A semantical and operational account of call-by-value solvability

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Abstract. In Plotkin's call-by-value lambda-calculus, solvable terms are characterized syntactically by means of call-by-name reductions and there is no neat semantical characterization of such terms. Preserving confluence, we extend Plotkin's original reduction without adding extra syntactical constructors, and we get a call-by-value operational characterization of solvable terms. Moreover, we give a semantical characterization of solvable terms in a relational model, based on Linear Logic, satisfying the Taylor expansion formula. As a technical tool, we also use a resource-sensitive calculus (with tests) in which the elements of the model are definable.

Keywords: (resource) call-by-value lambda calculus, tests, potential valuability, solvability, relational semantics, weak and stratified reductions

1 Introduction

In the theory of ordinary (i.e. untyped call-by-name) λ-calculus, the notion of solvability plays a crucial role. A λ-term M is *solvable* if there is a *head context* $\mathtt{H}$ such that $\mathtt{H}(\!|M|\!) \twoheadrightarrow_\beta \lambda x.x = \mathbf{I}$ (the identity); M is *unsolvable* if it is not solvable. Solvability (see [1]) underlies the fundamental notions of approximants, Böhm-trees and separability; moreover, it is possible to encode partial recursive functions in λ-calculus in such a way that undefinedness is represented by unsolvable λ-terms ([1, Ch. 8]). Enforcing the idea of unsolvable-as-meaningless, it is consistent to equate all unsolvable λ-terms (but not all λ-terms having no β-normal form, [1, Ch. 16]). A fundamental theorem for ordinary λ-calculus (see [2,3]) states that for every λ-term M the following are equivalent: (1) M is solvable; (2) the head reduction of M terminates; (3) the semantics of M in the Scott's model D_∞ is not the least element. Equivalence (1)⇔(2) (resp. (1)⇔(3)) gives a *semantical* (resp. *syntactical* or *operational*) characterization of solvability in ordinary λ-calculus.

The most common parameter passing policy for programming languages is call-by-value (CBV). Plotkin [4] introduced the λ_v-calculus in order to grasp the CBV paradigm in a pure λ-calculus setting. The λ_v-calculus (without constants) has the same syntax as ordinary λ-calculus but its β_v-reduction rule allows the contraction of a β-redex only if the argument is a λ-value, i.e. a variable or an abstraction. As argued in [5], a good CBV λ-calculus should enjoy an *internal* operational characterization (i.e. by using CBV reduction rules) of CBV-solvability.

This is not the case for Plotkin's λ_{v}-calculus and the weakness of β_v-reduction is widely recognized and accepted. Following [6,7], a λ-term M is λ_{v}-*solvable* if there is a head context $\mathtt{H}$ such that $\mathtt{H}(\!|M|\!) \twoheadrightarrow_{\beta_v} \mathbf{I}$. Let $\Delta = \lambda x.xx$: there is no head context sending (via β_v-reduction) $N = (\lambda y.\Delta)(x\mathbf{I})\Delta$ to $\mathbf{I}$, thus N is λ_{v}-unsolvable and hence it should be divergent, whereas it is β_v-normal. An operational characterization of λ_{v}-solvability has been provided in [6,7] but through a *call-by-name* reduction; this result is improved in [8] where the characterization is built upon strong normalization of the (call-by-name) lazy β-reduction.

There are many proposals of alternative CBV λ-calculi (see [9,10,11,12,5]) extending Plotkin's one by using explicit substitutions (constructors of the form `let...in`). In particular, Accattoli and Paolini [5] introduced recently the $\lambda_{\mathtt{vsub}}$-calculus where the reduction rule acts at a distance by extending the notion of β_v-redex (with explicit substitutions). In this setting they give an internal operational characterization of solvability and this characterization lifts to Herbelin and Zimmermann's λ_{CBV}-calculus, another CBV λ-calculus with explicit substitutions introduced in [9] (without rules acting at a distance but with commutation rules for explicit substitutions).

Paolini and Ronchi Della Rocca [6,7] made major contributions to the study of CBV-solvability through denotational semantics. In [6] they showed an intersection type system that characterizes λ_{v}-*potentially valuable*[3] (Thm. 6.4) and λ_{v}-solvable λ-terms (Thm. 6.5). We quote from [6, p. 28]: "The type assignment system presented here is strongly related to the system presented in [13] for reasoning on the denotational semantics of the [Plotkin's] λ_{v}-calculus. [...] The two systems have the same typability power". It is not shown whether this type system is "legal" (see [7, Def. 10.1.5]), which is substantially a sufficient condition to turn the type system into a *filter model* (i.e. a true domain model). In [7, Ch. 12] the same authors exhibit two models, $\mathcal{V}$ (§ 12.1) and $\mathcal{VV}$ (§ 12.2), both built from intersection type systems. The model $\mathcal{V}$ comes from a legal type system and it is shown to be isomorphic to the one of [13]. All and only λ_{v}-potentially valuable λ-terms have non trivial interpretation in $\mathcal{V}$, but $\mathcal{V}$ gives only a *partial* semantical characterization of λ_{v}-solvable λ-terms (Thm. 12.1.19). The model $\mathcal{VV}$ characterizes observational equivalence (Thm. 12.2.14) but it is not a filter model. Recently, Ehrhard [14] used a relational model of the λ_{v}-calculus, based on Linear Logic, to show that if the semantics of a λ-term M is not empty, then M is strongly normalizing for the lazy β_v-reduction (which does not reduce under abstractions); the converse is false (the aforesaid λ-term N is a counterexample).

The starting points of our work are [6,5,14]. We introduce the $\lambda_{\mathsf{v}}^{\sigma}$-calculus, a CBV λ-calculus having the same syntax as ordinary (and hence Plotkin's CBV) λ-calculus (there are no explicit substitutions) and extending the β_v-reduction by adding two reduction rules, σ_1 and σ_3. For the $\lambda_{\mathsf{v}}^{\sigma}$-calculus we give a semantical and an internal operational characterization of solvability and potential valuability. We use the relational model of [14], which can also be seen as a model of ordinary

[3] Following [6,7], a λ-term is λ_{v}-potentially valuable if there is a substitution sending it (via $\twoheadrightarrow_{\beta_v}$) into a λ-value. This notion is important for a CBV λ-calculus because if we want to manipulate some subterms, we need first to transform them into λ-values.

λ-calculus (unlike the model $\mathcal{V}$ of [7]) and satisfies a version of the Taylor formula (see [14]). We also introduce a resource-sensitive calculus with tests in which the elements of the relational model are definable: this is a promising tool to face the CBV full abstraction problem, along the lines of [15].

Our $\lambda_{\mathsf{v}}^{\sigma}$-calculus springs from Girard's call-by-value "boring" translation $(\cdot)^v$ of λ-calculus into Intuitionistic Multiplicative Exponential Linear Logic (IMELL) proof-nets, identified by $(A \Rightarrow B)^v = !A^v \multimap !B^v$ (see [16]). The images of a σ_1- or σ_3-redex and its contractum under $(\cdot)^v$ are equal modulo some specified "immediate" steps of cut-elimination. Our σ-rules are related to (but partly different from) Regnier's σ-reduction defined in [17,18] for the ordinary λ-calculus. Moreover, σ_1 and σ_3 correspond respectively to the commutation rules let_{app} and (a generalization of) let_{let} in λ_{CBV}-calculus (see [9,5]). In some sense, they can be seen as a finer (and local) decomposition of the reduction rules acting at a distance in $\lambda_{\mathtt{vsub}}$-calculus (it is possible to simulate $\lambda_{\mathtt{vsub}}$- and λ_{CBV}-calculus in our $\lambda_{\mathsf{v}}^{\sigma}$-calculus), but the absence of explicit substitutions in $\lambda_{\mathsf{v}}^{\sigma}$-calculus prevents from lifting the internal operational characterization of CBV-solvability from $\lambda_{\mathtt{vsub}}$- or λ_{CBV}-calculus to our $\lambda_{\mathsf{v}}^{\sigma}$-calculus.

Outline. In §2 we introduce our $\lambda_{\mathsf{v}}^{\sigma}$-calculus. Then, §3, §4 and §5 are devoted to the technical notions which are necessary in order to state our main results: in §3 we present two sub-reductions in the $\lambda_{\mathsf{v}}^{\sigma}$-calculus, called `w`- and `s`-reduction; in §4 and §5 we present a resource-sensitive version of the $\lambda_{\mathsf{v}}^{\sigma}$-calculus and the relational model of the (resource) $\lambda_{\mathsf{v}}^{\sigma}$-calculus. In §6 we state and prove our main theorems: the semantical (via the relational model) and syntactical (via `w`- and `s`-reductions) characterization of potential valuability and solvability; they say also that weak and strong normalizations coincide for both `w`- and `s`-reductions.

2 A CBV lambda-calculus with sigma-like-reductions

In this section we introduce $\lambda_{\mathsf{v}}^{\sigma}$, our version of CBV λ-calculus. The syntax of $\lambda_{\mathsf{v}}^{\sigma}$ is the same as the one of ordinary λ-calculus. Given a countable set of *variables* (denoted by $x, y, z, \ldots$), the language of $\lambda_{\mathsf{v}}^{\sigma}$ is defined by the following grammar:

$$\begin{array}{lrl} (\Lambda^{\mathsf{v}}) & V, U ::= x \mid \lambda x.M & \lambda\textit{-values} \\ (\Lambda) & M, N, L ::= V \mid MN & \lambda\textit{-terms} \end{array}$$

All λ-terms are considered up to α-conversion. The set of free variables of a λ-term M is denoted by $\mathsf{fv}(M)$. Given pairwise distinct variables $x_1, \ldots, x_n$, we denote by $M\{V_1/x_1, \ldots, V_n/x_n\}$ the λ-term obtained by the *capture-avoiding simultaneous substitution* of each free occurrence of x_i in the λ-term M by the λ-value V_i (for $1 \leq i \leq n$). Notice that, for all λ-values $V, V_1, \ldots, V_n$ and pairwise distinct variables $x_1, \ldots, x_n$, $V\{V_1/x_1, \ldots, V_n/x_n\}$ is a λ-value.

Contexts (with exactly one hole) are defined as usual via the grammar:

$$\mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x.\mathtt{C} \mid \mathtt{C}M \mid M\mathtt{C}\,.$$

We use $\mathtt{C}(\!|M|\!)$ for the λ-term obtained by the capture-allowing substitution of the λ-term M for $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

Definition 1. *We define the following binary relations from Λ to Λ:*

$$\begin{array}{lll} (\lambda x.M)V & \mapsto_{\beta_v} M\{V/x\} & \textit{with } V \in \Lambda^{\mathsf{v}} \\ (\lambda x.M)NL & \mapsto_{\sigma_1} (\lambda x.ML)N & \textit{with } x \notin \mathsf{fv}(L) \\ V((\lambda x.L)N) & \mapsto_{\sigma_3} (\lambda x.VL)N & \textit{with } x \notin \mathsf{fv}(V) \textit{ and } V \in \Lambda^{\mathsf{v}} \end{array}$$

For $R \in \{\beta_v, \sigma_1, \sigma_3\}$, if $M \mapsto_R M'$ then M is called R-redex.
We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ and $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

The side conditions on $\mapsto_\sigma$ in Def. 1 can be always fulfilled by α-renaming.

Notation. Let $\mapsto_R \subseteq \Lambda \times \Lambda$. We use $\to_R$ (called *R-reduction*) for the closure of $\mapsto_R$ under all contexts; we denote by $\twoheadrightarrow_R$ (resp. $\to_R^+$) the reflexive-transitive (resp. transitive) closure of $\to_R$. Let M be a λ-term: M is *R-normal* if there is no λ-term N such that $M \to_R N$; M is *R-normalizable* if there is a R-normal λ-term N such that $M \twoheadrightarrow_R N$; M is *strongly R-normalizing* if there is no sequence $(N_i)_{i\in\mathbf{N}}$ such that $M = N_0$ and $N_i \to_R N_{i+1}$ for every $i \in \mathbf{N}$.

Notice that, for any λ-value V, if $V \to_{\mathsf{v}} M$, then M is a λ-value.

The $\lambda_{\mathsf{v}}^{\sigma}$*-calculus* is the set Λ of λ-terms endowed with the v-reduction $\to_{\mathsf{v}}$. The set Λ endowed with $\to_{\beta_v}$ is Plotkin's CBV λ-calculus ([4]) without constants.

Informally, σ-rules unblock β_v-redexes which are hidden by the "hyper-sequential structure" of λ-terms. This approach is alternative to the one in [5] where hidden β_v-redexes are reduced thanks to a rule acting at a distance.

Example. $N = (\lambda y.\Delta)(x\mathbf{I})\Delta \to_{\sigma_1} (\lambda y.\Delta\Delta)(x\mathbf{I}) \to_{\beta_v} (\lambda y.\Delta\Delta)(x\mathbf{I}) \to_{\beta_v} \dots$ is the only possible v-reduction path from N: N is not v-normalizable but β_v-normal.

2.1 Confluence of our CBV lambda-calculus

Our goal here is to prove that the v-reduction is confluent.

Proof at p. 16

Proposition 2. *The reduction $\to_\sigma$ is strongly normalizing.*

Proof. First, we define two sizes $\mathsf{s}(M)$ and $\#M$ by induction on the λ-term M:

$$\begin{array}{ll} \mathsf{s}(x) = 2; & \#x = 1; \\ \mathsf{s}(\lambda x.M) = \mathsf{s}(M) + 1; & \#\lambda x.M = \#M + \mathsf{s}(M); \\ \mathsf{s}(MN) = \mathsf{s}(M) + \mathsf{s}(N). & \#MN = \#M + \#N + 2\mathsf{s}(M)\mathsf{s}(N) - 1. \end{array}$$

It is sufficient to show that if $N \to_\sigma N'$ then $\mathsf{s}(N) = \mathsf{s}(N')$ and $\#N > \#N'$. □

Proof at p. 16

Proposition 3. *The reduction $\to_\sigma$ is (not strongly) confluent.*

See Remarks 41 and 42 at p. 18

Proof. By Newman's Lemma and Prop. 2, it is sufficient to show that $\to_\sigma$ is locally confluent. The proof of local confluence is by induction on M. The λ-term $\Xi = (\lambda x.x')\big((\lambda y.y'\mathbf{I})(z\mathbf{I})\big)(z'\mathbf{I})$ is an objection to strong confluence of $\to_\sigma$. □

Lemma 4 (Hindley–Rosen, [1, p. 64]). *Let $\to_1, \to_2 \subseteq X^2$ (for any set X). If they are both confluent and they commute, i.e. if $t \twoheadrightarrow_1 u_1$ and $t \twoheadrightarrow_2 u_2$ then there exists s such that $u_1 \twoheadrightarrow_2 s$ and $u_2 \twoheadrightarrow_1 s$, then $\to_1 \cup \to_2$ is confluent.*

Proof at p. 17

Lemma 5. *Let* $M, M' \in \Lambda$, $V, V', V_1, \dots, V_m \in \Lambda^{\mathsf{v}}$ *and* $R \in \{\beta_v, \sigma, \mathsf{v}\}$.

(i) *If* $V \to_R V'$ *then* $M\{V/x\} \twoheadrightarrow_R M\{V'/x\}$.
(ii) *If* $M \to_R M'$ *then* $M\{V_1/x_1, \dots, V_m/x_m\} \to_R M'\{V_1/x_1, \dots, V_m/x_m\}$.

Lemma 6. *The reductions* $\to_{\beta_v}$ *and* $\to_\sigma$ *commute.* Proof at p. 18

Proof. It suffices to prove that if $M \to_\sigma N_1$ and $M \to_{\beta_v} N_2$ then there is L s.t. $N_2 \twoheadrightarrow_\sigma L$ and $N_1 \to_{\beta_v} L$. The proof of this statement is by induction on M. $\square$

By Lemmas 4 and 6, Prop. 3 and confluence of $\to_{\beta_v}$ (see [4]), we conclude:

Theorem 7. *The reduction* $\to_{\mathsf{v}}$ *is (not strongly) confluent.*

The λ-term Ξ (see proof of Prop. 3) is an objection to strong confluence of $\to_{\mathsf{v}}$. See Remarks 41 and 42 at p. 18

If in the definition of $\mapsto_{\sigma_3}$ (Def. 1) we replace the λ-value V with any λ-term M then $\to_\sigma$ and $\to_{\mathsf{v}}$ are not (locally) confluent: consider $(\lambda x.x')(z\mathbf{I})\big((\lambda y.y')(z'\mathbf{I})\big)$. See Remark 43 at p. 18

3 Weak and stratified CBV reductions

In this section we introduce two sub-reductions of $\to_{\mathsf{v}}$: *weak* (or $\mathtt{w}$-)*reduction* and *stratified* (or $\mathtt{s}$-)*reduction*. We will show in §6 that they give an operational characterization of potential valuability and solvability: they are the "CBV counterpart" of head reduction for ordinary λ-calculus. Whereas head reduction is strictly deterministic (any λ-term has at most one head redex), a λ-term might have several (overlapping) $\mathtt{w}$- or $\mathtt{s}$-redexes. Anyway, both $\mathtt{w}$- and $\mathtt{s}$-reductions are confluent (Prop. 10) and for them weak and strong normalization coincide (Thm. 24 and 25). We have gathered our definition of $\mathtt{w}$- and $\mathtt{s}$-reductions from [5].

Definition 8. Weak *and* stratified contexts *(denoted respectively by* $\mathtt{W}$ *and* $\mathtt{S}$*) are contexts defined via the grammar:*

$$\mathtt{W} ::= (\!|\cdot|\!) \mid \mathtt{W}M \mid M\mathtt{W} \mid (\lambda x.\mathtt{W})M \qquad \mathtt{S} ::= \mathtt{W} \mid \lambda x.\mathtt{S} \mid \mathtt{S}M$$

Notation. Let $\mapsto_R \subseteq \Lambda \times \Lambda$: we use $\to_{\mathtt{w}[R]}$ (resp. $\to_{\mathtt{s}[R]}$) for the closure under weak (resp. stratified) contexts of $\mapsto_R$. We set $\mathtt{w} = \mathtt{w}[\mathsf{v}]$ and $\mathtt{s} = \mathtt{s}[\mathsf{v}]$; for instance, $\to_{\mathtt{w}} = \to_{\mathtt{w}[\mathsf{v}]}$ (called $\mathtt{w}$-*reduction*) and $\to_{\mathtt{s}} = \to_{\mathtt{s}[\mathsf{v}]}$ (called $\mathtt{s}$-*reduction*).

Note that $\to_{\mathtt{w}} \subsetneq \to_{\mathtt{s}} \subsetneq \to_{\mathsf{v}}$. In weak contexts, if the hole is under an abstraction then this abstraction is the left-hand side of an application. Stratified contexts never contain the hole under an abstraction which is in the right-hand side of some application, unless the abstraction is the left-hand side of an application.

Example. Let $\Omega = \Delta\Delta$: one has $\Omega \to_{\mathtt{w}} \Omega \to_{\mathtt{w}} \dots$, $\lambda y.\Omega \to_{\mathtt{s}} \lambda y.\Omega \to_{\mathtt{s}} \dots$, and $x(\lambda y.\Omega) \to_{\mathsf{v}} x(\lambda y.\Omega) \to_{\mathsf{v}} \dots$, whereas $\lambda y.\Omega$ (resp. $x(\lambda y.\Omega)$) is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal.

We will now prove that the $\mathtt{w}$- and $\mathtt{s}$-reductions are confluent.

Lemma 9. (i) *The reductions* $\to_{\mathtt{w}[\beta_v]}$ *and* $\to_{\mathtt{s}[\beta_v]}$ *are strongly confluent.* Proof at p. 19
(ii) *The reductions* $\to_{\mathtt{w}[\sigma]}$ *and* $\to_{\mathtt{s}[\sigma]}$ *are confluent.*
(iii) *The reductions* $\to_{\mathtt{w}[\beta_v]}$ *and* $\to_{\mathtt{w}[\sigma]}$ *(resp.* $\to_{\mathtt{s}[\beta_v]}$ *and* $\to_{\mathtt{s}[\sigma]}$*) commute.*

By Lemmas 4 and 9 we can conclude:

Proposition 10. *The reductions* $\to_{\mathtt{w}}$ *and* $\to_{\mathtt{s}}$ *are (not strongly) confluent.*

The λ-term Ξ (see p. 4) is an objection to strong confluence of $\to_{\mathtt{w}}$ and $\to_{\mathtt{s}}$.

3.1 Characterization of $\mathtt{w}$- and $\mathtt{s}$-normal forms

Our goal here is to characterize $\mathtt{w}$- and $\mathtt{s}$-normal forms. Having no explicit substitutions, our characterization appears more concise than the one in [5].

Definition 11. *We define the subsets $\mathsf{a}_{\mathtt{nf}}$, $\mathsf{s}_{\mathtt{nf}}$ and $\mathsf{w}_{\mathtt{nf}}$ of Λ as follows:*

$$
\begin{array}{lrcl}
(\mathsf{a}_{\mathtt{nf}}) & A_{\mathtt{nf}} & ::= & xV \mid xA_{\mathtt{nf}} \mid A_{\mathtt{nf}}W_{\mathtt{nf}} \\
(\mathsf{w}_{\mathtt{nf}}) & W_{\mathtt{nf}} & ::= & V \mid (\lambda x.W_{\mathtt{nf}})A_{\mathtt{nf}} \mid A_{\mathtt{nf}} \\
(\mathsf{s}_{\mathtt{nf}}) & S_{\mathtt{nf}} & ::= & x \mid \lambda x.S_{\mathtt{nf}} \mid (\lambda x.S_{\mathtt{nf}})A_{\mathtt{nf}} \mid A_{\mathtt{nf}}
\end{array}
$$

A β-redex is a λ-term of shape $(\lambda x.M)L$. Notice that $\mathsf{a}_{\mathtt{nf}} \subsetneq \mathsf{s}_{\mathtt{nf}} \subsetneq \mathsf{w}_{\mathtt{nf}}$ and if $N \in \mathsf{a}_{\mathtt{nf}}$ then N has a free "head variable" and it is neither a value nor a β-redex.

Proof at p. 20

Proposition 12. *Let M be a λ-term.*

(i) *M is $\mathtt{w}$-normal iff $M \in \mathsf{w}_{\mathtt{nf}}$.*
(ii) *M is $\mathtt{s}$-normal iff $M \in \mathsf{s}_{\mathtt{nf}}$.*
(iii) *M is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal and is neither a value nor a β-redex iff $M \in \mathsf{a}_{\mathtt{nf}}$.*

4 A resource CBV lambda-calculus

We now introduce the *resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus*, a valuable tool to prove some parts of our main results. It is an extension of the resource CBV λ-calculus introduced in [14, §5.2]. Its syntax is defined by the following grammar (the same as in [14]):

$$
\begin{array}{lrcll}
(\mathsf{r}\Lambda^{\mathsf{v}}) & u, v & ::= & x \mid \lambda x.t & \textit{resource values} \\
(\mathsf{r}\Lambda^{\mathsf{t}}) & s, t & ::= & st \mid [v_1, \dots, v_k] \quad (k \geq 0) & \textit{resource terms} \\
(\mathsf{r}\Lambda) & e, f & ::= & v \mid t & \textit{expressions}
\end{array}
$$

A resource term like $[v_1, \dots, v_k]$ is a multiset of resource values (called *bag*).

The resource-version of the β_v-rule makes use of *linear substitution*, which requires to enrich the syntax of the calculus with finite sets of resource terms.

Notation. Since the set $\mathcal{P}_{\mathrm{f}}(A)$ of all finite subsets of a set A is the free module $\mathbf{2}\langle A\rangle$ generated by A over the boolean semiring $\{0, 1\}$ with $1 + 1 = 1$, we will use algebraic notations for operations on its elements (+ for set unions, 0 for the empty set), as done in [15,14].

We denote by $\mathsf{deg}_x(e)$ the number of free occurrences of the variable x in the expression e. Given $e \in \mathsf{r}\Lambda$, $v_1, \dots, v_k \in \mathsf{r}\Lambda^{\mathsf{v}}$ and an enumeration of the free occurrences of variable x in e, if $\mathsf{deg}_x(e) = k$ then by $\sum_{f \in \mathfrak{S}_k} e\{v_{f(1)}/x^1, \dots, v_{f(k)}/x^k\}$ we mean the sum of all expressions obtained by substituting $v_{f(i)}$ for the i-th free occurrence of x in e, as f varies over all elements of the set $\mathfrak{S}_k$ of permutations of $\{1, \dots, k\}$. Finally, the linear substitution of $[v_1, \dots, v_k]$ for x in e is

$$
e\langle [v_1, \dots, v_k]/x \rangle = \begin{cases} \sum_{f \in \mathfrak{S}_k} e\{v_{f(1)}/x^1, \dots, v_{f(k)}/x^k\} & \text{if } \mathsf{deg}_x(e) = k \\ 0 & \text{otherwise} \end{cases}
$$

Notice that, for $\mathtt{n} \in \{\mathtt{v}, \mathtt{t}\}$, if $e \in \mathsf{r}\Lambda^{\mathtt{n}}$ then $e\langle [v_1, \ldots, v_k]/x\rangle \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{n}}\rangle$.

Resource contexts (with exactly one hole) are defined via the grammar:

$$\mathtt{R} ::= (\!|\cdot|\!) \mid \mathtt{R}t \mid t\mathtt{R} \mid [\lambda x.\mathtt{R}, v_1, \ldots, v_k] \quad (k \geq 0)$$

Let $\mathtt{R}$ be a resource context. We use $\mathtt{R}(\!|t|\!)$ for the resource term obtained by the capture-allowing substitution of the resource term t for the hole $(\!|\cdot|\!)$ in $\mathtt{R}$. If $\mathbb{T} = \sum_{i=1}^{n} t_i$ (with $t_1, \ldots, t_n \in \mathsf{r}\Lambda^{\mathtt{t}}$), then $\mathtt{R}(\!|\mathbb{T}|\!) = \sum_{i=1}^{n} \mathtt{R}(\!|t_i|\!) \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}}\rangle$ (see also [14, §5.2] and [15, §2.1]). For example, $\mathtt{R}(\!|0|\!) = 0$ and $[\lambda x.[x](\!|[y][z] + [z][y]|\!), y] = [\lambda x.[x]([y][z]), y] + [\lambda x.[x]([z][y]), y]$.

Definition 13. *We define the following binary relations from* $\mathsf{r}\Lambda^{\mathtt{t}}$ *to* $\mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}}\rangle$*:*

$$[\lambda x.t][v_1, \ldots, v_k] \mapsto_{\beta_v} t\langle [v_1, \ldots, v_k]/x\rangle \qquad [\lambda x.t]ss' \mapsto_{\sigma_1} [\lambda x.ts']s \quad \text{if } x \notin \mathsf{fv}(s')$$
$$[v_1, \ldots, v_n]t \mapsto_0 0 \quad \text{if } n \neq 1 \qquad [v]([\lambda x.t]s) \mapsto_{\sigma_3} [\lambda x.[v]t]s \quad \text{if } x \notin \mathsf{fv}(v)$$

We set $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3} \cup \mapsto_0$.

According to the convention of §2, $\to_{\mathsf{v}} \subseteq \mathsf{r}\Lambda^{\mathtt{t}} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}}\rangle$ is the reduction obtained by resource-contextual closure of $\mapsto_{\mathsf{v}}$.

The *resource* $\lambda_{\mathsf{v}}^{\sigma}$*-calculus* consists of the language $\mathsf{r}\Lambda^{\mathtt{t}}$ and the reduction $\to_{\mathsf{v}}$: it is the resource CBV λ-calculus of [14] plus the σ_1- and σ_3-rules.

As a technical simplification, we extend $\to_{\mathsf{v}}$ to a binary relation on $\mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}}\rangle$ by linearity, i.e. $(\sum_{i=1}^{n} t_i) + \mathbb{S} \to_{\mathsf{v}} (\sum_{i=1}^{n} \mathbb{T}_i) + \mathbb{S}$ iff $t_i \to_{\mathsf{v}} \mathbb{T}_i$ for every $i = 1, \ldots, n$ ($n \geq 1$). With this extension we can concisely state the following theorem:

Theorem 14. *Reduction* $\to_{\mathsf{v}}$ *on* $\mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}}\rangle$ *is strongly normalizing and confluent.*

We omit the proof of Thm. 14. Strong normalization is evident (see [14] for a proof for the resource-contextual closure of $\mapsto_{\beta_v} \cup \mapsto_0$). The proof of local confluence for the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus is analogous to the one for v-reduction on λ-terms (see §2). Finally, confluence is obtained by Newman's Lemma.

5 A relational model of (resource) CBV lambda-calculus

In this section we present a relational model for both the $\lambda_{\mathsf{v}}^{\sigma}$-calculus and the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus. This model is to be found in the category **Rel** of sets and relations (i.e. $\mathbf{Rel}(X, Y) = \mathcal{P}(X \times Y)$). In **Rel** identities are diagonal relations and composition of morphisms is the standard composition of relations. This category has a symmetric monoidal structure given by $\mathbf{1} = \{1\}$ (arbitrary singleton set) and $X \otimes Y = X \times Y$. This symmetric monoidal category is closed, with $X \multimap Y = X \times Y$, and $*$-autonomous with dualizing object $\bot = \mathbf{1}$. Category **Rel** is cartesian, with $X \,\&\, Y = (\{1\} \times X) \cup (\{2\} \times Y)$, and has an exponential functor ! defined by $!X = \mathcal{M}_{\mathrm{f}}(X)$ (the set of finite multisets on X) and $!f = \{([\alpha_1, \ldots, \alpha_n], [\beta_1, \ldots, \beta_n]) : n \geq 0,\ (\alpha_i, \beta_i) \in f \ \forall\, 1 \leq i \leq n\}$ for $f \in \mathbf{Rel}(X, Y)$.

All this structure makes **Rel** a new-Seely category and hence a categorical model of Linear Logic (LL). For more details we refer the reader to [19,14].

The model. We build inductively a family of sets $(U_n)_{n \in \mathbf{N}}$ given by $U_0 = \emptyset$ and $U_{n+1} = \mathcal{M}_{\mathrm{f}}(U_n) \times \mathcal{M}_{\mathrm{f}}(U_n)$. Finally, we set $U = \bigcup_{n \in \mathbf{N}} U_n$. Notice that $U_n \subsetneq U_{n+1}$ for all $n \in \mathbf{N}$, and $U = \mathcal{M}_{\mathrm{f}}(U) \times \mathcal{M}_{\mathrm{f}}(U) = \,!U \multimap \,!U$.

5.1 Interpreting the CBV lambda-calculus

Using the fact that **Rel** has the structure of a LL model, we can give a concrete interpretation of λ-terms as morphisms from $\mathcal{M}_{\mathrm{f}}(U)^n$ to $\mathcal{M}_{\mathrm{f}}(U)$ in **Rel** (where $\mathcal{M}_{\mathrm{f}}(U)^n$ is the n-fold set-theoretic power of $\mathcal{M}_{\mathrm{f}}(U)$). This semantics can also be described by type judgements (see [14]). With $a \uplus b$ we indicate the union of the multisets a and b (accounting for repetitions); if $\vec{a}$ and $\vec{b}$ are two finite sequences (of the same length) of multisets, $\vec{a} \uplus \vec{b}$ is their component-wise union.

Definition 15. *For every λ-term M and repetition-free list $\vec{x} \supseteq \mathsf{fv}(M)$, we define, by induction on M, its* interpretation $[\![M]\!]_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$ *(where n is the length of $\vec{x}$), as follows:*

$$
\begin{aligned}
[\![x_i]\!]_{\vec{x}} &= \{(\vec{a}, a_i) : a_i \in \mathcal{M}_{\mathrm{f}}(U),\ a_j = [\,]\ \textit{for all}\ 1 \leq j \leq n\ \textit{with}\ j \neq i\} \\
[\![\lambda y.N]\!]_{\vec{x}} &= \{(\biguplus_{i=1}^{k} \vec{a}_i, \biguplus_{i=1}^{k}[(b_i, c_i)]) : k \geq 0,\ \forall i = 1, \ldots, k.\,((\vec{a}_i, b_i), c_i) \in [\![N]\!]_{\vec{x},y}\} \\
[\![MN]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1, c) : \exists\, b \in \mathcal{M}_{\mathrm{f}}(U).\ (\vec{a}_0, [(b, c)]) \in [\![M]\!]_{\vec{x}},\ (\vec{a}_1, b) \in [\![N]\!]_{\vec{x}}\}\,.
\end{aligned}
$$

Notation. Hereafter, whenever we write $[\![M]\!]_{\vec{x}}$ we suppose that $\vec{x}$ is a repetition-free list of variables containing $\mathsf{fv}(M)$. Moreover, we will sometimes silently use the fact that $[\![M]\!]_{\vec{x},y} = \{((\vec{a}, [\,]), b) : (\vec{a}, b) \in [\![M]\!]_{\vec{x}}\}$ whenever $y \notin \vec{x}$.

Proof at p. 21 **Theorem 16 (soundness).** *Let $M, N \in \Lambda$. If $M \to_{\mathsf{v}} N$, then $[\![M]\!]_{\vec{x}} = [\![N]\!]_{\vec{x}}$.*

5.2 Interpreting the resource CBV lambda-calculus

In addition to the structure mentioned above, **Rel** is additive, and more precisely its hom-sets are enriched over the category of complete lattices, with set-theoretic union as join operation. The category **Rel** is a *weak differential* LL *model* (see [14]). Using this structure we can give the concrete interpretation of expressions as morphisms from $\mathcal{M}_{\mathrm{f}}(U)^n$ to $\mathcal{M}_{\mathrm{f}}(U)$ in **Rel**.

Definition 17. *For every expression e and repetition-free list $\vec{x} \supseteq \mathsf{fv}(e)$, we define, by induction on e, its* interpretation $[\![e]\!]_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$ *(where n is the length of $\vec{x}$), as follows:*

$$
\begin{aligned}
[\![x_i]\!]_{\vec{x}} &= \{(\vec{a}, [\alpha]) : \alpha \in U,\ a_i = [\alpha],\ a_j = [\,]\ \textit{for all}\ 1 \leq j \leq n\ \textit{with}\ j \neq i\} \\
[\![\lambda z.t]\!]_{\vec{x}} &= \{(\vec{a}, [(b, c)]) : ((\vec{a}, b), c) \in [\![t]\!]_{\vec{y},z}\} \\
[\![st]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U).\ (\vec{a}_0, [(b, c)]) \in [\![s]\!]_{\vec{x}},\ (\vec{a}_1, b) \in [\![t]\!]_{\vec{x}}\} \\
[\![[v_1, \ldots, v_k]]\!]_{\vec{x}} &= \{(\biguplus_{i=1}^{k} \vec{a}_i, \biguplus_{i=1}^{k} b_i) : k \geq 0,\ \forall i = 1, \ldots, k.\ (\vec{a}_i, b_i) \in [\![v_i]\!]_{\vec{x}}\}\,.
\end{aligned}
$$

Finally, sums of expressions are interpreted by setting $[\![\sum_{i=1}^{n} e_i]\!]_{\vec{x}} = \bigcup_{i=1}^{n} [\![e_i]\!]_{\vec{x}}$.

Notation. As for λ-terms, whenever we write $[\![e]\!]_{\vec{x}}$ we suppose that $\vec{x}$ is a repetition-free list of variables containing $\mathsf{fv}(e)$, and similarly for the sums. Note that $[\![[\,]]\!]_{\vec{x}} = \{([\,]^n, [\,])\} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$, where $[\,]^n = (\underbrace{[\,], \ldots, [\,]}_{n \text{ times}})$.

Proof at p. 21

Theorem 18 (soundness). *Let* $\mathbb{S}, \mathbb{T} \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathsf{t}} \rangle$. *If* $\mathbb{S} \to_{\mathsf{v}} \mathbb{T}$, *then* $[\![\mathbb{S}]\!]_{\vec{x}} = [\![\mathbb{T}]\!]_{\vec{x}}$.

The following notion of CBV Taylor expansion has been introduced in [14].

Definition 19 ([14], Taylor expansion). *Given a* λ*-term* M*, we inductively define a set* $\mathcal{T}(M)$ *of resource terms, called the* Taylor expansion *of* M*, as follows:*

$$\mathcal{T}(x) = \{[x^n] : n \geq 0\} \quad \textit{where } [x^n] = [\overbrace{x, \ldots, x}^{n \text{ times}}]$$
$$\mathcal{T}(\lambda x.M) = \{[\lambda x.t_1, \ldots, \lambda x.t_n] : n \geq 0,\ \forall i.\ t_i \in \mathcal{T}(M)\}$$
$$\mathcal{T}(MN) = \{st : s \in \mathcal{T}(M),\ t \in \mathcal{T}(N)\}\,.$$

Theorem 20 ([14]). *Let* M *be a* λ*-term. Then* $[\![M]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}(M)} [\![t]\!]_{\vec{x}}$.

Thm. 20 shows the semantical connection between λ-terms and their Taylor expansion. In the next section (§6) it will be applied in Thm. 39.1, which is in turn a fundamental part of one of our main results Thm. 24.

Definition 21. *For every expression* e *we define by induction the set* $\mathsf{strat}(e)$ *of multisets of resource values that occur in* e *in* stratified position*, as follows:*

$\mathsf{strat}(x) = \emptyset$; $\quad\mathsf{strat}([v_1, \ldots, v_n]) = \{[v_1, \ldots, v_n]\} \cup \bigcup_{i=1}^{n} \mathsf{strat}(v_i)\ (n \geq 0)$;
$\mathsf{strat}(st) = \mathsf{strat}(s)$; $\quad\mathsf{strat}(\lambda x.t) = \mathsf{strat}(t)$.

We set $\mathsf{Strat} = \{t \in \mathsf{r}\Lambda^{\mathsf{t}} : [\,] \notin \mathsf{strat}(t)\}$*, whose elements are called* stratified resource terms.

A stratified resource term t does not contain any $[\,]$ in stratified position, i.e. every occurrence of $[\,]$ in t is a subterm of some subterm of t in argument position. For instance: $[x][\,]$, $[x]([\lambda z.[\,]][\,]) \in \mathsf{Strat}$ but $[\,]$, $[\,][z]$, $[\lambda z.[\,][x, y]] \notin \mathsf{Strat}$.

Stratified resource terms are not closed under v-reduction. For example, the stratified resource term $[\lambda x.[x]][\lambda y.[\,]]$ v-reduces to the non-stratified $[\lambda y.[\,]]$.

Definition 22 (stratified Taylor expansion). *Given a* λ*-term* M*, we define its* stratified Taylor expansion $\mathcal{T}_{\mathsf{s}}(M) = \{t \in \mathcal{T}(M) :$ *if* $t \twoheadrightarrow_{\mathsf{v}} \mathbb{T}$*, then* $\mathbb{T} \subseteq \mathsf{Strat}\}$.

Example. The λ-term $M = (\lambda xy.x)\Omega$ is neither $\mathtt{w}$- nor $\mathtt{s}$-normalizable and every resource term in $\mathcal{T}(M)$ v-reduces to 0. Instead the non-$\mathtt{s}$-normalizable (but $\mathtt{w}$-normal) λ-term $N = (\lambda xy.\Omega)(zz')$ has infinitely many resource terms in $\mathcal{T}(N)$ that do not v-reduce to 0, like $t = [\lambda x.[\,]]([z][z'])$ for example. However $t \notin \mathcal{T}_{\mathsf{s}}(N)$ and $\mathcal{T}_{\mathsf{s}}(N)$ contains only resource terms that v-reduce to 0, because all resource terms in $\mathcal{T}(N)$ not v-reducing to 0 contain at least one $[\,]$ in stratified position.

The semantical connection between λ-terms and their stratified Taylor expansion is illustrated in one of our main results, Thm. 25. In particular, Thm. 39.2 is the step in which it is proved that the interpretation of $\mathcal{T}_{\mathsf{s}}(M)$ actually witnesses the strong $\mathtt{s}$-normalization of M. Intuitively, if $t \in \mathcal{T}_{\mathsf{s}}(M)$ then the v-normal form of t is a sum $\sum_{i=1}^{n} t_i$ $(n \geq 0)$ of stratified resource terms, each of which does not contain $[\,]$ in stratified position: a subterm $[\,]$ inside a t_i does not "hide" a non-$\mathtt{s}$-normalizable λ-term N such that $M = \mathtt{S}(\!|N|\!)$. So, by Lemma 38.ii one can prove that if $t \neq 0$ then M is strongly $\mathtt{s}$-normalizing.

6 The main theorems

In this section we will present our main results: the semantical and internal operational characterization of *potential valuability* (Thm. 24) and *solvability* (Thm. 25) for the $\lambda_{\mathsf{v}}^{\sigma}$-calculus. See §1 for a overview of these notions.

Definition 23 (Potential valuability, solvability). *Let M be a λ-term:*

– *M is* potentially valuable *if there exist variables $x_1, \ldots, x_m$ and λ-values $V, V_1, \ldots, V_m$ (with $m \geq 0$) such that $M\{V_1/x_1, \ldots, V_m/x_m\} \twoheadrightarrow_{\mathsf{v}} V$;*
– *M is* solvable *if there exist variables $x_1, \ldots, x_m$ and λ-terms $N_1, \ldots, N_n$ (for some $n, m \geq 0$) such that $(\lambda x_1 \ldots x_m.M)N_1 \cdots N_n \twoheadrightarrow_{\mathsf{v}} \mathbf{I}$.*

We state now the two main theorems. In particular, Thm. 24 says that $\mathtt{w}$-normalizability (i.e. potential valuability) plays a role analogous to that of head-normalizability for many call-by-name models, like Scott's D_{∞}.

Theorem 24. *Let M be a λ-term with $\vec{x} \supseteq \mathsf{fv}(M)$. The following are equivalent:*
(i) M is $\mathtt{w}$-normalizable;
(ii) M is potentially valuable;
(iii) $[\![M]\!]_{\vec{x}} \neq \emptyset$;
(iv) M is strongly $\mathtt{w}$-normalizing.

Theorem 25. *Let M be a λ-term with $\vec{x} \supseteq \mathsf{fv}(M)$. The following are equivalent:*
(i) M is $\mathtt{s}$-normalizable;
(ii) M is solvable;
(iii) $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$;
(iv) M is strongly $\mathtt{s}$-normalizing.

An immediate corollary of Thm. 24 and 25 is that every solvable (i.e. $\mathtt{s}$-normalizable) λ-term is also potentially valuable (i.e. $\mathtt{w}$-normalizable).

The proofs of Thm. 24 and 25 are divided into parts, which are detailed separately in the next subsections, due to the different techniques used for each one of them. The splitting of the two proofs follows the same pattern. The implications (i) ⇒ (ii) of both theorems are proved in §6.1 by purely syntactical means. The implication (ii) ⇒ (iii) of Thm. 24 is shown in §6.2 using the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus of §4; for this implication of Thm. 25 we use an extension of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus presented in §6.3. The implication (iii) ⇒ (iv) of both theorems is proved in §6.4 using simulations of $\mathtt{w}$- and $\mathtt{s}$-reductions in $\lambda_{\mathsf{v}}^{\sigma}$-calculus by the v-reduction of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus. Finally, (iv) ⇒ (i) are trivial in both cases.

6.1 From weak and stratified normalization to solvability and potential valuability

Our goal here is to prove the implication (i) ⇒ (ii) of Thm. 24 and 25. Our approach is largely inspired by [6,7,5].

For every $n \in \mathbf{N}$, we set $\mathsf{o}^n = \lambda x_n \ldots x_0.x_0$. Notice that $\mathsf{o}^0 = \mathbf{I}$ and o^n is a closed value for any $n \in \mathbf{N}$. Moreover, $\mathsf{o}^n V \mapsto_{\beta_v} \mathsf{o}^{n-1}$ for any $n > 0$ and $V \in \Lambda^{\mathsf{v}}$.

Proof at p. 21

Lemma 26. *Let $M \in \mathsf{w}_{\mathtt{nf}}$ with $\mathsf{fv}(M) \subseteq \{x_1, \ldots, x_m\}$ and let $j \in \mathbf{N}$. Then there exists $h > 0$ such that for all $n_1, \ldots, n_m \geq j + h$ there exists a λ-term N such that $M\{\mathsf{o}^{n_1}/x_1, \ldots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.N$ and $\lambda x.N$ is closed.*

Proof at p. 22

Lemma 27. *Let $M \in \mathsf{s_{nf}}$ with $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m\}$ and let $j \in \mathbf{N}$. Then there exist $h, k \in \mathbf{N}$ such that for all $n_1, \dots, n_{m+k} \geq j + h$ there exists $n \geq j$ such that $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$.*

Theorem 28. *Let M be a λ-term.*

1. [(i)⇒(ii) of Thm. 24] *If M is $\mathtt{w}$-normalizable then M is potentially valuable.*
2. [(i)⇒(ii) of Thm. 25] *If M is $\mathtt{s}$-normalizable then M is solvable.*

Proof. For point 1 (resp. 2), hypothesis means that there is a $\mathtt{w}$-(resp. $\mathtt{s}$-)normal form M' such that $M \twoheadrightarrow_{\mathtt{w}} M'$ (resp. $M \twoheadrightarrow_{\mathtt{s}} M'$), moreover $M' \in \mathsf{w_{nf}}$ (resp. $M' \in \mathsf{s_{nf}}$) by Prop. 12. Let $\mathsf{fv}(M) = \{x_1, \dots, x_m\}$ and thus $\mathsf{fv}(M') \subseteq \{x_1, \dots, x_m\}$.

1. By Lemma 26 (taking $j = 0$) there exists $h > 0$ such that: $M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.N$, for some closed λ-value $\lambda x.N$. One has $M\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\} \twoheadrightarrow_{\mathsf{v}} M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\}$ by Lemma 5.ii, so that M is potentially valuable because $\lambda x.N$ is a closed λ-value.
2. By Lemma 27 (taking $j = 0$), there exist $h, k, n \in \mathbf{N}$ such that: $(M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\})\mathsf{o}^h \dots \mathsf{o}^h \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$ (o^h is applied k times). We conclude that M is solvable because if we set $\mathtt{H} = (\lambda x_1 \dots x_m.(\!|\cdot|\!))\ \underbrace{\mathsf{o}^h \dots \mathsf{o}^h}_{m+k \text{ times}}\ \underbrace{\mathbf{I} \dots \mathbf{I}}_{n \text{ times}}$, then
$$\mathtt{H}(\!|M|\!) \twoheadrightarrow_{\mathsf{v}} \mathtt{H}(\!|M'|\!) \twoheadrightarrow_{\mathsf{v}} (M'\{\mathsf{o}^h/x_1, \dots, \mathsf{o}^h/x_m\})\mathsf{o}^h \dots \mathsf{o}^h\mathbf{I} \dots \mathbf{I} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n\mathbf{I} \dots \mathbf{I} \twoheadrightarrow_{\mathsf{v}} \mathbf{I}.$$ □

6.2 From potential valuability to non-emptyness

The following theorem proves the implication (ii) ⇒ (iii) of Thm. 24.

Theorem 29. *Let M be a λ-term with $\vec{x} \supseteq \mathsf{fv}(M)$. If M is potentially valuable, then $[\![M]\!]_{\vec{x}} \neq \emptyset$.*

Proof. If M is potentially valuable (see Def. 23) there exist variables $x_1, \dots, x_m$ and λ-values $V, V_1, \dots, V_m$ (for some $m \geq 0$) s.t. $M\{V_1/x_1, \dots, V_m/x_m\} \twoheadrightarrow_{\mathsf{v}} V$. Since variables are λ-values, we can suppose without loss of generality that $\vec{x} = (x_1, \dots, x_m) \supseteq \mathsf{fv}(M)$. Let $\vec{y} = \mathsf{fv}(V) \cup \bigcup_{i=1}^m \mathsf{fv}(V_i)$. One can prove by induction on M that See Lemma 48 at p. 24

$$[\![M\{V_1/x_1, \dots, V_m/x_m\}]\!]_{\vec{y}} = \big\{(\textstyle\biguplus_{i=1}^m \vec{a}_i, c) :\ \exists\, b_1, \dots, b_m \in \mathcal{M}_{\mathrm{f}}(U) : ((b_1, \dots, b_m), c) \in [\![M]\!]_{\vec{x}},\ (\vec{a}_i, b_i) \in [\![V_i]\!]_{\vec{y}} \text{ for all } 1 \leq i \leq m\big\}.$$

Since $[\![V]\!]_{\vec{y}} \neq \emptyset$ (this can be proved by simple inspection), by Thm. 16 we obtain that $[\![M\{V_1/x_1, \dots, V_m/x_m\}]\!]_{\vec{y}} \neq \emptyset$ also holds, so that $[\![M]\!]_{\vec{x}} \neq \emptyset$. □ See Lemma 47 at p. 23

6.3 From solvability to non-emptyness of stratified Taylor expansion

The implication (ii) ⇒ (iii) of Thm. 25 seems much more difficult to prove. To accomplish this task we introduce the *resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests*, a CBV version of the resource calculus with tests defined in [15]. In this syntax all elements of the relational model are definable (see Def. 34).

The language extends that of resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus (see §4, p. 6) as follows:

$$\begin{array}{lll} (\mathsf{r}\Lambda^{\mathtt{v}}) & u, v ::= x \mid \lambda x.t & \textit{resource values} \\ (\mathsf{r}\Lambda^{\mathtt{t}}) & s, t ::= t * p \mid st \mid [v_1, \dots, v_k] \quad (k \geq 0) & \textit{resource terms} \\ (\mathsf{r}\Lambda^{\tau}) & p, q ::= \tau[t_1, \dots, t_k] \quad (k \geq 0) & \textit{tests} \end{array}$$

Note the overloaded use of $\mathsf{r}\Lambda^{\mathtt{v}}$ and $\mathsf{r}\Lambda^{\mathtt{t}}$, which now (and until Lemma 36) indicate larger sets than those introduced in §4. We will use this extension to prove Lemma 36 (whose statement concerns only resource terms without tests).

Tests are – formally – multisets of resource terms, the "τ" being a tag for distinguishing them from bags of values. Intuitively, they are constructions which can produce either *success*, represented by $\tau[\,]$, or *failure*, represented by 0.

Notation. We set $\varepsilon = \tau[\,]$ and $\tau[t_1, \dots, t_k] \parallel \tau[t_{k+1}, \dots, t_n] = \tau[t_1, \dots, t_n]$ $(k \leq n)$.

The test $p \parallel q$ represents the (must-)parallel composition of p and q (i.e., $p \parallel q$ succeeds iff both p and q succeed). The composition is parallel in the sense that the order of evaluation is inessential (remember that they are multisets). The binary operator $*$ allows to build a resource term out of a resource term and a test: intuitively, the resource term $t * p$ may be thought of as something that outputs the result of t only if p succeeds. Dually, the "cork construction" $\tau[t]$ may be thought of as a check that tests whether or not t v-reduces to $[\,]$.

Resource, *test-resource* and *test-test contexts* (with exactly one hole), denoted resp. by $\mathtt{R}$, $\mathtt{Q}$ and $\mathtt{P}$, are defined by mutual induction via the grammar ($k \geq 0$):

$$\mathtt{R} ::= (\!|\cdot|\!) \mid \mathtt{R}t \mid t\mathtt{R} \mid t * \mathtt{Q} \mid [\lambda x.\mathtt{R}, v_1, \dots, v_k] \quad (\textit{resource contexts});$$
$$\mathtt{Q} ::= \tau[\mathtt{R}, t_1, \dots, t_k] \ (\textit{test-resource c.}); \qquad \mathtt{P} ::= (\!|\cdot|\!) \parallel \tau[t_1, \dots, t_k] \ (\textit{test-test c.}).$$

Let $t, t_1, \dots, t_n \in \mathsf{r}\Lambda^{\mathtt{t}}$ (resp. $p, p_1, \dots, p_n \in \mathsf{r}\Lambda^{\tau}$). We use $\mathtt{Q}(\!|t|\!)$ (resp. $\mathtt{P}(\!|p|\!)$) for the test obtained by the capture-allowing substitution of t (resp. p) for the hole $(\!|\cdot|\!)$ in $\mathtt{Q}$ (resp. $\mathtt{P}$); similarly for $\mathtt{R}(\!|t|\!)$ (see p. 7). As usual, $\mathtt{R}(\!|\sum_i t_i|\!) = \sum_i \mathtt{R}(\!|t_i|\!)$, $\mathtt{Q}(\!|\sum_i t_i|\!) = \sum_i \mathtt{Q}(\!|t_i|\!)$ and $\mathtt{P}(\!|\sum_i p_i|\!) = \sum_i \mathtt{P}(\!|p_i|\!)$. E.g., $t * 0 = t * \mathtt{Q}(\!|0|\!) = \mathtt{R}(\!|0|\!) = 0$.

Definition 30. *The operational semantics of the resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests extends the set of rules listed in Def. 13 with the following ones:*

$$\begin{array}{rl} t(s * p) \mapsto_{\tau_1} ts * p & \qquad \tau[t * p] \mapsto_{\tau_4} \tau[t] \parallel p \\ (t * p)s \mapsto_{\tau_2} ts * p & \qquad \tau[[v_1, \dots, v_n]] \mapsto_{\tau_5} \begin{cases} \varepsilon & \textit{if } n = 0 \\ 0 & \textit{otherwise} \end{cases} \\ (t * p) * q \mapsto_{\tau_3} t * (p \parallel q) & \end{array}$$

We set $\mapsto_{\mathsf{v}\tau} = \mapsto_{\mathsf{v}} \cup (\bigcup_{i=1}^{5} \mapsto_{\tau_i}) \subseteq (\mathsf{r}\Lambda^{\mathtt{t}} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}} \rangle) \cup (\mathsf{r}\Lambda^{\tau} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\tau} \rangle)$. Then, according to the convention of §2, $\to_{\mathsf{v}\tau} \subseteq \mathsf{r}\Lambda^{\tau} \times \mathbf{2}\langle \mathsf{r}\Lambda^{\tau} \rangle$ is the reduction obtained by test-contextual closure[4] of $\mapsto_{\mathsf{v}\tau}$. The *resource $\lambda_{\mathsf{v}}^{\sigma}$-calculus with tests* consists of the language $\mathsf{r}\Lambda^{\tau}$ and the reduction $\to_{\mathsf{v}\tau}$.

---

[4] This means that, for every $p \in \mathsf{r}\Lambda^{\tau}$ and $p' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\tau} \rangle$, if $p \to_{\mathsf{v}\tau} p'$ then either there exist a test-test context $\mathtt{P}$, $q \in \mathsf{r}\Lambda^{\tau}$ and $q' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\tau} \rangle$ such that $p = \mathtt{P}(\!|q|\!)$, $p' = \mathtt{P}(\!|q'|\!)$ and $q \mapsto_{\tau_i} q'$ with $i \in \{4, 5\}$; or there exist a test-resource context $\mathtt{Q}$, $t \in \mathsf{r}\Lambda^{\mathtt{t}}$ and $t' \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathtt{t}} \rangle$ such that $p = \mathtt{Q}(\!|t|\!)$, $p' = \mathtt{Q}(\!|t'|\!)$ and $t \mapsto_{\mathsf{v}\tau'} t'$ with $\mapsto_{\mathsf{v}\tau'} = \mapsto_{\mathsf{v}} \cup (\bigcup_{i=1}^{3} \mapsto_{\tau_i})$.

As a technical simplification, we extend $\to_{\mathsf{v}\tau}$ to a binary relation on $\mathbf{2}\langle \mathsf{r}\Lambda^\tau\rangle$ by linearity, i.e., $(\sum_{i=1}^n q_i)+\mathbb{P} \to_{\mathsf{v}\tau} (\sum_{i=1}^n \mathbb{Q}_i)+\mathbb{P}$ iff $q_i \to_{\mathsf{v}\tau} \mathbb{Q}_i$ for every $i=1,\dots,n$ ($n \geq 1$). With this extension we can concisely state the following theorem:

Theorem 31. *Reduction $\to_{\mathsf{v}\tau}$ on $\mathbf{2}\langle \mathsf{r}\Lambda^\tau\rangle$ is strongly normalizing and confluent.*

Definition 32. *For every test p and repetition-free list $\vec{x} \supseteq \mathsf{fv}(p)$, we define the* interpretation $[\![p]\!]_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(U)^n \times \mathbf{1}$ *of p, where n is the length of $\vec{x}$, by mutual induction with Def. 17 as follows:*

$$[\![\varepsilon]\!]_{\vec{x}} = \{([\,]^n, 1)\} \qquad [\![p \,\|\, q]\!]_{\vec{x}} = \{(\vec{a} \uplus \vec{b}, 1) : (\vec{a},1) \in [\![p]\!]_{\vec{x}},\ (\vec{b},1) \in [\![q]\!]_{\vec{x}}\}$$
$$[\![\tau[t]]\!]_{\vec{x}} = \{(\vec{a},1) : (\vec{a},[\,]) \in [\![t]\!]_{\vec{x}}\} \qquad [\![t * p]\!]_{\vec{x}} = \{(\vec{a} \uplus \vec{b}, c) : (\vec{a},c) \in [\![t]\!]_{\vec{x}},\ (\vec{b},1) \in [\![p]\!]_{\vec{x}}\}.$$

Finally, sums of tests are interpreted by setting $[\![\sum_{i=1}^n p_i]\!]_{\vec{x}} = \bigcup_{i=1}^n [\![p_i]\!]_{\vec{x}}$.

Theorem 33 (soundness). *Let $\mathbb{P},\mathbb{Q} \in \mathbf{2}\langle \mathsf{r}\Lambda^\tau\rangle$. If $\mathbb{P} \to_{\mathsf{v}\tau} \mathbb{Q}$, then $[\![\mathbb{P}]\!]_{\vec{x}} = [\![\mathbb{Q}]\!]_{\vec{x}}$.* Proof at p. 25

A key tool to connect the semantics with the $\mathsf{v}\tau$-reduction is the following transformation of elements of $\mathcal{M}_{\mathrm{f}}(U)$ into resource terms and test contexts. The role of this transformation is made clear in Lemma 35, used to prove Lemma 36.

Definition 34. *Let $c = [(a_1,b_1),\dots,(a_n,b_n)] \in \mathcal{M}_{\mathrm{f}}(U)$ ($n \geq 0$). We define:*

- *the closed resource term $c^- = [\lambda y_1.{b_1}^- * {a_1}^+ (\![[y_1^{m_1}]]\!), \dots, \lambda y_n.{b_n}^- * {a_n}^+ (\![[y_n^{m_n}]]\!)]$, where m_i is the cardinality of the multiset a_i (for $i=1,\dots,n$);*
- *the test-resource context $c^+ = \tau[[\lambda x.[\,] * \|_{i=1}^n \tau[[\lambda y.[\,] * {b_i}^+ (\![[y^{k_i}]]\!)]([x]{a_i}^-)]] (\!\cdot\!)]$, where k_i is the cardinality if the multiset b_i (for $i=1,\dots,n$).*

Notation. For any $a \in \mathcal{M}_{\mathrm{f}}(U)$, $\#a$ indicates its cardinality. For $\vec{a} = (a_1,\dots,a_n) \in \mathcal{M}_{\mathrm{f}}(U)^n$ and $t \in \mathsf{r}\Lambda^{\mathsf{t}}$, we write $t\langle \vec{a}^-/\vec{x}\rangle$ as a shorthand for $t\langle {a_1}^-/x_1\rangle \cdots \langle {a_n}^-/x_n\rangle$.

Lemma 35. *Let $(\vec{a},b) \in \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$, $k = \#b$ and $t \in \mathsf{r}\Lambda^{\mathsf{t}}$ with $\vec{x} \supseteq \mathsf{fv}(t)$. Then $(\vec{a},b) \in [\![t]\!]_{\vec{x}}$ iff $\tau[[\lambda y.[\,] * b^+ (\![[y^k]]\!)](t\langle \vec{a}^-/\vec{x}\rangle)] \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$.* Proof at p. 26

Lemma 36. *Let s and t be v-normal resource terms without tests (i.e., generated by the grammar on §4, p. 6). If $s \in \mathsf{Strat}$ and $t \notin \mathsf{Strat}$, then $[\![s]\!]_{\vec{x}} \cap [\![t]\!]_{\vec{x}} = \emptyset$.*

Proof. Let $(\vec{a},b) \in \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$ and $\mathsf{Q}(\!\cdot\!) = \tau[[\lambda y.[\,] * b^+ (\![[y^k]]\!)]((\!\cdot\!)\langle \vec{a}^-/\vec{x}\rangle)]$, with $k = \#b$. One can prove by induction on the v-normal resource terms (without tests) that: either $\mathsf{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ and $\mathsf{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$; or $\mathsf{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ and $\mathsf{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$; or $\mathsf{Q}(\!s\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$ and $\mathsf{Q}(\!t\!) \twoheadrightarrow_{\mathsf{v}\tau} 0$. Hence, by Lemma 35, $(\vec{a},b) \notin [\![s]\!]_{\vec{x}} \cap [\![t]\!]_{\vec{x}}$. □

Hereafter, when we will mention resource terms, we will refer to the ones without test (i.e., generated by the grammar on §4, p. 6).

The following theorem proves the implication (ii) ⇒ (iii) of Thm. 25.

Theorem 37. *Let M be a λ-term and let $\vec{x} \supseteq \mathsf{fv}(M)$. If M is solvable, then $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$.*

Proof. If M is solvable then there exists a context $\mathtt{C} = (\lambda x_1 \dots x_m.(\!|\cdot|\!))N_1 \cdots N_n$ (for some $n, m \geq 0$) such that $\mathtt{C}(\!|M|\!) \twoheadrightarrow_{\mathsf{v}} \mathbf{I}$. By Thm. 16 and 20, $\bigcup_{t \in \mathcal{T}(\mathtt{C}(\!|M|\!))} [\![t]\!]_{\vec{x}} = [\![\mathtt{C}(\!|M|\!)]\!]_{\vec{x}} = [\![\mathbf{I}]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}(\mathbf{I})} [\![t]\!]_{\vec{x}}$. Using Lemma 36 we infer that $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))} [\![t]\!]_{\vec{x}} = \bigcup_{t \in \mathcal{T}_{\mathsf{s}}(\mathbf{I})} [\![t]\!]_{\vec{x}}$. Therefore $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))} [\![t]\!]_{\vec{x}} \neq \emptyset$ because it is easy to check that $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(\mathbf{I})} [\![t]\!]_{\vec{x}} \neq \emptyset$. By Thm. 18 and 14, $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))} [\![t]\!]_{\vec{x}} \neq \emptyset$ implies that there is a resource term in $\mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))$ that v-reduces to a non-zero v-normal form. Now all resource terms in $\mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))$ are of the shape $\mathtt{R}(\!|s|\!)$ for some $s \in \mathcal{T}_{\mathsf{s}}(M)$ (because the hole of $\mathtt{C}$ is in stratified position), so that if all resource terms in $\mathcal{T}_{\mathsf{s}}(M)$ v-reduced to 0, then all resource terms in $\mathcal{T}_{\mathsf{s}}(\mathtt{C}(\!|M|\!))$ would v-reduce to 0. Thus, there is $t \in \mathcal{T}_{\mathsf{s}}(M)$ that v-reduces to a v-normal form $\mathbb{T} \neq 0$. It is easy to prove that $[\![t']\!]_{\vec{x}} \neq \emptyset$ for every v-normal form t', hence $[\![t]\!]_{\vec{x}} = [\![\mathbb{T}]\!]_{\vec{x}} \neq \emptyset$ by Thm. 18. □

6.4 From non-emptyness to strong normalization

Our goal here is to prove the implication (iii) ⇒ (iv) of Thm. 24 and 25.

Proof at p. 27 **Lemma 38.** *Let M, M' be λ-terms.*

(i) *If $M \to_{\mathtt{w}} M'$ and $t \in \mathcal{T}(M)$, then there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ such that $t \to_{\mathsf{v}} \mathbb{T}$.*
(ii) *If $M \to_{\mathtt{s}} M'$ and $s \in \mathcal{T}_{\mathsf{s}}(M)$, then there exists $\mathbb{S} \subseteq \mathcal{T}_{\mathsf{s}}(M')$ such that $s \to_{\mathsf{v}}^{+} \mathbb{S}$.*

Lemma 38.i is false if we replace the hypothesis $M \to_{\mathtt{w}} M'$ with $M \to_{\mathtt{s}} M'$. For instance, take $M = \lambda x.\Omega$: then $[\,] \in \mathcal{T}(M)$ and $M \to_{\mathtt{s}} M$, but $[\,]$ is v-normal.

Theorem 39. *Let M be a λ-term and let $\vec{x} \supseteq \mathsf{fv}(M)$.*

1. [(iii)⇒(iv) of Thm. 24] *If $[\![M]\!]_{\vec{x}} \neq \emptyset$ then M is strongly $\mathtt{w}$-normalizing.*
2. [(iii)⇒(iv) of Thm. 25] *If $\bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}} \neq \emptyset$, then M is strongly $\mathtt{s}$-normalizing.*

Proof. Let $(\vec{a}, b) \in [\![M]\!]_{\vec{x}}$ (resp. $(\vec{a}, b) \in \bigcup_{t \in \mathcal{T}_{\mathsf{s}}(M)} [\![t]\!]_{\vec{x}}$). By Thm. 20 (resp. Then) there exists $t \in \mathcal{T}(M)$ (resp. $t \in \mathcal{T}_{\mathsf{s}}(M)$) such that $(\vec{a}, b) \in [\![t]\!]_{\vec{x}}$. If $M \to_{\mathtt{w}} M'$ (resp. $M \to_{\mathtt{s}} M'$), then by Lemma 38.i (resp. Lemma 38.ii) there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ (resp. $\mathbb{T} \subseteq \mathcal{T}_{\mathsf{s}}(M')$) such that $t \to_{\mathsf{v}}^{+} \mathbb{T}$. According to Thm. 18, $(\vec{a}, b) \in [\![\mathbb{T}]\!]_{\vec{x}}$, hence $\mathbb{T} \neq \emptyset$ and so there exists $t' \in \mathbb{T}$ such that $(\vec{a}, b) \in [\![t']\!]_{\vec{x}}$. Therefore, if there was an infinite reduction $M \to_{\mathtt{w}} M_1 \to_{\mathtt{w}} M_2 \to_{\mathtt{w}} \dots$ (resp. $M \to_{\mathtt{s}} M_1 \to_{\mathtt{s}} M_2 \to_{\mathtt{s}} \dots$) then there would also be a an infinite reduction $t \to_{\mathsf{v}}^{+} \mathbb{T}_1 \to_{\mathsf{v}}^{+} \mathbb{T}_2 \to_{\mathsf{v}}^{+} \dots$, which is impossible by Thm. 14. □

Conclusions and future work

Our approach, that exploits the validity of the Taylor formula for a resource CBV λ-calculus, makes use of purely combinatorial proofs, rather than more standard approaches based on reducibility or some specific machines. The interesting feature of this approach is that it can be used for many different calculi always using a similar relational model and a suitable resource calculus.

We think that using the ordinary syntax of λ-calculus with our reduction will allow to develop a reasonable theory of CBV Böhm trees, never defined before (Paolini's separability result in [20] for λ_{v}-calculus does not use Böhm trees), together with connections between equivalence of Böhm trees and observational equivalence. A future challenge is that of finding other fully abstract denotational models, in view of Paolini and Ronchi Della Rocca's proof of absence of fully abstract filter models (see [7, Thm. 12.1.25]) built from legal type systems.

Another direction is relating two equivalence relations on λ-terms, the one generated by our σ-rules and the one induced by Girard's CBV "boring" translation $(\cdot)^v$ of λ-calculus into IMELL proof-nets (along the lines of [17,18,21]).

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A Technical appendix

A.1 Proofs and remarks of Section 2

Stated at p. 4

Proposition 2. *The reduction $\to_\sigma$ is strongly normalizing.*

Proof. First we define two sizes $\mathsf{s}(M)$ and $\#M$ by induction on the λ-term M:

- $\mathsf{s}(x) = 2$; — $\#x = 1$;
- $\mathsf{s}(\lambda x.M) = \mathsf{s}(M) + 1$; — $\#\lambda x.M = \#M + \mathsf{s}(M)$;
- $\mathsf{s}(MN) = \mathsf{s}(M) + \mathsf{s}(N)$. — $\#MN = \#M + \#N + 2\mathsf{s}(M)\mathsf{s}(N) - 1$.

Notice that $\mathsf{s}(N) \geq 2$ and $\#N \geq 1$ for any λ-term N.

In order to prove that σ-reduction is strongly normalizing, it suffices to show that if $M \to_\sigma M'$ then $\mathsf{s}(M) = \mathsf{s}(M')$ and $\#M > \#M'$. We proceed by induction on the definition of $M \to_\sigma M'$.

If $M \mapsto_{\sigma_1} M'$ then $M = (\lambda x.M_0)NL$ and $M' = (\lambda x.M_0L)N$, hence

$$\begin{aligned}\#M &= \#M_0 + \#L + \#N + \mathsf{s}(M_0) + 2\,\mathsf{s}(N) + 2\,\mathsf{s}(L)\\ &\quad + 2\,\mathsf{s}(M_0)\mathsf{s}(N) + 2\,\mathsf{s}(M_0)\mathsf{s}(L) + 2\,\mathsf{s}(L)\mathsf{s}(N) - 2\\ \#M' &= \#M_0 + \#L + \#N + \mathsf{s}(M_0) + 2\,\mathsf{s}(N) + \mathsf{s}(L)\\ &\quad + 2\,\mathsf{s}(M_0)\mathsf{s}(N) + 2\,\mathsf{s}(M_0)\mathsf{s}(L) + 2\,\mathsf{s}(L)\mathsf{s}(N) - 2\end{aligned}$$

so $\#M = \#M' + \mathsf{s}(L) > \#M'$. Moreover $\mathsf{s}(M) = \mathsf{s}(M_0) + \mathsf{s}(L) + \mathsf{s}(N) + 1 = \mathsf{s}(M')$.

If $M \mapsto_{\sigma_3} M'$ then $M = V((\lambda xL)N)$ and $M' = (\lambda x.VL)N$, hence

$$\begin{aligned}\#M &= \#V + \#L + \#N + 2\,\mathsf{s}(V) + 2\,\mathsf{s}(N) + \mathsf{s}(L)\\ &\quad + 2\,\mathsf{s}(V)\mathsf{s}(N) + 2\,\mathsf{s}(V)\mathsf{s}(L) + 2\,\mathsf{s}(L)\mathsf{s}(N) - 2\\ \#M' &= \#V + \#L + \#N + \mathsf{s}(V) + 2\,\mathsf{s}(N) + \mathsf{s}(L)\\ &\quad + 2\,\mathsf{s}(V)\mathsf{s}(N) + 2\,\mathsf{s}(V)\mathsf{s}(L) + 2\,\mathsf{s}(L)\mathsf{s}(N) - 2\end{aligned}$$

so $\#M = \#M' + \mathsf{s}(V) > \#M'$. Moreover $\mathsf{s}(M) = \mathsf{s}(V) + \mathsf{s}(L) + \mathsf{s}(N) + 1 = \mathsf{s}(M')$.

If $M = \lambda x.N$ and $M' = \lambda x.N'$ with $N \to_\sigma N'$, then $\#M = \#N + \mathsf{s}(N)$ and $\#M' = \#N' + \mathsf{s}(N')$; by induction hypothesis, $\#N > \#N'$ and $\mathsf{s}(N) = \mathsf{s}(N')$, hence $\#M > \#N' + \mathsf{s}(N) = \#M'$ and $\mathsf{s}(M) = \mathsf{s}(N) + 1 = \mathsf{s}(M')$.

If $M = LN$ and $M' = L'N$ (resp. $M = NL$ and $M' = NL'$) with $L \to_\sigma L'$, then $\#M = \#L + \#N + 2\,\mathsf{s}(L)\mathsf{s}(N) - 1$ and $\#M' = \#L' + \#N + 2\,\mathsf{s}(L')\mathsf{s}(N) - 1$; by induction hypothesis, $\#L > \#L'$ and $\mathsf{s}(L) = \mathsf{s}(L')$, hence $\#M > \#L + \#N + 2\,\mathsf{s}(L')\mathsf{s}(N) - 1 = \#M'$ and $\mathsf{s}(M) = \mathsf{s}(L) + \mathsf{s}(N) = \mathsf{s}(M')$. □

Stated at p. 4

Proposition 3. *The reduction $\to_\sigma$ is confluent.*

Proof. By Newman's lemma and Prop. 2, it suffices to show that $\to_\sigma$ is locally confluent: if $M \to_\sigma N_1$ and $M \to_\sigma N_2$ then there is M' s.t. $N_1 \twoheadrightarrow_\sigma M'$ and $N_2 \twoheadrightarrow_\sigma M'$. We proceed by induction on M, the only interesting cases are:

- if $M = (\lambda x.M_0)\big((\lambda y.L)N\big)L'$ with $M \to_{\sigma_1} (\lambda x.M_0L')\big((\lambda y.L)N\big) = N_1$ and $M \to_{\sigma_3} (\lambda y.(\lambda x.M_0)L)NL' = N_2$, then $N_2 \to_{\sigma_1} (\lambda y.(\lambda x.M_0)LL')N \to_{\sigma_1} (\lambda y.(\lambda x.M_0L')L)N = M'$ and $N_1 \to_{\sigma_3} M'$;

– if $M = V\big((\lambda x.L)((\lambda x'.L')N)\big)$ with $M \to_{\sigma_3} V\big((\lambda x'.(\lambda x.L)L')N\big) = N_1$ and $M \to_{\sigma_3} (\lambda x.VL)\big((\lambda x'.L')N\big) = N_2$, then $N_1 \to_{\sigma_3} \big(\lambda x'.V((\lambda x.L)L')\big)N \to_{\sigma_3} (\lambda x'.(\lambda x.VL)L')N = M'$ and $N_2 \to_{\sigma_3} M'$. □

Lemma 5. *Let $M, M' \in \Lambda$, $V, V', V_1, \dots, V_m \in \Lambda^{\mathsf{v}}$ and $R \in \{\beta_v, \sigma, \mathsf{v}\}$.* Stated at p. 5

(i) *If $V \to_R V'$ then $M\{V/x\} \twoheadrightarrow_R M\{V'/x\}$.*
(ii) *If $M \to_R M'$ then $M\{V_1/x_1, \dots, V_m/x_m\} \to_R M'\{V_1/x_1, \dots, V_m/x_m\}$.*

Proof. For $R = \beta_v$, the proofs of (i) and (ii) are in [4]. For $R = \mathsf{v}$, the proof of (i) (resp. (ii)) is a consequence of the property (i) (resp. (ii)) for both $\to_{\beta_v}$ and $\to_\sigma$, since $\to_{\mathsf{v}} = \to_{\beta_v} \cup \to_\sigma$. Let us prove (i) and (ii) for $R = \sigma$.

(i) The proof is by induction on the λ-term M. If $M = x$, then $M\{V/x\} = V$ and $M\{V'/x\} = V'$, so $M\{V/x\} \twoheadrightarrow_\sigma M\{V'/x\}$ by hypothesis.
If $M = y \neq x$, then $M\{V/x\} = y = M\{V'/x\}$, then $M\{V/x\} \twoheadrightarrow_\sigma M\{V'/x\}$ by reflexivity of $\twoheadrightarrow_\sigma$.
If $M = \lambda y.N$ for some λ-term N, then we can suppose without loss of generality that $y \notin \mathsf{fv}(V) \cup \{x\}$, so $M\{V/x\} = \lambda y.N\{V/x\}$ and $M\{V'/x\} = \lambda y.N\{V'/x\}$; by induction hypothesis, $N\{V/x\} \twoheadrightarrow_\sigma N\{V'/x\}$ and therefore $M\{V/x\} \twoheadrightarrow_\sigma M\{V'/x\}$.
If $M = LN$ for some λ-terms L and N, then $M\{V/x\} = L\{V/x\}N\{V/x\}$ and $M\{V'/x\} = L\{V'/x\}N\{V'/x\}$; $L\{V/x\} \twoheadrightarrow_\sigma L\{V'/x\}$ and $N\{V/x\} \twoheadrightarrow_\sigma N\{V'/x\}$ by induction hypothesis, thus $M\{V/x\} \twoheadrightarrow_\sigma L\{V'/x\}N\{V/x\} \twoheadrightarrow_\sigma M\{V'/x\}$.

(ii) The proof is by induction on the definition of $M \to_\sigma M'$. For any λ-term N we set $N^* = N\{V_1/x_1, \dots, V_m/x_m\}$.
If $M \mapsto_{\sigma_1} M'$ then $M = (\lambda y.M_0)NL$ and $M' = (\lambda y.M_0L)N$ with $y \notin \mathsf{fv}(L)$; we can suppose without loss of generality that $y \notin \bigcup_{i=1}^m(\mathsf{fv}(V_i) \cup \{x_i\})$, so $M^* = (\lambda y.M_0^*)N^*L^*$ and $M'^* = (\lambda y.M_0^*L^*)N^*$, hence $M^* \to_{\sigma_1} M'^*$ since $y \notin (\mathsf{fv}(L) \setminus \{x_1, \dots, x_m\}) \cup \bigcup_{i=1}^m \mathsf{fv}(V_i) = \mathsf{fv}(L^*)$.
If $M \mapsto_{\sigma_3} M'$ then $M = V((\lambda y.L)N)$ and $M' = (\lambda y.VL)N$ with $y \notin \mathsf{fv}(V)$; we can suppose without loss of generality that $y \notin \bigcup_{i=1}^m(\mathsf{fv}(V_i) \cup \{x_i\})$, thus $M^* = V^*((\lambda y.L^*)N^*)$ and $M'^* = (\lambda y.V^*L^*)N^*$, so $M^* \to_{\sigma_3} M'^*$ since $y \notin (\mathsf{fv}(V) \setminus \{x_1, \dots, x_m\}) \cup \bigcup_{i=1}^m \mathsf{fv}(V_i) = \mathsf{fv}(V^*)$.
If $M = \lambda y.N$ and $M' = \lambda y.N'$ with $N \to_\sigma N'$ then we can suppose without loss of generality that $y \notin \bigcup_{i=1}^m(\mathsf{fv}(V_i) \cup \{x_i\})$, hence $M^* = \lambda y.N^*$ and $M'^* = \lambda y.N'^*$; by induction hypothesis, $N^* \to_\sigma N'^*$ and so $M^* \to_\sigma M'^*$.
If $M = LN$ and $M' = L'N$ (resp. $M = NL$ and $M' = NL'$) with $L \to_\sigma L'$, then $M^* = L^*N^*$ and $M'^* = L'^*N^*$ (resp. $M^* = N^*L^*$ and $M'^* = N^*L'^*$); by induction hypothesis, $L^* \to_\sigma L'^*$, so $M^* \to_\sigma M'^*$. □

Lemma 40. *Let $\to_1, \to_2 \subseteq X^2$ (for any set X) be such that if $t \to_1 u_1$ and $t \to_2 u_2$ then there is $v \in X$ such that $u_2 \twoheadrightarrow_1 v$ and $u_1 \to_2 v$. Then they commute (i.e. if $t \twoheadrightarrow_1 u_1$ and $t \twoheadrightarrow_2 u_2$ then there is $s \in X$ such that $u_1 \twoheadrightarrow_2 s$ and $u_2 \twoheadrightarrow_1 s$).*

Proof. For every $t, u \in X$, $\to \subseteq X^2$ and $n \in \mathbf{N}$, we write $t \to^n u$ if there exist $v_0, \dots, v_n \in X$ such that $t = v_0$, $u = v_n$ and $v_i \to v_{i+1}$ for all $0 \leq i \leq n$. We prove the following stronger statement, in order to apply the right induction hypothesis: if $t \twoheadrightarrow_1 u_1$ and $t \to_2^m u_2$ then there exists $t' \in X$ such that $u_2 \twoheadrightarrow_1 t'$ and $u_1 \to_2^m t'$. Let $t \to_1^n u_1$: the proof is by induction on (m, n) with the lexicographical order on $\mathbf{N}^2$. If $m = 0$ or $n = 0$, we conclude easily.

Let $m, n > 0$: there exist $u_1', u_2' \in X$ such that $t \to_1 u_1'$, $t \to_2 u_2'$, $u_1' \to_1^{n-1} u_1$ and $u_2' \to_2^{m-1} u_2$. By hypothesis, there is $v \in X$ such that $u_1' \to_2 v$ and $u_2' \twoheadrightarrow_1 v$. By induction hypothesis applied to u_2', there is $v' \in X$ such that $u_2 \twoheadrightarrow_1 v'$ and $v \twoheadrightarrow_2^{m-1} v'$; thus $u_1' \to_2^m v'$, so there exists $s \in X$ such that $v' \twoheadrightarrow_1 s$ and $u_1 \to_2^m s$ by applying the induction hypothesis to u_1', therefore $u_2 \twoheadrightarrow_1 s$. □

Stated at p. 5 **Lemma 6.** *The reductions $\to_{\beta_v}$ and $\to_\sigma$ commute.*

Proof. By lemma 40, it suffices to prove that if $M \to_\sigma N_1$ and $M \to_{\beta_v} N_2$ then there is L such that $N_2 \twoheadrightarrow_\sigma L$ and $N_1 \to_{\beta_v} L$. The proof of this property is by induction on M. The only interesting cases are:

– if $M = (\lambda x.N)VL'$ with $M \to_{\sigma_1} (\lambda x.NL')V = N_1$ and $M \to_{\beta_v} N\{V/x\}L' = N_2$, then $N_1 \to_{\beta_v} N_2$ since $x \notin \mathsf{fv}(L')$.
– if $M = U((\lambda x.N)V)$ with $M \to_{\sigma_3} (\lambda x.UN)V = N_1$ and $M \to_{\beta_v} U(N\{V/x\}) = N_2$, then $N_1 \to_{\beta_v} N_2$ since $x \notin \mathsf{fv}(U)$.
– if $M = (\lambda y.N')\big((\lambda x.N)V\big)L'$ with $M \to_{\sigma_1} (\lambda y.N'L')(\lambda x.N)V = N_1$ and $M \to_{\beta_v} (\lambda y.N')N\{V/x\}L' = N_2$, then $N_1 \to_{\beta_v} (\lambda y.N'L')N\{V/x\} = L$ and $N_2 \to_{\sigma_1} L$.
– if $M = (\lambda x.N)V$ with $M \to_\sigma N_1 = (\lambda x.N)V'$ (resp. $N_1 = (\lambda x.N')V$), $M \to_{\beta_v} N\{V/x\} = N_2$ and $V \to_\sigma V'$ (resp. $N \to_\sigma N'$), then we conclude by Lemma 5.i (resp. 5.ii). □

Remark 41. The reductions $\to_\sigma$ and $\to_{\mathsf{v}}$ are not strongly confluent. For instance take $N = z\mathbf{I}$, $L = z'\mathbf{I}$ and $M = (\lambda x.x')\big((\lambda y.y'\mathbf{I})N\big)L$: one has $M \to_{\sigma_1} (\lambda x.x'L)\big((\lambda y.y'\mathbf{I})N\big) = N_3$ and $M \to_{\sigma_3} \big(\lambda y.(\lambda x.x')(y'\mathbf{I})\big)NL = N_1$ (with $N_1 \neq N_3$) but for $i \in \{1, 3\}$ the only v-redex in N_i is the σ_i-redex N_i itself; therefore $N_1 \to_{\sigma_1} \big(\lambda y.(\lambda x.x')(y'\mathbf{I})L\big)N \neq \big(\lambda y.(\lambda x.x'L)(y'\mathbf{I})\big)N \;{}_{\sigma_3}\!\leftarrow N_3$.

Remark 42. If we define a Tait–Martin-Löf parallel reduction $\to_\rho$ of $\to_{\mathsf{v}}$ in the obvious way, then $\to_\rho$ is not strongly confluent. For instance, take $M = (\lambda x.L)\big((\lambda y.N)((\lambda z.N')N'')\big)L'$, $M_1 = (\lambda x.LL')\big((\lambda y.N)((\lambda z.N')N'')\big)$ and $M_2 = \big((\lambda y.(\lambda x.L)N)((\lambda z.N')N'')\big)L'$: then $M \to_\rho M_1$ and $M \to_\rho M_2$ but there is no λ-term M' such that $M_1 \to_\rho M' \;{}_\rho\!\leftarrow M_2$. Informally, $\to_\rho$ is not able to reduce in one step several "subsequent" σ_1-redexes created by one σ_3-step. Therefore, we cannot adapt the Tait–Martin-Löf technique in a natural way in order to prove that $\to_{\mathsf{v}}$ is confluent.

Remark 43. If in definition of $\mapsto_{\sigma_3}$ (Def. 1) we replace the λ-value V with any λ-term M then $\to_\sigma$ and $\to_{\mathsf{v}}$ are not (locally) confluent. For instance, take $N_i = (z_i)\mathbf{I}$ for $i \in \{1, 2\}$ and $M = (\lambda x_1.y_1)N_1\big((\lambda x_2.y_2)N_2\big)$: one would have $M \to_{\sigma_3} (\lambda x_2.(\lambda x_1.y_1)N_1y_2)N_2 = L_1$ and $M \to_{\sigma_1} \big(\lambda x_1.y_1((\lambda x_2.y_2)N_2)\big)N_1 = L_3$ (with $L_1 \neq L_3$) but the only v-redex in L_1 (resp. L_3) is the σ_1-redex $(\lambda x_1.y_1)N_1y_2$ (resp. σ_3-redex $y_1((\lambda x_2.y_2)N_2)$); so $L_1 \to_{\sigma_1} \big(\lambda x_2.(\lambda x_1.y_1y_2)N_1\big)N_2 = N_{12}$ and $L_3 \to_{\sigma_3} \big(\lambda x_1.(\lambda x_2.y_1y_2)N_2\big)N_1 = N_{21}$, with $N_{12} \neq N_{21}$ and N_{12}, N_{21} v-normal.

A.2 Proofs and remarks of Section 3

Lemma 44. *Let $M, M' \in \Lambda$, $V_1, \dots, V_m \in \Lambda^{\mathrm{v}}$ and $R \in \{\mathtt{w}[\beta_v], \mathtt{w}[\sigma], \mathtt{s}[\beta_v], \mathtt{s}[\sigma]\}$. If $M \to_R M'$ then $M\{V_1/x_1, \dots, V_m/x_m\} \to_R M'\{V_1/x_1, \dots, V_m/x_m\}$.*

Proof. All the proofs are by induction on the definition of $M \to_R M'$.
For $R \in \{\mathtt{w}[\sigma], \mathtt{s}[\sigma]\}$, the proof is analogous to that one for Lemma 5.ii.
For $R = \mathtt{w}[\beta_v]$ (for any λ-term N we set $N^* = N\{V_1/x_1, \dots, V_m/x_m\}$):

- If $M \mapsto_{\beta_v} M'$ then $M = (\lambda y.N)V$ and $M' = N\{V/y\}$, moreover we can suppose without loss of generality that $y \notin \bigcup_{i=1}^m (\mathsf{fv}(V_i) \cup \{x_i\})$; hence, $M^* = (\lambda y.N^*)V^* \to_{\beta_v} N^*\{V^*/y\} = M'^*$ (since V^* is a value).
- If $M = LN$ and $M' = L'N$ (resp. $M = NL$ and $M' = NL'$) with $L \to_{\mathtt{w}[\beta_v]} L'$, then $M^* = L^*N^*$ and $M'^* = L'^*N^*$ (resp. $M^* = N^*L^*$ and $M'^* = N^*L'^*$); by induction hypothesis, $L^* \to_{\mathtt{w}[\beta_v]} L'^*$, so $M^* \to_{\mathtt{w}[\beta_v]} M'^*$.
- If $M = (\lambda y.N)L$ and $M' = (\lambda y.N')L$ with $N \to_{\mathtt{w}[\beta_v]} N'$, moreover we can suppose without loss of generality that $y \notin \bigcup_{i=1}^m (\mathsf{fv}(V_i) \cup \{x_i\})$; hence, $M^* = (\lambda y.N^*)L^*$ and $M'^* = (\lambda y.N'^*)L^*$; by induction hypothesis, $N^* \to_{\mathtt{w}[\beta_v]} N'^*$ and thus $M^* \to_{\mathtt{w}[\beta_v]} M'^*$.

For $R = \mathtt{s}[\beta_v]$ (for any λ-term N we set $N^* = N\{V_1/x_1, \dots, V_m/x_m\}$):

- If $M \to_{\mathtt{w}[\beta_v]} M'$ then we have just proved that $M^* \to_{\mathtt{w}[\beta_v]} M'^*$ and so $M^* \to_{\mathtt{s}[\beta_v]} M'^*$ (since $\to_{\mathtt{w}[\beta_v]} \subseteq \to_{\mathtt{s}[\beta_v]}$).
- If $M = LN$ and $M' = L'N$ (resp. $M = \lambda y.L$ and $M' = \lambda y.L'$) with $L \to_{\mathtt{s}[\beta_v]} L'$, then $M^* = L^*N^*$ and $M'^* = L'^*N^*$ (resp. $M^* = \lambda y.L^*$ and $M'^* = \lambda y.L'^*$, by supposing without loss of generality that $y \notin \bigcup_{i=1}^m (\mathsf{fv}(V_i) \cup \{x_i\})$); by induction hypothesis, $L^* \to_{\mathtt{s}[\beta_v]} L'^*$, so $M^* \to_{\mathtt{s}[\beta_v]} M'^*$. □

Remark 45. There are no λ-values V and V' such that $V \to_{\mathtt{w}} V'$.
For $R \in \{\mathtt{s}[\beta_v], \mathtt{s}[\sigma]\}$, if $V \to_R V'$ then $V = \lambda z.N$, $V' = \lambda z.N'$ and $N \to_R N'$.

Remark 46. For $R \in \{\mathtt{s}[\beta_v], \mathtt{s}[\sigma]\}$, the analogous of Lemma 5.i does not hold: $V \to_R V'$ does not imply $M\{V/x\} \twoheadrightarrow_R M\{V'/x\}$. For instance, take $M = yx$: by Remark 45, for any λ-value V, $M\{V/x\} = yV$ is R-normal.

Lemma 9. (i) *The reductions $\to_{\mathtt{w}[\beta_v]}$ and $\to_{\mathtt{s}[\beta_v]}$ are strongly confluent.* Stated at p. 5
(ii) *The reductions $\to_{\mathtt{w}[\sigma]}$ and $\to_{\mathtt{s}[\sigma]}$ are confluent.*
(iii) *The reductions $\to_{\mathtt{w}[\beta_v]}$ and $\to_{\mathtt{w}[\sigma]}$ (resp. $\to_{\mathtt{s}[\beta_v]}$ and $\to_{\mathtt{s}[\sigma]}$) commute.*

Proof.

(i) Let $R \in \{\mathtt{w}[\beta_v], \mathtt{s}[\beta_v]\}$. We prove by induction on $M \in \Lambda$ that if $M \to_R N_1$ and $M \to_R N_2$ then there is $M' \in \Lambda$ such that $N_1 \to_R M'$ and $N_2 \to_R M'$. The only interesting case is when $M = (\lambda x.N)V$ with $M \to_R N\{V/x\} = N_1$, $M \to_R (\lambda x.N')V = N_2$ and $N \to_R N'$: $N_2 \to_R N'\{V/x\}$ and, by Lemma 44, $N_1 \to_R N'\{V/x\}$.

(ii) By Newman's lemma and Prop. 2 (since $\to_{\mathtt{w}[\sigma]}, \to_{\mathtt{s}[\sigma]} \subseteq \to_\sigma$), it suffices to show that $\to_{\mathtt{w}[\sigma]}$ and $\to_{\mathtt{s}[\sigma]}$ are locally confluent. The proof that $\to_{\mathtt{w}[\sigma]}$ and $\to_{\mathtt{s}[\sigma]}$ are locally confluent are analogous to that one for the local confluence of $\to_\sigma$ seen in Prop. 3.

(iii) The proof is analogous to that one for Lemma 6 (in particular, we use Lemma 40). The only notable difference is that here there is not a case where, for $R \in \{\mathtt{w}, \mathtt{s}\}$, $M = (\lambda x.N)V$, $M \to_{R[\beta_v]} N\{V/x\} = N_2$ and $M \to_{R[\sigma]} (\lambda x N)V' = N_1$ with $V \to_{R[\sigma]} V'$ (see Remarks 45 and 46). □

Notice that every $\mathtt{s}$-normal forms is also a $\mathtt{w}$-normal form, since $\to_\mathtt{w} \subseteq \to_\mathtt{s}$. Obviously, every β_v-redex is also a β-redex (a λ-term of the form $(\lambda x.M)N$).

Stated at p. 6

Proposition 12. *Let M be a λ-term.*

(i) *M is $\mathtt{w}$-normal iff $M \in \mathsf{w}_{\mathtt{nf}}$.*
(ii) *M is $\mathtt{s}$-normal iff $M \in \mathsf{s}_{\mathtt{nf}}$.*
(iii) *M is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal and is neither a value nor a β-redex iff $M \in \mathsf{a}_{\mathtt{nf}}$.*

Proof.

$\Rightarrow$: We prove simultaneously the left-to-right part of the three statements, by induction on the λ-term M.
If M is a λ-value then $M \in \mathsf{w}_{\mathtt{nf}}$. Furthermore, if M is a variable then $M \in \mathsf{s}_{\mathtt{nf}}$; if $M = \lambda x.N$ is $\mathtt{s}$-normal (for some λ-term N) then N is $\mathtt{s}$-normal, hence $N \in \mathsf{s}_{\mathtt{nf}}$ by induction hypothesis, and so $M \in \mathsf{s}_{\mathtt{nf}}$.
If M is not a λ-value then $M = M_1M_2$ for some λ-terms M_1 and M_2. By simple inspection of the definition of $\to_\mathtt{w}$ (resp. $\to_\mathtt{s}$), the fact that M is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal implies that M_1 is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal and M_2 is $\mathtt{w}$-normal, moreover M_1 is not a β-redex (otherwise M would be a σ_1-redex) and M is neither a β_v- nor a σ_3-redex. There are only three cases:

1. M_1 is not a value: by induction hypothesis $M_1 \in \mathsf{a}_{\mathtt{nf}}$ and $M_2 \in \mathsf{w}_{\mathtt{nf}}$, therefore $M \in \mathsf{a}_{\mathtt{nf}}\mathsf{w}_{\mathtt{nf}} \subseteq \mathsf{a}_{\mathtt{nf}}$.
2. $M_1 = \lambda x.N$: then M_2 is neither a β-redex (otherwise M would be a σ_3-redex) nor a value (otherwise M would be a β_v-redex), so $M_2 \in \mathsf{a}_{\mathtt{nf}}$ by induction hypothesis. Moreover, the fact that M_1 is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal entails that N is $\mathtt{w}$-(resp. $\mathtt{s}$-)normal and thus $N \in \mathsf{w}_{\mathtt{nf}}$ (resp. $N \in \mathsf{s}_{\mathtt{nf}}$), by induction hypothesis. Hence $M \in \mathsf{a}_{\mathtt{nf}}$.
3. M_1 is a variable: if M_2 is a value then $M \in \mathsf{a}_{\mathtt{nf}}$; if M_2 is not a value then M_2 is not a β-redex (otherwise M would be a σ_3-redex) and thus $M_2 \in \mathsf{a}_{\mathtt{nf}}$ by induction hypothesis, therefore $M \in \mathsf{a}_{\mathtt{nf}}$.

$\Leftarrow$: The proof right-to-left part of the statement (i) (resp. (ii)) is by induction on $M \in \mathsf{w}_{\mathtt{nf}}$ (resp. $M \in \mathsf{s}_{\mathtt{nf}}$). The right-to-left part of the statement (iii) is an immediate consequence of (i) and (ii), since $\mathsf{a}_{\mathtt{nf}} \subseteq \mathsf{s}_{\mathtt{nf}} \subseteq \mathsf{w}_{\mathtt{nf}}$ and if $M \in \mathsf{a}_{\mathtt{nf}}$ then M is neither a value nor a β-redex. □

A.3 Proofs of Section 5

Stated at p. 8

Theorem 16 (soundness). *Let $M, N \in \Lambda$. If $M \to_{\mathsf{v}} N$, then $[\![M]\!]_{\vec{x}} = [\![N]\!]_{\vec{x}}$.*

Proof. For soundness w.r.t. the β_v-rule we refer to [14] (see also Lemma 48).
Regarding the σ-rules we have:

$$\begin{aligned}[\![(\lambda y.M)NL]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1 \uplus \vec{a}_2, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U).\ \exists d \in \mathcal{M}_{\mathrm{f}}(U).\\ &\qquad ((\vec{a}_1, d), [(b,c)]) \in [\![M]\!]_{\vec{x},y},\ (\vec{a}_0, b) \in [\![L]\!]_{\vec{x}},\ (\vec{a}_2, d) \in [\![N]\!]_{\vec{x}}\}\\ &= [\![(\lambda y.ML)N]\!]_{\vec{x}}\end{aligned}$$

which validates the rule σ_1, and

$$\begin{aligned}[\![V((\lambda y.L)N)]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1 \uplus \vec{a}_2, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U).\ \exists d \in \mathcal{M}_{\mathrm{f}}(U).\\ &\qquad (\vec{a}_0, [(b,c)]) \in [\![V]\!]_{\vec{x}},\ ((\vec{a}_1, d), b) \in [\![L]\!]_{\vec{x},y},\ (\vec{a}_2, d) \in [\![N]\!]_{\vec{x}}\}\\ &= [\![(\lambda y.VL)N]\!]_{\vec{x}}\end{aligned}$$

which validates the rule σ_3. Finally it is easy to check that the interpretation is contextual. □

Theorem 18 (soundness). *Let $\mathbb{S}, \mathbb{T} \in \mathbf{2}\langle \mathsf{r}\Lambda^{\mathsf{t}} \rangle$. If $\mathbb{S} \to_{\mathsf{v}} \mathbb{T}$, then $[\![\mathbb{S}]\!]_{\vec{x}} = [\![\mathbb{T}]\!]_{\vec{x}}$.* Stated at p. 9

Proof. For soundness w.r.t. the β_v-rule and 0-rule, we refer to [14].
Regarding the σ-rules we have:

$$\begin{aligned}[\![[\lambda y.t]ss']\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1 \uplus \vec{a}_2, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U).\ \exists d \in \mathcal{M}_{\mathrm{f}}(U).\\ &\qquad ((\vec{a}_1, d), [(b,c)]) \in [\![t]\!]_{\vec{x},y},\ (\vec{a}_0, b) \in [\![s']\!]_{\vec{x}},\ (\vec{a}_2, d) \in [\![s]\!]_{\vec{x}}\}\\ &= [\![[\lambda y.ts']s]\!]_{\vec{x}}\end{aligned}$$

which validates the rule σ_1, and

$$\begin{aligned}[\![[v]([\lambda y.t]s)]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1 \uplus \vec{a}_2, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U).\ \exists d \in \mathcal{M}_{\mathrm{f}}(U).\\ &\qquad (\vec{a}_0, [(b,c)]) \in [\![v]\!]_{\vec{x}},\ ((\vec{a}_1, d), b) \in [\![t]\!]_{\vec{x},y},\ (\vec{a}_2, d) \in [\![s]\!]_{\vec{x}}\}\\ &= [\![[\lambda y.[v]t]s]\!]_{\vec{x}}\end{aligned}$$

which validates the rule σ_3. Finally it is easy to check that the interpretation is contextual. □

A.4 Proofs of Subsection 6.1

Lemma 26. *Let M be a λ-term with $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m\}$ and let $j \in \mathbf{N}$.* Stated at p. 10

- *If $M \in \mathsf{a}_{\mathtt{nf}}$ then there exists $h \in \mathbf{N}^*$ such that for all $n_1, \dots, n_m \geq j + h$ one has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^k$ for some $k \geq j$.*
- *If $M \in \mathsf{w}_{\mathtt{nf}}$ then there is $h \in \mathbf{N}^*$ such that for all $n_1, \dots, n_m \geq j + h$ one has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.N$ for some λ-term N s.t. $\lambda x.N$ is closed.*

Proof. By mutual induction on $M \in \mathsf{a_{nf}}$ and $M \in \mathsf{w_{nf}}$. Notice that if $M \in \mathsf{a_{nf}}$ then $\mathsf{fv}(M) \neq \emptyset$ and thus $m > 0$.

If $M = xV$ for some variable x and λ-value V, then $x = x_i$ for some $1 \leq i \leq m$. Let $n_1, \dots, n_m \geq j+1$. One has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} = \mathsf{o}^{n_i} V\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \to_{\mathsf{v}} \mathsf{o}^{n_i - 1}$ (since $V\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\}$ is a λ-value) where $n_i - 1 \geq j$. Hence we conclude by taking $h = 1$ and $k = n_i - 1$.

If $M = xN$ for some variable x and $N \in \mathsf{a_{nf}}$, then $x = x_i$ for some $1 \leq i \leq m$. By induction hypothesis there exists $h \in \mathbf{N}^*$ such that for all $n_1, \dots, n_m \geq j+h$ one has $N\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^k$ for some $k \geq j$. Hence for all $n_1, \dots, n_m \geq j+h$ one has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^{n_i}\mathsf{o}^k \to_{\mathsf{v}} \mathsf{o}^{n_i - 1}$ where $n_i - 1 \geq j + h - 1 \geq j$.

If $M = N'N''$ for some $N' \in \mathsf{a_{nf}}$ and $N'' \in \mathsf{w_{nf}}$, then by induction hypothesis there are $h', h'' \in \mathbf{N}^*$ s.t. for all $n'_1, \dots, n'_m \geq j+1+h'$ and $n''_1, \dots, n''_m \geq j+h''$ one has $N'\{\mathsf{o}^{n'_1}/x_1, \dots, \mathsf{o}^{n'_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^k$ and $N''\{\mathsf{o}^{n''_1}/x_1, \dots, \mathsf{o}^{n''_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.L$ for some $k \geq j+1$ and λ-term L such that $\lambda x.L$ is closed. Let $h = \max\{h'+1, h''\}$: for all $n_1, \dots, n_m \geq j+h$ one has $N\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^k \lambda x.L \to_{\mathsf{v}} \mathsf{o}^{k-1}$ where $k - 1 \geq j$.

If M is a variable then $M = x_i$ for some $1 \leq i \leq m$, hence for all $n_1, \dots, n_m \geq j+1$ one has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} = \mathsf{o}^{n_i}$ which is a closed abstraction.

If $M = \lambda x.N$ for some λ-term N, then we can suppose without loss of generality that $x \neq x_i$ for any $1 \leq i \leq m$, so $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} = \lambda x.N\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\}$ which is closed because the o^{n_i}'s are closed and $\mathsf{fv}(N) \subseteq \{x, x_1, \dots, x_n\}$.

If $M = (\lambda x.N')N''$ for some λ-terms $N' \in \mathsf{w_{nf}}$ and $N'' \in \mathsf{a_{nf}}$ then we can suppose without loss of generality that $x \neq x_i$ for any $1 \leq i \leq m$, moreover $\mathsf{fv}(N') \subseteq \{x, x_1, \dots, x_m\}$. By induction hypothesis, there exist $h', h'' \in \mathbf{N}^*$ such that for all $n', n'_1, \dots, n'_m \geq j+h'$ and $n''_1, \dots, n''_m \geq j+h'+h''$ one has

$$N'\{\mathsf{o}^{n'}/x, \mathsf{o}^{n'_1}/x_1 \dots, \mathsf{o}^{n'_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x L \quad \text{and} \quad N''\{\mathsf{o}^{n''_1}/x_1, \dots, \mathsf{o}^{n''_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^k$$

for some $k \geq j+h'$ and λ-term L s.t. $\lambda x.L$ is closed. If $h = h' + h''$ then, for all $n_1, \dots, n_m \geq j+h$, $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} (\lambda x.N'\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^k$ $\to_{\mathsf{v}} N'\{\mathsf{o}^k/x, \mathsf{o}^{n_1}/x_1 \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \lambda x.L$. □

Stated at p. 11

Lemma 27. *Let $M \in \mathsf{s_{nf}}$ with $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m\}$ and let $j \in \mathbf{N}$. Then there exist $h, k \in \mathbf{N}$ such that for all $n_1, \dots, n_{m+k} \geq j+h$ there exists $n \geq j$ such that*

$$M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n.$$

Proof. By induction on $M \in \mathsf{s_{nf}}$.

If M is a variable then $M = x_i$ with $1 \leq i \leq m$, so $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} = \mathsf{o}^{n_i}$ where $n_i \geq j$, hence we conclude by taking $h = 0 = k$.

If $M = \lambda x.N$ for some $N \in \mathsf{s_{nf}}$ then we can suppose without loss of generality that $x \neq x_i$ for any $1 \leq i \leq m$, moreover $\mathsf{fv}(N) \subseteq \{x, x_1, \dots, x_m\}$. By induction hypothesis, there exist $h, k' \in \mathbf{N}$ such that for all $n', n_1, \dots, n_{m+k} \geq j+h$ one has

$(N\{\mathsf{o}^{n'}/x, \mathsf{o}^{n_1}/x_1 \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k'}} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$ for some $n \geq j$. Hence

$$\begin{aligned}(M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n'}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k'}} &= \\ (\lambda x N\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n'}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k'}} &\to_{\mathsf{v}} \\ (N\{\mathsf{o}^{n'}/x, \mathsf{o}^{n_1}/x_1 \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k'}} &\twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n\end{aligned}$$

where $n \geq j$, thus we conclude by taking $k = k' + 1$.

If $M = (\lambda x.N')N''$ for some λ-terms $N' \in \mathsf{s}_{\mathtt{nf}}$ and $N'' \in \mathsf{a}_{\mathtt{nf}}$ then we can suppose without loss of generality that $x \neq x_i$ for any $1 \leq i \leq m$, moreover $\mathsf{fv}(N') \subseteq \{x, x_1, \dots, x_m\}$. By induction hypothesis, there exist $h', k \in \mathbf{N}$ such that for all $n', n'_1, \dots, n'_{m+k} \geq j + h'$ one has

$$(N'\{\mathsf{o}^{n'}/x, \mathsf{o}^{n'_1}/x_1 \dots, \mathsf{o}^{n'_m}/x_m\})\mathsf{o}^{n'_{m+1}} \dots \mathsf{o}^{n'_{m+k}} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$$

for some $n \geq j$. By lemma 26 there exists $h'' \in \mathbf{N}^*$ such that for all $n''_1, \dots, n''_m \geq j + h' + h''$ one has $N''\{\mathsf{o}^{n''_1}/x_1, \dots, \mathsf{o}^{n''_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^{n''}$ for some $n'' \geq j + h'$. Let $h = h' + h''$: for all $n_1, \dots, n_{m+k'} \geq j + h$ one has (where $n \geq j$)

$$\begin{aligned}(M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} &\twoheadrightarrow_{\mathsf{v}} \\ (\lambda x N'\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n''}\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} &\to_{\mathsf{v}} \\ (N'\{\mathsf{o}^{n''}/x, \mathsf{o}^{n_1}/x_1 \dots, \mathsf{o}^{n_m}/x_m\})\mathsf{o}^{n_{m+1}} \dots \mathsf{o}^{n_{m+k}} &\twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n\end{aligned}$$

If $M \in \mathsf{a}_{\mathtt{nf}}$ then there exists $h \in \mathbf{N}^*$ such that for all $n_1, \dots, n_m \geq j + h$ one has $M\{\mathsf{o}^{n_1}/x_1, \dots, \mathsf{o}^{n_m}/x_m\} \twoheadrightarrow_{\mathsf{v}} \mathsf{o}^n$ for some $n \geq j$ by lemma 26, thus we conclude by taking $k = 0$. □

A.5 Proofs of Subsection 6.2

The two following lemmas are used in the proof of Thm. 29 at p. 11.

Lemma 47. *Let V be a λ-value and $\vec{x} = (x_1, \dots, x_n) \supseteq \mathsf{fv}(V)$ (with $n \in \mathbf{N}$).*

(i) *For every $((a_1, \dots, a_n), [\,]) \in [\![V]\!]_{\vec{x}}$ one has $a_i = [\,]$ for any $1 \leq i \leq n$;*
(ii) *For any $m \in \mathbf{N}$, if $((a_1^1, \dots, a_n^1), a_0^1), \dots, ((a_1^m, \dots, a_n^m), a_0^m) \in [\![V]\!]_{\vec{x}}$ and $a_j = \biguplus_{i=1}^m a_j^i$ for any $0 \leq j \leq n$, then $((a_1, \dots, a_n), a_0) \in [\![V]\!]_{\vec{x}}$ (in particular, $([\,]^n, [\,]) \in [\![V]\!]_{\vec{x}}$).*

Proof. We prove simultaneously points (i) and (ii) by simple inspection.

- If V is a variable then $V = x_k$ for some $1 \leq k \leq n$, thus $((a_1, \dots, a_n), [\,]) \in [\![V]\!]_{\vec{x}}$ entails $a_j = [\,]$ for any $1 \leq j \leq n$. If $((a_1^1, \dots, a_n^1), a_0^1), \dots, ((a_1^m, \dots, a_n^m), a_0^m) \in [\![x_k]\!]_{\vec{x}}$ then, for every $1 \leq i \leq m$, one has $a_0^i = a_k^i$ (and so $a_0 = \biguplus_{i=1}^m a_0^i = \biguplus_{i=1}^m a_k^i = a_k$) and $a_j^i = [\,]$ (and so $a_j = \biguplus_{i=1}^m a_j^i = [\,]$) for any $1 \leq j \leq n$ with $j \neq k$, therefore $((a_1, \dots, a_n), a_0) = (([\,], \dots, a_k, \dots, [\,]), a_k) \in [\![x_k]\!]_{\vec{x}} = [\![V]\!]_{\vec{x}}$.

– If $V = \lambda y.N$ for some λ-term N then $[\![V]\!]_{\vec{x}} = \{(\biguplus_{k=1}^{p} \vec{d_k}, [(b_1, c_1), \dots, (b_p, c_p)]) : p \geq 0,\ \forall k = 1, \dots, p.\ ((\vec{d_k}, b_k), c_k) \in [\![N]\!]_{\vec{x},y}\}$. If $((a_1, \dots, a_n), [\,]) \in [\![V]\!]_{\vec{x}}$ then $p = 0$ and so $a_j = [\,]$ for any $1 \leq j \leq n$. If $((a_1^1, \dots, a_n^1), a_0^1), \dots, ((a_1^m, \dots, a_n^m), a_0^m) \in [\![V]\!]_{\vec{x}}$ then, for any $1 \leq i \leq m$, there exist $p_i \in \mathbf{N}$, $b_1^i, \dots, b_{p_i}^i, c_1^i, \dots, c_{p_i}^i \in \mathcal{M}_{\mathrm{f}}(U)$ and $(a_{1,1}^i, \dots, a_{n,1}^i), \dots, (a_{1,p_i}^i, \dots, a_{n,p_i}^i) \in \mathcal{M}_{\mathrm{f}}(U)^n$ such that $a_0^i = [(b_1^i, c_1^i), \dots, (b_{p_i}^i, c_{p_i}^i)]$, $a_j^i = \biguplus_{k=1}^{p_i} a_{j,k}^i$ for any $1 \leq j \leq n$ and $((a_{1,k}^i, \dots, a_{n,k}^i, b_k^i), c_k^i) \in [\![N]\!]_{\vec{x},y}$ for any $1 \leq i \leq m$ and $1 \leq k \leq p_i$; since $a_j = \biguplus_{i=1}^{m} a_j^i = \biguplus_{i=1}^{m} \biguplus_{k=1}^{p_i} a_{j,k}^i$ for any $1 \leq j \leq n$ and $a_0 = \biguplus_{i=1}^{m} a_0^i = \biguplus_{i=1}^{m} \biguplus_{k=1}^{p_i} [(b_k^i, c_k^i)]$, one has $((a_1, \dots, a_n), a_0) \in [\![\lambda y.N]\!]_{\vec{x}} = [\![V]\!]_{\vec{x}}$. □

Lemma 48. *Let M be a λ-term, let $\vec{x} = (x_1, \dots, x_m)$ and $\vec{y} = (y_1, \dots, y_n)$ be two finite sequences of pairwise distinct variables such that $\mathsf{fv}(M) \subseteq \{x_1, \dots, x_m, y_1, \dots, y_n\}$. If $V_1, \dots, V_m$ are λ-values such that $\bigcup_{i=1}^{m} \mathsf{fv}(V_i) \subseteq \{y_1, \dots, y_n\}$, then*

$$[\![M\{V_1/x_1, \dots, V_m/x_m\}]\!]_{\vec{y}} = \big\{(\biguplus_{i=0}^{m} \vec{a}_i, c) : \exists\, b_1, \dots, b_m \in \mathcal{M}_{\mathrm{f}}(U). \\ ((\vec{a}_0, b_1, \dots, b_m), c) \in [\![M]\!]_{\vec{y},\vec{x}} \text{ and } (\vec{a}_i, b_i) \in [\![V_i]\!]_{\vec{y}} \text{ for all } 1 \leq i \leq m\big\} \quad (1)$$

Proof. By induction on M. Let us denote by S the set in the right-hand side of relation (1), and by N^* the λ-term $N\{V_1/x_1, \dots, V_m/x_m\}$, for any λ-term N.

If $M = x_i$ for some $1 \leq i \leq m$, then $[\![M^*]\!]_{\vec{y}} = [\![V_i]\!]_{\vec{y}} = S$ by Lemma 47 and since $[\![x_i]\!]_{\vec{y},\vec{x}} = \{(([\,]^n, [\,], \dots, c, \dots, [\,]), c) : c \in \mathcal{M}_{\mathrm{f}}(U)\}$.

If M is a variable different from all x_i's, then $M = y_j$ for some $1 \leq j \leq n$, hence $[\![M^*]\!]_{\vec{y}} = [\![y_j]\!]_{\vec{y}} = \{(([\,], \dots, c, \dots, [\,]), c) : c \in \mathcal{M}_{\mathrm{f}}(U)\} = S$ by Lemma 47 and since $[\![y_j]\!]_{\vec{y},\vec{x}} = \{(([\,], \dots, c, \dots, [\,], [\,]^m), c) : c \in \mathcal{M}_{\mathrm{f}}(U)\}$.

If $M = \lambda z N$ for some λ-term N then $\mathsf{fv}(N) \subseteq \{x_1, \dots, x_m, y_1, \dots, y_n, z\}$ and we can suppose without loss of generality that $z \notin \{x_1, \dots, x_m, y_1, \dots, y_n\}$, so

$$[\![M^*]\!]_{\vec{y}} = [\![\lambda z.N^*]\!]_{\vec{y}} = \big\{(\biguplus_{j=1}^{k} \vec{a}^j, [(d^1, c^1), \dots, (d^k, c^k)]) : k \geq 0, \\ ((\vec{a}^j, d^j), c^j) \in [\![N^*]\!]_{\vec{y},z} \text{ for all } 1 \leq j \leq k\big\}. \quad (2)$$

By induction hypothesis, for every $1 \leq j \leq k$, $((\vec{a}^j, d^j), c^j) \in [\![N^*]\!]_{\vec{y},z}$ if and only if there exist $b_1^j, \dots, b_m^j \in \mathcal{M}_{\mathrm{f}}(U)$ and $\vec{a}_0^j, \dots, \vec{a}_m^j \in \mathcal{M}_{\mathrm{f}}(U)^n$ such that $\vec{a}^j = \biguplus_{i=0}^{m} \vec{a}_i^j$, $((\vec{a}_0^j, d^j, b_1^j, \dots, b_m^j), c^j) \in [\![N]\!]_{\vec{y},z,\vec{x}}$ and $((\vec{a}_i^j, [\,]), b_i^j) \in [\![V_i]\!]_{\vec{y},z}$ (which is equivalent to $(\vec{a}_i^j, b_i^j) \in [\![V_i]\!]_{\vec{y}}$ because $z \notin \mathsf{fv}(V_i)$) for any $1 \leq i \leq m$; let $c = [(d^1, c^1), \dots, (d^k, c^k)]$, $\vec{a}_i = \biguplus_{j=1}^{k} \vec{a}_i^j$ and $b_i = \biguplus_{j=1}^{k} b_i^j$: one has $\biguplus_{i=0}^{m} \vec{a}_i = \biguplus_{i=0}^{m} \biguplus_{j=1}^{k} \vec{a}_i^j = \biguplus_{j=1}^{k} \vec{a}^j$, $((\vec{a}_0, b_1, \dots, b_m), c) \in [\![\lambda z.N]\!]_{\vec{y},\vec{x}}$ and, by Lemma 47, $(\vec{a}_i, b_i) \in [\![V_i]\!]_{\vec{y}}$ for all $1 \leq i \leq m$. Therefore, according to relation (2), $[\![M^*]\!]_{\vec{y}} = S$.

If $M = NL$ then $\mathsf{fv}(N), \mathsf{fv}(L) \subseteq \{y_1, \dots, y_n\}$ and

$$[\![M^*]\!]_{\vec{y}} = [\![N^* L^*]\!]_{\vec{y}} = \{(\vec{a'} \uplus \vec{a''}, c) \mid \exists\, b \in \mathcal{M}_{\mathrm{f}}(U) : (\vec{a'}, [(b, c)]) \in [\![N^*]\!]_{\vec{y}} \\ \text{and } (\vec{a''}, b) \in [\![L^*]\!]_{\vec{y}}\}. \quad (3)$$

By induction hypothesis, $(\vec{a'}, [(b, c)]) \in [\![N^*]\!]_{\vec{y}}$ iff there exist $b'_1, \dots, b'_m \in \mathcal{M}_{\mathrm{f}}(U)$ and $\vec{a'_0}, \dots, \vec{a'_m} \in \mathcal{M}_{\mathrm{f}}(U)^n$ such that $a' = \biguplus_{i=0}^{m} \vec{a'_i}$, $((\vec{a'_0}, b'_1, \dots, b'_m), [(b, c)]) \in$

$[\![N]\!]_{\vec{y},\vec{x}}$ and $(\vec{a_i'}, b_i') \in [\![V_i]\!]_{\vec{y}}$ for all $1 \le i \le m$; and $(\vec{a''}, b) \in [\![L^*]\!]_{\vec{y}}$ iff there exist $b_1'', \ldots, b_m'' \in \mathcal{M}_{\mathrm{f}}(U)$ and $\vec{a_0''}, \ldots, \vec{a_m''} \in \mathcal{M}_{\mathrm{f}}(U)^n$ such that $a'' = \biguplus_{i=0}^{m} \vec{a_i''}$, $((\vec{a_0''}, b_1'', \ldots, b_m''), b) \in [\![L]\!]_{\vec{y},\vec{x}}$ and $(\vec{a_i''}, b_i'') \in [\![V_i]\!]_{\vec{y}}$ for all $1 \le i \le m$; let $\vec{a}_i = \vec{a_i'} \uplus \vec{a_i''}$ and $b_i = b_i' \uplus b_i''$: one has $\biguplus_{i=0}^{m} \vec{a}_i = \biguplus_{i=0}^{m} \vec{a_i'} \uplus \vec{a_i''} = \vec{a}' \uplus \vec{a}''$, $((\vec{a}_0, b_1, \ldots, b_m), c) \in [\![NL]\!]_{\vec{y}}$ and, by Lemma 47, $(\vec{a}_i, b_i) \in [\![V_i]\!]_{\vec{y}}$ for all $1 \le i \le m$. Therefore, according to relation (3), $[\![M^*]\!]_{\vec{y}} = S$. □

A.6 Proofs of Subsection 6.3

Theorem 33 (soundness). *Let $\mathbb{P}, \mathbb{Q} \in \mathbf{2}\langle \mathsf{r}\Lambda^\tau \rangle$. If $\mathbb{P} \to_{\mathsf{v}\tau} \mathbb{Q}$, then $[\![\mathbb{P}]\!]_{\vec{x}} = [\![\mathbb{Q}]\!]_{\vec{x}}$.* Stated at p. 13

Proof. By Thm. 18 it suffices to prove that the τ_i-rules are sound. For example

$$\begin{aligned} [\![t(s * p)]\!]_{\vec{x}} &= \{(\vec{a}_0 \uplus \vec{a}_1 \uplus \vec{a}_2, c) : \exists b \in \mathcal{M}_{\mathrm{f}}(U). \\ &\qquad (\vec{a}_0, [(b, c)]) \in [\![t]\!]_{\vec{x}},\ (\vec{a}_1, b) \in [\![s]\!]_{\vec{x}},\ (\vec{a}_2, 1) \in [\![p]\!]_{\vec{x}}\} \\ &= [\![ts * p]\!]_{\vec{x}} \end{aligned}$$

If $n > 0$, then

$$[\![\tau[[v_1, \ldots, v_n]]]\!]_{\vec{x}} = \{(\vec{a}, 1) : (\vec{a}, [\,]) \in [\![[v_1, \ldots, v_n]]\!]_{\vec{x}}\} = \emptyset$$

because the interpretation of a non-empty bag of values which does not reduce to 0 never contains an element like $(\vec{a}, [\,])$. Instead

$$[\![\tau[[\,]]]\!]_{\vec{x}} = \{(\vec{a}, 1) : (\vec{a}, [\,]) \in \{([\,]^n, [\,])\}\} = [\![\varepsilon]\!]_{\vec{x}}$$

This shows that the interpretation is invariant w.r.t. the rules $\mapsto_{\tau_1}$ and $\mapsto_{\tau_5}$. The other τ_i-rules are proved similarly. □

Lemma 49. *Let p be a closed test. Then:*

(i) either $p \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ or $p \twoheadrightarrow_{\mathsf{v}\tau} 0$;
(ii) $[\![p]\!] \neq \emptyset$ iff $p \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$.

Proof. (i) It suffices to show that for every closed resource term t, either $\tau[t] \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$ or $\tau[t] \twoheadrightarrow_{\mathsf{v}\tau} 0$. As the $\mathsf{r}\tau\lambda_{\mathsf{v}}^{\sigma}$-calculus is strongly normalizing, we have that $t \twoheadrightarrow_{\mathsf{v}\tau} \sum_{i=1}^{k} s_i$, where each s_i is a closed normal form. If $k = 0$ then $\tau[t] \twoheadrightarrow_{\mathsf{v}\tau} 0$ since $\tau[0] = 0$. Otherwise for each s_i there are two possibilities:
- $s_i = [v_1, \ldots, v_m]$ with v_j not a variable. Then $\tau[s_i]$ reduces either to ε or to 0, depending on the value of m.
- $s_i = \tau[[v_1, \ldots, v_m] * ([v_1', \ldots, v_{m'}'] \parallel q)]$ with v_j, v_l' not a variable. Then $s_i \to_{\mathsf{v}\tau} \tau[[v_1, \ldots, v_m]] \parallel ([v_1', \ldots, v_{m'}'] \parallel q)$ that, using the induction hypothesis, can only reduce to 0 or ε.

We conclude since $\tau[t] \twoheadrightarrow_{\mathsf{v}\tau} \sum_{i=1}^{k} \tau[s_i]$, and this latter expression reduces to a finite (possibly empty) sum of ε's, which is thus equal to 0 or ε.

(ii) By the soundness of the model (Thm. 33) and item (i). □

The set U admits a well-founded ordering via the notion of *rank of an element* $\alpha \in U$: the rank of α, notation $\mathsf{rk}(\alpha)$, is the smallest $n \in \mathbf{N}$ such that $\alpha \in U_n$. The *rank of a multiset* $b \in \mathcal{M}_{\mathrm{f}}(U)$, denoted by $\mathsf{rk}(b)$, is the greatest among the ranks of its elements, if it is non-empty; the empty multiset has rank 0.

Lemma 50. *Let* $a \in \mathcal{M}_{\mathrm{f}}(U)$. *Then:*

(i) $\llbracket a^- \rrbracket = \{(1, a)\}$,
(ii) $\llbracket a^+ (\!|[x^n]|\!) \rrbracket_x = \{(a, 1)\}$, *where* $n = \#a$.

Proof. The points (i) and (ii) are proved simultaneously by induction on $\mathsf{rk}(a)$. We write IH(i) and IH(ii) for the induction hypotheses concerning (i) and (ii), respectively.

If $\mathsf{rk}(a) = 0$ then $a = [\,]$, hence $\llbracket a^- \rrbracket = \llbracket [\,] \rrbracket = \{[\,]\}$ and $\llbracket a^+ (\!|[x^0]|\!) \rrbracket_x = \llbracket \tau[[\lambda y.[\,] * \varepsilon][\,]] \rrbracket_x = \llbracket \varepsilon \rrbracket_x = \{([\,], 1)\}$. This proves the base cases for (i) and (ii).
For the inductive step, suppose $\mathsf{rk}(a) > 0$ and $\#a = r$, so that $a = [(b_1, c_1), \ldots, (b_r, c_r)]$.
We prove (i). By definition $\llbracket a^- \rrbracket = \llbracket [\lambda y_1.{c_1}^- * {b_1}^+ (\!|[y_1^{m_1}]|\!), \ldots, \lambda y_r.{c_r}^- * {b_r}^+ (\!|[y_r^{m_r}]|\!)] \rrbracket$. So we have $(1, a') \in \llbracket a^- \rrbracket$ iff $a' = [(b'_1, c'_1), \ldots, (b'_r, c'_r)]$ and for all $1 \leq j \leq r$ $(b'_j, c'_j) \in \llbracket {c_j}^- * {b_j}^+ (\!|[y_j^{m_j}]|\!) \rrbracket$, i.e., $(1, c'_j) \in \llbracket {c_j}^- \rrbracket$ and $(b'_j, 1) \in \llbracket {b_j}^+ (\!|[y_j^{m_j}]|\!) \rrbracket_{y_j}$. By IH(i) and IH(ii) we have $c'_j = c_j$ and $b'_j = b_j$. Therefore $a' = a$.
We prove (ii). By definition $a^+ (\!|[x^r]|\!) = \tau[[\lambda z.[\,] * \|_{i=1}^r \tau[[\lambda y.[\,] * {c_i}^+ (\!|[y^{k_i}]|\!)]([z]{b_i}^-)]][x^r]]$, where $k_i = \#c_i$ for all $1 \leq i \leq r$. Hence, $a^+ (\!|[x^r]|\!) \twoheadrightarrow_{\mathsf{v}\tau} \|_{i=1}^r \tau[[\lambda y.[\,] * {c_i}^+ (\!|[y^{k_i}]|\!)]([x]{b_i}^-)]$. Using IH(i) and IH(ii) we have that $\llbracket {b_i}^- \rrbracket = \{(1, b_i)\}$ and $\llbracket {c_i}^+ (\!|[y^{k_i}]|\!) \rrbracket_y = \{(c_i, 1)\}$ and therefore $\llbracket \tau[[\lambda y.[\,] * {c_i}^+ (\!|[y^{k_i}]|\!)]([x]{b_i}^-)] \rrbracket_x = \{([(b_i, c_i)], 1)\}$, for each $i = 1, \ldots, r$. In conclusion $\llbracket a^+ (\!|[x^r]|\!) \rrbracket_x = \{([(b_1, c_1), \ldots, (b_r, c_r)], 1)\}$. □

Lemma 51. *Let* $b \in \mathcal{M}_{\mathrm{f}}(U)$ *and let* t *be a resource term with* $\mathsf{fv}(t) \subseteq \vec{x}$. *Then* $\llbracket b^+ (\!|t|\!) \rrbracket_{\vec{x}} = \{(\vec{a}, 1) : (\vec{a}, b) \in \llbracket t \rrbracket_{\vec{x}}\}$.

Proof. By induction on the structure of t. □

Lemma 52. *Let* $\vec{a} \in \mathcal{M}_{\mathrm{f}}(U)^n$ *and let* p *be a test with* $\mathsf{fv}(p) \subseteq \vec{x}$. *Then* $\llbracket p\langle \vec{a}^- / \vec{x} \rangle \rrbracket \neq \emptyset$ *iff* $(\vec{a}, 1) \in \llbracket p \rrbracket_{\vec{x}}$.

Proof. By induction on the structure of p. □

Lemma 53. *Let* $b \in \mathcal{M}_{\mathrm{f}}(U)$ *and let* $\#b = n$. *If* s *is a resource term with* $\mathsf{fv}(s) \subseteq \vec{x}$, *then* $\llbracket b^+ (\!|s|\!) \rrbracket_{\vec{x}} \neq \emptyset$ *iff* $\llbracket \tau[[\lambda y.[\,] * b^+ (\!|[y^n]|\!)]s] \rrbracket_{\vec{x}} \neq \emptyset$.

Proof. By induction on the rank of b. □

Stated at p. 13

Lemma 35. *Let* $(\vec{a}, b) \in \mathcal{M}_{\mathrm{f}}(U)^n \times \mathcal{M}_{\mathrm{f}}(U)$, $\#b = r$ *and let* t *be a resource term. Then* $(\vec{a}, b) \in \llbracket t \rrbracket_{\vec{x}}$ *iff* $\tau[[\lambda y.[\,] * b^+ (\!|[y^r]|\!)](t\langle \vec{a}^- / \vec{x} \rangle)] \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon$.

Proof. We have the following chain of equivalences:

$$
\begin{aligned}
(\vec{a}, b) \in \llbracket t \rrbracket_{\vec{x}} &\Leftrightarrow (\vec{a}, 1) \in \llbracket b^+ (\!|t|\!) \rrbracket_{\vec{x}}, \quad \text{by Lemma 51} \\
&\Leftrightarrow \llbracket b^+ (\!|t|\!) \langle \vec{a}^- / \vec{x} \rangle \rrbracket \neq \emptyset, \quad \text{by Lemma 52} \\
&\Leftrightarrow \llbracket b^+ (\!|t\langle \vec{a}^- / \vec{x} \rangle|\!) \rrbracket \neq \emptyset, \quad \text{since } \mathsf{fv}(b^+ (\!|\cdot|\!)) = \emptyset \\
&\Leftrightarrow \llbracket \tau[[\lambda y.[\,] * b^+ (\!|[y^r]|\!)](t\langle \vec{a}^- / \vec{x} \rangle)] \rrbracket \neq \emptyset, \quad \text{by Lemma 53} \\
&\Leftrightarrow \tau[[\lambda y.[\,] * b^+ (\!|[y^r]|\!)](t\langle \vec{a}^- / \vec{x} \rangle)] \twoheadrightarrow_{\mathsf{v}\tau} \varepsilon, \quad \text{by Lemma 49}
\end{aligned}
$$
□

A.7 Proofs of Subsection 6.4

Lemma 54. *Let $M \in \Lambda$, $V \in \Lambda^{\mathsf{v}}$, $t \in \mathcal{T}(M)$ and $v_1, \ldots, v_m \in \mathcal{T}(V)$. For every $f \in \mathfrak{S}_m$, if $\mathsf{deg}_x(t) = m$ then $t\{v_{f(1)}/x^1, \ldots, v_{f(m)}/x^m\} \in \mathcal{T}(N\{V/x\})$.*

Proof. See Lemma 16 in [14]. □

The following lemma is a generalization of Lemma 18 in [14].

Lemma 38. *Let M, M' be λ-terms.* Stated at p. 14

(i) *If $M \to_{\mathtt{w}} M'$ and $t \in \mathcal{T}(M)$, then there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ such that $t \to_{\mathsf{v}} \mathbb{T}$.*
(ii) *If $M \to_{\mathtt{s}} M'$ and $s \in \mathcal{T}_{\mathsf{s}}(M)$, then there exists $\mathbb{S} \subseteq \mathcal{T}_{\mathsf{s}}(M')$ such that $s \to_{\mathsf{v}}^{+} \mathbb{S}$.*

Proof (of Lemma 38.i). By induction on the definition of $M \to_{\mathtt{w}} M'$.

If $M = (\lambda x.N)V \to_{\mathtt{w}} N\{V/x\} = M'$ then $t = [\lambda x.t_1, \ldots, \lambda x.t_n][v_1, \ldots, v_m]$ for some $n, m \in \mathbf{N}$, with $t_i \in \mathcal{T}(N)$ and $v_j \in \mathcal{T}(V)$ for all $1 \le i \le n$ and $1 \le j \le m$. If $n \neq 1$, or $n = 1$ but $\mathsf{deg}_x(t_1) \neq m$, then $t \to_{\mathsf{v}} 0 \subseteq \mathcal{T}_{\mathsf{s}}(M')$. Otherwise, $n = 1$ and $\mathsf{deg}_x(t_1) = m$, so $t = [\lambda x.t_1][v_1, \ldots, v_m] \to_{\mathsf{v}} \sum_{f \in \mathfrak{S}_m} t_1\{v_{f(1)}/x^1, \ldots, v_{f(m)}/x^m\} = \mathbb{T}$. By Lemma 54, $\mathbb{T} \subseteq \mathcal{T}(M')$.

If $M = (\lambda x.M_0)NL \to_{\mathtt{w}} (\lambda x.M_0 L)N = M'$, then $t = [\lambda x.t_1, \ldots, \lambda x.t_n]s_1 s_2$ for some $s_1 \in \mathcal{T}(N)$, $s_2 \in \mathcal{T}(L)$ and $n \in \mathbf{N}$, with $t_i \in \mathcal{T}(M_0)$ for all $1 \le i \le n$. If $n \neq 1$ then $t \to_{\mathsf{v}} 0 \subseteq \mathcal{T}(M')$, otherwise $t = [\lambda x.t_1]s_1 s_2 \to_{\mathsf{v}} [\lambda x.t_1 s_2]s_1 \subseteq \mathcal{T}(M')$.

If $M = V((\lambda x.L)N) \to_{\mathtt{w}} (\lambda x.VL)N = M'$, then $t = [v_1, \ldots, v_n]([\lambda x.t_1, \ldots, \lambda x.t_n]s)$ for some $s \in \mathcal{T}(N)$ and $n, m \in \mathbf{N}$, with $v_i \in \mathcal{T}(V)$ and $t_j \in \mathcal{T}(L)$ for all $1 \le i \le n$ and $1 \le j \le m$. If $n \neq 1$ or $m \neq 1$ then $t \to_{\mathsf{v}} 0 \subseteq \mathcal{T}(M')$, otherwise $t = [v_1]([\lambda x.t_1]s') \to_{\mathsf{v}} [\lambda x.v_1 t_1]s' \subseteq \mathcal{T}(M')$.

If $M = M_0 M_1 \to_{\mathtt{w}} M_0' M_1 = M'$ with $M_0 \to_{\mathtt{w}} M_0'$, then $t = s_0 s_1$ where $s_i \in \mathcal{T}(M_i)$ for $i \in \{1, 2\}$. By induction hypothesis, there exists $\mathbb{T}_0 \subseteq \mathcal{T}(M_0')$ such that $s_0 \to_{\mathsf{v}} \mathbb{T}_0$. Therefore $t \to_{\mathsf{v}} \mathbb{T}_0 s_1 \subseteq \mathcal{T}(M')$.

The case where $M = M_0 M_1 \to_{\mathtt{w}} M_0 M_1' = M'$ with $M_1 \to_{\mathtt{w}} M_1'$ is perfectly similar to the previous one.

If $M = (\lambda x.L)N \to_{\mathtt{w}} (\lambda x.L')N = M'$ with $L \to_{\mathtt{w}} L'$, then $t = [\lambda x.t_1, \ldots, \lambda x.t_n]s$ for some $s \in \mathcal{T}(N)$ and $n \in \mathbf{N}$, with $t_i \in \mathcal{T}(L)$ for all $1 \le i \le n$. If $n \neq 1$ then $t \to_{\mathsf{v}} 0 \subseteq \mathcal{T}(M')$. Otherwise $t = [\lambda x.t_1]s$ and, by induction hypothesis, there exists $\mathbb{L} \subseteq \mathcal{T}(L')$ such that $t_1 \to_{\mathsf{v}} \mathbb{L}$. Hence $t \to_{\mathsf{v}} (\lambda x.\mathbb{L})s \subseteq \mathcal{T}(M')$. □

Proof (of Lemma 38.ii). By induction on the definition of $M \to_{\mathtt{s}} M'$.

If $M \to_{\mathtt{w}} M'$ then there exists $\mathbb{T} \subseteq \mathcal{T}(M')$ such that $t \to_{\mathsf{v}} \mathbb{T}$, by Lemma 38.i. Since $t \in \mathcal{T}_{\mathsf{s}}(M)$, for every $\mathbb{T}'$ if $\mathbb{T} \to_{\mathsf{v}}^{*} \mathbb{T}'$ then $\mathbb{T}' \subseteq \mathsf{Strat}$. Therefore $\mathbb{T} \subseteq \mathcal{T}_{\mathsf{s}}(M')$.

If $M = M_0 M_1 \to_{\mathtt{s}} M_0' M_1 = M'$ with $M_0 \to_{\mathtt{s}} M_0'$, then $t = s_0 s_1$ where $s_0 \in \mathcal{T}_{\mathsf{s}}(M_0)$ and $s_1 \in \mathcal{T}(M_1)$. By induction hypothesis, there exists $\mathbb{S}_0 \subseteq \mathcal{T}_{\mathsf{s}}(M_0')$ such that $s_0 \to_{\mathsf{v}}^{+} \mathbb{S}_0$, thus $t \to_{\mathsf{v}}^{+} \mathbb{S}_0 s_1 \subseteq \mathcal{T}(M')$. Since $t \in \mathcal{T}_{\mathsf{s}}(M)$, for every $\mathbb{T}'$ if $\mathbb{S}_0 s_1 \to_{\mathsf{v}}^{*} \mathbb{T}'$ then $\mathbb{T}' \subseteq \mathsf{Strat}$. Therefore $\mathbb{S}_0 s_1 \subseteq \mathcal{T}_{\mathsf{s}}(M')$.

If $M = \lambda x.L \to_{\mathtt{s}} \lambda x.L' = M'$ with $L \to_{\mathtt{s}} L'$, then $t = [\lambda x.t_1, \ldots, \lambda x.t_n]$ for some $n \ge 1$ with $t_i \in \mathcal{T}_{\mathsf{s}}(L)$ for all $1 \le i \le n$. By induction hypothesis, for all $1 \le i \le n$ there exists $\mathbb{L}_i \subseteq \mathcal{T}_{\mathsf{s}}(L')$ such that $t_i \to_{\mathsf{v}}^{+} \mathbb{L}_i$. So, $t \to_{\mathsf{v}}^{+} [\lambda x.\mathbb{L}_1, \lambda x.t_2, \ldots, \lambda x.t_n] \to_{\mathsf{v}}^{+} \ldots \to_{\mathsf{v}}^{+} [\lambda x.\mathbb{L}_1, \ldots, \lambda x.\mathbb{L}_n] \subseteq \mathcal{T}(M')$. Since $t \in \mathcal{T}_{\mathsf{s}}(M)$, for every $\mathbb{T}'$ if $[\lambda x.\mathbb{L}_1, \ldots, \lambda x.\mathbb{L}_n] \to_{\mathsf{v}}^{*} \mathbb{T}'$ then $\mathbb{T}' \subseteq \mathsf{Strat}$. Therefore $[\lambda x.\mathbb{L}_1, \ldots, \lambda x.\mathbb{L}_n] \subseteq \mathcal{T}_{\mathsf{s}}(M')$. □