L}$-box-connected with $? \in \mathcal{L}$ and $! \notin \mathcal{L}$, $\mathsf{card}(\mathsf{P}_\rho^{\mathsf{aux}}((l_m, a^-))) \geq 2$ and ρ is R-fat, then: $\mathsf{P}_R^{\mathsf{aux}}(l_m) = \{p\}$ iff $\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)) = \mathcal{P}_\rho{}^a$ iff $\mathsf{inbox}_R(l_m) = \overline{\mathsf{forget}_{\mathcal{P}}^{\rho,R}}(\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)))$.*

Proof. First, $p \in \mathcal{P}_R$ by Lemma 28.1, $n_1, \ldots, n_m \in \mathbb{N}^+$ and $\{l_1, \ldots, l_m\} = \{l \in \mathcal{C}_R^{\mathsf{box}} \mid p \in \mathsf{inbox}_R(l)\}$ by Lemma 28.3 (with $\mathsf{inbox}_R(l_j) \subsetneq \mathsf{inbox}_R(l_i)$ for any $1 \leq i < j \leq m$).

1. Suppose $\mathsf{P}_R^{\mathsf{aux}}(l_m) \neq \{p\}$: then $\mathsf{P}_R^{\mathsf{aux}}(l_m) = \{p_m\}$ with $p \neq p_m \in \mathcal{P}_R$; by Lemmas 28.2-3 and the nesting condition, $(p_m, a) \in \mathcal{P}_\rho$ (and thus $(p_m, a) \in \mathcal{P}_\rho{}^a$), since $\mathsf{inbox}_R(l_j) \subseteq \mathsf{inbox}_R(l_i)$ for any $1 \leq i < j \leq m$ (because $(p, ((l_1, n_1), \ldots, (l_m, n_m))) \in \mathcal{P}_\rho$). If $(p_m, a) \notin \mathcal{L}\text{-}\mathsf{acces}_\rho((p, a))$ then (since we just proved that $(p_m, a) \in \mathcal{P}_\rho{}^a$) $\mathcal{P}_\rho{}^a \neq \mathcal{L}\text{-}\mathsf{acces}_\rho((p, a))$, which is against the hypothesis.

   If $(p_m, a) \in \mathcal{L}\text{-}\mathsf{acces}_\rho((p, a))$, let p'_m be such that $\mathsf{P}_R^{\mathsf{pri}}(l_m) = \{p'_m\}$. One has $(p'_m, b') \in \mathcal{P}_\rho$ with $b' = ((l_1, n_1), \ldots, (l_{m-1}, n_{m-1}))$ by Lemmas 28.2-3 (since $p'_m \notin \mathsf{inbox}_R(l_m)$) and $(p'_m, b') \in \mathcal{L}\text{-}\mathsf{acces}_\rho((p, a))$ by Definition 36 (as $p \neq p_m$ and $! \notin \mathcal{L}$). Since on the other hand $(p'_m, b') \notin \mathcal{P}_\rho{}^a$, one concludes $\mathcal{L}\text{-}\mathsf{acces}_\rho((p, a)) \neq \mathcal{P}_\rho{}^a$, which is again against the hypothesis.

2. We prove that $?\text{-}\mathsf{acces}_\rho((p, a)) \subseteq \mathcal{P}_\rho{}^a$ by induction on the definition of $(q, b) \in ?\text{-}\mathsf{acces}_\rho((p, a))$. The statement of Lemma 41.2 follows because of Remark 40.1.

   - If $(q, b) = (p, a)$ then $a = b$ and hence $(q, b) \in \mathcal{P}_\rho{}^a$.

[14] Recall: for any $1 \leq i \leq m$, $l_i \in \mathcal{C}_R^{\mathsf{box}}$ and $(l_i, (l_1, n_1) \cdot \ldots \cdot (l_{i-1}, n_{i-1})) \in \mathcal{C}_\rho^!$ by Lemma 28.3.

- If there exist $(l, b'') \in \mathcal{C}_\rho$ and $(q', b') \in \mathcal{P}_\rho$ such that $(q, b) \in \mathsf{P}^{\mathsf{aux}}_\rho((l, b''))$ $\cap$?-$\mathsf{acces}_\rho((p, a))$ and $(q', b') \in \mathsf{P}^{\mathsf{pri}}_\rho((l, b'')) \cap$?-$\mathsf{acces}_\rho((p, a))$, then $b' = b''$ by Lemma 28.1 and $(q', b') \in \mathcal{P}_\rho{}^a$ by induction hypothesis; hence $l \neq l_m$ (otherwise $q' \notin \mathsf{inbox}_R(l_m)$ and thus $b' \neq a \cdot c$ for any finite sequence c, by Lemmas 28.1 and 28.3, that is impossible), so $q', q \in \mathsf{inbox}_R(l_m)$ and $b = a \cdot c$ for some finite sequence c, by Lemma 28.3 and the nesting condition. Therefore $(q, b) \in \mathcal{P}_\rho{}^a$.
- If there exist $(l, b'') \in \mathcal{C}_\rho \smallsetminus \mathcal{C}^?_\rho$ and $(q', b') \in \mathcal{P}_\rho$ such that $(q, b) \in \big(\mathsf{P}^{\mathsf{pri}}_\rho((l, b''))\cup\mathsf{P}^{\mathsf{aux}}_\rho((l, b''))\big) \cap$?-$\mathsf{acces}_\rho((p, a))$ and $(q', b') \in (\mathsf{P}^{\mathsf{aux}}_\rho((l, b''))$ $\smallsetminus \{(p, a)\}) \cap$?-$\mathsf{acces}_\rho((p, a))$, then $(q', b') \in \mathcal{P}_\rho{}^a$ by induction hypothesis and $l \neq l_m$ (otherwise by Lemma 28.3 $b' = a$ and then $q' = p$ by Lemma 28.1 and because $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_R(l)) = 1$, but $(q', b') = (p, a)$ is impossible); hence, similarly to the previous point, we can conclude that $(q, b) \in \mathcal{P}_\rho{}^a$.
- If there exist $(l, b'') \in \mathcal{C}^?_\rho$ with $\emptyset \neq \mathsf{P}^{\mathsf{aux}}_\rho((l, b'')) \subseteq$?-$\mathsf{acces}_\rho((p, a))$ and $(q, b) \in \mathsf{P}^{\mathsf{pri}}_\rho((l, b'')) \cap$?-$\mathsf{acces}_\rho((p, a))$, then $b = b''$ by Lemma 28.1 and $\mathsf{P}^{\mathsf{aux}}_\rho((l, b)) \subseteq \mathcal{P}_\rho{}^a$ by induction hypothesis, hence for every $(q', b') \in \mathsf{P}^{\mathsf{aux}}_\rho((l, b))$ there exists a finite sequence c such that $b' = a \cdot c'$ and thus $q' \in \mathsf{inbox}_R(l_m)$ by Lemma 28.3 but $q' \notin \mathsf{auxd}_R(l_m)$ (otherwise by Lemma 28.5, since $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\rho((l_m, a^-))) \geq 2$, there would be $(q'', b'') \in \mathsf{P}^{\mathsf{aux}}_\rho((l, b))$ such that $b'' \neq a \cdot c''$ for any finite sequence c'', that is impossible). So, also $q \in \mathsf{inbox}_R(l_m)$ and then $b = a \cdot c$ for some finite sequence c by Lemma 28.3, therefore $(q, b) \in \mathcal{P}_\rho{}^a$.

3. Let R_{l_m} be the content of the box of l_m in R. By Lemmas 28.1-3, $\mathcal{P}_\rho{}^a = \{(q, a \cdot b) \in \mathcal{P}_\rho \mid q \in \mathsf{inbox}_R(l_m), b \text{ is a finite sequence}\}$. By Remark 40.2 and since R is $\mathcal{L}$-box-connected, $\{(q, a{\cdot}b) \in \mathcal{P}_\rho \mid q \in \mathsf{inbox}_R(l_m), b \text{ is a finite sequence}\} \subseteq \{(q, a \cdot b) \in \mathcal{P}_\rho \mid b \text{ is a finite sequence}, q \in \mathcal{L}\text{-}\mathsf{acces}_{R_{l_m}}(p)\}$. We are done thanks to Lemma 38.

4. Immediate consequence of Lemmas 41.1-3 and (for the last equivalence) Lemma 29.

□

Lemma 42 (Box-cells, *pri*-doors and nesting preservation)**.** *Let R and S be some $\mathcal{L}$-box-connected* MELL*-ps, where $? \in \mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}} \smallsetminus \{!\}$. Let $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$ be strongly fat. Let $\varphi = (\varphi_\mathcal{P}, \varphi_\mathcal{C})\colon \rho \simeq \sigma$.*

1. *Every $l \in \mathcal{C}^{\mathsf{box}}_R$ φ-*maps to *a unique $o \in \mathcal{C}^{\mathsf{box}}_S$: if $(l, a), (l, a') \in \mathcal{C}^!_\rho$ and $\varphi_\mathcal{C}((l, a)) = (o, b)$ and $\varphi_\mathcal{C}((l, a')) = (o', b')$, then $o = o' \in \mathcal{C}^{\mathsf{box}}_S$. So, φ induces a bijection $\varphi^{\mathsf{box}}_\mathcal{C}\colon \mathcal{C}^{\mathsf{box}}_R \to \mathcal{C}^{\mathsf{box}}_S$ defined by:*

$$\varphi^{\mathsf{box}}_\mathcal{C}(l) = o \iff l \in \mathcal{C}^{\mathsf{box}}_R,\ o \in \mathcal{C}^{\mathsf{box}}_S \text{ and there are some finite sequences } a, b \text{ such that } (l, a) \in \mathcal{C}_\rho,\ (o, b) \in \mathcal{C}_\sigma,\ \varphi_\mathcal{C}((l, a)) = (o, b).$$

2. *If $l \in \mathcal{C}^{\mathsf{box}}_R$ φ-maps to $o \in \mathcal{C}^{\mathsf{box}}_S$, then the principal door of l φ-*maps to *the principal door of o: if $(p, a), (p, a') \in \mathcal{P}_\rho$ where $\mathsf{prid}_R(l) = p$ and*

$\varphi_{\mathcal{P}}((p,a)) = (q,b)$ *and* $\varphi_{\mathcal{P}}((p,a')) = (q',b')$*, then* $q = q' = \mathsf{prid}_S(o)$*. So,* φ *induces a bijection* $\varphi^{\mathsf{box}}_{\mathcal{P}} \colon \mathcal{P}ridoors_R \to \mathcal{P}ridoors_S$ *defined by:*

$$\varphi^{\mathsf{box}}_{\mathcal{P}}(p) = q \iff \textit{there are } l \in \mathcal{C}^{\mathsf{box}}_R,\ o \in \mathcal{C}^{\mathsf{box}}_S : \mathsf{prid}_R(l) = p,\ \mathsf{prid}_S(o) = q.$$

3. *The nesting tree-order of* $!$*-cells is preserved by* φ*: for all* $p, p' \in \mathcal{P}ridoors_R$*, if* $p \preccurlyeq_R p'$ *then* $\varphi^{\mathsf{box}}_{\mathcal{P}}(p) \preccurlyeq_S \varphi^{\mathsf{box}}_{\mathcal{P}}(p')$*; for all* $l, l' \in \mathcal{C}^{\mathsf{box}}_R$*, if* $l \preccurlyeq_{\mathcal{C}^{\mathsf{box}}_R} l'$ *then* $\varphi^{\mathsf{box}}_{\mathcal{C}}(l) \preccurlyeq_{\mathcal{C}^{\mathsf{box}}_S} \varphi^{\mathsf{box}}_{\mathcal{C}}(l')$*; for every* $p \in \mathcal{P}ridoors_R$*,* $\mathsf{boxesof}_S(\varphi^{\mathsf{box}}_{\mathcal{P}}(p)) = \widehat{\varphi^{\mathsf{box}}_{\mathcal{C}}}(\mathsf{boxesof}_R(p))$*.*
4. *The nesting tree-order is preserved by* φ *for all ports and cells: for every* $(p,a) \in \mathcal{P}_\rho$ *and* $(q,b) \in \mathcal{P}_\sigma$ *with* $\varphi_{\mathcal{P}}((p,a)) = (q,b)$*,* $\mathsf{boxesof}_S(q) = \widehat{\varphi^{\mathsf{box}}_{\mathcal{C}}}(\mathsf{boxesof}_R(p))$*; for every* $(l,a) \in \mathcal{C}_\rho$ *and* $(o,b) \in \mathcal{C}_\sigma$ *where* $\varphi_{\mathcal{C}}((l,a)) = (o,b)$*,* $\mathsf{boxesof}_S(o) = \widehat{\varphi^{\mathsf{box}}_{\mathcal{C}}}(\mathsf{boxesof}_R(l))$*.*

Proof. Are strongly fatness and $\mathcal{L}$-box-connectedness necessary? Yes! ☐

Lemma 43 (Copies preservation)**.** *Let* R *and* S *be some* MELL*-ps, let* $\rho \in \mathcal{T}_R$ *and* $\sigma \in \mathcal{T}_S$*. Let* $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$*, let* $(p,a),(p',a') \in \mathcal{P}_\rho$ *and* $(q,b),(q',b') \in \mathcal{P}_\sigma$ *such that* $\varphi_{\mathcal{P}}((p,a)) = (q,b)$ *and* $\varphi_{\mathcal{P}}((p',a')) = (q',b')$*. If* R *and* S *are* $\mathcal{L}$*-box-connected* MELL*-ps, where* $? \in \mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}} \smallsetminus \{!\}$*, and* ρ *and* σ *are strongly fat, then one has:* $a = a'$ *if and only if* $b = b'$*.*

Proof. Immediate consequence of Lemma 41.4 and Remark 37. It is false in general if R or S is a DiLL-ps. ☐

Reckoning with Lemma 28, Lemma 43 says that, given two $\mathcal{L}$-box-connected MELL-ps R and S and $\rho \in \mathcal{T}_R$ and $\sigma \in \mathcal{T}_S$, every isomorphism φ between ρ and σ in Lemma 43 preserves the copies of the content of a box, i.e., if two ports of ρ (resp. σ) are in the same copy of the content of a box in R (resp. S) or correspond to ports with depth 0 in R (resp. S), then their images in σ (resp. preimages in ρ) via φ are in the same copy of a box in S (resp. R) or correspond to ports with depth 0 in S (resp. R).

Lemma 44 (Building isomorphism)**.** *Let* $R, S \in \mathbf{PS}_{\mathsf{MELL}}$ *be* $\mathcal{L}$*-box-connected where* $? \in \mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}} \smallsetminus \{!\}$*. Let* ρ *(resp.* σ*) be the* k*-diffnet of* R *(resp.* S*). Suppose* $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \to \sigma$*. Let* $\phi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ *and* $\phi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ *be defined as follows:*

$$\phi_{\mathcal{P}}(p) = \mathsf{forget}^{\rho,R}_{\mathcal{P}}(\varphi_{\mathcal{P}}(p,a)) \quad \textit{for every } p \in \mathcal{P}_R \textit{ where } (p,a) \in \mathcal{P}_\rho \textit{ with } |a| = 1;$$
$$\phi_{\mathcal{C}}(l) = \mathsf{forget}^{\rho,R}_{\mathcal{C}}(\varphi_{\mathcal{C}}(l,a)) \quad \textit{for every } l \in \mathcal{C}_R \textit{ where } (l,a) \in \mathcal{C}_\rho \textit{ with } |a| = 1.$$

Then, $\phi = (\phi_{\mathcal{P}}, \phi_{\mathcal{C}}) \colon R \simeq S$*.*

Proof. First, observe that the functions $\phi_{\mathcal{P}}$ and $\phi_{\mathcal{C}}$ are well-defined by Fact 33.

We prove that $\phi_{\mathcal{P}} \colon \mathcal{P}_R \to \mathcal{P}_S$ is bijective. The proof that $\phi_{\mathcal{C}} \colon \mathcal{C}_R \to \mathcal{C}_S$ is bijective is perfectly analogous and it is left to the reader.

Injectivity: Let $p, p' \in \mathcal{P}_R$ with $p \neq p'$. Then, for the unique finite sequences a and a' such that $(p, a), (p', a') \in \mathcal{P}_R$ and $|a| = 1 = |a'|$ (see Fact 33), one has $(p, a) \neq (p', a')$. Let $\varphi_{\mathcal{P}}(p, a) = (q, b) \in \mathcal{P}_\sigma$ and $\varphi_{\mathcal{P}}(p', a') = (q', b') \in \mathcal{P}_\sigma$: by definition of ϕ, $\phi_{\mathcal{P}}(p) = q$ and $\phi_{\mathcal{P}}(p') = q'$. Since $\varphi_{\mathcal{P}}$ is injective, $(q, b) \neq (q', b')$. There are only two possibilities:

- either $a = a'$, then $b = b'$ by Lemma 43 (since k-wideness for any $k \geq 2$ implies strong fatness) and thus $q \neq q'$;
- or $a \neq a'$, then $\mathsf{boxesof}_R(p) \neq \mathsf{boxesof}_R(p')$ by Lemma 28.3 and since $|a| = 1 = |a'|$. According to Lemma 42.1, $\widehat{\varphi_{\mathcal{C}}^{\mathsf{box}}}(\mathsf{boxesof}_R(p)) \neq \widehat{\varphi_{\mathcal{C}}^{\mathsf{box}}}(\mathsf{boxesof}_R(p'))$, hence $\mathsf{boxesof}_S(q) \neq \mathsf{boxesof}_S(q')$ since $\mathsf{boxesof}_S(q) = \widehat{\varphi_{\mathcal{C}}^{\mathsf{box}}}(\mathsf{boxesof}_R(p))$ and $\widehat{\varphi_{\mathcal{C}}^{\mathsf{box}}}(\mathsf{boxesof}_R(p')) = \mathsf{boxesof}_S(q')$ by Lemma 42.4. Therefore $q \neq q'$.

In both cases, $\phi_{\mathcal{P}}(p) = q \neq q' = \phi_{\mathcal{P}}(p')$. Therefore, ϕ is injective.

Surjectivity:

□

Finally, we can claim and prove our main result.

Theorem 45. *Let R and S be some $\mathcal{L}$-box-connected* MELL*-ps, where $? \in \mathcal{L} \subseteq \mathcal{L}_{\mathsf{MELL}} \smallsetminus \{!\}$. Let $\rho_0 \in \mathcal{T}_R^{\simeq}$ and $\sigma_0 \in \mathcal{T}_S^{\simeq}$ be k-wide for some $k \geq 2$.*[15] *If $\rho_0 = \sigma_0$ then $R \simeq S$.*

Proof. According to Definition 27, $\rho_0 = \sigma_0$ implies that there are $\rho \in \mathcal{T}_R \cap \rho_0$, $\sigma \in \mathcal{T}_S \cap \sigma_0$ and $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) \colon \rho \simeq \sigma$. By Remarks 32.1-2, ρ (resp. σ) is the k-diffnet of R (resp. S). By Lemma 44, there is $\phi \colon R \simeq S$. □

Theorem 45 can be generalized by replacing the hypothesis that ρ_0 and σ_0 are k-wide for some $k \geq 2$ with the hypothesis that ρ_0 and σ_0 are strongly fat.

5. Relational semantics

5.1. Relational Experiments

The general goal of denotational semantics is to give a "mathematical" counterpart to syntactical devices such as proofs and programs, bringing to the fore their essential properties: the basic pattern is to associate with every formula/type an object of some category and with every proof/program a morphism of this category (its interpretation or semantics).

We introduce here, in a concrete way, the relational semantics, a widely studied denotational model of LL and λ-calculus (see for example [20, 2, 3, 6, 4]) which can be seen as a "degenerate case" of Girard's coherent semantics [10, 11].

[15] According to Definition 2, the notion of k-wideness is not defined for a set of DiLL-ps, but it can be extended to a set of isomorphic DiLL-ps thanks to Remark 4. Similarly for (strongly) fatness.

Definition 46 (Web of a MELL formula)**.** *Let $|\cdot|$ be the function associating with every $A \in \mathcal{F}_{\mathsf{MELL}}$ a set (called* web of A, *whose elements are the* points of A*), defined by induction on $A \in \mathcal{F}_{\mathsf{MELL}}$ by:*

$$|X| = |X^{\perp}| = \mathcal{A}t, \text{ for any } X \in \mathcal{V}_{\mathsf{MELL}}; \qquad |1| = |\perp| = \{()\};$$
$$|A \otimes B| = |A \parr B| = |A| \times |B|; \qquad |!A| = |?A| = \mathscr{M}_{\text{fin}}(|A|),$$

where $()$ is the empty sequence.

So $|A^{\perp}| = |A|$ for any $A \in \mathcal{F}_{\mathsf{MELL}}$: relational semantics is a degenerate denotational model of DiLL.

Definition 47 (Experiment of a DiLL-proof structure)**.** *Let Φ be a DiLL-ps.*

An experiment e of Φ, denoted by $\mathsf{e} : \Phi$, is a function associating with every $p \in \mathcal{P}_{\Phi}$ with $\mathsf{tp}_{\Phi}(p) = A$ for some $A \in \mathcal{F}_{\mathsf{MELL}}$ an element of $\mathscr{M}_{\text{fin}}(|A|)$ and with every $l \in \mathcal{C}_{\Phi}^{\mathsf{box}}$ a finite multiset of finite multisets of experiments of Φ_l, where Φ_l is the content of the box of l in Φ.

The definition is by induction on $\mathsf{depth}(\Phi) \in \mathbb{N}$, and we ask that $\mathsf{card}(\mathsf{e}(l)) = 1$ for any $l \in \mathcal{C}_{\Phi}^{\mathsf{box}_0}$, and $\mathsf{card}(\mathsf{e}(p)) = 1$ for any $p \in \mathcal{P}_{\Phi}$ such that $\mathsf{depth}_{\Phi}(p) = 0$. Furthermore the following conditions are to be fulfilled, for every $l \in \mathcal{C}_{\Phi}$ with $\mathsf{depth}_{\Phi}(l) = 0$:

- *if $l \in \mathcal{C}_{\Phi}^{ax}$ (resp. $l \in \mathcal{C}_{\Phi}^{cut}$) with $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{p, q\}$ (resp. $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p, q\}$), then $\mathsf{e}(p) = \mathsf{e}(q)$;*
- *if $l \in \mathcal{C}_{\Phi}^{1,\perp}$ with $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$, then $\mathsf{e}(q) = [()]$;*
- *if $l \in \mathcal{C}_{\Phi}^{\otimes,\parr}$ with $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_1, p_2\}$, $\mathsf{P}_{\Phi}^{\mathsf{left}}(l) = \{p_1\}$, $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$ and $\mathsf{e}(p_i) = [a_i]$ for any $i \in \{1, 2\}$, then $\mathsf{e}(q) = [(a_1, a_2)]$;*
- *if $l \in \mathcal{C}_{\Phi}^{!,?} \smallsetminus \mathcal{C}_{\Phi}^{\mathsf{box}_0}$, $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_1, \ldots, p_n\}$ for some $n \in \mathbb{N}$ and $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$, then $\mathsf{e}(q) = \sum_{i=1}^{n} \mathsf{e}(p_i)$;*
- *if $l \in \mathcal{C}_{\Phi}^{\mathsf{box}_0}$ with $\mathsf{P}_{\Phi}^{\mathsf{pri}}(l) = \{q\}$, $\mathsf{P}_{\Phi}^{\mathsf{aux}}(l) = \{p_l\}$ and $\mathsf{e}(l) = [[\mathsf{e}_1, \ldots, \mathsf{e}_n]]$ for some $n \in \mathbb{N}$, then $\mathsf{e}(q) = \sum_{i=1}^{n} [\mathsf{e}_i(p_l)]$, $\mathsf{e}(p) = \sum_{i=1}^{n} \mathsf{e}_i(p)$ for any $p \in \mathsf{inbox}_{\Phi}(l)$, and $\mathsf{e}(l') = \sum_{i=1}^{n} \mathsf{e}_i(l')$ for any $l' \in \mathcal{C}_{\Phi_l}^{\mathsf{box}}$ where Φ_l is the content of the box of l in Φ.*

We set $\mathsf{Exps}(\Phi)$ the set of experiments of Φ.

Experiments are actually defined on equivalence classes of DiLL-proof structures. Precisely,

Fact 48. *Let Φ be a DiLL-ps, and e an experiment of Φ. Let Ψ be a DiLL-ps so that $\varphi = (\varphi_{\mathcal{P}}, \varphi_{\mathcal{C}}) : \Phi \simeq \Phi'$. φ transports naturally e to an experiment, noted $\varphi_* \mathsf{e}$, of Ψ by induction on $\mathsf{depth}(\Psi)$, defined by, for all $p \in \mathcal{P}_{\Psi}$ and $l \in \mathcal{C}_{\Psi}$, $\varphi_* \mathsf{e}(p) = \mathsf{e}(\varphi_{\mathcal{P}}^{-1}(p))$ and $\varphi_* \mathsf{e}(l) = \mathsf{e}(\varphi_{\mathcal{C}}^{-1}(l))$*

Remark 49. An isomorphism $\varphi : \Phi \simeq \Phi'$ between two DiLL-ps induces a bijection φ_* between the set of experiments of Φ and of Φ'.

Roughly speaking, an experiment has two purposes: to coherently give names to all axioms; and to fix a number of copies for each box. This reminds us of an element of the Taylor expansion, that copies each box a certain number of times. The remainder of this section is devoted to clearing the relation between the experiments of DiLL-ps and its Taylor expansion.

Lemma 50. *Let Φ be a* DiLL*-ps. Let $\rho \in \mathcal{T}_\Phi$ and $\mathsf{e}^\rho : \rho$.*
Let $\mathsf{forget}_{\mathcal{P}}^{\rho,\Phi}$ and $\mathsf{forget}_{\mathcal{C}}^{\rho,\Phi}$ be the forgetful functions from ρ to Φ.
e^ρ defines an experiment e^Φ of Φ, by induction on $\mathsf{depth}(\Phi)$:

- *if Φ is of depth 0, by Fact 35.1, the forgetful functions are bijections. We set*

$$\forall l \in \mathcal{P}_\Phi, \mathsf{e}^\Phi(l) = \mathsf{e}^\rho_{\mathcal{P}} \circ \left(\mathsf{forget}_{\mathcal{P}}^{\rho,\Phi}\right)^{-1}(l),$$
$$\forall c \in \mathcal{C}_\Phi, \mathsf{e}^\Phi(c) = \mathsf{e}^\rho_{\mathcal{C}} \circ \left(\mathsf{forget}_{\mathcal{C}}^{\rho,\Phi}\right)^{-1}(c).$$

- *if Φ is of depth $n+1$, let $l \in \mathcal{P}_\Phi$ (respectively $c \in \mathcal{C}$) of depth 0. We set, as in the previous case,*

$$\mathsf{e}^\Phi(l) = \mathsf{e}^\rho_{\mathcal{P}} \circ \left(\mathsf{forget}_{\mathcal{P}}^{\rho,\Phi}\right)^{-1}(l),$$
$$\mathsf{e}^\Phi(c) = \mathsf{e}^\rho_{\mathcal{C}} \circ \left(\mathsf{forget}_{\mathcal{C}}^{\rho,\Phi}\right)^{-1}(c).$$

  Let $l \in \mathcal{C}_\Phi^{\mathsf{box}_0}$. Let Φ_l be the content of the box of l in Φ. Let $\left(\mathsf{forget}_{\mathcal{P}}^{\rho,\Phi}\right)^{-1}(\{l\}) = \{l_1, \ldots, l_n\}$ and let $\rho_1, \ldots, \rho_n$ be the subnets of ρ such that for all $i \in \{1, \ldots, n\}, l_i$ is a port of ρ_i, and $\rho_i \in \mathcal{T}_{\Phi_l}$. We set e^{ρ_i} the restriction of e to each ρ_i. By induction hypothesis, each e^{ρ_i} define an experiment e_i of Φ_l.

  We set $\mathsf{P}_\Phi^{\mathsf{pri}}(l) = \{q\}$, $\mathsf{P}_\Phi^{\mathsf{aux}}(l) = \{p_l\}$ and $\mathsf{e}(l) = [[\mathsf{e}_1, \ldots, \mathsf{e}_n]]$ for some $n \in \mathbb{N}$, then $\mathsf{e}(q) = \sum_{i=1}^n [\mathsf{e}_i(p_l)]$, $\mathsf{e}(p) = \sum_{i=1}^n \mathsf{e}_i(p)$ for any $p \in \mathsf{inbox}_\Phi(l)$, and $\mathsf{e}(l') = \sum_{i=1}^n \mathsf{e}_i(l')$ for any $l' \in \mathcal{C}_{\Phi_l}^{\mathsf{box}}$.

  As either port (respectively cell) is either of depth 0 or inside a box, e^Φ is defined on the whole of Φ.

Proof. By induction on the depth of Φ. □

So, we have a map

$$\bigcup_{\rho \in \mathcal{T}_\Phi} \mathsf{Exps}(\rho) \to \mathsf{Exps}(\Phi)$$

that transports experiments of elements of the Taylor expansion of Φ to experiments of Φ, by glueing together, for each box of Φ, the experiments induced by e^ρ on the different copies of this box, in a multiset.

We will now go on to define a map in the opposite direction.

5.3. DiLL_0*-ps induced by an experiment*

Representatives of protonets fix an order to every instance of the content of a box. Choosing a representative of an experiment allows to "draw it", that is, to produce a proto-net from it.

Definition 51 (Proto-net induced by a representative)**.** *Let Φ be a* DiLL*-ps and* $n = \mathsf{card}(\mathcal{C}_{\Phi}^{\mathsf{box}_0})$*.*

Let e *be an experiment of Φ and Φ' be a representative of Φ. We define* $\alpha_{\mathsf{e}}^{\Phi'}$ the proto-net induced by Φ' by induction on $\mathsf{depth}(\Phi)$*:*

$$\alpha_{\mathsf{e}}^{\Phi'} = (\langle \alpha_{\mathsf{e}}^{\Phi'}(1,1), \ldots, \alpha_{\mathsf{e}}^{\Phi'}(1, \mathsf{card}(\mathsf{box}_{\Phi'}^{0}(1)))\rangle, \ldots, \\ \langle \alpha_{\mathsf{e}}^{\Phi'}(n,1), \ldots, \alpha_{\mathsf{e}}^{\Phi'}(n, \mathsf{card}(\mathsf{box}_{\Phi'}^{0}(n)))\rangle)$$

where $\alpha_{\mathsf{e}}^{\mathsf{e}'}(i, k_i) = \alpha_{\mathsf{e}\restriction_{\mathsf{inbox}(i)}}^{\mathsf{e}'\restriction_{\mathsf{baux}_{\mathsf{e}'}(\mathsf{box}_{\Phi'}^{0}(i))(k_i)}}$ *is the proto-net defined on the subnet selected as the k_i-th interior of the i-th box (following the enumeration of* e'*) for* $1 \leq i \leq \mathsf{card}(n)$ *and* $1 \leq k_i \leq \mathsf{card}(\mathsf{box}_{\Phi'}^{0}(i))$*.* rephrase

Fact 52. *Let Φ be a* DiLL*-ps and Φ' a representative of Φ. Let* e *be an experiment of Φ.*

The protonet defined by Φ' is part of the proto-Taylor expansion of Φ.

Proof. By induction on the number on the depth of Φ. □

Definition 53 (DiLL_0-ps induced by a representative of an experiment)**.** *Let Φ be a* DiLL*-ps. Let* e *be an experiment of Φ.*

A representative e' *of* e *defines a proto-net that is part of the proto-Taylor expansion of Φ; and thus a ps in the Taylor expansion of Φ. We call it the* DiLL_0-ps induced by e'*.*

Fact 54. *Let Φ be a* DiLL*-ps. Let* e *be an experiment of Φ.*

Let Φ'_1 and Φ'_2 be two representatives of Φ. Let ρ_1 be the ps induced by Φ'_1, ρ_2 be the ps induced by Φ'_2. ρ_1 and ρ_2 are isomorphic.

Proof. To be done. □

So an experiment of a DiLL-ps Φ defines an equivalence class on $\mathbf{PS}_{\mathsf{DiLL}_0}$ of elements of the Taylor expansion of Φ: this equivalence class is exactly an element of $\mathcal{T}_{\Phi}^{\simeq}$.

Definition 55 (DiLL_0-ps induced by an experiment)**.** *Let Φ be a* DiLL*-ps. Let* e *be an experiment of Φ.*

The DiLL_0-ps induced by e *is the proof structure induced by any representative of Φ. It is an element of the Taylor expansion of Φ.*

Lemma 56. *if Φ is of depth $n+1$, let $l \in \mathcal{C}_{\Phi}^{\mathsf{box}_0}$. Let Φ_l be the content of the box of l in Φ. Let* $\left(\mathsf{forget}_{\mathcal{P}}^{\rho;\Phi}\right)^{-1}(\{l\}) = \{l_1, \ldots, l_n\}$ *and let $\rho_1, \ldots, \rho_n$ be the subnets of ρ such that for all $i \in \{1, \ldots, n\}, l_i$ is a port of ρ_i, and $\rho_i \in \mathcal{T}_{\Phi_l}$. We* *set* e^{ρ_i} *the restriction of* e *to each* ρ_i*. By induction hypothesis, each* e^{ρ_i} *define an experiment* e_i *of* Φ_l*.*

We set $\mathsf{P}^{\mathsf{pri}}_{\Phi}(l) = \{q\}$, $\mathsf{P}^{\mathsf{aux}}_{\Phi}(l) = \{p_l\}$ *and* $\mathsf{e}(l) = [[\mathsf{e}_1, \ldots, \mathsf{e}_n]]$ *for some* $n \in \mathbb{N}$*, then* $\mathsf{e}(q) = \sum_{i=1}^{n} [\mathsf{e}_i(p_l)]$, $\mathsf{e}(p) = \sum_{i=1}^{n} \mathsf{e}_i(p)$ *for any* $p \in \mathsf{inbox}_{\Phi}(l)$*, and* $\mathsf{e}(l') = \sum_{i=1}^{n} \mathsf{e}_i(l')$ *for any* $l' \in \mathcal{C}^{\mathsf{box}}_{\Phi_l}$*.*

Lemma 57. *Let* Φ *be a* DiLL*-ps. Let* $\mathsf{e}^{\Phi} = (\mathsf{e}^{\Phi}_{\mathcal{P}}, \mathsf{e}^{\Phi}_{\mathcal{C}})$ *be an experiment of* Φ*.*

Let ρ *be the* DiLL_0*-ps induced by* e*. Let* $\mathsf{forget}^{\rho,\Phi}_{\mathcal{P}}$ *and* $\mathsf{forget}^{\rho,\Phi}_{\mathcal{C}}$ *be the forgetful functions from* ρ *to* Φ*.*

e^{Φ} *defines an experiment* e^{ρ} *of* ρ*, by induction on* $\mathsf{depth}(\Phi)$*:*

- *if* Φ *is of depth 0, we set*

$$\mathsf{e}^{\rho} = \left(\mathsf{e}^{\Phi}_{\mathcal{P}} \circ \mathsf{forget}^{\rho,\Phi}_{\mathcal{P}}, \mathsf{e}^{\Phi}_{\mathcal{C}} \circ \mathsf{forget}^{\rho,\Phi}_{\mathcal{C}}\right).$$

- *if* $\mathsf{depth}(\Phi) = n+1$*, let* $l \in \mathcal{C}^{\mathsf{box}_0}_{\Phi}$*. Let* Φ_l *be the content of the box of* l *in* Φ*. Let* $\left(\mathsf{forget}^{\rho,\Phi}_{\mathcal{P}}\right)^{-1}(\{l\}) = \{l_1, \ldots, l_n\}$ *and let* $\rho_1, \ldots, \rho_n$ *be the subnets of* ρ *such that for all* $i \in \{1, \ldots, n\}, l_i$ *is a port of* ρ_i*, and* $\rho_i \in \mathcal{T}_{\Phi_l}$*.*
  $\mathsf{e}^{\Phi}(l) = [[\mathsf{e}^{\Phi}_1, \ldots, \mathsf{e}^{\Phi}_n]]$

So, we have a map

$$\mathsf{Exps}(\Phi) \to \bigcup_{\rho \in \mathcal{T}_{\Phi}} \mathsf{Exps}(\rho)$$

that sends experiments of elements of Φ *to experiments of the* DiLL_0*-ps in its Taylor expansion.*

Proof. By induction on $\mathsf{depth}(\Phi)$. ☐

We can restate it as a reciprocal to 50:

Lemma 58. *Let* Φ *be a* DiLL *ps. Let* e *be an experiment of* Φ*.*

e *(pre-composed with a forgetful function) is an experiment of an element of the Taylor expansion of* Φ*.*

5.4. Relational Semantics

In order to define the interpretation of a DiLL-ps Φ in the relational model, we have to enumerate, i.e. to fix an order on, the conclusions of Φ, engendering indexed DiLL-ps. The notion of isomorphism extends to indexed DiLL-ps: an isomorphism between two indexed DiLL-ps is a isomorphism between DiLL-ps preserving the order of the conclusions. Since a DiLL-ps Φ and any element of $\mathcal{T}_{\Phi}$ are "morally interfaced" (Remark 25), an order on the conclusions of Φ extends to any element of $\mathcal{T}_{\Phi}$ and we can define the Taylor expansion of an indexed DiLL-ps.

Definition 59 (Indexed DiLL-proof structure)**.** *An* indexed DiLL-ps *is a pair* $R = (\Phi_R, \mathsf{concl}_R)$ *where* Φ_R *is a* DiLL*-ps and* concl_R *is an enumeration of* $\mathcal{P}^{\mathsf{free}}_{\Phi_R}$. *The* type of the conclusions of R *is* $(A_1, \ldots, A_n)$ *where* $n = \mathsf{card}(\mathcal{P}^{\mathsf{free}}_{\Phi_R})$ *and* $\mathsf{tp}_{\Phi_R}(\mathsf{concl}_R(i)) = A_i$ *for any* $1 \leq i \leq n$.

Let $R = (\Phi_R, \mathsf{concl}_R)$ *and* $S = (\Phi_S, \mathsf{concl}_S)$ *be indexed* DiLL*-ps. We say that* φ *is an* isomorphism from R to S *if* $\varphi\colon \Phi_R \simeq \Phi_S$ *and* $\varphi_{\mathcal{P}}(\mathsf{concl}_R(i)) = \mathsf{concl}_S(i)$ *for any* $1 \leq i \leq \mathsf{card}(\mathcal{P}^{\mathsf{free}}_{\Phi_R})$*; we then write* $\varphi\colon R \simeq S$. *We say that* R *and* S *are* isomorphic *(and we then write* $R \simeq S$*) if there exists* $\varphi\colon R \simeq S$.

Let $R = (\Phi_R, \mathsf{concl}_R)$ *be an indexed* DiLL*-ps. The* Taylor expansion of R *is* $\mathcal{T}_R = (\mathcal{T}_{\Phi_R}, \mathsf{concl}_R)$. *The* quotiented Taylor expansion of R *is* $\mathcal{T}_R^{\simeq} = \{\{\mu \mid \mu \simeq (\pi, \mathsf{concl}_R)\} \mid \pi \in \mathcal{T}_{\Phi_R}\}$.

Definition 60 (Result of an experiment and interpretation of an indexed DiLL-ps)**.** *Let* $R = (\Phi_R, \mathsf{concl}_R)$ *be an indexed* DiLL*-ps with* $\mathsf{card}(\mathcal{P}^{\mathsf{free}}_{\Phi_R}) = n \in \mathbb{N}$.

Let e *be an experiment of* Φ_R*: the* result of e in R *is* $|\mathsf{e}|_{\mathsf{concl}_R} = (a_1, \ldots, a_n)$ *where* $\mathsf{e}(\mathsf{concl}_R(i)) = [a_i]$ *for any* $1 \leq i \leq n$. *The* interpretation*, or* semantics*,* of R *is:*

$$[\![R]\!] = \{|\mathsf{e}|_{\mathsf{concl}_R} \mid \mathsf{e} \text{ is an experiment of } \Phi_R\}.$$

It easy to verify that if R is an indexed DiLL-ps where Γ is the type of its conclusions, then $[\![R]\!] \subseteq |\parr\Gamma|$, hence $[\![R]\!]$ can be seen as a morphism from an arbitrary singleton set to $|\parr\Gamma|$ in the category **Rel** of sets and relations; moreover $[\![R]\!]$ can be shown to be invariant under cut-elimination and η-expansion.

Lemma 61. *Let* $R = (\Phi_R, \mathsf{concl}_R)$ *and* $S = (\Phi_S, \mathsf{concl}_S)$ *be isomorphic indexed* DiLL*-ps. Then*

$$[\![S]\!] = [\![R]\!]$$

Proof. Let φ an isomorphism such that $\varphi\colon R \simeq S$. We already remarked (remark 49) that φ induces a bijection between the set of experiments of Φ_R and the set of experiments of Φ_S. So,

$$[\![S]\!] = \{|\varphi_*\mathsf{e}|_{\mathsf{concl}_S} \mid \mathsf{e} \text{ is an experiment of } \Phi_R\}.$$

Let e be an experiment of Φ_R. By definition, $|\varphi_*\mathsf{e}|_{\mathsf{concl}_S} = (a_1, \ldots, a_n)$ where $\varphi_*\mathsf{e}(\mathsf{concl}_S(i)) = [a_i]$ for any $1 \leq i \leq n$ and $n = \mathsf{card}(\mathcal{P}^{\mathsf{free}}_{\Phi_R})$.

But, as φ is an isomorphism of indexed DiLL-ps, $\varphi_{\mathcal{P}}(\mathsf{concl}_R(i)) = \mathsf{concl}_S(i)$ for any $1 \leq i \leq \mathsf{card}(\mathcal{P}^{\mathsf{free}}_{\Phi_R})$. Moreover, by definition of φ_*, for any port $p \in \mathcal{P}_{\Phi_R}$, $\varphi_*\mathsf{e}(\varphi_{\mathcal{P}}p) = \mathsf{e}(p)$.

So, $|\varphi_*\mathsf{e}|_{\mathsf{concl}_S} = |\mathsf{e}|_{\mathsf{concl}_R}$, which concludes. □

Definition 62 (ps induced by a point)**.** *Let* $R = (\Phi_R, \mathsf{concl}_R)$ *be a normal indexed* DiLL*-ps. Let* x *be a point in* $[\![R]\!]$.

The DiLL_0-ps induced by x *is the proof structure induced by any experiment* e *such that* $|\mathsf{e}|_{\mathsf{concl}_R} = x$. *It is an element of the Taylor expansion of* Φ.

Lemma 63. *Let* $R = (\Phi_R, \mathsf{concl}_R)$ *be a normal indexed* DiLL*-ps. Let* x *be a point in* $[\![R]\!]$. *Let* e *and* e' *be two experiments of* R.

If $|\mathsf{e}|_{\mathsf{concl}_R} = |\mathsf{e}'|_{\mathsf{concl}_R} = x$*, then the ps induced by* e *and the ps induced by* e' *are isomorphic.*

Proof. To be done. □

5.5. Injective Semantics

Definition 64 (Atomic substitution, injective point)**.** *Let $A \in \mathcal{F}_{\mathsf{MELL}}$.*

Let $\sigma\colon \mathcal{A}t \to \mathcal{A}t$ be a bijection: the atomic substitution on A induced by σ *is a bijection $\sigma_A\colon |A| \to |A|$ defined by induction on $A \in \mathcal{F}_{\mathsf{MELL}}$ as follows:*

$$
\begin{aligned}
\sigma_1 = \sigma_\bot &= \mathsf{id}_{\{()\}} && \\
\sigma_X = \sigma_{X^\bot} &= \sigma && \textit{for any } X \in \mathcal{V}_{\mathsf{MELL}}; \\
\sigma_{A \otimes B}((a,b)) &= (\sigma_A(a), \sigma_B(b)) && \textit{for any } a \in |A| \textit{ and } b \in |B|; \\
\sigma_{A \parr B}((a,b)) &= (\sigma_A(a), \sigma_B(b)) && \textit{for any } a \in |A| \textit{ and } b \in |B|; \\
\sigma_{!A}([a_1, \ldots, a_n]) &= ([\sigma_A(a_1), \ldots, \sigma_A(a_n)]) && \textit{for any } n \in \mathbb{N} \textit{ and } a_1, \ldots, a_n \in |A|; \\
\sigma_{?A}([a_1, \ldots, a_n]) &= ([\sigma_A(a_1), \ldots, \sigma_A(a_n)]) && \textit{for any } n \in \mathbb{N} \textit{ and } a_1, \ldots, a_n \in |A|.
\end{aligned}
$$

We set $\mathcal{S}_A = \{\sigma_A \mid \sigma\colon \mathcal{A}t \to \mathcal{A}t$ is a bijection$\}$. We denote by $\sim_A$ the relation on $|A|$ defined by: $a \sim_A a'$ iff there exists $\sigma_A \in \mathcal{S}_A$ such that $a = \sigma_A(a')$.

A point a of A is injective *if every atom occurring in a occurs exactly twice. For every $\mathcal{A} \subseteq |A|$, we set $\mathcal{A}_{\text{inj}} = \{a \in \mathcal{A} \mid a$ is injective$\}$.*

For example, $[\,] \in |!A|_{\text{inj}} = |?A|_{\text{inj}}$ for any $A \in \mathcal{F}_{\mathsf{MELL}}$.

Remark 65. Let $A \in \mathcal{F}_{\mathsf{MELL}}$. If $a \in |A|_{\text{inj}}$ and $\sigma_A \in \mathcal{S}_A$, then $\sigma_A(a) \in |A|_{\text{inj}}$. Moreover, the relation $\sim_A$ on $|A|$ is an equivalence.

Roughly speaking, given a MELL formula A, the equivalence relation $\sim_A$ on $|A|_{\text{inj}}$ identifies any two injective points of A that are equal up to renaming of their atoms.

Definition 66 (Injective semantics)**.** *Let R be an indexed DiLL-proof structure. We define its injective semantics as the set of injective points of its semantics:*

$$[\![R]\!]_{\text{inj}} = [\![R]\!] \cap |\parr \Gamma|_{\text{inj}}$$

where Γ is the type of the conclusions of R.

Fact 67. *Let ρ be a indexed normal DiLL_0-ps with conclusion Γ. The points of the injective semantics of ρ are $\sim_\Gamma$-equivalent.*

Proof. By induction on the number of cells of ρ. □

Lemma 68. *Let R be an indexed normal MELL-ps of conclusion Γ.*

Let $\rho, \rho' \in \mathcal{T}_R$, $x \in [\![\rho]\!]_{\text{inj}}$, $x' \in [\![\rho']\!]_{\text{inj}}$.

If $x \sim_\Gamma x'$, then $\rho \simeq \rho'$.

Proof. Let us suppose $x \sim_\Gamma x'$. Let $\mathsf{e} : \rho$ and $\mathsf{e}' : \rho$ be such that $|\mathsf{e}|_{\text{concl}_R} = x$ and $|\mathsf{e}'|_{\text{concl}_R} = x'$.

As R — and in turn ρ and ρ' — is normal, all the atoms appearing in the image of e (respectively in e') appear in x (respectively in x'), so all the bijections

$\sigma : \mathcal{A}t \to \mathcal{A}t$ such that $\sigma(x) = x'$ lift to the same bijection (still noted σ) from the image of e to the image of e'. So, w.l.o.g., by replacing e with $\sigma \circ \mathsf{e}$, we can assume that $x = x'$.

We now prove that ρ and ρ' are isomorphic, by induction on the number of cells of ρ. If ρ has only one cell, all of its ports are free,...

If ρ has $n + 1$ cells, let l be a conclusion of ρ, and c be such that l is the principal port of c. □

By combining the different results, we obtain the main theorem of this section:

Theorem 69. *Let R be an indexed normal* MELL*-ps with conclusion Γ.*

The application $\mathrm{res} : \mathcal{T}_R \to \mathfrak{P}([\![R]\!]_{\mathrm{inj}})$ *that maps a* DiLL_0 *ps of the Taylor expansion of R to its set of injective results Factors through:*

- *the canonical inclusion* $[\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \hookrightarrow \mathfrak{P}([\![R]\!]_{\mathrm{inj}})$
- *the quotient* $\mathcal{T}_R \twoheadrightarrow \mathcal{T}_R^{\simeq}$

to a bijection $[\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \simeq \mathcal{T}_R^{\simeq}$.

$$\begin{array}{ccc} \mathcal{T}_R & \longrightarrow & \mathfrak{P}([\![R]\!]_{\mathrm{inj}}) \\ \downarrow & & \uparrow \\ \mathcal{T}_R^{\simeq} & \xrightarrow{\simeq} & [\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \end{array}$$

Proof. res is well-defined by Lemma 50.

Let $\rho \in \mathcal{T}_R$. By Fact 67, $\mathrm{res}(\rho)$ is included in a $\sim_\Gamma$ equivalence class. Let us prove that $\mathrm{res}(\rho)$ is closed by $\sim_\Gamma$. Let $x \in \mathrm{res}(\rho)$ and $y \sim_\Gamma x$. Let e be an experiment such that x is its result, and let σ be such that $y = \sigma(x)$. $\sigma \circ \mathsf{e}$ is an experiment of ρ of result y. So, $y \in \mathrm{res}(\rho)$.

So, res Factors through the canonical inclusion $[\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \hookrightarrow \mathfrak{P}([\![R]\!]_{\mathrm{inj}})$:

$$\begin{array}{ccc} \mathcal{T}_R & \longrightarrow & \mathfrak{P}([\![R]\!]_{\mathrm{inj}}) \\ & \searrow & \uparrow \\ & & [\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \end{array}$$

as $\overline{\mathrm{res}} : \mathcal{T}_R \to [\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma}$.

By Lemma 58, $\overline{\mathrm{res}}$ is surjective.

By Lemma 61, $\overline{\mathrm{res}}$ lifts through the quotient $\mathcal{T}_R \twoheadrightarrow \mathcal{T}_R^{\simeq}$

$$\begin{array}{ccc} \mathcal{T}_R & & \\ \downarrow & \searrow & \\ \mathcal{T}_R^{\simeq} & \longrightarrow & [\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \end{array}$$

to a surjective map

$$\widetilde{\mathrm{res}} : \mathcal{T}_R^{\simeq} \to [\![R]\!]_{\mathrm{inj}}/{\sim_\Gamma} \; .$$

By Lemma 68, $\widetilde{\mathrm{res}}$ is injective, which completes the proof. □

In informal terms, if R is a normal indexed MELL proof-structure, an injective experiment of R is a (recursive) choice of a number of copies of each box, together with a name for each axiom link in the said copies. As such, an equivalence class (up to renaming) of injective experiments of R is a (recursive) choice of a number of copies of each box, that is, an element of the quotiented Taylor expansion of R.

Another (imprecise) way to sum it up is that:

$$[\![R]\!]_{\mathrm{inj}} = \bigsqcup_{\rho \in \mathcal{T}_R} [\![\rho]\!]_{\mathrm{inj}},$$

with each $[\![\rho]\!]_{\mathrm{inj}}$ being an equivalence class under the substitution of atoms.

This theorem is folklore and can be seen as a mere side remark, as relational experiments and differential nets inside the Taylor expansion are variations around the idea that a box enshrines a subnet that can be copied an arbitrary number of time. We nonetheless decide to emphasize its precise formulation which is far from trivial.

5.6. Injectivity of the relational model

Theorem 70. *The relational model is injective for connected* MELL*-proof structures*

Compare and contrast with the current work of Daniel.

6. A graphical model of linear logic

6.1. The model of injective relations

The relational model of linear logic is the simplest (and probably the most canonical) of the models of linear logic. For our purposes, the category **Rel** can be defined as the category with MELL formulæ as objects and relations $f \subseteq |A| \times |B|$ as morphisms $f : A \to B$. It is a $*$-autonomous category, with the set-theoretic product as tensor product. **Rel** has finite products, with the set-theoretic sum as cartesian product. It admits a free comonoid modality as the functor that maps a set A to the set

$$\mathcal{M}_{\mathrm{fin}}(A) = \{[[a_1, \ldots, a_n]], n \in \mathbf{N}, a_1, \ldots, a_n \in A\}$$

of finite multisets over A. Equipped with all this structure, **Rel** is a model of linear logic.

Let R be an indexed MELL proof structure of conclusion $A^\perp \parr B$ (every proof structure can be seen as having a conclusion of this type). The relational interpretation defined in the last section $[\![R]\!]$ is a morphism

$$[\![R]\!] : A \to B.$$

We introduced a tamer subset of $[\![R]\!]$: its injective semantics $[\![R]\!]_{\mathrm{inj}}$. It seems interesting to define a smaller model of linear logic, contained in **Rel**, whose objects will be injective relations, that is, relations containing only injective points.

As a first try, we define the graph $\mathfrak{R}$ as the graph with objects the sets $|A|$, for A a formula of MELL and morphisms $f : A \to B$ the *injective relations* $f \subseteq |A^\perp \parr B|_{\mathrm{inj}}$. The first difficulty arise when trying to define the composition. The composition in **Rel** is given by, for $s : A \to B$ and $t : B \to C$:

$$t \circ s = \{(a, c) \mid \exists b \in B, (a, b) \in s, (b, c) \in t\}$$

But defining it in the same way in $\mathbf{Rel}_{\mathrm{inj}}$ would not define a model of linear logic. Let's consider the two ps: draw them

Their respective interpretation are

$$\begin{aligned}
[\![R]\!]_{\mathrm{inj}} &= \{((a, a'), (a, a')), a, a' \in |A|\},\\
[\![S]\!]_{\mathrm{inj}} &= \{(a, a), a \in |A|\}.
\end{aligned}$$

The two ps can be composed, and the injective interpretation of their composite is not empty, and is equal to $[\![S]\!]_{\mathrm{inj}}$. But $[\![S]\!]_{\mathrm{inj}}$ composed (in Rel) with $[\![R]\!]_{\mathrm{inj}}$ is empty. The composition needs to be relaxed in order to accommodate non-injective points during the composition. For $a \in |A|$, we write $\mathrm{At}(a)$ for the set of atoms that appear in x.

$$\begin{aligned}
t \circ_{\mathrm{inj}} s = \{(a, c) \in |A \parr C|_{\mathrm{inj}} \mid\, & (a, b) \in f_* s,\\
& (b, c) \in g_* t,\\
& x \in \mathrm{At}(a) \cap \mathrm{At}(c) \Rightarrow x \in \mathrm{At}(b)\}
\end{aligned}$$

The graph $\mathfrak{R}$ is not a category endowed with the composition $\circ_{\mathrm{inj}}$. Indeed, let A be an atomic formula and $a \in |A|$. The identity of A is not neutral:

$$\mathrm{id}_A \circ_{\mathrm{inj}} \{(a, a)\} = \{(a', a'), a' \in |A|\}.$$

In order for the identities to be neutral, all the relations must be closed for the substitution of atoms. Which motivates the definition:

Definition 71 (category of injective relations)**.** *The* category of injective relations $\mathbf{Rel}_{\mathrm{inj}}$ *is the category with* MELL *formulæ as objects, and saturated injective relations, that is relations* $f \subset |A^\perp \parr B|_{\mathrm{inj}}$ *such that,*

$$\forall a \in f, \forall a' \in |A^\perp \parr B|_{\mathrm{inj}}, a \sim_{A^\perp \parr B} a' \Rightarrow a' \in f$$

as morphisms $f : A \to B$*, with the composition given by:*

$$\begin{aligned}
t \circ_{\mathrm{inj}} s = \{(a, c) \in |A \parr C|_{\mathrm{inj}} \mid\, & (a, b) \in f_* s,\\
& (b, c) \in g_* t,\\
& x \in \mathrm{At}(a) \cap \mathrm{At}(c) \Rightarrow x \in \mathrm{At}(b)\}
\end{aligned}$$

Lemma 72. *Let* A*,* B *and* C *be three formulæ, and* $a \in |A|$*,* $b \in |B|$ *and* $c \in |C|$ *so that* $\overline{(a, b)} \in s : A \to B$ *and* $\overline{(b, c)} \in t : A \to B$*.*

There exists $a' \in |A|$ *so that* $(a', b) \in \overline{(a, b)}$ *and* $\mathrm{at}_2(a') \cap \mathrm{at}_2(c) = \varnothing$*.*

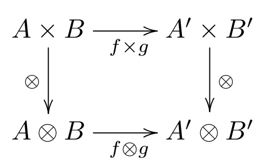
Proof. We prove it by induction on $n = \#(\mathrm{at}_2(a) \cap \mathrm{at}_2(c))$. If $n = 0$, it is obvious. Else, suppose it has been proved for $n - 1$.

Let $x \in \mathrm{at}_2(a) \cap \mathrm{at}_2(c)$ and X be an atomic formula so that $x \in |X|$. As $|(a, b)|_x = 2$, $x \notin \mathrm{at}(b)$. Let $x' \in |X| \setminus (\mathrm{at}(a, b) \cup \mathrm{at}(c))$. Let $\sigma \in \mathcal{S}_X$ be the permutation that exchange x and x'. $(\sigma a, b) \in \overline{(a, b)}$ and $\#(\mathrm{at}_2(\sigma a) \cap \mathrm{at}_2(c)) = n - 1$. □

This shows that we can always add conditions of non-intersection of atomic elements. This allows us to prove that the composition is associative, and that the diagonal relation is neutral for the composition, which is to say $\mathbf{Rel}_{\mathrm{inj}}$ is a category.

Proposition 73. *The category* $\mathbf{Rel}_{\mathrm{inj}}$ *is symmetric monoidal closed.*

Proof. We take the cartesian product $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as a tensorial product. The bifunctoriality is obvious, as it is the commutation of

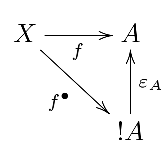


for every $f : A \to A', g : B \to B'$, but the two lines are equal, which makes the diagram trivial.

The other properties (lax associativity, neutrality of $|\mathbf{1}|$, symmetry) are well-known properties of the cartesian product. For the closedness, we take take the monoidal exponential equal to the cartesian product, $\multimap = \times$. □

Proposition 74. *The functor* ! *is the left adjoint of the forgetful functor of the category of* $\mathbf{Rel}_{\mathrm{inj}}$ *comonoids into* $\mathbf{Rel}_{\mathrm{inj}}$.

Proof. This propriety translates in the following way: for any object A of $\mathbf{Rel}_{\mathrm{inj}}$, there exists an object $!A$ and a morphism $\varepsilon_A : !A \to A$ (said DERELICTION MORPHISM) so that, for every comonoid X of $\mathbf{Rel}_{\mathrm{inj}}$, every morphism $f : X \to A$ lifts to a comonoid morphism $f^\bullet : X \to !A$ so that:



In our case, $!A = \mathcal{M}_{\mathrm{fin}}(A)$ and $\varepsilon_A = \{([a], a), a \in A\}$. The comonoid morphism defined by $f^\bullet : x \mapsto [f(x)]$ is the lifting desired. It is obviously unique. □

We deduce the main theorem of the section:

Theorem 75. $\mathbf{Rel}_{\mathrm{inj}}$ *is a model of the multiplicative/exponential fragment of linear logic.*

Proof. According to propositions 73 and 74, $\mathbf{Rel}_{\text{inj}}$ is a Lafont category, and, by proposition 23 of [16], a model of the multiplicative/exponential fragment of intuitionnistic linear logic. As the interpretation of every formula is isomorphic to its double negation, $\mathbf{Rel}_{\text{inj}}$ is a model of MELL. □

6.2. Recovering **Rel**

$\mathbf{Rel}_{\text{inj}}$ is not only a beautiful model, whose morphisms are sets of normal DiLL_0-ps, but it allows also to recover the full relational model in a canonical way, and is minimal for that construction.

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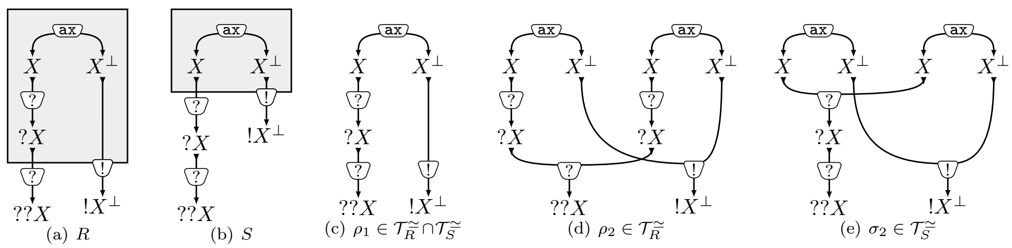

Figure A.5: Two non-isomorphic MELL-ps R (Figure 5(a)) and S (Figure 5(b)), having the same 1-diffnet ρ_1 (Figure 5(c)) but two different 2-diffnets, ρ_2 (Figure 5(d)) and σ_2 (Figure 5(e)) respectively.

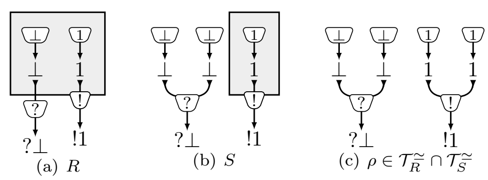

Figure A.6: Two non-isomorphic MELL-ps R (Figure 6(a)) and S (Figure 6(b)), having the same 2-diffnet ρ (Figure 6(c)).

AppendixA. Technical appendix

AppendixA.1. Examples

Some examples in several figures.