

A Deep Inference System for Differential Linear Logic

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Differential linear logic (DiLL) provides a fine analysis of resource consumption in cut-elimination and a logical basis for a theory of approximation via the notion of Taylor expansion. We investigate the subsystem of DiLL without promotion via the tools of deep inference in order to push cut-elimination at an atomic level. We prove that in our system every provable formula admits a derivation in normal form. Moreover, we provide a procedure to normalize derivations.

1 Introduction

Girard [12] introduced linear logic (LL) as a refinement of intuitionistic and classical logics, built around cut-elimination. In LL, a pair of dual modalities (the *exponentials* ! and ?) give a logical status to the operations of erasing and copying (sub-)proofs in the cut-elimination procedure. The idea is that *linear* proofs (*i.e.* proofs without exponentials) use their hypotheses exactly once, whilst *exponential* proofs may use their hypotheses at will. In particular, the *promotion rule* makes a (sub-)proof available to be erased or copied an unbounded number of times, provided that its hypotheses are as well (it is a contextual rule). Via Curry–Howard correspondence between programs and proofs, LL gives a logical status to the operations of erasing and copying data in the evaluation process. Linear proofs correspond to programs which call their arguments exactly once, exponential proofs to programs which call their arguments at will. The study of LL contributed to unveil the logical nature of resource consumption.

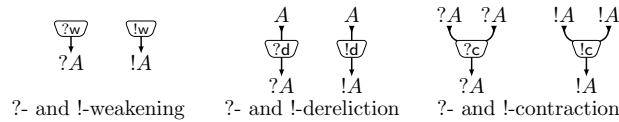
The importance of being differential. A further tool for the analysis of resource consumption came from Ehrhard and Regnier’s work on differential λ -calculus [6] and differential linear logic (DiLL, [8, 23]). Despite the fact that DiLL is inconsistent (every sequent $\vdash \Gamma$ can be proved), it has a cut-elimination theorem [23, 11] and internalizes notions from denotational models of LL into the syntax. In particular, DiLL₀ (the *promotion-free* fragment of DiLL, [8]) is a syntax corresponding to the semantic constructions defined by Ehrhard’s finiteness spaces [3]. Finiteness spaces interpret linear proofs as linear functions on certain topological vector spaces, on which one can define an operation of derivative. Exponential proofs are interpreted as analytic functions, in the sense that they can be arbitrarily approximated by the semantic equivalent of a Taylor expansion [3, 4], which becomes available thanks to the presence of a derivative operator. In syntactic terms, these constructions take an interesting form: they correspond to “symmetrizing” the exponential modalities, *i.e.* in DiLL₀ the rules handling the dual exponential modalities ! and ? are perfectly symmetrical, although the logic is not self-dual. Indeed, in LL, only the promotion rule introduces the ! modality, creating inputs that can be called an unbounded number of times. In DiLL₀ the rules handling the ! modality (!-dereliction !d, !-contraction !c, !-weakening !w) are the duals of the usual rules dealing with the ? modality (?-dereliction ?d, ?-contraction ?c, ?-weakening ?w). In particular, *!-dereliction* expresses in the syntax the semantic derivative: it releases inputs of type !A that must be called exactly *once*, so that executing a program f on a “!-derelicted” input x (*i.e.* performing cut-elimination on a proof f cut with a “!-derelicted” sub-proof x) amounts to compute the best linear approximation of f on x . This imposes non-deterministic choices: if in an evaluation the

program f needs several copies of the input x (*i.e.* if the proof f uses several times the hypothesis $!A$), then there are different executions of f on x , depending on which sub-routine (*i.e.* hypothesis) of f is fed with the unique available copy of x . Thus we get a formal *sum*, where each term represents a possibility. The rules $!$ -contraction and $!$ -weakening put together a finite (possibly 0) number of copies of an input, so that it can be called a *bounded* number of times during execution.

What is also interesting is that LL promotion rule can be encoded in DiLL_0 through the notion of syntactic *Taylor expansion* [7, 9, 22, 24]: a proof of LL can be decomposed into a possibly infinite set of (promotion-free) proofs of DiLL_0 . The idea, given a proof in LL with exactly one promotion rule $!p$, is to replace $!p$ (which makes the resource π available at will) with an infinite set of “differential” proofs of DiLL_0 , each of them taking $n \in \mathbb{N}$ copies of π so as to make the resource π available exactly n times.

The potential infinity of the promotion rule becomes an actual infinite via the Taylor expansion.

Nets vs. sequents. The system DiLL_0 is usually presented in two formalisms: sequent calculus and Lafont’s interaction nets [20] (a graphical representation of proofs similar to LL proof-nets). The *symmetry* of the rules handling the dual exponentials $!$ and $?$ in DiLL_0 is evident in interaction nets, but not at all in the sequent calculus. In interaction nets for DiLL_0 , the rules for $?$ and $!$ have the *same geometry*:



So, the distinction between $!$ and $?$ is given only by their different behaviors in correctness graphs (a geometrical characterization of the interaction nets corresponding to proofs in the DiLL_0 sequent calculus). But meaningful operations in DiLL_0 such as cut-elimination can be defined directly on interaction nets, regardless of being correct or not. The benefit is that DiLL_0 cut-elimination steps defined on interaction nets are perfectly symmetric: for instance, the step $?c/!d$ is exactly the *dual* of the step for a cut $!c/?d$, and similarly for the other steps (see [23, Fig. 4]).

This elegant symmetry in the presentation of cut-elimination steps is lost in DiLL_0 sequent calculus, see our Figure 3. Moreover, cut-elimination in DiLL_0 sequent calculus has to deal with (many!) uninteresting commutative steps, while interaction nets get rid of them. Thus, interaction nets allows one to express DiLL_0 cut-elimination with a sharper account than in sequent calculus. Not by chance, all papers dealing with DiLL_0 cut-elimination use only interaction nets [8, 23, 11, 25].

However, the interaction net presentation of DiLL_0 has some flaws that do not affect the sequent calculus: interaction nets do not have an *inductive tree-like structure* and so it is not easy to handle them. Moreover, not all interaction nets correspond to a derivation in DiLL_0 sequent calculus, a *global* correctness criterion is required to identify them.

Our contribution. We define a proof system for DiLL_0 in the formalism of the *open deduction* [16] following the principles of *deep inference* [17, 2, 15, 26, 31]. Such a formalism, which allows rules to be applied deep in a context, provides a more flexible composition of derivations and makes explicit the behavior of the cut-elimination process in DiLL_0 in a more fine-grained way, since it pushes cut-elimination at an *atomic* level. Besides, our deep inference system for DiLL_0 gathers good qualities of both sequent calculus and interaction nets formalisms: it restores the interaction net *symmetries* lost in the sequent calculus and its derivations keep a handy *inductive tree-like* (or better, *sequence-like*) structure as in the sequent calculus, without the need for a global correctness criterion like in interaction nets.

$$\begin{array}{c}
\frac{}{\vdash A, \bar{A}}^{\text{ax}} \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}^{\wp} \quad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}^{\otimes} \quad \frac{}{\vdash \bar{1}}^{\bar{1}} \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}^{\perp} \quad \frac{\vdash \Gamma, A \quad \vdash \bar{A}, \Delta}{\vdash \Gamma, \Delta}^{\text{cut}} \\
\frac{\vdash \Gamma, A}{\vdash \Gamma, !A}^{\text{!d}} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}^{\text{?d}} \quad \frac{\vdash \Gamma, !A \quad \vdash !A, \Delta}{\vdash \Gamma, !A, \Delta}^{\text{!c}} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}^{\text{?c}} \quad \frac{}{\vdash !A}^{\text{!w}} \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}^{\text{?w}} \quad \frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}^{\text{sum}} \quad \frac{}{\vdash \Gamma}^{\text{zero}} \quad \left| \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}^{\text{!p}}
\end{array}$$

Figure 1: Sequent calculus rules for DiLL₀ (on the left) and the promotion rule (on the right) [23].

A first attempt in the direction of a deep inference system for DiLL₀ is in [10] where, however, the sum-rule is absent and, as a consequence, it is not suitable to represent the dynamic behavior of DiLL₀.

To fully recover the expressiveness of this logic, we design our system to include a binary connective $+$ which represents the sum operation. The rules for $+$ (and for its unit 0) prevent the use of Guglielmi and Tubella’s general result [29, 30] to show cut-elimination. However, we are able to define a normalization procedure by rule permutations which fully captures the dynamics of DiLL₀ cut-elimination similar to the one in [27, 28].

Thanks to the symmetry of rules in our formalism, we are also able to reduce cut-rule to an *atomic* level. In particular, we can classify our rule permutations based on their behavior: some rule permutations correspond to multiplicative cut-elimination steps, other permutations correspond to “resource management” cut-elimination steps (involving the $?$ and $!$ rules), other permutations correspond to “slice management” operations (involving the propagation of $+$ and 0).

2 Differential Linear Logic

We present here the classical, propositional, one-sided sequent calculus for *differential linear logic* without promotion (DiLL₀). The formulas of DiLL₀ are exactly the same as in the multiplicative exponential fragment of linear logic (MELL). MELL *formulas* are defined by the grammar below, where a, b, c, \dots range over a countably infinite set of propositional variables:

$$A, B ::= a \mid \bar{a} \mid 1 \mid \perp \mid A \otimes B \mid A \wp B \mid !A \mid ?A$$

Linear negation $\bar{(\cdot)}$ is defined through De Morgan laws so as to be involutive ($\bar{\bar{A}} = A$ for any A):

$$\overline{(a)} = \bar{a} \quad \overline{(\bar{a})} = a \quad \overline{A \otimes B} = \bar{A} \wp \bar{B} \quad \overline{A \wp B} = \bar{A} \otimes \bar{B} \quad \bar{1} = \perp \quad \bar{\perp} = 1 \quad \overline{!A} = ?\bar{A} \quad \overline{?A} = !\bar{A}$$

Variables and their negations are *atomic*; \otimes, \wp are *multiplicative connectives* and $1, \perp$ are their respective *units*; $!, ?$ are *exponential modalities*. A MELL *sequent* is a (finite) multiset of MELL formulas A_1, \dots, A_n (for any $n \geq 0$), and it is ranged over by Γ, Δ, Σ .

Figure 1 gives the sequent calculus rules¹ for *differential linear logic* DiLL₀ (without *promotion* !p); the rules on the first line correspond to the *multiplicative linear logic* fragment MLL. We set:

$$\text{MELL} = \text{MLL} \cup \{?w, ?d, ?c, !p\} \quad \text{MELL}_0 = \text{MLL} \cup \{?w, ?d, ?c\} \quad \text{DiLL}_0^- = \text{DiLL}_0 \setminus \{\text{zero, sum}\}$$

We define \equiv as the equivalence on DiLL₀ derivations generated by the relations in Figure 2. Roughly, the rule zero plays the role of annihilating element with respect to all the other rules but sum, for which it is a neutral element; whilst the rule sum commutes with any rule below it. Clearly, \equiv preserves conclusions and can be oriented so as to define a terminating rewriting relation that pushes down the rules zero and sum in a derivation. As a consequence, every derivation in DiLL₀ can be rewritten in a *canonical form* (with the same conclusions).

¹Usually, in the literature on LL and DiLL, the rules $?w, ?d, ?c, !w, !d, !c$ are called respectively weakening, dereliction, contraction, co-weakening, co-dereliction and co-contraction. To avoid clashes with the usual terminology in deep inference (see Footnote 4), we call them $?-weakening, ?-dereliction, ?-contraction, !-weakening, !-dereliction$ and $!-contraction$, respectively.

$$\begin{array}{c}
\frac{}{\vdash \Gamma_{r_1} \text{zero} \equiv \overline{\vdash \Delta} \text{zero}} \quad \frac{}{\vdash \Gamma_{r_2} \text{zero} \equiv \overline{\vdash \Sigma} \text{zero} \equiv \overline{\vdash \Gamma} \text{zero} \vdash \Delta_{r_2}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{zero} \equiv \overline{\vdash \Gamma} \text{zero} \vdash \Gamma_{\text{sum}}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{zero} \equiv \overline{\vdash \Gamma} \text{zero} \vdash \Gamma_{\text{sum}}} \quad \frac{}{\vdash \Gamma_{r_1} \text{sum} \equiv \overline{\vdash \Delta} \text{sum} \vdash \Gamma_{r_1}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{sum} \equiv \overline{\vdash \Gamma} \text{sum} \vdash \Gamma_{\text{sum}}} \\
\frac{}{\vdash \Gamma_{\text{sum}} \text{sum} \equiv \overline{\vdash \Gamma} \text{sum} \vdash \Gamma_{\text{sum}}} \quad \frac{}{\vdash \Gamma_{r_2} \text{sum} \equiv \overline{\vdash \Sigma} \text{sum} \vdash \Delta_{r_2}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{sum} \equiv \overline{\vdash \Gamma} \text{sum} \vdash \Gamma_{\text{sum}}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{sum} \equiv \overline{\vdash \Gamma} \text{sum} \vdash \Gamma_{\text{sum}}} \quad \frac{}{\vdash \Gamma_{\text{sum}} \text{sum} \equiv \overline{\vdash \Gamma} \text{sum} \vdash \Gamma_{\text{sum}}}
\end{array}$$

Figure 2: The equivalence \equiv on derivations generated by the rules zero and sum in DiLL_0 , where r_1 is any unary rule in DiLL_0 , and r_2 is any binary rule in DiLL_0 but sum.

Definition 2.1 (Canonical form, slice). Let π be a derivation in DiLL_0 :

1. π is a *slice* if it is in DiLL_0^- (i.e. the rules zero and sum do not occur in π);
2. π is *canonical* or in *canonical form* if either it consists of a zero rule, or it is a slice, or if its last rule is sum with a canonical form as left premise and a slice as right premise.

A *canonical form* of π is any canonical derivation π' in DiLL_0 such that $\pi \equiv \pi'$.

Fact 2.2 (Canonicity). *Any derivation in DiLL_0 has a canonical form (with same conclusions).*

Intuitively, considering only canonical derivations, slices are the “real and meaningful” proofs in DiLL_0 (corresponding to simple nets in [8, 22, 23]), while the rules sum and zero are needed to define cut-elimination in DiLL_0 , in particular they ensure that the conclusions of a derivation is preserved after a cut-elimination step (see below). The rule sum puts together slices with the same conclusions $\vdash \Gamma$, similarly to multiset union: it expresses the possibility of several “real proofs” of $\vdash \Gamma$. The rule zero then corresponds to the empty multiset of “real proofs” of $\vdash \Gamma$: it claims $\vdash \Gamma$ without a proof (it is reminiscent of daimon in ludics [13]). Because of it, any MELL sequent (also the empty one) is provable in DiLL_0 .

A derivation in DiLL_0 is with *atomic axioms* if every instance of the rule ax introduces a MELL sequent of the form $\vdash a, \bar{a}$, where a is a propositional variable.

Proposition 2.3 (Atomic axioms). *For every derivation π in DiLL_0 with conclusion $\vdash \Gamma$, there exists a derivation π' in DiLL_0 with conclusion $\vdash \Gamma$ and atomic axioms. If, moreover, π is canonical (resp. a slice) then π' is canonical (resp. a slice).*

Proof. Rewrite any non-atomic instance of the rule ax according to the η -expansion relation \rightsquigarrow_η below:

$$\frac{}{\vdash A \otimes B, \bar{A} \wp \bar{B}} \text{ax} \rightsquigarrow_\eta \frac{\frac{}{\vdash A, \bar{A}} \text{ax} \quad \frac{}{\vdash B, \bar{B}} \text{ax}}{\vdash A \otimes B, \bar{A}, \bar{B}} \otimes \quad \frac{}{\vdash 1, \perp} \text{ax} \rightsquigarrow_\eta \frac{\frac{}{\vdash 1} 1}{\vdash 1, \perp} \perp \quad \frac{}{\vdash !A, ?A} \text{ax} \rightsquigarrow_\eta \frac{\frac{}{\vdash A, \bar{A}} \text{ax}}{\vdash A, !\bar{A}} \text{id} \quad \frac{}{\vdash A, \bar{A}} \text{id} \rightsquigarrow_\eta \frac{}{\vdash ?A, !\bar{A}} \text{?d} \quad (1)$$

It is immediate to prove that the relation \rightsquigarrow_η on the derivations of DiLL_0 is terminating. \square

Cut-elimination. Rewriting rules $\rightsquigarrow_{\text{cut}}$ for cut-elimination in DiLL_0 sequent calculus are defined in Figure 3. They are just the formulation in the sequent calculus formalism of the cut-elimination steps defined and studied in [8, 11] and [23, Fig. 4] within the interaction nets formalism. We represent in Figure 3 only the *key cases*, where the principal connectives in the cut formulas are dual (the pairs of dual connectives are \otimes/\wp , $1/\perp$, $!/?$). The way DiLL_0 deals with the *commutative cases* is omitted since is analogous to usual sequent calculi. With these cut-elimination steps it has been proved in [8, 23, 11] that the rule cut is admissible in DiLL_0 (and even in DiLL , i.e., the system DiLL_0 plus MELL promotion !p).

is allowed. The rules for $?$ ($?w$, $?d$, $?c$) *ask for* a number of resources of type $!A$ (0, 1, and the sum of the numbers asked by its premises, respectively), while the rules for $!$ ($!w$, $!d$, $!c$) *supply* a number of resources of type $!A$ (0, 1, and the sum of the numbers supplied by its premises, respectively). Cases:

1. If the numbers of demanded and supplied resources match, the cut-elimination proceeds normally (see the steps $?d/!d$ and $?w/!w$).
2. The step $?c/!c$ is slightly more complicated: it essentially connects the dual premises of a $?c$ -contraction and of a $!c$ -contraction in all possible ways.
3. The step $?c/!w$ duplicates the rule $!w$, spreading the information that there are no available resources to the premises of $?c$.
4. The step $?d/!w$ represents a *mismatch* in supply and demand: $?d$ -dereliction asks for a resource but $!w$ -weakening says that it is not available; the rule zero in the resulting derivation keeps track of this mismatch, as a sort of error in computation, and ensures that the conclusions are preserved.
5. In the step $?c/!d$, $?c$ -contraction represents two possible demands for a resource, but according to $!d$ -dereliction only one resource is available, so there is a *non-deterministic choice* on which request will be fed, the other one will receive a $!w$ -weakening; the rule sum has to be intended as a way to keep track of all possible choices, not as a way to duplicate resources; said differently, in the step $?c/!d$ a derivation reduces to a pair of derivations (of slices, if we consider their canonical forms).

By duality, the discussion above about resource management is similar for the steps $?w/!c$, $?w/!d$ and $?d/!c$, respectively. In Appendix A we provide some running examples of the cut-elimination procedure.

It is worth comparing cut-elimination steps as defined for DiLL_0 sequent calculus (Figure 3) and for DiLL_0 interaction nets ([8, Sect. 2], [23, Fig. 4]): symmetry and duality in the latter are lost in the former.

As there is not promotion rule $!p$, in DiLL_0 transforming a derivation in one with atomic axioms does not commute with cut-elimination. For instance, derivation π below reduces to π' via cut-elimination; but derivation π_η with atomic axioms, obtained from π through η -expansion (the procedure described in the proof of Proposition 2.3) reduces to $\pi'_\eta \neq \pi'$ via cut-elimination.

$$\pi = \frac{\frac{\overline{\vdash !\bar{a}, ?a}^{\text{ax}} \quad \overline{\vdash !\bar{a}}^{\text{!w}}}{\vdash !\bar{a}}^{\text{cut}}}{\vdash !\bar{a}} \rightsquigarrow_{\text{cut}} \overline{\vdash !\bar{a}}^{\text{!w}} = \pi' \qquad \pi_\eta = \frac{\frac{\overline{\vdash !\bar{a}, a}^{\text{!d}}}{\vdash !\bar{a}, ?a}^{\text{?d}} \quad \overline{\vdash !\bar{a}}^{\text{!w}}}{\vdash !\bar{a}}^{\text{cut}}}{\vdash !\bar{a}} \rightsquigarrow_{\text{cut}} \overline{\vdash !\bar{a}}^{\text{zero}} = \pi'_\eta$$

3 Calculus of Structures for DiLL_0

In this section, we introduce a deep inference system [17, 2, 15, 31] suitable for DiLL_0 , using the *open deduction* formalism [16]. As a first novelty, we internalize the rules zero and sum of DiLL_0 sequent calculus at the level of formulas. In fact, derivations in deep inference systems have a sequence structure instead of the more general tree-like structure of sequent calculus: every rule in deep inference has exactly one premise, consisting of one formula. This because the meta-connectives for sequent composition (the comma) and sequent juxtaposition (derivation branching) are internalized by \wp and \otimes , respectively. To internalize the DiLL_0 meta-connective for sum, together with its unit, we introduce the (commutative and associative) binary connective $+$ and its unit 0 ². In this way, the rule sum branches the derivation tree with a connective, $+$; and similarly, the rule zero has its own premise, 0 . Thus, *formulas* are defined by:

²Here, the new connective $+$ has nothing to do with the additive disjunction \oplus in LL; and the unit 0 for $+$ must not be confused with the additive unit 0 for \oplus in LL.

$$A, B ::= a \mid \bar{a} \mid A \otimes B \mid A \wp B \mid 1 \mid \perp \mid !A \mid ?B \mid 0 \mid A + B$$

where a, b, c, \dots range over the usual countably infinite set of propositional variables (so, a MELL formula is a formula with no occurrences of $+$ and 0). Formulas are identified up to the *equivalence* \simeq defined as the least congruence on formulas generated by the relations in (2).

$$\begin{array}{lll}
A \wp (B \wp C) \simeq (A \wp B) \wp C & A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C & A + (B + C) \simeq (A + B) + C \\
A \wp B \simeq B \wp A & A \otimes B \simeq B \otimes A & A + B \simeq B + A \\
A \wp \perp \simeq A & A \otimes 1 \simeq A & A + 0 \simeq A \\
A \wp (B + C) \simeq (A \wp B) + (A \wp C) & A \otimes (B + C) \simeq (A \otimes B) + (A \otimes C) & \\
!(A + B) \simeq !A + !B & ?(A + B) \simeq ?A + ?B & \\
0 \otimes A \simeq 0 & 0 \wp A \simeq 0 & ?0 \simeq 0 \quad !0 \simeq 0
\end{array} \tag{2}$$

[[Matteo: togliere !0 = ?0 = 0]]

Some equivalences in (2) correspond to well-known isomorphisms in MLL. With respect to \simeq , the formula 0 is an annihilating element for all other connectives but $+$, for which it is a neutral element; every connective other than $+$ distributes over $+$.

An *additive normal* formula A is a sum of MELL formulas, *i.e.* $A = A_1 + \dots + A_n$ ($n \in \mathbb{N}$) where all A_i 's are MELL formulas ($A = 0$ for $n = 0$). For any $n \in \mathbb{N}$, we set $n = \underbrace{1 + \dots + 1}_{n \text{ times}}$. Note that, by the

equivalences in (2), $n \otimes m = k$ where $k = n \times m$.

A *context* (resp. MELL *context*) $C\{\}$ is a formula (resp. MELL formula) with exactly one occurrence of the hole $\{\}$ (a special propositional variable). We write $C\{A\}$ for the formula obtained from the context $C\{\}$ by replacing its hole with the formula A .

Remark 3.1 (Additive normal form). By definition of \simeq , if $C\{\}$ is a context, $C\{A + 0\} \simeq C\{A\}$ and $C\{A + B\} \simeq C\{A\} + C\{B\}$. If $C\{\}$ is a MELL context, $C\{0\} \simeq 0$. In general, any formula A has an additive normal formula $A' \simeq A$: equivalences in (2) but the second line can be oriented to define a terminating rewriting relation whose normal forms are additive normal.

Derivations. A *deep inference system* \mathcal{S} is a set of inference rules. A *derivation* \mathcal{D} from a *premise* B

to a *conclusion* A in a deep inference system \mathcal{S} is denoted by $\mathcal{D} \parallel_{\mathcal{S}}$ or $\mathcal{D} \triangleright B \stackrel{\mathcal{S}}{\vdash} A$, and defined by:

- (*assumption*) a formula A is a derivation (denoted by A) with premise and conclusion A ;
- (*horizontal composition*) if for all $i \in \{1, 2\}$ \mathcal{D}_i is a derivation from B_i to A_i , then for any $\bullet \in \{\wp, \otimes, +\}$, $\mathcal{D}_1 \bullet \mathcal{D}_2$ is a derivation from $B_1 \bullet B_2$ to $A_1 \bullet A_2$ (see (3) below on the left);
- (*vertical composition*) if $\rho \frac{A_1}{B_2}$ and, for all $i \in \{1, 2\}$, \mathcal{D}_i is a derivation from B_i to A_i , then $\mathcal{D}_1 \circ_{\rho} \mathcal{D}_2$ is a derivation from B_1 to A_2 (see (3) below on the right).³

$$\begin{array}{ccc}
\begin{array}{c} B_1 \bullet B_2 \\ \mathcal{D}_1 \bullet \mathcal{D}_2 \parallel_{\mathcal{S}} \\ A_1 \bullet A_2 \end{array} = \begin{array}{|c|} \hline B_1 \\ \hline \mathcal{D}_1 \parallel_{\mathcal{S}} \\ \hline A_1 \end{array} \bullet \begin{array}{|c|} \hline B_2 \\ \hline \mathcal{D}_2 \parallel_{\mathcal{S}} \\ \hline A_2 \end{array} & \text{for } \bullet \in \{\wp, \otimes, +\} & \begin{array}{c} B_1 \\ \mathcal{D}_1 \parallel_{\mathcal{S}} \\ B_1 \\ \mathcal{D}_1 \circ_{\rho} \mathcal{D}_2 \parallel_{\mathcal{S}} = \rho \frac{A_1}{B_2} \\ A_2 \\ \mathcal{D}_2 \parallel_{\mathcal{S}} \\ A_2 \end{array} \text{ for } \rho \in \mathcal{S} \tag{3}
\end{array}$$

³ We can write $\simeq \frac{A}{B}$ as a rule in a derivation if $A \simeq B$, although formally its use is implicit as formulas are identified up to \simeq .

$$\begin{array}{cccccccccccc}
\text{ai}^\downarrow \frac{1}{a \wp \bar{a}} & \text{!d}^\downarrow \frac{A}{!A} & \text{?d}^\downarrow \frac{A}{?A} & \text{!w}^\downarrow \frac{1}{!A} & \text{?w}^\downarrow \frac{\perp}{?A} & \text{!c}^\downarrow \frac{!A \otimes !A}{!A} & \text{?c}^\downarrow \frac{?A \wp ?A}{?A} & \text{+}^\downarrow \frac{A+A}{A} & \text{0}^\downarrow \frac{0}{A} & & \frac{A \otimes (B \wp C)}{(A \otimes B) \wp C} \\
\text{ai}^\uparrow \frac{a \otimes \bar{a}}{\perp} & \text{!d}^\uparrow \frac{?A}{A} & \text{?d}^\uparrow \frac{!A}{A} & \text{!w}^\uparrow \frac{?A}{\perp} & \text{?w}^\uparrow \frac{!A}{1} & \text{!c}^\uparrow \frac{?A}{?A \wp ?A} & \text{?c}^\uparrow \frac{!A}{!A \otimes !A} & \text{+}^\uparrow \frac{A}{A+A} & \text{0}^\uparrow \frac{A}{0} & &
\end{array}$$

Figure 4: The rules of the deep inference system SDDI (A, B, C are MELL formulas). [\[\[Matteo: aggiungere regole strutturali per dereliction su 0\]\]](#)

We write $B \stackrel{s}{\vdash} A$ if there is a derivation $\mathcal{D} \triangleright B \stackrel{s}{\vdash} A$. A rule $\rho \frac{B}{A}$ is *derivable* in S if $B \stackrel{s}{\vdash} A$.

The system SDDI is defined by the rules in Figure 4. All rules in SDDI have exactly one premise, as usual in deep inference. The *down-fragment* and *up-fragment*⁴ of SDDI are the following sets of rules:

$$\text{DDI}^\downarrow = \{\text{ai}^\downarrow, \text{!d}^\downarrow, \text{?d}^\downarrow, \text{!w}^\downarrow, \text{?w}^\downarrow, \text{!c}^\downarrow, \text{?c}^\downarrow, \text{+}^\downarrow, \text{0}^\downarrow, \text{s}\} \quad \text{DDI}^\uparrow = \{\text{ai}^\uparrow, \text{!d}^\uparrow, \text{?d}^\uparrow, \text{!w}^\uparrow, \text{?w}^\uparrow, \text{!c}^\uparrow, \text{?c}^\uparrow, \text{+}^\uparrow, \text{0}^\uparrow, \text{s}\}$$

Note the perfect up/down symmetry between DDI^\downarrow and DDI^\uparrow , and that $\text{SDDI} = \text{DDI}^\downarrow \cup \text{DDI}^\uparrow$. We denote by $\text{DDI}^\downarrow_- = \text{DDI}^\downarrow \setminus \{\text{+}^\downarrow, \text{0}^\downarrow\}$. Note that in a DDI^\downarrow_- derivation only MELL formulas occur.

Roughly, rules in DDI^\downarrow somehow mimic the ones in $\text{DiLL}_0 \setminus \{\text{cut}\}$. Rules in DDI^\uparrow are their duals, turning them upside down. Derivations in DDI^\downarrow_- correspond to cut-free slices in DiLL_0^- (see Theorem 4.3).

Remark 3.2 (Deep). The idea of deep inference is that inference rules can be applied “deep” in any context: in a deep inference system S , if $\rho \frac{B}{A} \in S$ then, for any context $C\{\}$, $\rho \frac{C\{B\}}{C\{A\}}$ is derivable in S .

Therefore, a derivation in S can be seen as a finite *sequence* of “deep” rules: for instance, the derivation

$$\boxed{\text{?d}^\downarrow \frac{a}{?a}} \otimes \boxed{\text{!d}^\downarrow \frac{b}{!b}} \text{ in } \text{DDI}^\downarrow \text{ (with parallel } \text{?d}^\downarrow \text{ and } \text{!d}^\downarrow \text{)} \text{ can be “sequenced” as both } \text{!d}^\downarrow \frac{a \otimes b}{?a \otimes !b} \text{ and } \text{?d}^\downarrow \frac{a \otimes b}{!a \otimes !b}.$$

System SDDI has only the *atomic* introduction rules ai^\downarrow and ai^\uparrow (a is a propositional variable in Figure 4): they can be seen as the atomic version of ax- and cut-rules of sequent calculus, respectively. The *non-atomic* versions of the rules ai^\downarrow and ai^\uparrow are respectively:

$$\text{i}^\downarrow \frac{1}{A \wp \bar{A}} \quad \text{i}^\uparrow \frac{A \otimes \bar{A}}{\perp} \quad (\text{where } A \text{ is a MELL formula})$$

However, the rules i^\downarrow and i^\uparrow are derivable in SDDI (Lemma 3.3). Derivability of i^\downarrow is analogous to the fact that a derivation can be transformed to one with atomic axioms in DiLL_0 sequent calculus (Proposition 2.3), but derivability of i^\uparrow is a typical result in deep inference systems that does not have a corresponding result in the sequent calculus: it says that restricting cuts to an *atomic* level is not limiting.

Lemma 3.3 (Atomic axioms and atomic cuts). *The rule i^\downarrow is derivable in $\{\text{ai}^\downarrow, \text{s}, \text{?d}^\downarrow, \text{!d}^\downarrow\}$; and the rule i^\uparrow is derivable in $\{\text{ai}^\uparrow, \text{s}, \text{?d}^\uparrow, \text{!d}^\uparrow\}$.*

Proof. Concerning i^\downarrow , the proof is by induction on the size of the MELL formula A in $\text{i}^\downarrow \frac{1}{A \wp \bar{A}}$:

- if $A = a$ is a propositional variable (and similarly if $A = \bar{a}$), then $\text{ai}^\downarrow \frac{1}{a \wp \bar{a}}$;

⁴Usually in the literature on deep inference, the dual rule r^\uparrow of a rule r^\downarrow is called “co-r”. We avoid these names because they clash with the usual terminology in the literature on DiLL_0 , see Footnote 1.

- if $A = 1$ (and similarly if $A = \perp$), then $\simeq \frac{1}{1 \wp \perp}$;

- if $A = B \otimes C$ (and similarly for $A = B \wp C$), then
$$\frac{1}{\frac{1}{B \wp \bar{B}} \otimes \frac{1}{C \wp \bar{C}}} \simeq \frac{1}{(B \otimes C) \wp (\bar{B} \wp \bar{C})}$$
;

- if $A = !B$ (and similarly if $A = ?B$), then
$$\frac{1}{!d^\downarrow \frac{B}{!B}} \wp \frac{1}{?d^\downarrow \frac{\bar{B}}{?B}}.$$

The proof for i^\uparrow is dual, using ai^\uparrow , $!d^\uparrow$ and $?d^\uparrow$ instead of ai^\downarrow , $!d^\downarrow$ and $?d^\downarrow$, respectively. \square

The rule i^\uparrow plays a special role in deep inference systems, as the cut does in sequent calculi. Thanks to \simeq and 0^\downarrow , it makes superfluous all the rules in DDI^\uparrow (second line in Figure 4) but 0^\uparrow and s . Note that ai^\uparrow is not enough for that, because i^\uparrow needs $!d^\uparrow$ and $?d^\uparrow$ to be simulated by ai^\uparrow , as seen in Lemma 3.3.

Lemma 3.4 (Getting rid of up-rules via i^\uparrow and 0^\downarrow). *Any rule $\rho^\uparrow \in \{!d^\uparrow, ?d^\uparrow, !c^\uparrow, ?c^\uparrow, !w^\uparrow, ?w^\uparrow\}$ is derivable in $\{\rho^\downarrow, i^\uparrow, i^\downarrow, s\}$; the rule $+^\uparrow$ is derivable in $\{0^\downarrow\}$.*

Proof. For a rule $\rho^\uparrow \frac{\bar{B}}{\bar{A} \wp \bar{B}}$ with $\rho^\uparrow \in \{!d^\uparrow, ?d^\uparrow, !c^\uparrow, ?c^\uparrow, !w^\uparrow, ?w^\uparrow\}$, see (4). For the rule $+^\uparrow \frac{A}{A+A}$, see (5).

$$\begin{array}{c} \simeq \\ \frac{1}{i^\downarrow \frac{1}{\bar{A} \wp \bar{A}} \otimes \bar{B}} \\ s \\ \frac{\bar{A} \wp \frac{\rho^\downarrow \frac{A}{B} \otimes \bar{B}}{i^\uparrow \frac{\perp}{\bar{A}}}}{\bar{A}} \end{array} \quad (4) \quad \simeq \frac{A}{0^\downarrow \frac{0}{A} + A} \quad (5)$$

\square

[\[\[Matteo: lemma: regole strutturali sono eliminabili\]\]](#)

4 Correspondence between $DiLL_0$ and SDDI

[\[\[Matteo: sistemare soundness\]\]](#)

In this section we prove that SDDI is a sound and complete proof system for $DiLL_0$ sequent calculus. At first sight, this result is obvious because the rules zero in $DiLL_0$ and 0^\downarrow in SDDI make everything provable. But the interest is to show that the fragments without zero and 0^\downarrow correspond to each other.

If $\Gamma = A_1, \dots, A_n$ is a MELL sequent, we set $[\Gamma] = A_1 \wp \dots \wp A_n$ (so, $[\Gamma] = \perp$ for $n = 0$).

Theorem 4.1 (Completeness). *Let Γ be a MELL sequent.*

$$\text{If } \vdash^{DiLL_0} \Gamma \text{ then } n \vdash^{DDI^\downarrow \cup \{i^\uparrow\}} [\Gamma] \text{ (and so } n \vdash^{SDDI} [\Gamma]) \text{ for some } n \in \mathbb{N}.$$

Moreover, if $\vdash^{DiLL_0^-} \Gamma$ then $1 \vdash^{DDI^\downarrow \cup \{i^\uparrow\}} [\Gamma]$; if $\vdash^{DiLL_0^- \setminus \{cut\}} \Gamma$ then $1 \vdash^{DDI^\downarrow} [\Gamma]$; while if $\vdash^{\{zero\}} \Gamma$ then $0 \vdash^{\{0^\downarrow\}} [\Gamma]$.

$$\begin{array}{c}
\text{ai}^\dagger \frac{1}{a \wp \bar{a}} \rightarrow \frac{\overline{\vdash a, \bar{a}}^{\text{ax}}}{\overline{\vdash a \wp \bar{a}}^{\wp}} \perp \\
\text{i}^\dagger \frac{A \otimes \bar{A}}{\perp} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}}}{\overline{\vdash \bar{A} \wp A}^{\wp}} \perp \\
\text{!w}^\dagger \frac{1}{!A} \rightarrow \frac{\overline{\vdash !A}^{\text{!w}}}{\overline{\vdash !A, \perp}} \perp \quad \text{?w}^\dagger \frac{\perp}{?A} \rightarrow \frac{\overline{\vdash \perp}^{\text{!w}}}{\overline{\vdash \perp, ?A}^{\wp}} \\
\text{?d}^\dagger \frac{A}{?A} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}}}{\overline{\vdash \bar{A}, ?A}^{\wp \text{d}}} \quad \text{!d}^\dagger \frac{A}{!A} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}}}{\overline{\vdash \bar{A}, !A}^{\text{!d}}} \\
\text{s} \frac{A \otimes (B \wp C)}{(A \otimes B) \wp C} \rightarrow \frac{\overline{\vdash \bar{B}, B}^{\text{ax}} \quad \overline{\vdash \bar{C}, C}^{\text{ax}}}{\overline{\vdash B, C, \bar{B} \otimes \bar{C}}^{\otimes}} \otimes \\
\frac{\overline{\vdash A \otimes B, C, \bar{A}, \bar{B} \otimes \bar{C}}^{\otimes}}{\overline{\vdash (A \otimes B) \wp C, \bar{A} \wp (\bar{B} \otimes \bar{C})}^{\wp}} \otimes \\
\text{?c}^\dagger \frac{?A \wp ?A}{?A} \rightarrow \frac{\overline{\vdash \bar{!A}, ?A}^{\text{ax}} \quad \overline{\vdash \bar{!A}, ?A}^{\text{ax}}}{\overline{\vdash \bar{!A} \otimes \bar{!A}, ?A, ?A}^{\otimes}} \otimes \\
\frac{\overline{\vdash \bar{!A} \otimes \bar{!A}, ?A}^{\otimes}}{\overline{\vdash \bar{!A} \otimes \bar{!A}, ?A}^{\wp \text{c}}} \quad \text{!c}^\dagger \frac{!A \otimes !A}{!A} \rightarrow \frac{\overline{\vdash \bar{?A}, !A}^{\text{ax}} \quad \overline{\vdash \bar{?A}, !A}^{\text{ax}}}{\overline{\vdash \bar{?A}, ?A, !A}^{\text{!c}}} \text{!c} \\
\frac{\overline{\vdash \bar{?A}, ?A, !A}^{\text{!c}}}{\overline{\vdash \bar{!A}, ?A \wp ?A}^{\wp}} \\
\approx \frac{A \wp (B \wp C)}{(A \wp B) \wp C} \rightarrow \frac{\overline{\vdash \bar{B}, B}^{\text{ax}} \quad \overline{\vdash \bar{C}, C}^{\text{ax}}}{\overline{\vdash B, C, \bar{B} \otimes \bar{C}}^{\otimes}} \otimes \\
\frac{\overline{\vdash \bar{A} \otimes (\bar{B} \otimes \bar{C}), A, B, C}^{\otimes}}{\overline{\vdash \bar{A} \otimes (\bar{B} \otimes \bar{C}), (A \wp B) \wp C}^{\wp}} \otimes \\
\approx \frac{(A \wp B) \wp C}{A \wp (B \wp C)} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}} \quad \overline{\vdash \bar{B}, B}^{\text{ax}}}{\overline{\vdash B, A, \bar{A} \otimes \bar{B}}^{\otimes}} \otimes \\
\frac{\overline{\vdash \bar{A} \otimes \bar{B}, A, B, C}^{\otimes}}{\overline{\vdash \bar{A} \otimes \bar{B}, (A \wp B) \wp C}^{\wp}} \otimes \\
\approx \frac{A \wp B}{B \wp A} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}} \quad \overline{\vdash \bar{B}, B}^{\text{ax}}}{\overline{\vdash \bar{A} \otimes \bar{B}, B, A}^{\otimes}} \otimes \\
\frac{\overline{\vdash \bar{A} \otimes \bar{B}, B \wp A}^{\otimes}}{\overline{\vdash \bar{A} \otimes \bar{B}, B \wp A}^{\wp}} \quad \approx \frac{A}{A \wp \perp} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}}}{\overline{\vdash \bar{A}, A, \perp}^{\wp}} \perp \\
\frac{\overline{\vdash \bar{A}, A, \perp}^{\wp}}{\overline{\vdash \bar{A}, A \wp \perp}^{\wp}} \quad \approx \frac{A \wp \perp}{A} \rightarrow \frac{\overline{\vdash \bar{A}, A}^{\text{ax}} \quad \overline{\vdash \perp}^{\text{!w}}}{\overline{\vdash \bar{A} \otimes \perp, A}^{\wp}} \perp
\end{array}$$

Figure 6: Interpretation of the rules in $\text{DDI}_\perp^\dagger \cup \{\text{i}^\dagger\}$ and of \simeq as derivations in DiLL_0 sequent calculus.

many equivalences in (2) cannot occur in \mathcal{D} . The cases for $\rho = \simeq$ corresponding to $A \otimes B \simeq B \otimes A$ and $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ and $A \otimes 1 \simeq A$ are omitted in Figure 6 because analogous to the ones for \wp .

By induction on the MELL context $C\{\}$, if $\rho \frac{C\{B\}}{C\{A\}}$ then $\vdash^{\text{DiLL}_0^-} \overline{C\{B\}}, C\{A\}$:

$$\begin{array}{c}
C\{A\} = C \wp C'\{A\} \text{ or } C\{A\} = C \otimes C'\{B\} \\
\frac{\overline{\vdash C, \bar{C}}^{\text{ax}} \quad \overline{\vdash C'\{A\}, C'\{B\}}^{\text{IH}}}{\overline{\vdash C'\{A\}, C'\{B\} \otimes \bar{C}}^{\otimes}} \otimes \\
\frac{\overline{\vdash C'\{A\}, C'\{B\} \otimes \bar{C}}^{\otimes}}{\overline{\vdash C'\{A\} \wp C, \bar{C}'\{B\} \otimes \bar{C}}^{\wp}} \quad \left| \quad \begin{array}{c}
C\{A\} = !C'\{A\} \text{ or } C\{A\} = ?C'\{A\} \\
\overline{\vdash C'\{A\}, C'\{B\}}^{\text{IH}} \\
\overline{\vdash C'\{A\}, ?C'\{B\}}^{\wp \text{d}} \\
\overline{\vdash !C'\{A\}, ?C'\{B\}}^{\text{!d}}
\end{array}
\end{array}$$

We define a derivation of $\vdash \llbracket \Gamma \rrbracket$ in DiLL_0^- by induction on the number of rules in \mathcal{D} as follows:

$$1 \rightarrow \frac{1}{\vdash 1} \quad \rho \frac{\llbracket \Delta \rrbracket}{\llbracket \Gamma \rrbracket} \rightarrow \frac{\text{IH} \llbracket \text{DiLL}_0^- \rrbracket \quad \llbracket \text{DiLL}_0^- \rrbracket}{\vdash \llbracket \Delta \rrbracket \quad \vdash \llbracket \Delta \rrbracket, \llbracket \Gamma \rrbracket} \text{cut}$$

By reversibility of \wp (if $\vdash^{\text{DiLL}_0^-} A \wp B$ then $\vdash^{\text{DiLL}_0^-} A, B$), we have $\vdash^{\text{DiLL}_0^-} \Gamma$. \square

Let us sum up the correspondence between DiLL_0 sequent calculus and SDDI deep inference system.

Theorem 4.3 (Sequent calculus vs. deep inference). *Let Γ be a MELL sequent.*

1. DiLL_0 vs. SDDI: $\vdash^{\text{DiLL}_0} \Gamma$ if and only if $n \vdash^{\text{SDDI}} \llbracket \Gamma \rrbracket$ for some $n \in \mathbb{N}$.

2. DiLL_0 cut-free vs. DDI^\downarrow : $\text{DiLL}_0 \setminus \{\text{cut}\} \vdash \Gamma$ if and only if $n \vdash \llbracket \Gamma \rrbracket$ for some $n \in \mathbb{N}$.
3. DiLL_0^- cut-free vs. DDI_-^\downarrow : if $\text{DiLL}_0^- \setminus \{\text{cut}\} \vdash \Gamma$ then $n \vdash \llbracket \Gamma \rrbracket$ for some $n \in \mathbb{N}$.

Proof. 1. For \Rightarrow , by completeness (Theorem 4.1); for \Leftarrow , by soundness (Theorem 4.2).

2.–3. For \Rightarrow , see Theorem 4.1. For \Leftarrow (only for Item 2), by Theorems 2.4 and 4.3.1. \square

5 Normalization in SDDI

In this section we define a *standard* form for derivations in SDDI and a *normalization* procedure to obtain a “cut-free” standard derivation in DDI^\downarrow for any formula A provable in SDDI. The usual approach to prove normalization in deep inference system relies on the *splitting technique* [15, 19, 1, 18, 27]. However, the presence in our syntax of the connective $+$ and its unit 0 prevents us to use Guglielmi and Tubella’s normalization result [29, 30], which covers and generalizes the splitting proofs. This is mainly due to the fact that 0 is an absorbing element for \otimes , \wp , $?$ and $!$, together with the distributivity over $+$.

For this reason, following [27, 28], we define the normalization process in terms of *rule permutations*, which play the same role as cut-elimination steps in DiLL_0 . In some cases, their definition relies on the rules for the connective $+$ and its unit 0 . This behavior is coherent with the dynamics of cut-elimination in DiLL_0 [8, 23, 11], where the rules *sum* and *zero* step in to deal with non-deterministic choices or mismatches between “supply and demand”. Interestingly, these permutations in SDDI mimic the elegant symmetries of cut-elimination steps as defined for interaction nets [23, Fig. 4], instead of the awkward rewrite rules defined for the sequent calculus (Figure 3).

The fact that our syntax is more flexible and symmetric than the sequent calculus, and internalizes the connective $+$ and its unit 0 , allows for a more fine-grained analysis of the normalization process than in DiLL_0 . In particular, we can distinguish three kinds of rule permutations corresponding to three distinct phases in normalizing: *MLL cut-elimination steps* (involving ai^\uparrow , ai^\downarrow and s only), *resource management steps* (involving the $?$ - and the $!$ -rules only) and *slice operations* (the process of duplicating or removing a slice which is less evident in the DiLL_0 sequent calculus and DiLL_0 interaction nets).

Definition 5.1 (Permutation). A rule ρ *permutes over* a rule σ (or σ *permutes under* ρ) if, for any

derivation $\frac{\sigma \frac{C}{B}}{\rho \frac{A}{A}}$, one of the following holds:

$$A = C; \quad \sigma \frac{C}{A}; \quad \rho \frac{C}{A}; \quad \rho \frac{C}{B'} \text{ for some formula } B'; \quad 2 \times \rho \frac{C}{B'} \text{ for some formula } B'.$$

A rule ρ *permutes over* a rule σ (or σ *permutes under* ρ) by a set of rules S if ρ permutes over σ , or

$$\text{for any derivation } \frac{\sigma \frac{C}{B}}{\rho \frac{A}{A}}, \text{ one of the following holds: } \parallel_S, \text{ or } \frac{C}{A} \parallel_S \frac{\rho \frac{C}{B''}}{\sigma \frac{B'}{A}} \text{ for some formulas } B', B''.$$

Roughly, permuting σ under ρ means that σ can be pushed below ρ in a derivation with same premise and conclusion. In this operation, ρ or σ might disappear or other rules might appear in between. The definition of rule permutation is asymmetric: two ρ ’s can be above one σ , but not two σ ’s below one ρ .

We call *trivial* the rule permutations captured by the open deduction syntax such as the following:

$$\frac{\rho \frac{B_1 \otimes B_2}{A_1 \otimes B_2}}{\tau \frac{B_1 \otimes B_2}{A_1 \otimes A_2}} = \boxed{\rho \frac{B_1}{A_1}} \otimes \boxed{\tau \frac{B_2}{A_2}} = \tau \frac{B_1 \otimes B_2}{\rho \frac{B_1 \otimes A_2}{A_1 \otimes A_2}}$$

The following lemma is analogous to canonicity (Fact 2.2) for DiLL₀ sequent calculus. It means that, in SDDI, rules 0^\downarrow and $+^\downarrow$ can be pushed down in a derivation, and rules 0^\uparrow and $+^\uparrow$ can be pushed up.

Lemma 5.2 (Permuting 0 and +). *Any rule in SDDI permutes over 0^\downarrow and $+^\downarrow$, and under $+^\uparrow$ and 0^\uparrow .*

Proof. We define the rule permutations below, for $\rho, \tau, \sigma \in \text{SDDI}$ with $\rho \neq +^\uparrow$, $\tau \neq 0^\uparrow$ and $\sigma \notin \{+^\downarrow, 0^\downarrow\}$.

$$\begin{array}{ccc} \begin{array}{c} +^\downarrow \frac{B+B}{\boxed{\rho \frac{B}{A}}} \\ \rightsquigarrow_{\text{norm}} \\ \begin{array}{c} \boxed{\rho \frac{B}{A}} + \boxed{\rho \frac{B}{A}} \\ A \end{array} \end{array} & \begin{array}{c} +^\downarrow \frac{B+B}{B} \\ \rightsquigarrow_{\text{norm}} \\ B+B \\ +^\uparrow \frac{B+B}{B+B} \end{array} & \begin{array}{c} 0^\downarrow \frac{0}{B} \\ \rightsquigarrow_{\text{norm}} \\ 0^\downarrow \frac{0}{A} \\ \tau \frac{0}{A} \end{array} \\ \\ \begin{array}{c} 0^\downarrow \frac{0}{B} \\ \rightsquigarrow_{\text{norm}} \\ 0^\uparrow \frac{0}{0} \end{array} & \begin{array}{c} \boxed{\sigma \frac{B}{A}} \\ \rightsquigarrow_{\text{norm}} \\ \begin{array}{c} \boxed{\sigma \frac{B}{A}} + \boxed{\sigma \frac{B}{A}} \\ B \\ A+A \end{array} \\ +^\uparrow \frac{B}{\sigma \frac{B}{A} + \sigma \frac{B}{A}} \end{array} & \begin{array}{c} \frac{B}{\sigma \frac{B}{A}} \\ \rightsquigarrow_{\text{norm}} \\ 0^\uparrow \frac{B}{0} \end{array} \end{array} \quad (6)$$

Note that some of these rule permutations may require the use of formula equivalence in order to be applied, for example in the following rule permutations concerning ai^\downarrow and $+^\uparrow$, or ai^\downarrow and 0^\uparrow :

$$\begin{array}{ccc} \begin{array}{c} \text{ai}^\downarrow \frac{1}{\boxed{+^\uparrow \frac{a}{a+a}}} \\ \rightsquigarrow_{\text{norm}} \\ \begin{array}{c} \boxed{+^\uparrow \frac{a \wp \bar{a}}{(a+a) \wp \bar{a}}} \\ \simeq \\ \boxed{+^\uparrow \frac{a \wp \bar{a}}{(a \wp \bar{a}) + (a \wp \bar{a})}} \\ \rightsquigarrow_{\text{norm}} \\ \begin{array}{c} \boxed{\text{ai}^\downarrow \frac{1}{a \wp \bar{a}}} + \boxed{\text{ai}^\downarrow \frac{1}{a \wp \bar{a}}} \end{array} \end{array} \end{array} & \begin{array}{c} \text{ai}^\downarrow \frac{1}{\boxed{+^\uparrow \frac{a}{a}}} \\ \rightsquigarrow_{\text{norm}} \\ \begin{array}{c} \boxed{+^\uparrow \frac{a \wp \bar{a}}{0 \wp \bar{a}}} \\ \simeq \\ \boxed{+^\uparrow \frac{a \wp \bar{a}}{0}} \\ \rightsquigarrow_{\text{norm}} \\ 0^\uparrow \frac{1}{0} \end{array} \end{array} \end{array} \quad (7)$$

Similar permutations take place when ai^\downarrow is replaced with $!c^\uparrow$ or $?c^\uparrow$, and in their dual configurations. \square

As a consequence of Lemma 5.2, pushing up the rules 0^\uparrow , $+^\uparrow$ and down 0^\downarrow , $+^\downarrow$ can be interpreted as *slice management* operations: it duplicates and discard the “slices” (the subderivations without the rule 0^\uparrow , $+^\uparrow$, 0^\downarrow , $+^\downarrow$) and extend them as much as possible, propagating the non-deterministic choice $+$ and the resource mismatch 0 all along the derivation.

Corollary 5.3 (Slice management). *[[Matteo: fix usando structural]] If $A, A', B = C\{0\}$, C and C' are MELL formulas, then*

$$\begin{array}{ccc} \begin{array}{c} C \\ \mathcal{D}_2 \parallel \text{SDDI} \\ B \\ \rightsquigarrow_{\text{norm}}^m \\ \mathcal{D}_1 \parallel \text{SDDI} \\ A \end{array} & \begin{array}{c} n \\ \mathcal{D}_2 \parallel \text{SDDI} \\ B \\ \rightsquigarrow_{\text{norm}}^{m+1} \\ \mathcal{D}_1 \parallel \text{SDDI} \\ A \end{array} & \begin{array}{c} C' \\ \mathcal{D}_C \parallel \text{SDDI} \\ C \\ \rightsquigarrow_{\text{norm}}^k \\ \begin{array}{c} \boxed{\mathcal{D}_1 \parallel \text{SDDI} \frac{C}{A'}} + \boxed{\mathcal{D}_2 \parallel \text{SDDI} \frac{C}{A'}} \\ A' \\ \mathcal{D}_A \parallel \text{SDDI} \\ A \end{array} \\ \rightsquigarrow_{\text{norm}}^k \\ \begin{array}{c} \boxed{\mathcal{D}_A \circ \mathcal{D}_1 \circ \mathcal{D}_C \parallel \text{SDDI} \frac{C'}{A}} + \boxed{\mathcal{D}_A \circ \mathcal{D}_2 \circ \mathcal{D}_C \parallel \text{SDDI} \frac{C'}{A}} \\ A \end{array} \end{array} \end{array}$$

where $m = |\mathcal{D}_1| + |\mathcal{D}_2|$ and $k = |\mathcal{D}_A| + |\mathcal{D}_C|$ ($|\mathcal{D}|$ is the number of inference rules in the derivation \mathcal{D}).

Figure 7: Non-trivial permutations of rules for ! and ? by $\{0^\uparrow, 0^\downarrow, +^\uparrow, +^\downarrow\}$ in SDDI.

Actually, we can further structure SDDI derivations so as to separate an initial up-segment and a final down-segment (Theorem 5.7). To prove this, we use the two following lemmas.

Lemma 5.4 (Permutations of rules for ! and ?). *In SDDI, the following rule permutations hold:*

1. Interaction-net permutations: *The rules in $\{!d^\uparrow, ?d^\uparrow, !w^\uparrow, ?w^\uparrow, !c^\uparrow, ?c^\uparrow\}$ permute over the rules in $\{!d^\downarrow, ?d^\downarrow, !w^\downarrow, ?w^\downarrow, !c^\downarrow, ?c^\downarrow\}$ by the rules in $\{+^\downarrow, +^\uparrow, 0^\downarrow, 0^\uparrow\}$;*
2. *The rules in $\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}$ permutes under any rule in $\{ai^\uparrow, ai^\downarrow, s\}$;*
3. *The rules in $\{?d^\uparrow, !d^\uparrow, ?w^\uparrow, !w^\uparrow, ?c^\uparrow, !c^\uparrow\}$ permutes over any rule in $\{ai^\uparrow, ai^\downarrow, s\}$.*

Proof.

1. The rule permutations are either in Figure 7 or are their duals obtained by up/down symmetry.
2. First, note that all rule permutations involving ai^\downarrow in $\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}$ are trivial. Moreover, any $\rho \in \{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}$ permutes under s as follows.

$$\begin{array}{c} \boxed{\rho \frac{A'}{A}} \otimes (B \wp C) \\ \hline s \end{array} \xrightarrow{\text{norm}} \begin{array}{c} s \frac{A' \otimes (B \wp C)}{\left(\boxed{\rho \frac{A'}{A}} \otimes B \right) \wp C} \end{array} \quad (8)$$

We conclude by the following rule permutations for $?d^\downarrow$, $?c^\downarrow$ and $?w^\downarrow$ (permutations for $!d^\downarrow$, $!c^\downarrow$ and $!w^\downarrow$ are defined similarly).

$$\begin{array}{c} 1 \\ \hline ? \frac{1}{ai^\downarrow \frac{1}{a \wp \bar{a}}} \end{array} \xrightarrow{\text{norm}} \begin{array}{c} 1 \\ \hline ?d^\downarrow \frac{1}{a \wp \bar{a}} \end{array} \quad \begin{array}{c} ?1 \wp ?1 \\ \hline ? \frac{1}{ai^\downarrow \frac{1}{a \wp \bar{a}}} \end{array} \xrightarrow{\text{norm}} \begin{array}{c} ? \frac{1}{ai^\downarrow \frac{1}{a \wp \bar{a}}} \otimes \frac{1}{a \wp \bar{a}} \\ \hline ?c^\downarrow \frac{1}{(a \wp \bar{a})} \end{array} \quad \begin{array}{c} \perp \\ \hline ? \frac{1}{ai^\downarrow \frac{1}{a \wp \bar{a}}} \end{array} \xrightarrow{\text{norm}} \begin{array}{c} \perp \\ \hline ?w^\downarrow \frac{1}{(a \wp \bar{a})} \end{array}$$

3. It follows from Item 2 can be obtained dually by the up/down symmetry of rules. □

Permutations in Figure 7 and their duals correspond to the cut-elimination steps for modalities $?$ and $!$ in the *interaction-nets* presentation of DiLL_0 , see [23, Fig. 4] and [8, 11]. They take place when a down-rule for $!$ meets an up-rule for $?$, or vice-versa, and deal with the *resource management* (see Section 2). Akin to interaction-nets and unlike the sequent calculus, these permutations on SDDI are perfectly symmetric. Note the key role of the rules $+\uparrow$, $+\downarrow$, 0^\uparrow , 0^\downarrow in some permutations. In particular,

- formulas 0 appear when there is a mismatch between “supply and demand” ($?w^\downarrow/!d^\uparrow$ and $!w^\downarrow/?d^\uparrow$),
- formulas with $+$ appear when there is a non-deterministic choice on which request will be fed ($?c^\downarrow/!d^\uparrow$ and $!c^\downarrow/?d^\uparrow$).

Lemma 5.5 (Linear permutations). *Let A and B be MELL formulas. Then*

1. $B \stackrel{\{ai^\uparrow, ai^\downarrow, s\}}{\vdash} A$ if and only if $B \stackrel{ai^\downarrow}{\vdash} B' \stackrel{s}{\vdash} A' \stackrel{ai^\uparrow}{\vdash} A$ for some MELL formulas B' and A' ;
2. if $1 \stackrel{\{ai^\uparrow, ai^\downarrow, s\}}{\vdash} A$ then $1 \stackrel{ai^\downarrow}{\vdash} A' \stackrel{s}{\vdash} A$ for some MELL formula A' .

Proof (sketch). This is a standard result in deep inference systems. Nowadays, it is usually proved via splitting [30, 29, 31], as it is a consequence of cut elimination. In [14, 27, 28, 15], which were written before the splitting technique was found, we can find a proof of this result using some sort of rule permutations. To prove Item 1 it is enough to use the non-trivial rule permutations of ai^\downarrow over s shown in (9), and the dual rule permutations of ai^\uparrow under s obtained from (9) by up/down symmetry.

$$s \frac{1 \otimes (B \wp C)}{\left(\frac{1}{ai^\downarrow a \wp \bar{a}} \otimes B \right) \wp C} \rightsquigarrow_{\text{norm}} s \frac{\frac{1}{ai^\downarrow a \wp \bar{a}} \otimes (B \wp C)}{((a \wp \bar{a}) \otimes B) \wp C} \quad s \frac{A \otimes (B \wp 1)}{(A \otimes B) \wp \frac{1}{ai^\downarrow a \wp \bar{a}}} \rightsquigarrow_{\text{norm}} s \frac{A \otimes \left(B \wp \frac{1}{ai^\downarrow a \wp \bar{a}} \right)}{(A \otimes B) \wp 1} \quad (9)$$

To have an intuition for the proof of Item 2 it is enough to remark that if $1 \stackrel{ai^\downarrow}{\vdash} B'$, then there is a derivation of B' with *shallow* ai^\downarrow , that is, with ai^\downarrow applied only in \otimes -context as the one below on the left:

$$\begin{array}{c} \boxed{\frac{1}{ai^\downarrow a_1 \wp \bar{a}_1}} \otimes \dots \otimes \boxed{\frac{1}{ai^\downarrow a_n \wp \bar{a}_n}} \\ \parallel \{s\} \\ B' \end{array} \quad \left| \quad \begin{array}{c} 1 \\ \simeq \frac{\boxed{\frac{1}{ai^\downarrow a \wp \bar{a}}} \otimes \boxed{\frac{1}{ai^\downarrow a \wp \bar{a}}}}{2 \times s} \\ \rightsquigarrow_{\text{norm}} \frac{1}{ai^\downarrow a \wp \bar{a}} \\ \simeq \frac{a \wp \frac{a \wp \bar{a} \wp a}{ai^\uparrow \perp} \wp \bar{a}}{a \wp \bar{a}} \end{array}$$

Hence by Item 1, if $1 \stackrel{\{ai^\uparrow, ai^\downarrow, s\}}{\vdash} A$ then A is provable by using shallow ai^\downarrow rules and there is at least one ai^\uparrow which can be permuted up in the derivation until we can obtain a configuration as the one above on the right which can be replaced by an ai^\downarrow -rule. \square

Rule permutations involved in the proof of Lemma 5.5.2 essentially correspond to MLL *cut-elimination steps* in the DiLL_0 sequent calculus, indeed modalities $!$ and $?$ do not play any active role there.

Definition 5.6 (Normalization step). Any rewrite relation on SDDI derivations that is a non-trivial rule permutation of the proofs of Lemmas 5.2, 5.4 and 5.5 is a *normalization step* and denoted by $\rightsquigarrow_{\text{norm}}$.

Normalization steps rearrange rules in a DDI or SDDI derivation in a fixed order, leaving unchanged its premise and conclusion. So, derivability in DDI and SDDI can be decomposed in several segments. More precisely, every derivation in SDDI can be rearranged in a symmetrical way so that:

1. on the top there is a *up-segment* where:
 - (a) the first part consists of rules 0^\uparrow and $+^\uparrow$, which decompose the derivation in vertical slices;
 - (b) the second part consists of up-rules for $!$ and $?$, which deal with non-linear resources;
2. in the middle there is a *linear segment*, roughly corresponding to MLL and to linear resources;
3. on the bottom there is a *down-segment* where:
 - (a) the first part consists of down-rules for $!$ and $?$, which deal with non-linear resources;
 - (b) the second part consists of rules 0^\downarrow and $+^\downarrow$, which merge the vertical slices of the derivation.

The decomposition in DDI^\downarrow follows the same pattern but takes only down-rules, so there is no up-segment.

Theorem 5.7 (Decomposition). *Let A and B formulas.*

1. DDI-decomposition: *If $\mathcal{D} \triangleright n \stackrel{\text{DDI}^\downarrow}{\vdash} A$, then (for some MELL formulas A', A'', A''') there is a derivation \mathcal{D}' in DDI^\downarrow such that $\mathcal{D} \rightsquigarrow_{\text{norm}}^* \mathcal{D}'$ and*

$$\mathcal{D}' \triangleright n \stackrel{\{\text{ai}^\downarrow\}}{\vdash} A''' \stackrel{\{s\}}{\vdash} A'' \stackrel{\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}}{\vdash} A' \stackrel{\{0^\downarrow, +^\downarrow\}}{\vdash} A.$$

2. SDDI-decomposition: *If $\mathcal{D} \triangleright B \stackrel{\text{SDDI}}{\vdash} A$, then there is a derivation \mathcal{D}' (called standard) in SDDI from B to A such that $\mathcal{D} \rightsquigarrow_{\text{norm}}^* \mathcal{D}'$ and (for some MELL formulas $B', B'', B''', A''', A'', A'$):*

$$\mathcal{D}' \triangleright B \stackrel{\{0^\uparrow, +^\uparrow\}}{\vdash} B' \stackrel{\{?d^\uparrow, !d^\uparrow, !w^\uparrow, ?w^\uparrow, ?c^\uparrow, !c^\uparrow\}}{\vdash} B'' \stackrel{\{\text{ai}^\downarrow\}}{\vdash} B''' \stackrel{\{s\}}{\vdash} A''' \stackrel{\{\text{ai}^\uparrow\}}{\vdash} A'' \stackrel{\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}}{\vdash} A' \stackrel{\{0^\downarrow, +^\downarrow\}}{\vdash} A.$$

Proof. The decomposition of DDI derivations follows from Lemma 5.2, Lemma 5.4.2 and Lemma 5.5.1.

To prove decomposition of SDDI derivations, we alternating applications of Lemmas 5.2 and 5.4 until we obtain a derivation of the shape below. Then we conclude by applying Lemma 5.5.2.

$$B \stackrel{\{0^\uparrow, +^\uparrow\}}{\vdash} B' \stackrel{\{?d^\uparrow, !d^\uparrow, !w^\uparrow, ?w^\uparrow, ?c^\uparrow, !c^\uparrow\}}{\vdash} B'' \stackrel{\{\text{ai}^\downarrow, s, \text{ai}^\uparrow\}}{\vdash} A'' \stackrel{\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}}{\vdash} A' \stackrel{\{0^\downarrow, +^\downarrow\}}{\vdash} A. \quad \square$$

As a consequence, the up-fragment DDI^\uparrow of SDDI is *superfluous* (Corollary 5.8): all that can be proved in SDDI, is already derivable in the down-fragment DDI^\downarrow of SDDI by a standard derivation. The existence of a standard derivation in DDI^\downarrow is obvious because the rule 0^\downarrow makes every MELL formula derivable. The interesting part is that a standard derivation in DDI^\downarrow can be reached via normalization steps, hence in a computational way that is *internal* to SDDI. Indeed, normalization of SDDI derivations follows from SDDI decomposition (Theorem 5.7.2), so it relies on the normalization steps defined on SDDI derivations (Definition 5.6). This normalization result is the deep inference version of *cut-elimination*, since in DDI^\downarrow there is no analogue of the rule cut.

Corollary 5.8 (Normalization). *Let A be a formula and $n \in \mathbb{N}$. If $\mathcal{D} \triangleright n \stackrel{\text{SDDI}}{\vdash} A$ then, for some $n' \in \mathbb{N}$, there exists a standard $\mathcal{D}' \triangleright n \stackrel{\{0^\uparrow, +^\uparrow\}}{\vdash} n' \stackrel{\text{DDI}^\downarrow}{\vdash} A$ such that $\mathcal{D} \rightsquigarrow_{\text{norm}}^* \mathcal{D}'$. In particular, $n' \stackrel{\text{DDI}^\downarrow}{\vdash} A$ for some $n' \in \mathbb{N}$.*

Proof. By Theorem 5.7.2, if $n \stackrel{\text{SDDI}}{\vdash} A$ then there is a standard derivation

$$n \stackrel{\{0^\uparrow, +^\uparrow\}}{\vdash} B \stackrel{\{?d^\uparrow, !d^\uparrow, !w^\uparrow, ?w^\uparrow, ?c^\uparrow, !c^\uparrow\}}{\vdash} B' \stackrel{\{\text{ai}^\downarrow, s, \text{ai}^\uparrow\}}{\vdash} A'' \stackrel{\{?d^\downarrow, !d^\downarrow, ?w^\downarrow, !w^\downarrow, ?c^\downarrow, !c^\downarrow\}}{\vdash} A' \stackrel{\{0^\downarrow, +^\downarrow\}}{\vdash} A.$$

Moreover, $n \stackrel{\{0^\uparrow, +^\uparrow\}}{\vdash} B$ implies that $B = n'$ for some $n' \in \mathbb{N}$. As no rule in $\{?w^\uparrow, ?c^\uparrow, !c^\uparrow, ?d^\uparrow, !d^\uparrow\}$ can be applied on a formula of the form n' , we have $n' = B$ and we conclude by Lemma 5.5.2. \square

6 Conclusions and future work

In this paper we introduced the first sound and complete deep inference system, SDDI, for the promotion-free fragment of differential linear logic, DiLL_0 [8]. The deep inference syntax recovers the symmetry of this logic lacking in DiLL_0 sequent calculus—but which can be found in the interaction-net formalism for DiLL_0 [8]—and keeps the inductive and handy tree-like structure of sequent calculus derivations—missing in interaction nets. The deep inference formalism allows us to reduce cuts to atomic formulas, and provides a tool for a more fine-grained study of cut-elimination. Moreover, the syntax explicitly represents and internalizes the notion of slices of a derivation.

The inference rules of SDDI present an up/down symmetry and we proved that the up-fragment of SDDI is derivable from the down-fragment DDI^\downarrow . To prove this result we provided a normalization procedure based on rule permutations. In fact, the presence of the connective $+$ and its unit 0 prevent the use of the general normalization result for splittable systems [30]. In our normalization procedure for SDDI, we are able to distinguish different kinds of rule permutations depending on their computational behavior: some rule permutations correspond to linear (in terms of resource) cut-elimination steps, some to resource management cut-elimination steps and some to slice management operations. Thanks to Corollary 5.3 we could implement an optimal reduction strategy. The internal normalization procedure in SDDI to prove Corollary 5.8 provides “cut-free” derivations. And the translation $\llbracket \cdot \rrbracket$ defined in Figure 5 sends cut-free DiLL_0 derivations into DDI^\downarrow , the “cut-free” fragment of SDDI (Theorem 4.3.2).

Relation between cut-elimination in DiLL_0 and normalization in SDDI. We provide a translation $\llbracket \cdot \rrbracket$ of DiLL_0 derivations with atomic axioms to $\text{DDI}^\downarrow \cup \{i^\uparrow\}$ derivations (Figure 5). This translation exhibits a one-to-one correspondence between weakening, contraction and dereliction rules of the two systems.

However, the translation $\llbracket \cdot \rrbracket$ does not commute with cut-elimination/normalization: a derivation π in DiLL_0 sequent calculus might reduce to a cut-free derivation $\hat{\pi}$ via cut-elimination, but its translation $\llbracket \pi \rrbracket$ in SDDI normalizes to a DDI^\downarrow derivation $\llbracket \hat{\pi} \rrbracket$ other than $\llbracket \hat{\pi} \rrbracket$. That is, diagram (10) does *not commute*:

$$\begin{array}{ccc}
 \pi & \xrightarrow{\llbracket \cdot \rrbracket} & \llbracket \pi \rrbracket \\
 \text{cut} \downarrow & & \downarrow \text{norm} \\
 \hat{\pi} & \xrightarrow{\llbracket \cdot \rrbracket} & \llbracket \hat{\pi} \rrbracket \neq \widehat{\llbracket \pi \rrbracket}
 \end{array} \tag{10}$$

The technical reason for this lack of commutation is that the rule cut is translated as an instance of i^\uparrow , which is not a rule of SDDI (it is not an atomic cut) and hence has to be rewritten according to Lemma 3.3. But this rewriting in SDDI might not match the resource distribution of the corresponding cut. For instance, consider the derivation below in DiLL_0 (with π cut-free) and its translation \mathcal{D} in SDDI:

$$\begin{array}{c}
 \vdots \pi \\
 \frac{\vdash \Gamma}{\vdash \Gamma, ?a} \text{?w} \quad \frac{}{\vdash !\bar{a}} \text{!w} \\
 \frac{}{\vdash \Gamma} \text{cut}
 \end{array}
 \xrightarrow[\llbracket \cdot \rrbracket]{\text{Figure 5}}
 \begin{array}{c}
 \approx \frac{\text{[Diagram]}}{\text{[Diagram]}} \approx \text{[Diagram]} \otimes \text{[Diagram]} \xrightarrow{\text{Lemma 3.3}} \text{[Diagram]} = \mathcal{D}
 \end{array}$$

The diagram shows the translation of a cut rule from DiLL_0 to SDDI. On the left, a DiLL_0 derivation uses a cut rule to combine a weakening rule (with π) and a promotion rule. This is translated via Figure 5 into a SDDI derivation (middle) where the cut is represented as an i^\uparrow rule. This SDDI derivation is then rewritten using Lemma 3.3 into a final SDDI derivation \mathcal{D} (right), where the i^\uparrow rule is decomposed into a dereliction rule and a promotion rule, and the promotion rule is further simplified.

The DiLL_0 derivation reduces in one step of cut-elimination as follows:

$$\frac{\frac{\vdash \Gamma}{\vdash \Gamma, ?a} \text{?w} \quad \frac{}{\vdash !\bar{a}} \text{!w}}{\vdash \Gamma} \text{cut} \rightsquigarrow_{\text{cut}} \frac{\vdots \pi}{\vdash \Gamma}$$

However, the derivation \mathcal{D} in SDDI can be normalized as follows:

$$\begin{array}{c} \mathcal{D} \rightsquigarrow_{\text{norm}}^2 \\ \begin{array}{c} \approx \\ \begin{array}{c} \text{n} \\ \hline \begin{array}{c} \text{n} \\ \hline \begin{array}{c} [\pi] \\ \hline [\Gamma] \end{array} \\ \hline \begin{array}{c} [\Gamma] \text{?} \\ \hline \begin{array}{c} ?w^\downarrow \frac{\perp}{?a} \\ \hline ?d^\uparrow \frac{a}{a} \end{array} \end{array} \\ \hline \begin{array}{c} \otimes \\ \hline \begin{array}{c} !w^\downarrow \frac{1}{!\bar{a}} \\ \hline !d^\uparrow \frac{\bar{a}}{\bar{a}} \end{array} \end{array} \\ \hline \begin{array}{c} \text{s} \\ \hline \begin{array}{c} [\Gamma] \text{?} \\ \hline \begin{array}{c} a \otimes \bar{a} \\ \hline \perp \end{array} \end{array} \\ \hline \approx \\ \hline \begin{array}{c} [\Gamma] \end{array} \end{array} \end{array} \end{array} \rightsquigarrow_{\text{norm}} \begin{array}{c} \begin{array}{c} \text{n} \\ \hline \begin{array}{c} \text{n} \\ \hline \begin{array}{c} [\pi] \\ \hline [\Gamma] \end{array} \\ \hline \begin{array}{c} [\Gamma] \text{?} \\ \hline \begin{array}{c} 0^\uparrow \frac{\perp}{0} \\ \hline 0^\downarrow \frac{a}{a} \end{array} \end{array} \\ \hline \begin{array}{c} \otimes \\ \hline \begin{array}{c} !w^\downarrow \frac{1}{!\bar{a}} \\ \hline !d^\uparrow \frac{\bar{a}}{\bar{a}} \end{array} \end{array} \\ \hline \begin{array}{c} \text{s} \\ \hline \begin{array}{c} [\Gamma] \text{?} \\ \hline \begin{array}{c} a \otimes \bar{a} \\ \hline \perp \end{array} \end{array} \\ \hline \approx \\ \hline \begin{array}{c} [\Gamma] \end{array} \end{array} \end{array} \rightsquigarrow_{\text{norm}}^* \begin{array}{c} 0 \\ \hline [\Gamma] \end{array} \end{array}$$

We observe that the transformation of the general i^\uparrow -rule into ai^\uparrow (Lemma 3.3) transforms any potential interaction of weakening, contraction and dereliction up- and down-rules to an interaction of a down-rule with a dereliction-up rule. In our example, the translation creates the “mismatches” $?w^\downarrow/?d^\uparrow$ and $!w^\downarrow/!d^\uparrow$ even if in the original DiLL_0 derivation we had a “matched” interaction of a $?w$ with a $!w$.

Due to these mismatches, the normal form of the encoded derivation \mathcal{D} is $0^\downarrow \frac{0}{[\Gamma]}$, which is not the trans-

lation of the normalized (with respect to cut-elimination) original derivation in DiLL_0 , if $\pi \neq \overline{\vdash \Gamma}^{\text{zero}}$. More in general, this problem is related to the fact that in DiLL_0 we miss the promotion rule $!p$ (Figure 1), which would allow us to refine the relation \rightsquigarrow_η such a way it preserves resources, by replacing the last relation for η -expansion in (1) (Proposition 2.3) as follows:

$$\frac{}{\vdash !A, ?\bar{A}} \text{ax} \rightsquigarrow_\eta \frac{\frac{}{\vdash A, \bar{A}} \text{ax}}{\vdash ?A, \bar{A}} \text{?d} \quad \frac{}{\vdash ?A, !\bar{A}} \text{!p} \quad (11)$$

But promotion rule $!p$ makes a resource available at will [12], and it does not belong to the realm of DiLL_0 , where resources can be used only a finite number of times. In order to provide an *internal* solution (*i.e.*, in DiLL_0 without adding $!p$ to the system) of the commutation problem, we need to define a more sophisticated translation of cut-rules into SDDI. It goes beyond the scope of this paper and is ongoing work. This refined translation from DiLL_0 to SDDI commutes with cut-elimination/normalization.

Translation of DiLL proof-nets. Another ongoing work is to extend our deep inference system in order to represent the *full* differential linear logic $\text{DiLL} = \text{DiLL}_0 \cup \{!p\}$ [23, 25] (including the promotion rule), possibly with the rule *mix* which allows one to derive $A \text{?} B$ from $A \otimes B$. The presence of promotion $!p$ allows us to define a translation that commutes with cut-elimination for the reasons discussed above.

In this extended deep inference system, we can translate not only the DiLL sequent calculus but also DiLL proof-nets, via a direct embedding that does not pass through the sequent calculus. Indeed, open deduction formalism [16] allows a direct encoding of proof-nets, plus a handy and inductive syntax.

Computational meaning and non-determinism. In DiLL₀ interaction nets, when a cut-elimination step creates a new construct sum (expressing a non-deterministic choice) or zero (expressing mismatch on demanded and supplied resources), this construct is instantaneously propagated to all the interaction net where it is plugged in, without any computational step. It is like DiLL₀ interaction nets allow one to deal with canonical forms only, in the sense of Definition 2.1.

A feature of our deep inference formalism is that the constructs $+$ (non-determinism) and 0 (resource mismatch) are internalized in the syntax, and when they appear during the normalization process, they are propagated all along the derivation by means of normalization steps (slice management, Lemma 5.2 and Corollary 5.3). Is there a computational meaning in these kind of steps? Is it possible to interpret them in a model of computation which intrinsically represents non-determinism, parallelism and concurrency?

The π -calculus (a model of concurrent computation) can be encoded in DiLL₀ [5], but Mazza [21] pointed out that the non-determinism expressed by DiLL₀ is too weak for true concurrency. Maybe deep inference can shed new light on the quest for a convincing proof-theoretic counterpart of concurrency.

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