

Head reduction and normalization in a call-by-value lambda-calculus

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Abstract

Recently, a standardization theorem has been proven for a variant of Plotkin’s call-by-value lambda-calculus extended by means of two commutation rules (sigma-reductions): this result was based on a partitioning between head and internal reductions. We study the head normalization for this call-by-value calculus with sigma-reductions and we relate it to the weak evaluation of original Plotkin’s call-by-value lambda-calculus. We give also a (non-deterministic) normalization strategy for the call-by-value lambda-calculus with sigma-reductions.

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1 Introduction

The call-by-value λ -calculus (λ_v -calculus or λ_v for short) and the operational machine for its evaluation has been introduced by Plotkin [15] inspired by Landin’s seminal work [9] on the programming language ISWIM and the SECD machine. The λ_v -calculus is a paradigmatic language able to capture two features of many functional programming languages: call-by-value parameter passing policy (parameters are evaluated before being passed) and weak evaluation (the body of a function is evaluated only when parameters are supplied).

The syntax of λ_v is the same as that of the ordinary (i.e. call-by-name) λ -calculus (λ for short), but the reduction rule for λ_v , called β_v , is a restriction of the β -rule for λ : β_v allows the contraction of a redex $(\lambda x.M)N$ only in case the argument N is a value, i.e. a variable or an abstraction. Unfortunately, the semantic analysis of the λ_v -calculus has turned out to be more elaborate than that of ordinary λ -calculus. This is due essentially to the “weakness” of (full) β_v -reduction, a fact widely recognized: indeed, there are many proposals of alternative call-by-value λ -calculi extending Plotkin’s one [11, 10, 8, 2, 1]. To have an example of the “weakness” of the rewriting rules of λ_v , it is sufficient to consider that it is impossible to have an internal operational characterization (i.e. one that uses the β_v -reduction) of the semantically meaningful notions of call-by-value solvability and potential valuability, as shown in [13, 14, 2].

In this paper we will study the λ_v^σ -calculus (λ_v^σ for short), a call-by-value extension of λ_v recently proposed in [4]: it keeps the λ_v (and λ) syntax and it adds to the β_v -reduction two commutation rules, called σ_1 and σ_3 , which unblock “hidden” β_v -redexes that are concealed by the “hyper-sequential structure” of terms. The λ_v^σ -calculus enjoys some basic properties we expect from a calculus, namely confluence (see [4]) and standardization (see [7]). Moreover, λ_v^σ provides elegant characterizations of many semantic properties, e.g. solvability and potential valuability (see [4]), and it is conservative with respect to Plotkin’s λ_v : in particular, [7] shows that the notions of solvability and potential valuability for λ_v^σ coincide with those for λ_v .



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The \mathbf{v} -reduction (i.e. the reduction for λ_v^σ) can be partitioned into head \mathbf{v} -reduction and internal \mathbf{v} -reduction; the head \mathbf{v} -reduction is in turn decomposed into head β_v - and head σ -reduction. The head β_v -reduction is just the deterministic weak evaluation strategy for Plotkin's λ_v -calculus. According to a sequentialization theorem proven in [7, Theorem 22], any \mathbf{v} -reduction sequence can be sequentialized in an initial head β_v -reduction sequence followed by a head σ -reduction sequence followed by an internal \mathbf{v} -reduction sequence. Similar well-known results hold for λ and λ_v , and starting from them one can define a normalization strategy for λ and λ_v , i.e. a deterministic reduction strategy that reaches a normal form if and only if one exists: for example the leftmost reduction, see [19, Theorem 2.8] and [3, Theorem 13.2.2].

Is there a normalization strategy for λ_v^σ ? Theorem 24, one of the main results of this paper, proves that, starting from the sequentialization theorem mentioned above, a normalization strategy can be defined for λ_v^σ , based on the notions of head β_v - and head σ -reductions.

A first difference appears here between λ_v^σ and λ_v (or λ): the normalization strategy for λ_v^σ is not deterministic. Indeed, while the head β_v -reduction (or the call-by-name head reduction) is deterministic (i.e. a partial function), the head \mathbf{v} -reduction is non-deterministic and, still worse, non-confluent and there are terms having several head \mathbf{v} -normal forms: this might appear disappointing. So, three natural questions arise:

- With respect to head \mathbf{v} -reduction, do normalization and strong normalization coincide?¹
- Can we relate the termination of head β_v -reduction and head \mathbf{v} -reduction?
- Can we characterize the terms having a unique head \mathbf{v} -normal form?

Our Theorem 21 gives a positive answer to the first two questions. Observe that the lack of any form of confluence for head \mathbf{v} -reduction requires a more complex reasoning, passing through a syntactic characterization of head β_v - and head \mathbf{v} -normal forms. Theorem 21 not only shows that the head \mathbf{v} -reduction and the head β_v -reduction are deeply related (and hence, again, λ_v^σ is conservative with respect to λ_v) but also that both enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ -calculus.

Our Proposition 27 gives a partial answer to the third question above: it shows that in some cases (of interest) a head \mathbf{v} -normalizable term has a unique head \mathbf{v} -normal form; in particular, every closed head \mathbf{v} -normalizable term has a unique head \mathbf{v} -normal form.

So, λ_v^σ appears as an extension of Plotkin's λ_v -calculus that enjoys many meaningful conservation properties with respect to λ_v and therefore it is a useful tool for theoretical and semantic investigations about λ_v and call-by-value setting. See also conclusions in Section 6 for further and more precise motivations for this paper and future work.

Related work The λ_v^σ -calculus has been recently introduced in [4] and further investigated in [7]. It is an extension of Plotkin's λ_v -calculus inspired by the call-by-value translation of λ -terms into linear logic proof-nets [6]. Other variants of λ_v have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [11], Felleisen and Sabry [18], Maraist et al. [10], Herbelin and Zimmerman [8], Accattoli and Paolini [2] (the latter is inspired by the call-by-value translation of λ -terms into linear logic proof-nets, see [1]). All these proposals are based on the introduction of new constructs to the syntax of λ_v , so the comparison between them is not easy with respect to syntactical properties (some detailed comparison is given in [2]). We point out that the calculi introduced in [11, 18, 10, 8] present some variants of our σ_1

¹ The answer is trivially positive in the case of call-by-name head normalization (for λ) and head β_v -normalization, since these reductions are deterministic.

and/or σ_3 rules, often in a setting with explicit substitutions. Regnier [16, 17] used the rule σ_1 (but not σ_3) in ordinary (i.e. call-by-name) λ -calculus.

The head \mathbf{v} -reduction investigated here has been introduced in [7]. Some results of this paper are inspired by the Takahashi's results [19] on the ordinary (i.e. call-by-name) λ -calculus, partially adapted by Crary [5] for λ_v .

Outline In Section 2 we introduce the syntax and the reduction rules of the λ_v^σ -calculus. In Section 3 we define the head \mathbf{v} -reduction and the internal \mathbf{v} -reduction, and we recall some results already proven in [7] concerning them. Section 4 is devoted to proving the first main result of our paper: Theorem 21, which studies the normalization for the head \mathbf{v} -reduction and relates it to the weak evaluation strategy for Plotkin's λ_v -calculus. In Section 5 we show that the head \mathbf{v} -reduction can be used to define a normalization strategy for the λ_v^σ -calculus (Theorem 24), and moreover in some cases the head \mathbf{v} -normal form (if any) of a term is unique (Proposition 27). In Section 6 we summarize the findings and suggest future work.

2 The call-by-value lambda calculus with sigma-rules

In this section we present λ_v^σ , a call-by-value λ -calculus introduced in [4] that adds two σ -reduction rules to pure (i.e. without constants) call-by-value λ -calculus defined by Plotkin in [15].

The syntax of terms of λ_v^σ [4] is the same as the one of ordinary λ -calculus and Plotkin's call-by-value λ -calculus λ_v [15] (without constants). Given a countable set \mathcal{V} of *variables* (denoted by x, y, z, \dots), the sets Λ of *terms* and Λ_v of *values* are defined by mutual induction:

$$\begin{aligned} (\Lambda_v) \quad V, U &::= x \mid \lambda x.M && \text{values} \\ (\Lambda) \quad M, N, L &::= V \mid MN && \text{terms} \end{aligned}$$

Clearly, $\Lambda_v \subsetneq \Lambda$. All terms are considered up to α -conversion. The set of free variables of a term M is denoted by $\text{fv}(M)$. Given $V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables x_1, \dots, x_n , $M\{V_1/x_1, \dots, V_n/x_n\}$ denotes the term obtained by the *capture-avoiding simultaneous substitution* of V_i for each free occurrence of x_i in the term M (for all $1 \leq i \leq n$). Note that, for all $V, V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables x_1, \dots, x_n , $V\{V_1/x_1, \dots, V_n/x_n\} \in \Lambda_v$.

Contexts (with exactly one hole (\cdot)), denoted by \mathcal{C} , are defined as usual via the grammar:

$$\mathcal{C} ::= (\cdot) \mid \lambda x.\mathcal{C} \mid \mathcal{C}M \mid M\mathcal{C}.$$

We use $\mathcal{C}(M)$ for the term obtained by the capture-allowing substitution of the term M for the hole (\cdot) in the context \mathcal{C} .

► **Notation.** From now on, we set $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

The reduction rules of λ_v^σ consist of Plotkin's β_v -reduction rule, introduced in [15], and two simple commutation rules called σ_1 and σ_3 , studied in [4, 7].

► **Definition 1** (Reduction rules). *We define the following binary relations on Λ (for any $M, N, L \in \Lambda$ and any $V \in \Lambda_v$):*

$$\begin{aligned} (\lambda x.M)V &\mapsto_{\beta_v} M\{V/x\} \\ (\lambda x.M)NL &\mapsto_{\sigma_1} (\lambda x.ML)N \quad \text{with } x \notin \text{fv}(L) \\ V((\lambda x.L)N) &\mapsto_{\sigma_3} (\lambda x.VL)N \quad \text{with } x \notin \text{fv}(V). \end{aligned}$$

We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ and $\mapsto_{\mathbf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

For any $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathbf{v}\}$, if $M \mapsto_r M'$ then M is a r -redex and M' is its r -contractum. In this sense, a term of the shape $(\lambda x.M)N$ (for any $M, N \in \Lambda$) is a β -redex.

The side conditions on \mapsto_{σ_1} and \mapsto_{σ_3} in Definition 1 can be always fulfilled by α -renaming.

Obviously, any β_v -redex is a β -redex but the converse does not hold: $(\lambda x.z)(yI)$ is a β -redex but not a β_v -redex.

► **Example 2.** Redexes of different kind may overlap: for example, the term $\Delta I \Delta$ is a σ_1 -redex and it contains the β_v -redex ΔI ; the term $\Delta(I\Delta)(xI)$ is a σ_1 -redex and it contains the σ_3 -redex $\Delta(I\Delta)$, which contains in turn the β_v -redex $I\Delta$.

► **Notation.** Let R be a binary relation on Λ . We denote by R^* (resp. R^+ ; R^\equiv) the reflexive-transitive (resp. transitive; reflexive) closure of R .

► **Definition 3 (Reductions).** Let $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathbf{v}\}$.

The r -reduction \rightarrow_r is the contextual closure of \mapsto_r , i.e. $M \rightarrow_r M'$ iff there is a context \mathbf{C} and $N, N' \in \Lambda$ such that $M = \mathbf{C}(N)$, $M' = \mathbf{C}(N')$ and $N \mapsto_r N'$.

The r -equivalence \simeq_r is the reflexive-transitive and symmetric closure of \rightarrow_r .

Let M be a term: M is r -normal if there is no term N such that $M \rightarrow_r N$; M is r -normalizable if there is a r -normal term N such that $M \rightarrow_r^* N$; M is strongly r -normalizable if there is no sequence $(N_i)_{i \in \mathbb{N}}$ of terms such that $M = N_0$ and $N_i \rightarrow_r N_{i+1}$ for any $i \in \mathbb{N}$.

Obviously, $\rightarrow_\sigma = \rightarrow_{\sigma_1} \cup \rightarrow_{\sigma_3} \subsetneq \rightarrow_{\mathbf{v}}$ and $\rightarrow_{\beta_v} \subsetneq \rightarrow_{\mathbf{v}}$ and $\rightarrow_{\mathbf{v}} = \rightarrow_{\beta_v} \cup \rightarrow_\sigma$.

► **Remark 4.** For any $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathbf{v}\}$ (resp. $r \in \{\sigma_1, \sigma_3, \sigma\}$), values are closed under r -reduction (resp. r -expansion): for any $V \in \Lambda_v$, if $V \rightarrow_r M$ (resp. $M \rightarrow_r V$) then $M \in \Lambda_v$; more precisely, $V = \lambda x.N$ and $M = \lambda x.N'$ for some $N, N' \in \Lambda$ with $N \rightarrow_r N'$ (resp. $N' \rightarrow_r N$).

For any $r \in \{\beta_v, \mathbf{v}\}$, values are not closed under r -expansion: $I\Delta \rightarrow_{\beta_v} \Delta \in \Lambda_v$ but $I\Delta \notin \Lambda_v$.

► **Proposition 5 (See [4]).** The σ -reduction is confluent and strongly normalizing. The \mathbf{v} -reduction is confluent.

The λ_v^σ -calculus, λ_v^σ for short, is the set Λ of terms endowed with the \mathbf{v} -reduction $\rightarrow_{\mathbf{v}}$. The set Λ endowed with the β_v -reduction \rightarrow_{β_v} is the λ_v -calculus (λ_v for short), i.e. the Plotkin's call-by-value λ -calculus [15] (without constants), which is thus a sub-calculus of λ_v^σ .

► **Example 6.** $M = (\lambda y.\Delta)(xI)\Delta \rightarrow_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \dots$ and $N = \Delta((\lambda y.\Delta)(xI)) \rightarrow_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \dots$ are the only possible \mathbf{v} -reduction paths from M and N respectively: M and N are not \mathbf{v} -normalizable, and $M \simeq_{\mathbf{v}} N$. Meanwhile, M and N are β_v -normal and different, hence $M \not\equiv_{\beta_v} N$ (by confluence of \rightarrow_{β_v} , see [15]).

Informally, σ -rules unblock β_v -redexes which are hidden by the “hyper-sequential structure” of terms. This approach is alternative to the one in [2, 1] where hidden β_v -redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value λ -calculus with explicit substitution introduced in [2] can be embedded in λ_v^σ .

It is well-known that the β_v -reduction can be simulated by linear logic cut-elimination via the call-by-value translation $(\cdot)^v$ of λ -terms into proof-nets, called by Girard [6, pp. 81-82] “boring” and defined by $(A \Rightarrow B)^v = !A^v \multimap !B^v$ (see also [1]). The images under $(\cdot)^v$ of a σ -redex and its σ -contractum are equal modulo some non-structural cut-elimination steps.

3 Head and internal reductions

In this section we introduce the definitions of head \mathbf{v} -reduction (which is decomposed in head β_v - and head σ -reductions) and internal \mathbf{v} -reduction, then we recall some results proven in [7].

► **Notation.** From now on, we always assume that $V, V' \in \Lambda_v$.

Note that the generic form of a term is $VM_1 \dots M_m$ for some $m \in \mathbb{N}$ (in particular, values are obtained when $m = 0$). The sequentialization result is based on a partitioning of \mathbf{v} -reduction between head \mathbf{v} -reduction and internal \mathbf{v} -reduction.

► **Definition 7 (Head β_v -reduction).** *The head β_v -reduction $\xrightarrow{\beta_v}$ is the binary relation on Λ defined inductively by the following rules ($m \in \mathbb{N}$ in both rules):*

$$\frac{}{(\lambda x.M)VM_1 \dots M_m \xrightarrow{\beta_v} M\{V/x\}M_1 \dots M_m} \beta_v \quad \frac{N \xrightarrow{\beta_v} N'}{VNM_1 \dots M_m \xrightarrow{\beta_v} VN'M_1 \dots M_m} \text{right}$$

The head β_v -reduction $\xrightarrow{\beta_v}$ is exactly the (pure) “left reduction” defined in [15, p. 136] for λ_v and called “(weak) evaluation” in [18, 5]. If $N \xrightarrow{\beta_v} N'$ then N' is obtained from N by reducing the leftmost-outermost β_v -redex, not in the scope of a λ : thus, the head β_v -reduction is deterministic (i.e. it is a partial function from Λ to Λ) and does not reduce values.

► **Definition 8 (Head σ - and head \mathbf{v} -reductions).** *The head σ -reduction $\xrightarrow{\sigma}$ is the binary relation on Λ defined inductively by the following rules ($m \in \mathbb{N}$ in all the rules, $x \notin \text{fv}(L)$ in the rule σ_1 , $x \notin \text{fv}(V)$ in the rule σ_3):*

$$\frac{}{(\lambda x.M)NLM_1 \dots M_m \xrightarrow{\sigma} (\lambda x.ML)NM_1 \dots M_m} \sigma_1 \quad \frac{N \xrightarrow{\sigma} N'}{VNM_1 \dots M_m \xrightarrow{\sigma} VN'M_1 \dots M_m} \text{right}$$

$$\frac{}{V((\lambda x.L)N)M_1 \dots M_m \xrightarrow{\sigma} (\lambda x.VL)NM_1 \dots M_m} \sigma_3$$

The head \mathbf{v} -reduction is $\xrightarrow{\mathbf{v}} = \xrightarrow{\beta_v} \cup \xrightarrow{\sigma}$.

Let $r \in \{\beta_v, \sigma, \mathbf{v}\}$ and $N \in \Lambda$: N is head r -normal if there is no $N' \in \Lambda$ such that $N \xrightarrow{r} N'$; N is head r -normalizable if there is a r -normal term N' such that $N \xrightarrow{r}^* N'$; N is strongly head r -normalizable if there is no $(N_i)_{i \in \mathbb{N}}$ such that $N = N_0$ and $N_i \xrightarrow{r} N_{i+1}$ for any $i \in \mathbb{N}$.

Notice that $\mapsto_{\beta_v} \subsetneq \xrightarrow{\beta_v} \subsetneq \rightarrow_{\beta_v}$ and $\mapsto_{\sigma} \subsetneq \xrightarrow{\sigma} \subsetneq \rightarrow_{\sigma}$ and $\mapsto_{\mathbf{v}} \subsetneq \xrightarrow{\mathbf{v}} \subsetneq \rightarrow_{\mathbf{v}}$.

Informally, if $N \xrightarrow{\sigma} N'$ then N' is obtained from N by reducing “one of the leftmost” σ_1 - or σ_3 -redexes, not in the scope of a λ : in general, a term may contain several head σ_1 - and σ_3 -redexes. Indeed, differently from $\xrightarrow{\beta_v}$, the head σ -reduction $\xrightarrow{\sigma}$ is not deterministic, for example the leftmost-outermost σ_1 - and σ_3 -redexes may overlap: if $M = (\lambda y.y')(\Delta(xI))I$ then $M \xrightarrow{\sigma} (\lambda y.y'I)(\Delta(xI)) = N_1$ by applying the rule σ_1 and $M \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz))(xI)I = N_2$ by applying the rule σ_3 . Note that N_1 contains only a head σ_3 -redex and $N_1 \xrightarrow{\sigma} (\lambda z.(\lambda y.y'I)(zz))(xI) = N$ which is head \mathbf{v} -normal; meanwhile N_2 contains only a head σ_1 -redex and $N_2 \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz)I)(xI) = N'$ which is head \mathbf{v} -normal: $N \neq N'$, so the head σ - and head \mathbf{v} -reductions are not (locally) confluent and a term may have several head \mathbf{v} -normal forms (this example does not contradict the confluence of σ -reduction because $N' \rightarrow_{\sigma} N$ but by performing an internal \mathbf{v} -reduction step, see next Definition 9).

The head \mathbf{v} -reduction $\xrightarrow{\mathbf{v}}$ is non-deterministic not only because the head σ -reduction $\xrightarrow{\sigma}$ is non-deterministic, but also because the leftmost-outermost β_v -redex of a term may overlap with “one of its leftmost” σ_1 - or σ_3 -redexes, as seen in Example 2.

► **Definition 9 (Internal \mathbf{v} -reduction).** *The internal \mathbf{v} -reduction $\xrightarrow{\text{int}}_{\mathbf{v}}$ is the binary relation on Λ defined inductively by the following rules:*

$$\frac{(m \in \mathbb{N}) \quad N \rightarrow_{\mathbf{v}} N'}{(\lambda x.N)M_1 \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} (\lambda x.N')M_1 \dots M_m} \lambda \quad \frac{(m \in \mathbb{N}) \quad N \xrightarrow{\text{int}}_{\mathbf{v}} N'}{VNM_1 \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} VN'M_1 \dots M_m} \text{right}$$

$$\frac{(m \in \mathbb{N}^+) \quad M_i \rightarrow_{\mathbf{v}} M'_i \quad \text{for some } 1 \leq i \leq m}{VNM_1 \dots M_i \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} VNM_1 \dots M'_i \dots M_m} @ .$$

The following fact collects many minor properties which can be easily proved by inspection of the rules of Definitions 7-9.

- **Fact 10. 1.** *The head β_v -reduction $\xrightarrow{h}_{\beta_v}$ does not reduce a value (in particular, does not reduce under λ 's), i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \not\xrightarrow{h}_{\beta_v} M$.*
2. *The head σ -reduction \xrightarrow{h}_{σ} does neither reduce a value nor reduce to a value, i.e., for any $M \in \Lambda$ and any $V \in \Lambda_v$, one has $V \not\xrightarrow{h}_{\sigma} M$ and $M \not\xrightarrow{h}_{\sigma} V$.*
 3. *Values are closed under \xrightarrow{int}_v -expansion, i.e., for all $M \in \Lambda$ and $V \in \Lambda_v$, if $M \xrightarrow{int}_v V$ then $M \in \Lambda_v$; more precisely, $M = \lambda x.N$ and $V = \lambda x.N'$ for some $N, N' \in \Lambda$ where $N \rightarrow_v N'$.*
 4. *If $R \in \{\xrightarrow{h}_{\beta_v}, \xrightarrow{h}_{\sigma}, \xrightarrow{h}_v, \xrightarrow{int}_v\}$ and $M R M'$, then $MN R M'N$ for any $N \in \Lambda$.*

Clearly, $\xrightarrow{int}_v \subseteq \rightarrow_v$. Next Proposition 11 (whose proof uses Fact 10.4) relates \xrightarrow{int}_v and \xrightarrow{h}_v .

- **Proposition 11.** *One has $\xrightarrow{int}_v = \rightarrow_v \setminus \xrightarrow{h}_v$.*

Proof.

⊆: The proof that $\xrightarrow{int}_v \subseteq \rightarrow_v$ is trivial. The proof that $M \xrightarrow{int}_v M'$ implies $M \not\xrightarrow{h}_v M'$ is by induction on the derivation of $M \xrightarrow{int}_v M'$. Let us consider its last rule r . If $r \in \{\lambda, @\}$, then it is evident that there is no last rule to derive $M \xrightarrow{h}_v M'$. If $r = \textit{right}$ then $M = VNM_1 \dots M_m$ and $M' = VN'M_1 \dots M_m$ with $m \in \mathbb{N}$ and $N \xrightarrow{int}_v N'$; by induction hypothesis, $N \not\xrightarrow{h}_v N'$ and hence there is no last rule to derive $M \xrightarrow{h}_v M'$.

⊇: We show that $M \rightarrow_v M'$ and $M \not\xrightarrow{h}_v M'$ implies $M \xrightarrow{int}_v M'$, for all $M, M' \in \Lambda$. Since $M \rightarrow_v M'$, there exist a context C and terms N and N' such that $M = C(N)$, $M' = C(N')$ and $N \mapsto_{\beta_v} N'$. We proceed by induction on C .

If $C = (\cdot)$ then $M = N \mapsto_{\beta_v} N' = M'$ and thus $M \xrightarrow{h}_v M'$ since $\mapsto_{\beta_v} \subseteq \xrightarrow{h}_v$, which contradicts the hypothesis.

If $C = \lambda x.C'$ for some context C' , then $M \xrightarrow{int}_v M'$ by applying the rule λ for \xrightarrow{int}_v , since $C'(N) \rightarrow_v C'(N')$.

If $C = C'L$ for some context C' and term L , then $C'(N) \rightarrow_v C'(N')$ and $C'(N') \not\xrightarrow{h}_v C'(N')$ (by Fact 10.4, since $C'(N)L \not\xrightarrow{h}_v C'(N')L$). By induction hypothesis, $C'(N) \xrightarrow{int}_v C'(N')$, then $M = C'(N)L \xrightarrow{int}_v C'(N')L = M'$ by Fact 10.4.

If $C = VC'$ for some context C' and value V , then $C'(N) \rightarrow_v C'(N')$. There are two cases:

- either $C'(N') \xrightarrow{h}_v C'(N')$, hence $M = VC'(N) \xrightarrow{h}_v VC'(N') = M'$ by the rule *right* for $\xrightarrow{h}_{\beta_v}$ or \xrightarrow{h}_{σ} , which contradicts the hypothesis;
- or $C'(N') \not\xrightarrow{h}_v C'(N')$, hence $C'(N') \xrightarrow{int}_v C'(N')$ by induction hypothesis, thus $M = VC'(N) \xrightarrow{int}_v VC'(N') = M'$ by applying the rule *right* for \xrightarrow{int}_v .

Finally, if $C = LC'$ for some context C' and term $L \notin \Lambda_v$, then $L = VN_0 \dots N_n$ for some $n \in \mathbb{N}$, thus $M = VN_0 \dots N_n C'(N) \xrightarrow{int}_v VN_0 \dots N_n C'(N') = M'$ by the rule $@$ for \xrightarrow{int}_v . ◀

We end this section by recalling three results proven in [7] concerning head v -reduction and internal v -reduction: they will be used to prove the main results in Sections 4-5.

The following lemma (proven in [7, Lemma 14]) shows that a head σ -reduction step can be postponed after a head β_v -reduction step, and hence every head v -reduction sequence can be rearranged into a head β_v -reduction sequence followed by a head σ -reduction sequence.

- **Lemma 12** (Commutation of head β_v - and head σ -reductions, see [7]).

1. *If $M \xrightarrow{h}_{\sigma} L \xrightarrow{h}_{\beta_v} N$ then there exists $L' \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}_{\sigma} N$.*
2. *If $M \xrightarrow{h}_v^* M'$ then there exists $N \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^* N \xrightarrow{h}_{\sigma}^* M'$.*

Next Lemma 13 (proven in [7, Corollary 21]) says that internal \mathbf{v} -reduction can be shifted after head \mathbf{v} -reductions.²

► **Lemma 13** (Postponement, see [7]). *If $M \xrightarrow{\mathbf{v}}^{\text{int}} L$ and $L \xrightarrow{\beta_{\mathbf{v}}} N$ (resp. $L \xrightarrow{\sigma} N$), then there exist $L', L'' \in \Lambda$ such that $M \xrightarrow{\beta_{\mathbf{v}}}^+ L' \xrightarrow{\sigma}^* L'' \xrightarrow{\mathbf{v}}^{\text{int}*} N$ (resp. $M \xrightarrow{\beta_{\mathbf{v}}}^* L' \xrightarrow{\sigma}^* L'' \xrightarrow{\mathbf{v}}^{\text{int}*} N$).*

Next Theorem 14 is one of the main result proven in [7, Theorem 22] by adapting Takahashi's method [19, 5]: any \mathbf{v} -reduction sequence can be sequentialized into a head $\beta_{\mathbf{v}}$ -reduction sequence followed by a head σ -reduction sequence, followed by an internal \mathbf{v} -reduction sequence. In ordinary λ -calculus, the well-known result corresponding to our Theorem 14 states that a β -reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [19, Corollary 2.6]).

► **Theorem 14** (Sequentialization, see [7]). *If $M \xrightarrow{\mathbf{v}}^* M'$ then there exist $L, N \in \Lambda$ such that $M \xrightarrow{\beta_{\mathbf{v}}}^* L \xrightarrow{\sigma}^* N \xrightarrow{\mathbf{v}}^{\text{int}*} M'$.*

The sequentialization of Theorem 14 imposes no order between head σ -reductions. Indeed, the example in [7, p. 10] shows that it is impossible to sequentialize them by giving way to head σ_1 - or head σ_3 -redexes: a head σ_1 -reduction step can create a head σ_3 -redex, and vice versa.

In [7, Definition 27 and Corollary 29] it has also been proven that the \mathbf{v} -equivalence (and in particular the σ -equivalence) is contained in the call-by-value observational equivalence.

4 Head normalization

In this section we prove the first main result of our paper: Theorem 21, which studies the normalization for head \mathbf{v} -reduction and relates it to the head $\beta_{\mathbf{v}}$ -reduction (i.e. the weak evaluation strategy for Plotkin's $\lambda_{\mathbf{v}}$ -calculus). Let us start with a preliminary remark.

► **Remark 15.** According to Facts 10.1-2, every $V \in \Lambda_{\mathbf{v}}$ is head $\beta_{\mathbf{v}}$ - and head σ -normal, and hence is head \mathbf{v} -normal. The converse does not hold: xI is head \mathbf{v} -normal but $xI \notin \Lambda_{\mathbf{v}}$.

First, we give a syntactic characterization of head \mathbf{v} - and head $\beta_{\mathbf{v}}$ -normal forms.

► **Definition 16.** *We define the subsets Λ_a , Λ_b and Λ_c (whose elements are denoted by A , B and C respectively) of Λ as follows (for any variable x , any $V \in \Lambda_{\mathbf{v}}$ and any $N \in \Lambda$):*

$$(\Lambda_a) \quad A ::= xV \mid xA \mid AN \quad (\Lambda_b) \quad B ::= (\lambda x.N)A \quad (\Lambda_c) \quad C ::= xV \mid VC \mid CN$$

Notice that $\Lambda_a \cup \Lambda_b \subsetneq \Lambda_c$ and $M, N \in \Lambda_c \setminus (\Lambda_a \cup \Lambda_b)$ where $M = (\lambda y.\Delta)(xI)\Delta$ and $N = \Delta((\lambda y.\Delta)(xI))$ (as in Example 6). Moreover, $\Lambda_{\mathbf{v}} \cap \Lambda_a = \Lambda_{\mathbf{v}} \cap \Lambda_b = \Lambda_{\mathbf{v}} \cap \Lambda_c = \Lambda_a \cap \Lambda_b = \emptyset$ and all terms in $\Lambda_a \cup \Lambda_b \cup \Lambda_c$ are not closed. All terms in Λ_b are β -redexes that are not $\beta_{\mathbf{v}}$ -redexes; all terms in Λ_a have a free ‘‘head variable’’ and are neither a value nor a β -redex.

► **Proposition 17** (Characterization of head $\beta_{\mathbf{v}}$ -normal forms). *Let M be a term.*

1. M is head $\beta_{\mathbf{v}}$ -normal and is not a λ -value if and only if $M \in \Lambda_c$.
2. M is head $\beta_{\mathbf{v}}$ -normal if and only if $M \in \Lambda_{\mathbf{v}} \cup \Lambda_c$.

Proof. Statement (2) is an immediate consequence of statement (1) and Remark 15.

² In [7, Corollary 21] there is a more informative statement of our Lemma 13, involving a notion of internal parallel reduction $\xrightarrow{\mathbf{v}}^{\text{int}}$. Our Lemma 13 follows immediately from [7, Corollary 21] since $\xrightarrow{\mathbf{v}}^{\text{int}} \subseteq \xrightarrow{\mathbf{v}}^{\text{int}*} \subseteq \xrightarrow{\mathbf{v}}^{\text{int}*}$.

- \Rightarrow : We prove the left-to-right direction of statement (1), by induction on $M \in \Lambda$.
 The case where $M \in \Lambda_v$ is impossible by hypothesis.
 If $M = M_1 M_2$ (for some $M_1, M_2 \in \Lambda$) is head β_v -normal then M is not a λ -value and M_1 and M_2 are head β_v -normal, moreover either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v -redex). Therefore, there are only three cases:
- either $M_1 \notin \Lambda_v$, thus $M_1 \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$;
 - or $M_1 \in \Lambda_v$ and $M_2 \notin \Lambda_v$, so $M_2 \in \Lambda_c$ by induction hypothesis, and thus $M \in \Lambda_c$;
 - or M_1 is a variable and $M_2 \in \Lambda_v$, hence $M \in \Lambda_c$ (this is the base case).
- \Leftarrow : The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_c$. \blacktriangleleft

A consequence of Proposition 17 is that all closed head β_v -normal forms are abstractions.

► **Proposition 18** (Characterization of head v-normal forms). *Let $M \in \Lambda$.*

1. *M is head v-normal and is neither a λ -value nor a β -redex if and only if $M \in \Lambda_a$.*
2. *M is head v-normal and is a β -redex if and only if $M \in \Lambda_b$.*
3. *M is head v-normal if and only if $M \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$.*

Proof. Statement (3) is an immediate consequence of statements (1)-(2) and Remark 15.

- \Rightarrow : We prove simultaneously the left-to-right direction of statements (1) and (2), by induction on $M \in \Lambda$. The case where $M \in \Lambda_v$ is impossible by hypothesis.
 If $M = M_1 M_2$ (for some $M_1, M_2 \in \Lambda$) is head v-normal then M is not a λ -value and M_1 and M_2 are head v-normal, moreover M_1 is not a β -redex (otherwise M would be a head σ_1 -redex), and either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v -redex), and either $M_1 \notin \Lambda_v$ or M_2 is not a β -redex (otherwise M would be a head σ_3 -redex). There are only three cases:
- either M_1 is a variable and M_2 is not a β -redex, so M is not a β -redex; if $M_2 \in \Lambda_v$ then $M \in \Lambda_a$ (this is the base case); otherwise $M_2 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
 - or $M_1 \notin \Lambda_v$, thus M is not a β -redex and $M_1 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
 - or $M_1 = \lambda x.N$ for some $N \in \Lambda$ and M_2 is neither a λ -value nor a β -redex, so M is a β -redex, furthermore $M_2 \in \Lambda_a$ by induction hypothesis, and thus $M \in \Lambda_b$.
- \Leftarrow : The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_a$.
 Let us prove the right-to-left direction of statement (2): if $M \in \Lambda_b$ then $M = (\lambda x.N)A$ for some $N \in \Lambda$ and $A \in \Lambda_a$, thus M is a β -redex. For any $M' \in \Lambda$, the last rule of the derivation of $M \xrightarrow{h}_v M'$ might be neither σ_1 nor σ_3 (because A is not a β -redex by statement 1) nor β_v (because $A \notin \Lambda_v$ by statement 1 again) nor *right* (because A is head v-normal, by statement 1 again). Therefore, M is head v-normal. \blacktriangleleft

As a consequence of Proposition 18, all closed head v-normal forms are abstractions.

The sets of terms Λ_a , Λ_b and Λ_c of Definition 16 enjoy the closure properties summarized in Lemma 19 below. Together with the syntactic characterizations of head β_v -normal forms (Proposition 17) and head v-normal forms (Proposition 18), these closure properties allow one to reason about head v-reduction in spite of its non-confluence: they will be used to prove our main results, Theorems 21 and 24 and Proposition 27.

- **Lemma 19** (Closure properties). **1.** *The set Λ_a is closed under v-internal reduction and expansion, i.e., for any $N' \in \Lambda$ and $N \in \Lambda_a$, if $N' \xrightarrow{int}_v N$ or $N \xrightarrow{int}_v N'$ then $N' \in \Lambda_a$.*
- 2.** *The set Λ_b is closed under v-internal reduction and expansion, i.e., for any $N' \in \Lambda$ and $N \in \Lambda_b$, if $N' \xrightarrow{int}_v N$ or $N \xrightarrow{int}_v N'$ then $N' \in \Lambda_b$.*
- 3.** *Head v-normal forms are closed under v-internal reduction and expansion, i.e., for any $N, N' \in \Lambda$ where N is head v-normal, if $N' \xrightarrow{int}_v N$ or $N \xrightarrow{int}_v N'$ then N' is head v-normal.*

4. *Head β_v -normal forms are closed under head σ -reduction and expansion, i.e., for any $N, N' \in \Lambda$ where N is head β_v -normal, if $N' \xrightarrow{h}_\sigma N$ or $N \xrightarrow{h}_\sigma N'$ then N' is head β_v -normal.*

Proof. 1. We show that if $N \in \Lambda_a$ and $N' \xrightarrow{int}_v N$ (resp. $N \xrightarrow{int}_v N'$) then $N' \in \Lambda_a$, by induction on the derivation of $N' \xrightarrow{int}_v N$ (resp. $N \xrightarrow{int}_v N'$). Let us consider its last rule r . Since $N \in \Lambda_a$ (see Definition 16), $N = xLN_1 \dots N_n$ for some $n \in \mathbb{N}$, some variable x , some $L \in \Lambda_v \cup \Lambda_a$ and some $N_1, \dots, N_n \in \Lambda$, thus $r \neq \lambda$ and hence either $r = right$ or $r = @$. If $r = right$ then $N' = xL'N_1 \dots N_n$ where $L' \xrightarrow{int}_v L$ (resp. $L \xrightarrow{int}_v L'$). Since $L \in \Lambda_v \cup \Lambda_a$, there are two cases:

- either $L \in \Lambda_a$ and then $L' \in \Lambda_a$ by induction hypothesis, so $N' = xL'N_1 \dots N_n \in \Lambda_a$;
- or $L \in \Lambda_v$ and then $L' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\xrightarrow{int}_v \subseteq \rightarrow_v$), therefore $N' = xL'N_1 \dots N_n \in \Lambda_a$.

Finally, if $r = @$ then $n \in \mathbb{N}^+$ and $N' = xLN_1 \dots N'_i \dots N_n$ for some $1 \leq i \leq n$ with $N'_i \rightarrow_v N_i$ (resp. $N_i \rightarrow_v N'_i$), hence $N' \in \Lambda_a$ because $xL \in \Lambda_a$.

2. We show that if $N \in \Lambda_b$ and $N' \xrightarrow{int}_v N$ (resp. $N \xrightarrow{int}_v N'$) then $N' \in \Lambda_b$, by induction on the derivation of $N' \xrightarrow{int}_v N$ (resp. $N \xrightarrow{int}_v N'$). Let us consider its last rule r . Since $N \in \Lambda_b$, then $N = (\lambda x.M)A$ for some $M \in \Lambda$ and $A \in \Lambda_a$, hence $r \neq @$ because N has not the shape $VLN_1 \dots N_m$ for any $m \in \mathbb{N}^+$; therefore either $r = \lambda$ or $r = right$:
- if $r = \lambda$, then $N' = (\lambda x.M')A$ where $M' \rightarrow_v M$ (resp. $M \rightarrow_v M'$), hence $N' \in \Lambda_b$;
 - if $r = right$, then $N' = (\lambda x.M)A'$ where $A' \xrightarrow{int}_v A$ (resp. $A \xrightarrow{int}_v A'$), thus $A' \in \Lambda_a$ by Lemma 19.1, hence $N' \in \Lambda_b$;
3. Thanks to Proposition 18.3, it is sufficient to show that if $N \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ and $N' \xrightarrow{int}_v N$ (resp. $N \xrightarrow{int}_v N'$) then $N' \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. If $N \in \Lambda_v$ then $N' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\xrightarrow{int}_v \subseteq \rightarrow_v$). If $N \in \Lambda_a$ then $N' \in \Lambda_a$ by Lemma 19.1. Finally, if $N \in \Lambda_b$ then $N' \in \Lambda_b$ by Lemma 19.2.
4. By Proposition 17.2, $N \in \Lambda_v \cup \Lambda_c$. Since $M \xrightarrow{h}_\sigma N$ or $N \xrightarrow{h}_\sigma M$, $N \notin \Lambda_v$ by Fact 10.2. We prove by induction on $N \in \Lambda_c$ that $M \in \Lambda_c$. By Definition 16, there are only two cases:
- either $N = xVN_1 \dots N_n$ for some $n \in \mathbb{N}$, variable x , $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$, but this is impossible since the last rule of the derivation of $M \xrightarrow{h}_\sigma N$ or $N \xrightarrow{h}_\sigma M$ can be neither σ_1 nor σ_3 (because of the subterm xV) nor $right$ (because of Fact 10.2);
 - or $N = VLN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$, $L \in \Lambda_c$ and $N_1, \dots, N_n \in \Lambda$, and then there are three sub-cases, depending on the last rule r of the derivation of $M \xrightarrow{h}_\sigma N$ (resp. $N \xrightarrow{h}_\sigma M$):
 - if $r = \sigma_1$ then $V = \lambda x.N'N_0$ (resp. $\lambda x.N'$) and $M = (\lambda x.N')LN_0 \dots N_n$ (resp. $M = (\lambda x.N'N_1)LN_2 \dots N_n$ with $n > 0$) for some $N', N_0 \in \Lambda$, hence $M \in \Lambda_c$;
 - if $r = \sigma_3$ then $V = \lambda x.V'N'$ (resp. $L = (\lambda x.N')L'$) and $M = V'((\lambda x.N')L)N_1 \dots N_n$ (resp. $M = (\lambda x.VN')L'N_1 \dots N_n$) for some $V' \in \Lambda_v$ (resp. $L' \in \Lambda_c$) and $N' \in \Lambda$, thus $(\lambda x.N')L \in \Lambda_c$ (resp. $(\lambda x.VN')L' \in \Lambda_c$) and hence $M \in \Lambda_c$;
 - if $r = right$ then $M = VL'N_1 \dots N_n$ for some $L' \in \Lambda$ such that $L' \xrightarrow{h}_\sigma L$ (resp. $L \xrightarrow{h}_\sigma L'$), so $L' \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$.

◀

Lemma 19.4 is a formalization of the two following facts: (a) a head σ -reduction step may create a new β_v -redex but in this case it is not a head β_v -redex; (b) when $M \xrightarrow{h}_\sigma N$, the head β_v -redex of M (if any) has a residual in N which is the head β_v -redex of N .

► **Lemma 20.** *There exists no infinite head v -reduction sequence with finitely many head β_v -reduction steps.*

Proof. Suppose the opposite holds: then there would exist $m \in \mathbb{N}$ and an infinite sequence of terms $(M_i)_{i \in \mathbb{N}}$ such that $M_i \xrightarrow{h}_v M_{i+1}$ for any $1 \leq i \leq m$, $M_m \xrightarrow{h}_{\beta_v} M_{m+1}$ and $M_i \xrightarrow{h}_\sigma M_{i+1}$ for any $i > m$ (since $\xrightarrow{h}_v = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma$). But this is impossible because \xrightarrow{h}_σ is strongly normalizing (by Proposition 5 and since $\xrightarrow{h}_\sigma \subseteq \rightarrow_\sigma$). Contradiction. \blacktriangleleft

Now we can state and prove our main result about head β_v - and head v -normalization.

► **Theorem 21 (Head normalization).** *Let $M \in \Lambda$. The following are equivalent:*

1. *there exists a head β_v -normal form N such that $M \simeq_{\beta_v} N$;*
2. *there exists a head v -normal form N such that $M \simeq_v N$;*
3. *M is head v -normalizable;*
4. *M is head β_v -normalizable;*
5. *there is no v -reduction sequence from M with infinitely many head β_v -reduction steps;*
6. *M is strongly head v -normalizable.*

Proof.

- (1) \Rightarrow (2) By hypothesis, there exists a head β_v -normal $N \in \Lambda$ such that $M \simeq_{\beta_v} N$, thus $M \simeq_v N$. Since \xrightarrow{h}_σ is strongly normalizing (by Proposition 5 and because $\xrightarrow{h}_\sigma \subseteq \rightarrow_\sigma$), there exists a head σ -normal $N' \in \Lambda$ such that $N \xrightarrow{h}_\sigma^* N'$, therefore $M \simeq_v N'$ since $\xrightarrow{h}_\sigma \subseteq \rightarrow_v$. By Lemma 19.4, N' is also head β_v -normal and hence head v -normal.
- (2) \Rightarrow (3) Since $M \simeq_v N$, there is $L \in \Lambda$ such that $M \rightarrow_v^* L$ and $N \rightarrow_v^* L$, by confluence of \rightarrow_v (Proposition 5). By Theorem 14, there are $M_1, M_2, N_1, N_2 \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^* M_1 \xrightarrow{h}_\sigma^* M_2 \xrightarrow{h}_v^* L$ and $N \xrightarrow{h}_{\beta_v}^* N_1 \xrightarrow{h}_\sigma^* N_2 \xrightarrow{h}_v^* L$. As N is head v -normal, $N = N_1 = N_2 \xrightarrow{h}_v^* L$. By Lemma 19.3, L and M_2 are v -head normal. So, $M \xrightarrow{h}_v^* M_2$ with M_2 head v -normal.
- (3) \Rightarrow (4) By hypothesis, there is $N \in \Lambda$ head v -normal such that $M \xrightarrow{h}_v^* N$. By Lemma 12.2, there is $L \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_\sigma^* N$. Since N is head v -normal and in particular head β_v -normal, L is head β_v -normal according to Lemma 19.4. So M is head β_v -normalizable.
- (4) \Rightarrow (5) Lemma 12.1 says that if $N \xrightarrow{h}_\sigma L \xrightarrow{h}_{\beta_v} N'$ then there exists $L' \in \Lambda$ such that $N \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}_\sigma N'$; Lemma 13 and Fact 10.3 show that if $N \xrightarrow{h}_v L \xrightarrow{h}_{\beta_v} N'$ then there exist $L', L'' \in \Lambda$ such that $N \xrightarrow{h}_{\beta_v}^+ L' \xrightarrow{h}_\sigma^* L'' \xrightarrow{h}_v^* N'$. Since $\rightarrow_v = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma \cup \xrightarrow{h}_v$, this means that if there is an infinite v -reduction sequence from M with infinitely many head β_v -reduction steps, then for any $n \in \mathbb{N}$ there is a head β_v -reduction sequence from M whose length is at least n . Therefore, M is not head β_v -normalizable, since the head β_v -reduction is deterministic.
- (5) \Rightarrow (6) If M is not strongly head v -normalizable then there exists an infinite head v -reduction sequence. By Lemma 20, this head v -reduction (and hence v -reduction, since $\xrightarrow{h}_v \subseteq \rightarrow_v$) sequence has infinitely many head β_v -reduction steps.
- (6) \Rightarrow (1) As M is strongly head v -normalizable, in particular is head v -normalizable, hence there exists $N \in \Lambda$ head v -normal and in particular head β_v -normal such that $M \xrightarrow{h}_v^* N$. By Lemma 12.2, there exists $L \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_\sigma^* N$. Therefore $M \simeq_{\beta_v} L$ since $\xrightarrow{h}_{\beta_v} \subseteq \rightarrow_{\beta_v}$. According to Lemma 19.4, L is head β_v -normal. \blacktriangleleft

In Theorem 21, the equivalence (3) \Leftrightarrow (6) means that (weak) normalization and strong normalization are equivalent for head v -reduction (for head β_v -reduction they are trivially equivalent since the head β_v -reduction is deterministic), therefore if one is interested in studying the termination of head v -reduction, no difficulty arises from its non-determinism. The equivalence (4) \Leftrightarrow (3) or (4) \Leftrightarrow (6) says that the weak evaluation process defined for Plotkin's λ_v -calculus (the head β_v -reduction) terminates if and only if the weak evaluation process defined for λ_v^σ (the head v -reduction) terminates: σ -rules play no role in deciding the

termination of a head \mathbf{v} -reduction sequence. The equivalence (3) \Leftrightarrow (2) (resp. (4) \Leftrightarrow (1)) is the version for λ_v^σ (resp. λ_v) of a well-known theorem for ordinary λ -calculus (see for example [3, Theorem 8.3.11]): in some sense, it claims that the head \mathbf{v} -reduction (resp. head β_v -reduction) is complete with respect to the \mathbf{v} -equivalence (resp. β_v -equivalence). The equivalence (5) \Leftrightarrow (2) (resp. (5) \Leftrightarrow (1)) can be seen as the version for λ_v^σ (resp. λ_v) of the Quasi-Head Reduction Theorem [19, Theorem 2.10] stated by Takahashi for ordinary λ -calculus.

5 Normalization strategy and other results

Theorems 14 and 21 strengthen the idea that, in spite of non-determinism and non-confluence of head \mathbf{v} -reduction and non-sequentiability of head σ -reduction steps, the head \mathbf{v} -reduction can be used to define a normalization strategy for the λ_v^σ -calculus, as proven in next Theorem 24, the second main result of our paper: given a term M , one starts the (unique) head β_v -head reduction sequence from M as long as a head β_v -normal form N is reached (recall that, according to Theorem 21, a term is (strongly) head \mathbf{v} -normalizable if and only if it is head β_v -normalizable); then, one starts a head σ -reduction sequence from N (where head σ_1 - and head σ_3 -reduction steps can be performed in whatever order) as long as a head σ -normal form N' is reached (such a N' always exists because \xrightarrow{h}_σ is strongly normalizing, and it is head \mathbf{v} -normal by Lemma 19.4); finally, one performs the internal \mathbf{v} -reduction steps starting from N' by iterating the head β_v -reduction sequences and then the head σ -reduction sequences as above on the subterms of N' , from the left to the right. More precisely:

► **Definition 22** (Successors path). *Let $M \in \Lambda$.*

A successor of M is a $M' \in \Lambda$ defined by induction on $M \in \Lambda$ as follows:

- *if M is not head β_v -normal, then M' is such that $M \xrightarrow{h}_{\beta_v} M'$;*
- *if M is head β_v -normal but not head σ -normal, then M' is such that $M \xrightarrow{h}_\sigma M'$;*
- *if M is head \mathbf{v} -normal then:*
 - *if M is a variable then $M' = M$,*
 - *if $M = \lambda x.N$ for some $N \in \Lambda$, then $M' = \lambda x.N'$ for some successor N' of N ,*
 - *if $M = NL$ for some $N, L \in \Lambda$, then either N is not \mathbf{v} -normal and $M' = N'L$ where N' is a successor of N , or N is \mathbf{v} -normal and $M' = NL'$ where L' is a successor of L .*

A successors path of M is an infinite sequence $(M_i)_{i \in \mathbb{N}}$ of terms such that $M_0 = M$ and M_{i+1} is a successor of M_i , for any $i \in \mathbb{N}$.

Clearly, for every term M there is at least one successor M' of M ; moreover, this successor M' is unique when M is not head β_v -normal, since the head β_v -reduction is deterministic, and $M = M'$ when M is \mathbf{v} -normal.

► **Remark 23.** *Let $M \in \Lambda$ and let $(M_i)_{i \in \mathbb{N}}$ be a successors path of M .*

1. *For every $i \in \mathbb{N}$, there exist $0 \leq j \leq k \leq i$ such that $M \xrightarrow{h}_{\beta_v}^* M_j \xrightarrow{h}_\sigma^* M_k \xrightarrow{int}_{\mathbf{v}}^* M_i$.*
2. *For every $i \in \mathbb{N}$, if M_i is \mathbf{v} -normal then M_j is \mathbf{v} -normal for any $j \geq i$.*

A successors path of a term M is a call-by-value left-to-right \mathbf{v} -evaluation strategy starting from M that can reduce under a λ only when a head \mathbf{v} -normal form is reached. Due to the non-determinism of the head σ -reduction, a term M may have several successors paths. We cannot get rid of the non-determinism of the successors path of M because of the non-sequentiability of head σ -reductions, see p. 7 and [7, p. 10].

► **Theorem 24** (Normalization strategy). *Let $M \in \Lambda$. Every successors path $(M_i)_{i \in \mathbb{N}}$ of M is a normalization strategy for M , i.e. if M is \mathbf{v} -normalizable then there exists $j, k, \ell \in \mathbb{N}$ such that $j \leq k \leq \ell$, M_j is head β_v -normal, M_k is head \mathbf{v} -normal and M_ℓ is \mathbf{v} -normal.*

Proof. Let $(M_i)_{i \in \mathbb{N}}$ be a successors path of M and $N \in \Lambda$ be such that N is \mathbf{v} -normal and $M \rightarrow_v^* N$: we prove by induction on $N \in \Lambda$ that there exist $j, k, \ell \in \mathbb{N}$ such that M_j is head β_v -normal, M_k is head \mathbf{v} -normal and M_ℓ is \mathbf{v} -normal.

Since M is \mathbf{v} -normalizable, then it is head β_v -normalizable (because $\xrightarrow{\beta_v} \subseteq \rightarrow_v$), thus there exists $j \in \mathbb{N}$ such that M_j is head β_v -normal because $\xrightarrow{\beta_v}$ is deterministic. As $\xrightarrow{\beta_v}$ is strongly normalizing (by Proposition 5, since $\xrightarrow{\beta_v} \subseteq \rightarrow_v$), there exists $k \in \mathbb{N}$ with $j \leq k$ such that M_k is head σ -normal. According to Lemma 19.4, M_k is also head β_v -normal, hence M_k is head \mathbf{v} -normal. Certainly, $M_k = VN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$. By confluence of \rightarrow_v (Proposition 5) and since N is \mathbf{v} -normal and M_k is head \mathbf{v} -normal, one has $M_k \xrightarrow{int}_v^* N$ and hence $N = V'N'_1 \dots N'_n$ for some \mathbf{v} -normal $V' \in \Lambda_v$ and some \mathbf{v} -normal $N'_1, \dots, N'_n \in \Lambda$ such that $V \rightarrow_v^* V'$ and $N_r \rightarrow_v^* N'_r$ for any $1 \leq r \leq n$. By induction hypothesis, for every successors path $(V_i)_{i \in \mathbb{N}}$ of V and, for any $1 \leq r \leq n$, for every successors path $(L_i^r)_{i \in \mathbb{N}}$ of N^r there exist $p, p_1, \dots, p_n \in \mathbb{N}$ such that $V_p, L_{p_1}^1, \dots, L_{p_n}^n$ are \mathbf{v} -normal: by confluence of \rightarrow_v (Proposition 5), $V_p = V'$ and $N'_r = L_{p_r}^r$ for any $1 \leq r \leq n$.

Let us consider the infinite sequence of terms $s = (M = M_0, \dots, M_k = VN_1 \dots N_n = V_0N_1 \dots N_n, \dots, V_pN_1 \dots N_n = V'L_0^1N_2 \dots N_n, \dots, V'L_{p_1}^1N_2 \dots N_n = V'N'_1L_0^2 \dots N_n, \dots, V'N'_1N'_2 \dots N'_n = N, N, \dots)$: this is a successors path of M and, for an opportune choice of the successors paths $(V_i)_{i \in \mathbb{N}}$, $(L_i^1)_{i \in \mathbb{N}}$, \dots , $(L_i^n)_{i \in \mathbb{N}}$, one has that $s = (M_i)_{i \in \mathbb{N}}$, in particular there exists $\ell \in \mathbb{N}$ such that $j \leq k \leq \ell$ and $M_\ell = N$. \blacktriangleleft

In ordinary λ -calculus, the well-known theorem corresponding to our Theorem 24 is the Leftmost Reduction Theorem, see [19, Theorem 2.8] or [3, Theorem 13.2.2]. Differently from the leftmost reduction of ordinary λ -calculus, our normalization strategy is not deterministic, i.e., our Theorem 24 provides a family of normalization strategies.

Finally, we have shown at p. 5 that the head σ - and head \mathbf{v} -reductions are not (locally) confluent and a term may have several head \mathbf{v} -normal forms. Nevertheless, the characterization of head \mathbf{v} -normal forms given by Proposition 18 allows us to claim that (see next Proposition 27) in some cases (of interest), more precisely when a term has a head \mathbf{v} -normal form which is a value or an element of Λ_a , the head \mathbf{v} -normal form is unique (Proposition 27.1): all terms having several head \mathbf{v} -normal forms are such that all their head \mathbf{v} -normal forms are in Λ_b . In particular, every head \mathbf{v} -normalizable closed term has a unique head \mathbf{v} -normal form, which is an abstraction and coincides with its head β_v -normal form (Proposition 27.2).

► **Remark 25.** By inspection on the rules of Definition 8, it is easy to check that the head σ -reduction does not reduce to a term in Λ_a , i.e., for any $M \in \Lambda$ and $N \in \Lambda_a$, one has $M \not\rightarrow_\sigma N$.

Remark 25 does not hold if we replace $\xrightarrow{\sigma}$ with $\xrightarrow{\beta_v}$: for instance, $x(II) \xrightarrow{\beta_v} xI \in \Lambda_a$.

► **Fact 26.** For every $N \in \Lambda_v \cup \Lambda_a$, one has $M \xrightarrow{\beta_v}^* N$ if and only if $M \xrightarrow{v}^* N$.

Proof. The left-to-right direction follows from $\xrightarrow{\beta_v} \subseteq \xrightarrow{v}$. The right-to-left direction is a consequence of Lemma 12.2 and either Fact 10.2 (if $N \in \Lambda_v$) or Remark 25 (if $N \in \Lambda_a$). \blacktriangleleft

Fact 26 means that, given a head \mathbf{v} -reduction sequence, the head σ -reduction plays no role not only in deciding its termination (as stated in Theorem 21), but also in reaching a particular value or term in Λ_a . Fact 26 will be used in the proof of Proposition 27.

► **Proposition 27** (Uniqueness of “some” head \mathbf{v} -normal forms). *Let $M \in \Lambda$ and $M \xrightarrow{v}^* N$.*

1. *If $N \in \Lambda_v \cup \Lambda_a$ then, for every head \mathbf{v} -normal $L \in \Lambda$, $M \xrightarrow{v}^* L$ implies $N = L$.*
2. *If M is closed and N is head \mathbf{v} -normal, then $M \xrightarrow{\beta_v}^* N$ and $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\text{fv}(N') \subseteq \{x\}$; moreover, for any head \mathbf{v} -normal $L \in \Lambda$, $M \xrightarrow{v}^* L$ implies $N = L$.*

Proof. 1. Since $N \in \Lambda_v \cup \Lambda_a$, $M \xrightarrow{h}_v^* N$ implies $M \xrightarrow{h}_{\beta_v}^* N$ by Fact 26. According to Proposition 18.3, N is head v -normal.

Let $L \in \Lambda$ be head v -normal and such that $M \xrightarrow{h}_v^* L$: by Proposition 18.3, $L \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. We claim that $L \notin \Lambda_b$. Otherwise, $L \in \Lambda_b$ and then, by confluence of \rightarrow_v there would exist $M' \in \Lambda$ such that $N \rightarrow_v^* M'$ and $L \rightarrow_v^* M'$. According to Proposition 11 and since N and L are head v -normal, $N \xrightarrow{int}_v^* M'$ and $L \xrightarrow{int}_v^* M'$. By Remark 4 (since $\xrightarrow{int}_v \subseteq \rightarrow_v$) and Lemma 19.1, $M' \in \Lambda_v \cup \Lambda_a$. By Lemma 19.2, $M' \in \Lambda_b$. But $\Lambda_v \cap \Lambda_b = \emptyset = \Lambda_a \cap \Lambda_b$: contradiction, therefore $L \notin \Lambda_b$.

So, $L \in \Lambda_v \cup \Lambda_a$ and thus $M \xrightarrow{h}_{\beta_v}^* L$ by Fact 26, hence $N = L$ since $\xrightarrow{h}_{\beta_v}$ is deterministic.

2. Since M is closed, N is closed too. Hence, by Proposition 18.3, $N \in \Lambda_v$ (since the terms in $\Lambda_a \cup \Lambda_b$ are not closed) and N is not a variable, therefore $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\text{fv}(N') \subseteq \{x\}$. By Fact 26, $M \xrightarrow{h}_{\beta_v}^* N$. According to Proposition 27.1, for every head v -normal $L \in \Lambda$, $M \xrightarrow{h}_v^* L$ implies $N = L$. ◀

Recall that all head v -normal terms are head β_v -normal, since $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_v$.

6 Conclusions and future work

In this paper, we have investigated the λ_v^σ -calculus introduced in [4], an extension of Plotkin's call-by-value λ -calculus λ_v [15] with the same syntax as λ_v (without constants) and ordinary (i.e. call-by-name) λ -calculus. The peculiarity of λ_v^σ is in its reduction rules: the v -reduction adds to Plotkin's β_v -reduction two commutation rules called σ_1 and σ_3 which unblock “hidden” β_v -redexes. We have studied the head v -reduction, a non-confluent sub-reduction of the v -reduction already introduced in [7]. We now summarize our main contributions:

1. Theorem 21 is about head v -normalization, it shows that:
 - for the head v -reduction, normalization coincides with strong normalization;
 - the head v -reduction is deeply related to Plotkin's deterministic weak evaluation strategy for λ_v (the former terminates if and only if the latter terminates);
 - both head v -reduction and weak evaluation strategy for λ_v enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ -calculus.
2. Theorem 24 is about v -normalization: it proves that a top-down extension of the head v -normalization provides a family of normalization strategies for the (full) v -reduction.
3. Proposition 27 is about the uniqueness of the head v -normal form: it shows that, even if there are terms having several head v -normal forms, in some case of interest (for instance, closed terms) the head v -normal form, if any, is unique.

These results, together with the results proven in [4, 7], show that λ_v^σ is a useful tool to study some theoretical and semantic properties of Plotkin's λ_v -calculus, for instance the notions of call-by-value solvability and potential valuability. This is hard (or impossible) to obtain directly in λ_v because of the “weakness” of Plotkin's β_v -reduction. In the case of ordinary (i.e. call-by-name) λ -calculus, head reduction and solvability are the starting point to investigate separability, semi-separability and Böhm's trees. Hence, it may reasonably be supposed that we have all the ingredients for tackling the question of separability, semi-separability and Böhm's trees in a call-by-value setting. In particular, one may reasonably hope to improve in λ_v^σ the separability theorem already proven by Paolini [12] for λ_v .

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