

# Head reduction and normalization in a call-by-value lambda-calculus

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## Abstract

Recently, a standardization theorem has been proven for a variant of Plotkin’s call-by-value lambda-calculus extended by means of two commutation rules (sigma-reductions): this result was based on a partitioning between head and internal reductions. We study the head normalization for this call-by-value calculus with sigma-reductions and we relate it to the weak evaluation of original Plotkin’s call-by-value lambda-calculus. We give also a (non-deterministic) normalization strategy for the call-by-value lambda-calculus with sigma-reductions.

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## 1 Introduction

The call-by-value  $\lambda$ -calculus ( $\lambda_v$ -calculus or  $\lambda_v$  for short) and the operational machine for its evaluation has been introduced by Plotkin [15] inspired by Landin’s seminal work [9] on the programming language ISWIM and the SECD machine. The  $\lambda_v$ -calculus is a paradigmatic language able to capture two features of many functional programming languages: call-by-value parameter passing policy (parameters are evaluated before being passed) and weak evaluation (the body of a function is evaluated only when parameters are supplied).

The syntax of  $\lambda_v$  is the same as that of the ordinary (i.e. call-by-name)  $\lambda$ -calculus ( $\lambda$  for short), but the reduction rule for  $\lambda_v$ , called  $\beta_v$ , is a restriction of the  $\beta$ -rule for  $\lambda$ :  $\beta_v$  allows the contraction of a redex  $(\lambda x.M)N$  only in case the argument  $N$  is a value, i.e. a variable or an abstraction. Unfortunately, the semantic analysis of the  $\lambda_v$ -calculus has turned out to be more elaborate than that of ordinary  $\lambda$ -calculus. This is due essentially to the “weakness” of (full)  $\beta_v$ -reduction, a fact widely recognized: indeed, there are many proposals of alternative call-by-value  $\lambda$ -calculi extending Plotkin’s one [11, 10, 8, 2, 1]. To have an example of the “weakness” of the rewriting rules of  $\lambda_v$ , it is sufficient to consider that it is impossible to have an internal operational characterization (i.e. one that uses the  $\beta_v$ -reduction) of the semantically meaningful notions of call-by-value solvability and potential valuability, as shown in [13, 14, 2].

In this paper we will study the  $\lambda_v^\sigma$ -calculus ( $\lambda_v^\sigma$  for short), a call-by-value extension of  $\lambda_v$  recently proposed in [4]: it keeps the  $\lambda_v$  (and  $\lambda$ ) syntax and it adds to the  $\beta_v$ -reduction two commutation rules, called  $\sigma_1$  and  $\sigma_3$ , which unblock “hidden”  $\beta_v$ -redexes that are concealed by the “hyper-sequential structure” of terms. The  $\lambda_v^\sigma$ -calculus enjoys some basic properties we expect from a calculus, namely confluence (see [4]) and standardization (see [7]). Moreover,  $\lambda_v^\sigma$  provides elegant characterizations of many semantic properties, e.g. solvability and potential valuability (see [4]), and it is conservative with respect to Plotkin’s  $\lambda_v$ : in particular, [7] shows that the notions of solvability and potential valuability for  $\lambda_v^\sigma$  coincide with those for  $\lambda_v$ .



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The  $\mathbf{v}$ -reduction (i.e. the reduction for  $\lambda_v^\sigma$ ) can be partitioned into head  $\mathbf{v}$ -reduction and internal  $\mathbf{v}$ -reduction; the head  $\mathbf{v}$ -reduction is in turn decomposed into head  $\beta_v$ - and head  $\sigma$ -reduction. The head  $\beta_v$ -reduction is just the deterministic weak evaluation strategy for Plotkin's  $\lambda_v$ -calculus. According to a sequentialization theorem proven in [7, Theorem 22], any  $\mathbf{v}$ -reduction sequence can be sequentialized in an initial head  $\beta_v$ -reduction sequence followed by a head  $\sigma$ -reduction sequence followed by an internal  $\mathbf{v}$ -reduction sequence. Similar well-known results hold for  $\lambda$  and  $\lambda_v$ , and starting from them one can define a normalization strategy for  $\lambda$  and  $\lambda_v$ , i.e. a deterministic reduction strategy that reaches a normal form if and only if one exists: for example the leftmost reduction, see [19, Theorem 2.8] and [3, Theorem 13.2.2].

Is there a normalization strategy for  $\lambda_v^\sigma$ ? Theorem 24, one of the main results of this paper, proves that, starting from the sequentialization theorem mentioned above, a normalization strategy can be defined for  $\lambda_v^\sigma$ , based on the notions of head  $\beta_v$ - and head  $\sigma$ -reductions.

A first difference appears here between  $\lambda_v^\sigma$  and  $\lambda_v$  (or  $\lambda$ ): the normalization strategy for  $\lambda_v^\sigma$  is not deterministic. Indeed, while the head  $\beta_v$ -reduction (or the call-by-name head reduction) is deterministic (i.e. a partial function), the head  $\mathbf{v}$ -reduction is non-deterministic and, still worse, non-confluent and there are terms having several head  $\mathbf{v}$ -normal forms: this might appear disappointing. So, three natural questions arise:

- With respect to head  $\mathbf{v}$ -reduction, do normalization and strong normalization coincide?<sup>1</sup>
- Can we relate the termination of head  $\beta_v$ -reduction and head  $\mathbf{v}$ -reduction?
- Can we characterize the terms having a unique head  $\mathbf{v}$ -normal form?

Our Theorem 21 gives a positive answer to the first two questions. Observe that the lack of any form of confluence for head  $\mathbf{v}$ -reduction requires a more complex reasoning, passing through a syntactic characterization of head  $\beta_v$ - and head  $\mathbf{v}$ -normal forms. Theorem 21 not only shows that the head  $\mathbf{v}$ -reduction and the head  $\beta_v$ -reduction are deeply related (and hence, again,  $\lambda_v^\sigma$  is conservative with respect to  $\lambda_v$ ) but also that both enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary  $\lambda$ -calculus.

Our Proposition 27 gives a partial answer to the third question above: it shows that in some cases (of interest) a head  $\mathbf{v}$ -normalizable term has a unique head  $\mathbf{v}$ -normal form; in particular, every closed head  $\mathbf{v}$ -normalizable term has a unique head  $\mathbf{v}$ -normal form.

So,  $\lambda_v^\sigma$  appears as an extension of Plotkin's  $\lambda_v$ -calculus that enjoys many meaningful conservation properties with respect to  $\lambda_v$  and therefore it is a useful tool for theoretical and semantic investigations about  $\lambda_v$  and call-by-value setting. See also conclusions in Section 6 for further and more precise motivations for this paper and future work.

**Related work** The  $\lambda_v^\sigma$ -calculus has been recently introduced in [4] and further investigated in [7]. It is an extension of Plotkin's  $\lambda_v$ -calculus inspired by the call-by-value translation of  $\lambda$ -terms into linear logic proof-nets [6]. Other variants of  $\lambda_v$  have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [11], Felleisen and Sabry [18], Maraist et al. [10], Herbelin and Zimmerman [8], Accattoli and Paolini [2] (the latter is inspired by the call-by-value translation of  $\lambda$ -terms into linear logic proof-nets, see [1]). All these proposals are based on the introduction of new constructs to the syntax of  $\lambda_v$ , so the comparison between them is not easy with respect to syntactical properties (some detailed comparison is given in [2]). We point out that the calculi introduced in [11, 18, 10, 8] present some variants of our  $\sigma_1$

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<sup>1</sup> The answer is trivially positive in the case of call-by-name head normalization (for  $\lambda$ ) and head  $\beta_v$ -normalization, since these reductions are deterministic.

and/or  $\sigma_3$  rules, often in a setting with explicit substitutions. Regnier [16, 17] used the rule  $\sigma_1$  (but not  $\sigma_3$ ) in ordinary (i.e. call-by-name)  $\lambda$ -calculus.

The head  $\mathbf{v}$ -reduction investigated here has been introduced in [7]. Some results of this paper are inspired by the Takahashi's results [19] on the ordinary (i.e. call-by-name)  $\lambda$ -calculus, partially adapted by Crary [5] for  $\lambda_v$ .

**Outline** In Section 2 we introduce the syntax and the reduction rules of the  $\lambda_v^\sigma$ -calculus. In Section 3 we define the head  $\mathbf{v}$ -reduction and the internal  $\mathbf{v}$ -reduction, and we recall some results already proven in [7] concerning them. Section 4 is devoted to proving the first main result of our paper: Theorem 21, which studies the normalization for the head  $\mathbf{v}$ -reduction and relates it to the weak evaluation strategy for Plotkin's  $\lambda_v$ -calculus. In Section 5 we show that the head  $\mathbf{v}$ -reduction can be used to define a normalization strategy for the  $\lambda_v^\sigma$ -calculus (Theorem 24), and moreover in some cases the head  $\mathbf{v}$ -normal form (if any) of a term is unique (Proposition 27). In Section 6 we summarize the findings and suggest future work.

## 2 The call-by-value lambda calculus with sigma-rules

In this section we present  $\lambda_v^\sigma$ , a call-by-value  $\lambda$ -calculus introduced in [4] that adds two  $\sigma$ -reduction rules to pure (i.e. without constants) call-by-value  $\lambda$ -calculus defined by Plotkin in [15].

The syntax of terms of  $\lambda_v^\sigma$  [4] is the same as the one of ordinary  $\lambda$ -calculus and Plotkin's call-by-value  $\lambda$ -calculus  $\lambda_v$  [15] (without constants). Given a countable set  $\mathcal{V}$  of *variables* (denoted by  $x, y, z, \dots$ ), the sets  $\Lambda$  of *terms* and  $\Lambda_v$  of *values* are defined by mutual induction:

$$\begin{aligned} (\Lambda_v) \quad & V, U ::= x \mid \lambda x.M \quad \text{values} \\ (\Lambda) \quad & M, N, L ::= V \mid MN \quad \text{terms} \end{aligned}$$

Clearly,  $\Lambda_v \subsetneq \Lambda$ . All terms are considered up to  $\alpha$ -conversion. The set of free variables of a term  $M$  is denoted by  $\text{fv}(M)$ . Given  $V_1, \dots, V_n \in \Lambda_v$  and pairwise distinct variables  $x_1, \dots, x_n$ ,  $M\{V_1/x_1, \dots, V_n/x_n\}$  denotes the term obtained by the *capture-avoiding simultaneous substitution* of  $V_i$  for each free occurrence of  $x_i$  in the term  $M$  (for all  $1 \leq i \leq n$ ). Note that, for all  $V, V_1, \dots, V_n \in \Lambda_v$  and pairwise distinct variables  $x_1, \dots, x_n$ ,  $V\{V_1/x_1, \dots, V_n/x_n\} \in \Lambda_v$ .

*Contexts* (with exactly one hole  $(\cdot)$ ), denoted by  $\mathcal{C}$ , are defined as usual via the grammar:

$$\mathcal{C} ::= (\cdot) \mid \lambda x.\mathcal{C} \mid \mathcal{C}M \mid M\mathcal{C}.$$

We use  $\mathcal{C}(M)$  for the term obtained by the capture-allowing substitution of the term  $M$  for the hole  $(\cdot)$  in the context  $\mathcal{C}$ .

► **Notation.** From now on, we set  $I = \lambda x.x$  and  $\Delta = \lambda x.xx$ .

The reduction rules of  $\lambda_v^\sigma$  consist of Plotkin's  $\beta_v$ -reduction rule, introduced in [15], and two simple commutation rules called  $\sigma_1$  and  $\sigma_3$ , studied in [4, 7].

► **Definition 1** (Reduction rules). *We define the following binary relations on  $\Lambda$  (for any  $M, N, L \in \Lambda$  and any  $V \in \Lambda_v$ ):*

$$\begin{aligned} (\lambda x.M)V &\mapsto_{\beta_v} M\{V/x\} \\ (\lambda x.M)NL &\mapsto_{\sigma_1} (\lambda x.ML)N \quad \text{with } x \notin \text{fv}(L) \\ V((\lambda x.L)N) &\mapsto_{\sigma_3} (\lambda x.VL)N \quad \text{with } x \notin \text{fv}(V). \end{aligned}$$

We set  $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$  and  $\mapsto_{\mathbf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$ .

For any  $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathbf{v}\}$ , if  $M \mapsto_r M'$  then  $M$  is a  $r$ -redex and  $M'$  is its  $r$ -contractum. In this sense, a term of the shape  $(\lambda x.M)N$  (for any  $M, N \in \Lambda$ ) is a  $\beta$ -redex.

The side conditions on  $\mapsto_{\sigma_1}$  and  $\mapsto_{\sigma_3}$  in Definition 1 can be always fulfilled by  $\alpha$ -renaming.

Obviously, any  $\beta_v$ -redex is a  $\beta$ -redex but the converse does not hold:  $(\lambda x.z)(yI)$  is a  $\beta$ -redex but not a  $\beta_v$ -redex.

► **Example 2.** Redexes of different kind may overlap: for example, the term  $\Delta I \Delta$  is a  $\sigma_1$ -redex and it contains the  $\beta_v$ -redex  $\Delta I$ ; the term  $\Delta(I\Delta)(xI)$  is a  $\sigma_1$ -redex and it contains the  $\sigma_3$ -redex  $\Delta(I\Delta)$ , which contains in turn the  $\beta_v$ -redex  $I\Delta$ .

► **Notation.** Let  $R$  be a binary relation on  $\Lambda$ . We denote by  $R^*$  (resp.  $R^+$ ;  $R^\equiv$ ) the reflexive-transitive (resp. transitive; reflexive) closure of  $R$ .

► **Definition 3 (Reductions).** Let  $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, v\}$ .

The  $r$ -reduction  $\rightarrow_r$  is the contextual closure of  $\mapsto_r$ , i.e.  $M \rightarrow_r M'$  iff there is a context  $C$  and  $N, N' \in \Lambda$  such that  $M = C(N)$ ,  $M' = C(N')$  and  $N \mapsto_r N'$ .

The  $r$ -equivalence  $\simeq_r$  is the reflexive-transitive and symmetric closure of  $\rightarrow_r$ .

Let  $M$  be a term:  $M$  is  $r$ -normal if there is no term  $N$  such that  $M \rightarrow_r N$ ;  $M$  is  $r$ -normalizable if there is a  $r$ -normal term  $N$  such that  $M \rightarrow_r^* N$ ;  $M$  is strongly  $r$ -normalizable if there is no sequence  $(N_i)_{i \in \mathbb{N}}$  of terms such that  $M = N_0$  and  $N_i \rightarrow_r N_{i+1}$  for any  $i \in \mathbb{N}$ .

Obviously,  $\rightarrow_\sigma = \rightarrow_{\sigma_1} \cup \rightarrow_{\sigma_3} \subsetneq \rightarrow_v$  and  $\rightarrow_{\beta_v} \subsetneq \rightarrow_v$  and  $\rightarrow_v = \rightarrow_{\beta_v} \cup \rightarrow_\sigma$ .

► **Remark 4.** For any  $r \in \{\beta_v, \sigma_1, \sigma_3, \sigma, v\}$  (resp.  $r \in \{\sigma_1, \sigma_3, \sigma\}$ ), values are closed under  $r$ -reduction (resp.  $r$ -expansion): for any  $V \in \Lambda_v$ , if  $V \rightarrow_r M$  (resp.  $M \rightarrow_r V$ ) then  $M \in \Lambda_v$ ; more precisely,  $V = \lambda x.N$  and  $M = \lambda x.N'$  for some  $N, N' \in \Lambda$  with  $N \rightarrow_r N'$  (resp.  $N' \rightarrow_r N$ ).

For any  $r \in \{\beta_v, v\}$ , values are not closed under  $r$ -expansion:  $I\Delta \rightarrow_{\beta_v} \Delta \in \Lambda_v$  but  $I\Delta \notin \Lambda_v$ .

► **Proposition 5 (See [4]).** The  $\sigma$ -reduction is confluent and strongly normalizing. The  $v$ -reduction is confluent.

The  $\lambda_v^\sigma$ -calculus,  $\lambda_v^\sigma$  for short, is the set  $\Lambda$  of terms endowed with the  $v$ -reduction  $\rightarrow_v$ . The set  $\Lambda$  endowed with the  $\beta_v$ -reduction  $\rightarrow_{\beta_v}$  is the  $\lambda_v$ -calculus ( $\lambda_v$  for short), i.e. the Plotkin's call-by-value  $\lambda$ -calculus [15] (without constants), which is thus a sub-calculus of  $\lambda_v^\sigma$ .

► **Example 6.**  $M = (\lambda y.\Delta)(xI)\Delta \rightarrow_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \dots$  and  $N = \Delta((\lambda y.\Delta)(xI)) \rightarrow_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} (\lambda y.\Delta\Delta)(xI) \rightarrow_{\beta_v} \dots$  are the only possible  $v$ -reduction paths from  $M$  and  $N$  respectively:  $M$  and  $N$  are not  $v$ -normalizable, and  $M \simeq_v N$ . Meanwhile,  $M$  and  $N$  are  $\beta_v$ -normal and different, hence  $M \not\equiv_{\beta_v} N$  (by confluence of  $\rightarrow_{\beta_v}$ , see [15]).

Informally,  $\sigma$ -rules unblock  $\beta_v$ -redexes which are hidden by the “hyper-sequential structure” of terms. This approach is alternative to the one in [2, 1] where hidden  $\beta_v$ -redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value  $\lambda$ -calculus with explicit substitution introduced in [2] can be embedded in  $\lambda_v^\sigma$ .

It is well-known that the  $\beta_v$ -reduction can be simulated by linear logic cut-elimination via the call-by-value translation  $(\cdot)^v$  of  $\lambda$ -terms into proof-nets, called by Girard [6, pp. 81-82] “boring” and defined by  $(A \Rightarrow B)^v = !A^v \multimap !B^v$  (see also [1]). The images under  $(\cdot)^v$  of a  $\sigma$ -redex and its  $\sigma$ -contractum are equal modulo some non-structural cut-elimination steps.

### 3 Head and internal reductions

In this section we introduce the definitions of head  $v$ -reduction (which is decomposed in head  $\beta_v$ - and head  $\sigma$ -reductions) and internal  $v$ -reduction, then we recall some results proven in [7].

► **Notation.** From now on, we always assume that  $V, V' \in \Lambda_v$ .

Note that the generic form of a term is  $VM_1 \dots M_m$  for some  $m \in \mathbb{N}$  (in particular, values are obtained when  $m = 0$ ). The sequentialization result is based on a partitioning of  $\mathbf{v}$ -reduction between head  $\mathbf{v}$ -reduction and internal  $\mathbf{v}$ -reduction.

► **Definition 7 (Head  $\beta_v$ -reduction).** *The head  $\beta_v$ -reduction  $\xrightarrow{\beta_v}$  is the binary relation on  $\Lambda$  defined inductively by the following rules ( $m \in \mathbb{N}$  in both rules):*

$$\frac{}{(\lambda x.M)VM_1 \dots M_m \xrightarrow{\beta_v} M\{V/x\}M_1 \dots M_m} \beta_v \quad \frac{N \xrightarrow{\beta_v} N'}{VNM_1 \dots M_m \xrightarrow{\beta_v} VN'M_1 \dots M_m} \text{right}$$

The head  $\beta_v$ -reduction  $\xrightarrow{\beta_v}$  is exactly the (pure) “left reduction” defined in [15, p. 136] for  $\lambda_v$  and called “(weak) evaluation” in [18, 5]. If  $N \xrightarrow{\beta_v} N'$  then  $N'$  is obtained from  $N$  by reducing the leftmost-outermost  $\beta_v$ -redex, not in the scope of a  $\lambda$ : thus, the head  $\beta_v$ -reduction is deterministic (i.e. it is a partial function from  $\Lambda$  to  $\Lambda$ ) and does not reduce values.

► **Definition 8 (Head  $\sigma$ - and head  $\mathbf{v}$ -reductions).** *The head  $\sigma$ -reduction  $\xrightarrow{\sigma}$  is the binary relation on  $\Lambda$  defined inductively by the following rules ( $m \in \mathbb{N}$  in all the rules,  $x \notin \text{fv}(L)$  in the rule  $\sigma_1$ ,  $x \notin \text{fv}(V)$  in the rule  $\sigma_3$ ):*

$$\frac{}{(\lambda x.M)NLM_1 \dots M_m \xrightarrow{\sigma} (\lambda x.ML)NM_1 \dots M_m} \sigma_1 \quad \frac{N \xrightarrow{\sigma} N'}{VNM_1 \dots M_m \xrightarrow{\sigma} VN'M_1 \dots M_m} \text{right}$$

$$\frac{}{V((\lambda x.L)N)M_1 \dots M_m \xrightarrow{\sigma} (\lambda x.VL)NM_1 \dots M_m} \sigma_3$$

The head  $\mathbf{v}$ -reduction is  $\xrightarrow{\mathbf{v}} = \xrightarrow{\beta_v} \cup \xrightarrow{\sigma}$ .

Let  $r \in \{\beta_v, \sigma, \mathbf{v}\}$  and  $N \in \Lambda$ :  $N$  is head  $r$ -normal if there is no  $N' \in \Lambda$  such that  $N \xrightarrow{r} N'$ ;  $N$  is head  $r$ -normalizable if there is a  $r$ -normal term  $N'$  such that  $N \xrightarrow{r}^* N'$ ;  $N$  is strongly head  $r$ -normalizable if there is no  $(N_i)_{i \in \mathbb{N}}$  such that  $N = N_0$  and  $N_i \xrightarrow{r} N_{i+1}$  for any  $i \in \mathbb{N}$ .

Notice that  $\mapsto_{\beta_v} \subsetneq \xrightarrow{\beta_v} \subsetneq \rightarrow_{\beta_v}$  and  $\mapsto_{\sigma} \subsetneq \xrightarrow{\sigma} \subsetneq \rightarrow_{\sigma}$  and  $\mapsto_{\mathbf{v}} \subsetneq \xrightarrow{\mathbf{v}} \subsetneq \rightarrow_{\mathbf{v}}$ .

Informally, if  $N \xrightarrow{\sigma} N'$  then  $N'$  is obtained from  $N$  by reducing “one of the leftmost”  $\sigma_1$ - or  $\sigma_3$ -redexes, not in the scope of a  $\lambda$ : in general, a term may contain several head  $\sigma_1$ - and  $\sigma_3$ -redexes. Indeed, differently from  $\xrightarrow{\beta_v}$ , the head  $\sigma$ -reduction  $\xrightarrow{\sigma}$  is not deterministic, for example the leftmost-outermost  $\sigma_1$ - and  $\sigma_3$ -redexes may overlap: if  $M = (\lambda y.y')(\Delta(xI))I$  then  $M \xrightarrow{\sigma} (\lambda y.y'I)(\Delta(xI)) = N_1$  by applying the rule  $\sigma_1$  and  $M \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz))(xI)I = N_2$  by applying the rule  $\sigma_3$ . Note that  $N_1$  contains only a head  $\sigma_3$ -redex and  $N_1 \xrightarrow{\sigma} (\lambda z.(\lambda y.y'I)(zz))(xI) = N$  which is head  $\mathbf{v}$ -normal; meanwhile  $N_2$  contains only a head  $\sigma_1$ -redex and  $N_2 \xrightarrow{\sigma} (\lambda z.(\lambda y.y')(zz)I)(xI) = N'$  which is head  $\mathbf{v}$ -normal:  $N \neq N'$ , so the head  $\sigma$ - and head  $\mathbf{v}$ -reductions are not (locally) confluent and a term may have several head  $\mathbf{v}$ -normal forms (this example does not contradict the confluence of  $\sigma$ -reduction because  $N' \rightarrow_{\sigma} N$  but by performing an internal  $\mathbf{v}$ -reduction step, see next Definition 9).

The head  $\mathbf{v}$ -reduction  $\xrightarrow{\mathbf{v}}$  is non-deterministic not only because the head  $\sigma$ -reduction  $\xrightarrow{\sigma}$  is non-deterministic, but also because the leftmost-outermost  $\beta_v$ -redex of a term may overlap with “one of its leftmost”  $\sigma_1$ - or  $\sigma_3$ -redexes, as seen in Example 2.

► **Definition 9 (Internal  $\mathbf{v}$ -reduction).** *The internal  $\mathbf{v}$ -reduction  $\xrightarrow{\text{int}}_{\mathbf{v}}$  is the binary relation on  $\Lambda$  defined inductively by the following rules:*

$$\frac{(m \in \mathbb{N}) \quad N \rightarrow_{\mathbf{v}} N'}{(\lambda x.N)M_1 \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} (\lambda x.N')M_1 \dots M_m} \lambda \quad \frac{(m \in \mathbb{N}) \quad N \xrightarrow{\text{int}}_{\mathbf{v}} N'}{VNM_1 \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} VN'M_1 \dots M_m} \text{right}$$

$$\frac{(m \in \mathbb{N}^+) \quad M_i \rightarrow_{\mathbf{v}} M'_i \quad \text{for some } 1 \leq i \leq m}{VNM_1 \dots M_i \dots M_m \xrightarrow{\text{int}}_{\mathbf{v}} VNM_1 \dots M'_i \dots M_m} \textcircled{\text{a}} .$$

The following fact collects many minor properties which can be easily proved by inspection of the rules of Definitions 7-9.

- **Fact 10. 1.** *The head  $\beta_v$ -reduction  $\xrightarrow{h}_{\beta_v}$  does not reduce a value (in particular, does not reduce under  $\lambda$ 's), i.e., for any  $M \in \Lambda$  and any  $V \in \Lambda_v$ , one has  $V \not\xrightarrow{h}_{\beta_v} M$ .*
2. *The head  $\sigma$ -reduction  $\xrightarrow{h}_{\sigma}$  does neither reduce a value nor reduce to a value, i.e., for any  $M \in \Lambda$  and any  $V \in \Lambda_v$ , one has  $V \not\xrightarrow{h}_{\sigma} M$  and  $M \not\xrightarrow{h}_{\sigma} V$ .*
  3. *Values are closed under  $\xrightarrow{int}_v$ -expansion, i.e., for all  $M \in \Lambda$  and  $V \in \Lambda_v$ , if  $M \xrightarrow{int}_v V$  then  $M \in \Lambda_v$ ; more precisely,  $M = \lambda x.N$  and  $V = \lambda x.N'$  for some  $N, N' \in \Lambda$  where  $N \rightarrow_v N'$ .*
  4. *If  $R \in \{\xrightarrow{h}_{\beta_v}, \xrightarrow{h}_{\sigma}, \xrightarrow{h}_v, \xrightarrow{int}_v\}$  and  $M R M'$ , then  $MN R M'N$  for any  $N \in \Lambda$ .*

Clearly,  $\xrightarrow{int}_v \subseteq \rightarrow_v$ . Next Proposition 11 (whose proof uses Fact 10.4) relates  $\xrightarrow{int}_v$  and  $\xrightarrow{h}_v$ .

- **Proposition 11.** *One has  $\xrightarrow{int}_v = \rightarrow_v \setminus \xrightarrow{h}_v$ .*

**Proof.**

⊆: The proof that  $\xrightarrow{int}_v \subseteq \rightarrow_v$  is trivial. The proof that  $M \xrightarrow{int}_v M'$  implies  $M \not\xrightarrow{h}_v M'$  is by induction on the derivation of  $M \xrightarrow{int}_v M'$ . Let us consider its last rule  $r$ . If  $r \in \{\lambda, @\}$ , then it is evident that there is no last rule to derive  $M \xrightarrow{h}_v M'$ . If  $r = \text{right}$  then  $M = VNM_1 \dots M_m$  and  $M' = VN'M_1 \dots M_m$  with  $m \in \mathbb{N}$  and  $N \xrightarrow{int}_v N'$ ; by induction hypothesis,  $N \not\xrightarrow{h}_v N'$  and hence there is no last rule to derive  $M \xrightarrow{h}_v M'$ .

⊇: We show that  $M \rightarrow_v M'$  and  $M \not\xrightarrow{h}_v M'$  implies  $M \xrightarrow{int}_v M'$ , for all  $M, M' \in \Lambda$ . Since  $M \rightarrow_v M'$ , there exist a context  $C$  and terms  $N$  and  $N'$  such that  $M = C(N)$ ,  $M' = C(N')$  and  $N \mapsto_{\beta_v} N'$ . We proceed by induction on  $C$ .

If  $C = (\cdot)$  then  $M = N \mapsto_{\beta_v} N' = M'$  and thus  $M \xrightarrow{h}_v M'$  since  $\mapsto_{\beta_v} \subseteq \xrightarrow{h}_v$ , which contradicts the hypothesis.

If  $C = \lambda x.C'$  for some context  $C'$ , then  $M \xrightarrow{int}_v M'$  by applying the rule  $\lambda$  for  $\xrightarrow{int}_v$ , since  $C'(N) \rightarrow_v C'(N')$ .

If  $C = C'L$  for some context  $C'$  and term  $L$ , then  $C'(N) \rightarrow_v C'(N')$  and  $C'(N') \not\xrightarrow{h}_v C'(N')$  (by Fact 10.4, since  $C'(N)L \not\xrightarrow{h}_v C'(N')L$ ). By induction hypothesis,  $C'(N) \xrightarrow{int}_v C'(N')$ , then  $M = C'(N)L \xrightarrow{int}_v C'(N')L = M'$  by Fact 10.4.

If  $C = VC'$  for some context  $C'$  and value  $V$ , then  $C'(N) \rightarrow_v C'(N')$ . There are two cases:

- either  $C'(N') \xrightarrow{h}_v C'(N')$ , hence  $M = VC'(N) \xrightarrow{h}_v VC'(N') = M'$  by the rule *right* for  $\xrightarrow{h}_{\beta_v}$  or  $\xrightarrow{h}_{\sigma}$ , which contradicts the hypothesis;
- or  $C'(N') \not\xrightarrow{h}_v C'(N')$ , hence  $C'(N') \xrightarrow{int}_v C'(N')$  by induction hypothesis, thus  $M = VC'(N) \xrightarrow{int}_v VC'(N') = M'$  by applying the rule *right* for  $\xrightarrow{int}_v$ .

Finally, if  $C = LC'$  for some context  $C'$  and term  $L \notin \Lambda_v$ , then  $L = VN_0 \dots N_n$  for some  $n \in \mathbb{N}$ , thus  $M = VN_0 \dots N_n C'(N) \xrightarrow{int}_v VN_0 \dots N_n C'(N') = M'$  by the rule  $@$  for  $\xrightarrow{int}_v$ . ◀

We end this section by recalling three results proven in [7] concerning head  $v$ -reduction and internal  $v$ -reduction: they will be used to prove the main results in Sections 4-5.

The following lemma (proven in [7, Lemma 14]) shows that a head  $\sigma$ -reduction step can be postponed after a head  $\beta_v$ -reduction step, and hence every head  $v$ -reduction sequence can be rearranged into a head  $\beta_v$ -reduction sequence followed by a head  $\sigma$ -reduction sequence.

- **Lemma 12** (Commutation of head  $\beta_v$ - and head  $\sigma$ -reductions, see [7]).

1. *If  $M \xrightarrow{h}_{\sigma} L \xrightarrow{h}_{\beta_v} N$  then there exists  $L' \in \Lambda$  such that  $M \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}_{\sigma} N$ .*
2. *If  $M \xrightarrow{h}_v^* M'$  then there exists  $N \in \Lambda$  such that  $M \xrightarrow{h}_{\beta_v}^* N \xrightarrow{h}_{\sigma}^* M'$ .*

Next Lemma 13 (proven in [7, Corollary 21]) says that internal  $\mathbf{v}$ -reduction can be shifted after head  $\mathbf{v}$ -reductions.<sup>2</sup>

► **Lemma 13** (Postponement, see [7]). *If  $M \xrightarrow{\text{int}}_{\mathbf{v}} L$  and  $L \xrightarrow{h}_{\beta_{\mathbf{v}}} N$  (resp.  $L \xrightarrow{h}_{\sigma} N$ ), then there exist  $L', L'' \in \Lambda$  such that  $M \xrightarrow{h^+}_{\beta_{\mathbf{v}}} L' \xrightarrow{h^*}_{\sigma} L'' \xrightarrow{\text{int}^*}_{\mathbf{v}} N$  (resp.  $M \xrightarrow{h^*}_{\beta_{\mathbf{v}}} L' \xrightarrow{h^*}_{\sigma} L'' \xrightarrow{\text{int}^*}_{\mathbf{v}} N$ ).*

Next Theorem 14 is one of the main result proven in [7, Theorem 22] by adapting Takahashi's method [19, 5]: any  $\mathbf{v}$ -reduction sequence can be sequentialized into a head  $\beta_{\mathbf{v}}$ -reduction sequence followed by a head  $\sigma$ -reduction sequence, followed by an internal  $\mathbf{v}$ -reduction sequence. In ordinary  $\lambda$ -calculus, the well-known result corresponding to our Theorem 14 states that a  $\beta$ -reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [19, Corollary 2.6]).

► **Theorem 14** (Sequentialization, see [7]). *If  $M \rightarrow^*_{\mathbf{v}} M'$  then there exist  $L, N \in \Lambda$  such that  $M \xrightarrow{h^*}_{\beta_{\mathbf{v}}} L \xrightarrow{h^*}_{\sigma} N \xrightarrow{\text{int}^*}_{\mathbf{v}} M'$ .*

The sequentialization of Theorem 14 imposes no order between head  $\sigma$ -reductions. Indeed, the example in [7, p. 10] shows that it is impossible to sequentialize them by giving way to head  $\sigma_1$ - or head  $\sigma_3$ -redexes: a head  $\sigma_1$ -reduction step can create a head  $\sigma_3$ -redex, and vice versa.

In [7, Definition 27 and Corollary 29] it has also been proven that the  $\mathbf{v}$ -equivalence (and in particular the  $\sigma$ -equivalence) is contained in the call-by-value observational equivalence.

## 4 Head normalization

In this section we prove the first main result of our paper: Theorem 21, which studies the normalization for head  $\mathbf{v}$ -reduction and relates it to the head  $\beta_{\mathbf{v}}$ -reduction (i.e. the weak evaluation strategy for Plotkin's  $\lambda_{\mathbf{v}}$ -calculus). Let us start with a preliminary remark.

► **Remark 15.** According to Facts 10.1-2, every  $V \in \Lambda_{\mathbf{v}}$  is head  $\beta_{\mathbf{v}}$ - and head  $\sigma$ -normal, and hence is head  $\mathbf{v}$ -normal. The converse does not hold:  $xI$  is head  $\mathbf{v}$ -normal but  $xI \notin \Lambda_{\mathbf{v}}$ .

First, we give a syntactic characterization of head  $\mathbf{v}$ - and head  $\beta_{\mathbf{v}}$ -normal forms.

► **Definition 16.** *We define the subsets  $\Lambda_a$ ,  $\Lambda_b$  and  $\Lambda_c$  (whose elements are denoted by  $A$ ,  $B$  and  $C$  respectively) of  $\Lambda$  as follows (for any variable  $x$ , any  $V \in \Lambda_{\mathbf{v}}$  and any  $N \in \Lambda$ ):*

$$(\Lambda_a) \quad A ::= xV \mid xA \mid AN \quad (\Lambda_b) \quad B ::= (\lambda x.N)A \quad (\Lambda_c) \quad C ::= xV \mid VC \mid CN$$

Notice that  $\Lambda_a \cup \Lambda_b \subsetneq \Lambda_c$  and  $M, N \in \Lambda_c \setminus (\Lambda_a \cup \Lambda_b)$  where  $M = (\lambda y.\Delta)(xI)\Delta$  and  $N = \Delta((\lambda y.\Delta)(xI))$  (as in Example 6). Moreover,  $\Lambda_{\mathbf{v}} \cap \Lambda_a = \Lambda_{\mathbf{v}} \cap \Lambda_b = \Lambda_{\mathbf{v}} \cap \Lambda_c = \Lambda_a \cap \Lambda_b = \emptyset$  and all terms in  $\Lambda_a \cup \Lambda_b \cup \Lambda_c$  are not closed. All terms in  $\Lambda_b$  are  $\beta$ -redexes that are not  $\beta_{\mathbf{v}}$ -redexes; all terms in  $\Lambda_a$  have a free ‘‘head variable’’ and are neither a value nor a  $\beta$ -redex.

► **Proposition 17** (Characterization of head  $\beta_{\mathbf{v}}$ -normal forms). *Let  $M$  be a term.*

1.  $M$  is head  $\beta_{\mathbf{v}}$ -normal and is not a  $\lambda$ -value if and only if  $M \in \Lambda_c$ .
2.  $M$  is head  $\beta_{\mathbf{v}}$ -normal if and only if  $M \in \Lambda_{\mathbf{v}} \cup \Lambda_c$ .

**Proof.** Statement (2) is an immediate consequence of statement (1) and Remark 15.

<sup>2</sup> In [7, Corollary 21] there is a more informative statement of our Lemma 13, involving a notion of internal parallel reduction  $\xrightarrow{\text{int}}$ . Our Lemma 13 follows immediately from [7, Corollary 21] since  $\xrightarrow{\text{int}}_{\mathbf{v}} \subseteq \xrightarrow{\text{int}} \subseteq \xrightarrow{\text{int}^*}_{\mathbf{v}}$ .

- $\Rightarrow$ : We prove the left-to-right direction of statement (1), by induction on  $M \in \Lambda$ .  
 The case where  $M \in \Lambda_v$  is impossible by hypothesis.  
 If  $M = M_1 M_2$  (for some  $M_1, M_2 \in \Lambda$ ) is head  $\beta_v$ -normal then  $M$  is not a  $\lambda$ -value and  $M_1$  and  $M_2$  are head  $\beta_v$ -normal, moreover either  $M_1 \neq \lambda x.N$  (for any  $N \in \Lambda$ ) or  $M_2 \notin \Lambda_v$  (otherwise  $M$  would be a head  $\beta_v$ -redex). Therefore, there are only three cases:
- either  $M_1 \notin \Lambda_v$ , thus  $M_1 \in \Lambda_c$  by induction hypothesis, and hence  $M \in \Lambda_c$ ;
  - or  $M_1 \in \Lambda_v$  and  $M_2 \notin \Lambda_v$ , so  $M_2 \in \Lambda_c$  by induction hypothesis, and thus  $M \in \Lambda_c$ ;
  - or  $M_1$  is a variable and  $M_2 \in \Lambda_v$ , hence  $M \in \Lambda_c$  (this is the base case).
- $\Leftarrow$ : The right-to-left direction of statement (1) can easily be proved by induction on  $M \in \Lambda_c$ .  $\blacktriangleleft$

A consequence of Proposition 17 is that all closed head  $\beta_v$ -normal forms are abstractions.

► **Proposition 18** (Characterization of head v-normal forms). *Let  $M \in \Lambda$ .*

1.  *$M$  is head v-normal and is neither a  $\lambda$ -value nor a  $\beta$ -redex if and only if  $M \in \Lambda_a$ .*
2.  *$M$  is head v-normal and is a  $\beta$ -redex if and only if  $M \in \Lambda_b$ .*
3.  *$M$  is head v-normal if and only if  $M \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ .*

**Proof.** Statement (3) is an immediate consequence of statements (1)-(2) and Remark 15.

- $\Rightarrow$ : We prove simultaneously the left-to-right direction of statements (1) and (2), by induction on  $M \in \Lambda$ . The case where  $M \in \Lambda_v$  is impossible by hypothesis.  
 If  $M = M_1 M_2$  (for some  $M_1, M_2 \in \Lambda$ ) is head v-normal then  $M$  is not a  $\lambda$ -value and  $M_1$  and  $M_2$  are head v-normal, moreover  $M_1$  is not a  $\beta$ -redex (otherwise  $M$  would be a head  $\sigma_1$ -redex), and either  $M_1 \neq \lambda x.N$  (for any  $N \in \Lambda$ ) or  $M_2 \notin \Lambda_v$  (otherwise  $M$  would be a head  $\beta_v$ -redex), and either  $M_1 \notin \Lambda_v$  or  $M_2$  is not a  $\beta$ -redex (otherwise  $M$  would be a head  $\sigma_3$ -redex). There are only three cases:
- either  $M_1$  is a variable and  $M_2$  is not a  $\beta$ -redex, so  $M$  is not a  $\beta$ -redex; if  $M_2 \in \Lambda_v$  then  $M \in \Lambda_a$  (this is the base case); otherwise  $M_2 \in \Lambda_a$  by induction hypothesis, so  $M \in \Lambda_a$ ;
  - or  $M_1 \notin \Lambda_v$ , thus  $M$  is not a  $\beta$ -redex and  $M_1 \in \Lambda_a$  by induction hypothesis, so  $M \in \Lambda_a$ ;
  - or  $M_1 = \lambda x.N$  for some  $N \in \Lambda$  and  $M_2$  is neither a  $\lambda$ -value nor a  $\beta$ -redex, so  $M$  is a  $\beta$ -redex, furthermore  $M_2 \in \Lambda_a$  by induction hypothesis, and thus  $M \in \Lambda_b$ .
- $\Leftarrow$ : The right-to-left direction of statement (1) can easily be proved by induction on  $M \in \Lambda_a$ .  
 Let us prove the right-to-left direction of statement (2): if  $M \in \Lambda_b$  then  $M = (\lambda x.N)A$  for some  $N \in \Lambda$  and  $A \in \Lambda_a$ , thus  $M$  is a  $\beta$ -redex. For any  $M' \in \Lambda$ , the last rule of the derivation of  $M \xrightarrow{h}_v M'$  might be neither  $\sigma_1$  nor  $\sigma_3$  (because  $A$  is not a  $\beta$ -redex by statement 1) nor  $\beta_v$  (because  $A \notin \Lambda_v$  by statement 1 again) nor *right* (because  $A$  is head v-normal, by statement 1 again). Therefore,  $M$  is head v-normal.  $\blacktriangleleft$

As a consequence of Proposition 18, all closed head v-normal forms are abstractions.

The sets of terms  $\Lambda_a$ ,  $\Lambda_b$  and  $\Lambda_c$  of Definition 16 enjoy the closure properties summarized in Lemma 19 below. Together with the syntactic characterizations of head  $\beta_v$ -normal forms (Proposition 17) and head v-normal forms (Proposition 18), these closure properties allow one to reason about head v-reduction in spite of its non-confluence: they will be used to prove our main results, Theorems 21 and 24 and Proposition 27.

- **Lemma 19** (Closure properties). **1.** *The set  $\Lambda_a$  is closed under v-internal reduction and expansion, i.e., for any  $N' \in \Lambda$  and  $N \in \Lambda_a$ , if  $N' \xrightarrow{int}_v N$  or  $N \xrightarrow{int}_v N'$  then  $N' \in \Lambda_a$ .*
- 2.** *The set  $\Lambda_b$  is closed under v-internal reduction and expansion, i.e., for any  $N' \in \Lambda$  and  $N \in \Lambda_b$ , if  $N' \xrightarrow{int}_v N$  or  $N \xrightarrow{int}_v N'$  then  $N' \in \Lambda_b$ .*
- 3.** *Head v-normal forms are closed under v-internal reduction and expansion, i.e., for any  $N, N' \in \Lambda$  where  $N$  is head v-normal, if  $N' \xrightarrow{int}_v N$  or  $N \xrightarrow{int}_v N'$  then  $N'$  is head v-normal.*

4. Head  $\beta_v$ -normal forms are closed under head  $\sigma$ -reduction and expansion, i.e., for any  $N, N' \in \Lambda$  where  $N$  is head  $\beta_v$ -normal, if  $N' \xrightarrow{h}_\sigma N$  or  $N \xrightarrow{h}_\sigma N'$  then  $N'$  is head  $\beta_v$ -normal.

**Proof. 1.** We show that if  $N \in \Lambda_a$  and  $N' \xrightarrow{int}_v N$  (resp.  $N \xrightarrow{int}_v N'$ ) then  $N' \in \Lambda_a$ , by induction on the derivation of  $N' \xrightarrow{int}_v N$  (resp.  $N \xrightarrow{int}_v N'$ ). Let us consider its last rule  $r$ . Since  $N \in \Lambda_a$  (see Definition 16),  $N = xLN_1 \dots N_n$  for some  $n \in \mathbb{N}$ , some variable  $x$ , some  $L \in \Lambda_v \cup \Lambda_a$  and some  $N_1, \dots, N_n \in \Lambda$ , thus  $r \neq \lambda$  and hence either  $r = right$  or  $r = @$ . If  $r = right$  then  $N' = xL'N_1 \dots N_n$  where  $L' \xrightarrow{int}_v L$  (resp.  $L \xrightarrow{int}_v L'$ ). Since  $L \in \Lambda_v \cup \Lambda_a$ , there are two cases:

- either  $L \in \Lambda_a$  and then  $L' \in \Lambda_a$  by induction hypothesis, so  $N' = xL'N_1 \dots N_n \in \Lambda_a$ ;
- or  $L \in \Lambda_v$  and then  $L' \in \Lambda_v$  by Fact 10.3 (resp. Remark 4, since  $\xrightarrow{int}_v \subseteq \rightarrow_v$ ), therefore  $N' = xL'N_1 \dots N_n \in \Lambda_a$ .

Finally, if  $r = @$  then  $n \in \mathbb{N}^+$  and  $N' = xLN_1 \dots N'_i \dots N_n$  for some  $1 \leq i \leq n$  with  $N'_i \rightarrow_v N_i$  (resp.  $N_i \rightarrow_v N'_i$ ), hence  $N' \in \Lambda_a$  because  $xL \in \Lambda_a$ .

2. We show that if  $N \in \Lambda_b$  and  $N' \xrightarrow{int}_v N$  (resp.  $N \xrightarrow{int}_v N'$ ) then  $N' \in \Lambda_b$ , by induction on the derivation of  $N' \xrightarrow{int}_v N$  (resp.  $N \xrightarrow{int}_v N'$ ). Let us consider its last rule  $r$ . Since  $N \in \Lambda_b$ , then  $N = (\lambda x.M)A$  for some  $M \in \Lambda$  and  $A \in \Lambda_a$ , hence  $r \neq @$  because  $N$  has not the shape  $VLN_1 \dots N_m$  for any  $m \in \mathbb{N}^+$ ; therefore either  $r = \lambda$  or  $r = right$ :
- if  $r = \lambda$ , then  $N' = (\lambda x.M')A$  where  $M' \rightarrow_v M$  (resp.  $M \rightarrow_v M'$ ), hence  $N' \in \Lambda_b$ ;
  - if  $r = right$ , then  $N' = (\lambda x.M)A'$  where  $A' \xrightarrow{int}_v A$  (resp.  $A \xrightarrow{int}_v A'$ ), thus  $A' \in \Lambda_a$  by Lemma 19.1, hence  $N' \in \Lambda_b$ ;
3. Thanks to Proposition 18.3, it is sufficient to show that if  $N \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$  and  $N' \xrightarrow{int}_v N$  (resp.  $N \xrightarrow{int}_v N'$ ) then  $N' \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ . If  $N \in \Lambda_v$  then  $N' \in \Lambda_v$  by Fact 10.3 (resp. Remark 4, since  $\xrightarrow{int}_v \subseteq \rightarrow_v$ ). If  $N \in \Lambda_a$  then  $N' \in \Lambda_a$  by Lemma 19.1. Finally, if  $N \in \Lambda_b$  then  $N' \in \Lambda_b$  by Lemma 19.2.
4. By Proposition 17.2,  $N \in \Lambda_v \cup \Lambda_c$ . Since  $M \xrightarrow{h}_\sigma N$  or  $N \xrightarrow{h}_\sigma M$ ,  $N \notin \Lambda_v$  by Fact 10.2. We prove by induction on  $N \in \Lambda_c$  that  $M \in \Lambda_c$ . By Definition 16, there are only two cases:
- either  $N = xVN_1 \dots N_n$  for some  $n \in \mathbb{N}$ , variable  $x$ ,  $V \in \Lambda_v$  and  $N_1, \dots, N_n \in \Lambda$ , but this is impossible since the last rule of the derivation of  $M \xrightarrow{h}_\sigma N$  or  $N \xrightarrow{h}_\sigma M$  can be neither  $\sigma_1$  nor  $\sigma_3$  (because of the subterm  $xV$ ) nor  $right$  (because of Fact 10.2);
  - or  $N = VLN_1 \dots N_n$  for some  $n \in \mathbb{N}$ ,  $V \in \Lambda_v$ ,  $L \in \Lambda_c$  and  $N_1, \dots, N_n \in \Lambda$ , and then there are three sub-cases, depending on the last rule  $r$  of the derivation of  $M \xrightarrow{h}_\sigma N$  (resp.  $N \xrightarrow{h}_\sigma M$ ):
    - if  $r = \sigma_1$  then  $V = \lambda x.N'N_0$  (resp.  $\lambda x.N'$ ) and  $M = (\lambda x.N')LN_0 \dots N_n$  (resp.  $M = (\lambda x.N'N_1)LN_2 \dots N_n$  with  $n > 0$ ) for some  $N', N_0 \in \Lambda$ , hence  $M \in \Lambda_c$ ;
    - if  $r = \sigma_3$  then  $V = \lambda x.V'N'$  (resp.  $L = (\lambda x.N')L'$ ) and  $M = V'((\lambda x.N')L)N_1 \dots N_n$  (resp.  $M = (\lambda x.VN')L'N_1 \dots N_n$ ) for some  $V' \in \Lambda_v$  (resp.  $L' \in \Lambda_c$ ) and  $N' \in \Lambda$ , thus  $(\lambda x.N')L \in \Lambda_c$  (resp.  $(\lambda x.VN')L' \in \Lambda_c$ ) and hence  $M \in \Lambda_c$ ;
    - if  $r = right$  then  $M = VL'N_1 \dots N_n$  for some  $L' \in \Lambda$  such that  $L' \xrightarrow{h}_\sigma L$  (resp.  $L \xrightarrow{h}_\sigma L'$ ), so  $L' \in \Lambda_c$  by induction hypothesis, and hence  $M \in \Lambda_c$ .

Lemma 19.4 is a formalization of the two following facts: (a) a head  $\sigma$ -reduction step may create a new  $\beta_v$ -redex but in this case it is not a head  $\beta_v$ -redex; (b) when  $M \xrightarrow{h}_\sigma N$ , the head  $\beta_v$ -redex of  $M$  (if any) has a residual in  $N$  which is the head  $\beta_v$ -redex of  $N$ .

► **Lemma 20.** *There exists no infinite head  $v$ -reduction sequence with finitely many head  $\beta_v$ -reduction steps.*

**Proof.** Suppose the opposite holds: then there would exist  $m \in \mathbb{N}$  and an infinite sequence of terms  $(M_i)_{i \in \mathbb{N}}$  such that  $M_i \xrightarrow{h}_v M_{i+1}$  for any  $1 \leq i \leq m$ ,  $M_m \xrightarrow{h}_{\beta_v} M_{m+1}$  and  $M_i \xrightarrow{h}_\sigma M_{i+1}$  for any  $i > m$  (since  $\xrightarrow{h}_v = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma$ ). But this is impossible because  $\xrightarrow{h}_\sigma$  is strongly normalizing (by Proposition 5 and since  $\xrightarrow{h}_\sigma \subseteq \rightarrow_\sigma$ ). Contradiction.  $\blacktriangleleft$

Now we can state and prove our main result about head  $\beta_v$ - and head  $v$ -normalization.

► **Theorem 21 (Head normalization).** *Let  $M \in \Lambda$ . The following are equivalent:*

1. *there exists a head  $\beta_v$ -normal form  $N$  such that  $M \simeq_{\beta_v} N$ ;*
2. *there exists a head  $v$ -normal form  $N$  such that  $M \simeq_v N$ ;*
3.  *$M$  is head  $v$ -normalizable;*
4.  *$M$  is head  $\beta_v$ -normalizable;*
5. *there is no  $v$ -reduction sequence from  $M$  with infinitely many head  $\beta_v$ -reduction steps;*
6.  *$M$  is strongly head  $v$ -normalizable.*

**Proof.**

- (1) $\Rightarrow$ (2) By hypothesis, there exists a head  $\beta_v$ -normal  $N \in \Lambda$  such that  $M \simeq_{\beta_v} N$ , thus  $M \simeq_v N$ . Since  $\xrightarrow{h}_\sigma$  is strongly normalizing (by Proposition 5 and because  $\xrightarrow{h}_\sigma \subseteq \rightarrow_\sigma$ ), there exists a head  $\sigma$ -normal  $N' \in \Lambda$  such that  $N \xrightarrow{h}_\sigma^* N'$ , therefore  $M \simeq_v N'$  since  $\xrightarrow{h}_\sigma \subseteq \rightarrow_v$ . By Lemma 19.4,  $N'$  is also head  $\beta_v$ -normal and hence head  $v$ -normal.
- (2) $\Rightarrow$ (3) Since  $M \simeq_v N$ , there is  $L \in \Lambda$  such that  $M \rightarrow_v^* L$  and  $N \rightarrow_v^* L$ , by confluence of  $\rightarrow_v$  (Proposition 5). By Theorem 14, there are  $M_1, M_2, N_1, N_2 \in \Lambda$  such that  $M \xrightarrow{h}_{\beta_v}^* M_1 \xrightarrow{h}_\sigma^* M_2 \xrightarrow{int}_v^* L$  and  $N \xrightarrow{h}_{\beta_v}^* N_1 \xrightarrow{h}_\sigma^* N_2 \xrightarrow{int}_v^* L$ . As  $N$  is head  $v$ -normal,  $N = N_1 = N_2 \xrightarrow{int}_v^* L$ . By Lemma 19.3,  $L$  and  $M_2$  are  $v$ -head normal. So,  $M \xrightarrow{h}_v^* M_2$  with  $M_2$  head  $v$ -normal.
- (3) $\Rightarrow$ (4) By hypothesis, there is  $N \in \Lambda$  head  $v$ -normal such that  $M \xrightarrow{h}_v^* N$ . By Lemma 12.2, there is  $L \in \Lambda$  such that  $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_\sigma^* N$ . Since  $N$  is head  $v$ -normal and in particular head  $\beta_v$ -normal,  $L$  is head  $\beta_v$ -normal according to Lemma 19.4. So  $M$  is head  $\beta_v$ -normalizable.
- (4) $\Rightarrow$ (5) Lemma 12.1 says that if  $N \xrightarrow{h}_\sigma L \xrightarrow{h}_{\beta_v} N'$  then there exists  $L' \in \Lambda$  such that  $N \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}_\sigma^* N'$ ; Lemma 13 and Fact 10.3 show that if  $N \xrightarrow{int}_v L \xrightarrow{h}_{\beta_v} N'$  then there exist  $L', L'' \in \Lambda$  such that  $N \xrightarrow{h}_{\beta_v}^+ L' \xrightarrow{h}_\sigma^* L'' \xrightarrow{int}_v^* N'$ . Since  $\rightarrow_v = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma \cup \xrightarrow{int}_v$ , this means that if there is an infinite  $v$ -reduction sequence from  $M$  with infinitely many head  $\beta_v$ -reduction steps, then for any  $n \in \mathbb{N}$  there is a head  $\beta_v$ -reduction sequence from  $M$  whose length is at least  $n$ . Therefore,  $M$  is not head  $\beta_v$ -normalizable, since the head  $\beta_v$ -reduction is deterministic.
- (5) $\Rightarrow$ (6) If  $M$  is not strongly head  $v$ -normalizable then there exists an infinite head  $v$ -reduction sequence. By Lemma 20, this head  $v$ -reduction (and hence  $v$ -reduction, since  $\xrightarrow{h}_v \subseteq \rightarrow_v$ ) sequence has infinitely many head  $\beta_v$ -reduction steps.
- (6) $\Rightarrow$ (1) As  $M$  is strongly head  $v$ -normalizable, in particular is head  $v$ -normalizable, hence there exists  $N \in \Lambda$  head  $v$ -normal and in particular head  $\beta_v$ -normal such that  $M \xrightarrow{h}_v^* N$ . By Lemma 12.2, there exists  $L \in \Lambda$  such that  $M \xrightarrow{h}_{\beta_v}^* L \xrightarrow{h}_\sigma^* N$ . Therefore  $M \simeq_{\beta_v} L$  since  $\xrightarrow{h}_{\beta_v} \subseteq \rightarrow_{\beta_v}$ . According to Lemma 19.4,  $L$  is head  $\beta_v$ -normal.  $\blacktriangleleft$

In Theorem 21, the equivalence (3) $\Leftrightarrow$ (6) means that (weak) normalization and strong normalization are equivalent for head  $v$ -reduction (for head  $\beta_v$ -reduction they are trivially equivalent since the head  $\beta_v$ -reduction is deterministic), therefore if one is interested in studying the termination of head  $v$ -reduction, no difficulty arises from its non-determinism. The equivalence (4) $\Leftrightarrow$ (3) or (4) $\Leftrightarrow$ (6) says that the weak evaluation process defined for Plotkin's  $\lambda_v$ -calculus (the head  $\beta_v$ -reduction) terminates if and only if the weak evaluation process defined for  $\lambda_v^\sigma$  (the head  $v$ -reduction) terminates:  $\sigma$ -rules play no role in deciding the

termination of a head  $\mathbf{v}$ -reduction sequence. The equivalence (3) $\Leftrightarrow$ (2) (resp. (4) $\Leftrightarrow$ (1)) is the version for  $\lambda_v^\sigma$  (resp.  $\lambda_v$ ) of a well-known theorem for ordinary  $\lambda$ -calculus (see for example [3, Theorem 8.3.11]): in some sense, it claims that the head  $\mathbf{v}$ -reduction (resp. head  $\beta_v$ -reduction) is complete with respect to the  $\mathbf{v}$ -equivalence (resp.  $\beta_v$ -equivalence). The equivalence (5) $\Leftrightarrow$ (2) (resp. (5) $\Leftrightarrow$ (1)) can be seen as the version for  $\lambda_v^\sigma$  (resp.  $\lambda_v$ ) of the Quasi-Head Reduction Theorem [19, Theorem 2.10] stated by Takahashi for ordinary  $\lambda$ -calculus.

## 5 Normalization strategy and other results

Theorems 14 and 21 strengthen the idea that, in spite of non-determinism and non-confluence of head  $\mathbf{v}$ -reduction and non-sequentiability of head  $\sigma$ -reduction steps, the head  $\mathbf{v}$ -reduction can be used to define a normalization strategy for the  $\lambda_v^\sigma$ -calculus, as proven in next Theorem 24, the second main result of our paper: given a term  $M$ , one starts the (unique) head  $\beta_v$ -head reduction sequence from  $M$  as long as a head  $\beta_v$ -normal form  $N$  is reached (recall that, according to Theorem 21, a term is (strongly) head  $\mathbf{v}$ -normalizable if and only if it is head  $\beta_v$ -normalizable); then, one starts a head  $\sigma$ -reduction sequence from  $N$  (where head  $\sigma_1$ - and head  $\sigma_3$ -reduction steps can be performed in whatever order) as long as a head  $\sigma$ -normal form  $N'$  is reached (such a  $N'$  always exists because  $\xrightarrow{h}_\sigma$  is strongly normalizing, and it is head  $\mathbf{v}$ -normal by Lemma 19.4); finally, one performs the internal  $\mathbf{v}$ -reduction steps starting from  $N'$  by iterating the head  $\beta_v$ -reduction sequences and then the head  $\sigma$ -reduction sequences as above on the subterms of  $N'$ , from the left to the right. More precisely:

► **Definition 22** (Successors path). *Let  $M \in \Lambda$ .*

*A successor of  $M$  is a  $M' \in \Lambda$  defined by induction on  $M \in \Lambda$  as follows:*

- *if  $M$  is not head  $\beta_v$ -normal, then  $M'$  is such that  $M \xrightarrow{h}_{\beta_v} M'$ ;*
- *if  $M$  is head  $\beta_v$ -normal but not head  $\sigma$ -normal, then  $M'$  is such that  $M \xrightarrow{h}_\sigma M'$ ;*
- *if  $M$  is head  $\mathbf{v}$ -normal then:*
  - *if  $M$  is a variable then  $M' = M$ ,*
  - *if  $M = \lambda x.N$  for some  $N \in \Lambda$ , then  $M' = \lambda x.N'$  for some successor  $N'$  of  $N$ ,*
  - *if  $M = NL$  for some  $N, L \in \Lambda$ , then either  $N$  is not  $\mathbf{v}$ -normal and  $M' = N'L$  where  $N'$  is a successor of  $N$ , or  $N$  is  $\mathbf{v}$ -normal and  $M' = NL'$  where  $L'$  is a successor of  $L$ .*

*A successors path of  $M$  is an infinite sequence  $(M_i)_{i \in \mathbb{N}}$  of terms such that  $M_0 = M$  and  $M_{i+1}$  is a successor of  $M_i$ , for any  $i \in \mathbb{N}$ .*

Clearly, for every term  $M$  there is at least one successor  $M'$  of  $M$ ; moreover, this successor  $M'$  is unique when  $M$  is not head  $\beta_v$ -normal, since the head  $\beta_v$ -reduction is deterministic, and  $M = M'$  when  $M$  is  $\mathbf{v}$ -normal.

► **Remark 23.** *Let  $M \in \Lambda$  and let  $(M_i)_{i \in \mathbb{N}}$  be a successors path of  $M$ .*

1. *For every  $i \in \mathbb{N}$ , there exist  $0 \leq j \leq k \leq i$  such that  $M \xrightarrow{h}_{\beta_v}^* M_j \xrightarrow{h}_\sigma^* M_k \xrightarrow{int}_\mathbf{v}^* M_i$ .*
2. *For every  $i \in \mathbb{N}$ , if  $M_i$  is  $\mathbf{v}$ -normal then  $M_j$  is  $\mathbf{v}$ -normal for any  $j \geq i$ .*

A successors path of a term  $M$  is a call-by-value left-to-right  $\mathbf{v}$ -evaluation strategy starting from  $M$  that can reduce under a  $\lambda$  only when a head  $\mathbf{v}$ -normal form is reached. Due to the non-determinism of the head  $\sigma$ -reduction, a term  $M$  may have several successors paths. We cannot get rid of the non-determinism of the successors path of  $M$  because of the non-sequentiability of head  $\sigma$ -reductions, see p. 7 and [7, p. 10].

► **Theorem 24** (Normalization strategy). *Let  $M \in \Lambda$ . Every successors path  $(M_i)_{i \in \mathbb{N}}$  of  $M$  is a normalization strategy for  $M$ , i.e. if  $M$  is  $\mathbf{v}$ -normalizable then there exists  $j, k, \ell \in \mathbb{N}$  such that  $j \leq k \leq \ell$ ,  $M_j$  is head  $\beta_v$ -normal,  $M_k$  is head  $\mathbf{v}$ -normal and  $M_\ell$  is  $\mathbf{v}$ -normal.*

**Proof.** Let  $(M_i)_{i \in \mathbb{N}}$  be a successors path of  $M$  and  $N \in \Lambda$  be such that  $N$  is  $\mathbf{v}$ -normal and  $M \rightarrow_v^* N$ : we prove by induction on  $N \in \Lambda$  that there exist  $j, k, \ell \in \mathbb{N}$  such that  $M_j$  is head  $\beta_v$ -normal,  $M_k$  is head  $\mathbf{v}$ -normal and  $M_\ell$  is  $\mathbf{v}$ -normal.

Since  $M$  is  $\mathbf{v}$ -normalizable, then it is head  $\beta_v$ -normalizable (because  $\xrightarrow{\beta_v} \subseteq \rightarrow_v$ ), thus there exists  $j \in \mathbb{N}$  such that  $M_j$  is head  $\beta_v$ -normal because  $\xrightarrow{\beta_v}$  is deterministic. As  $\xrightarrow{\beta_v}$  is strongly normalizing (by Proposition 5, since  $\xrightarrow{\beta_v} \subseteq \rightarrow_v$ ), there exists  $k \in \mathbb{N}$  with  $j \leq k$  such that  $M_k$  is head  $\sigma$ -normal. According to Lemma 19.4,  $M_k$  is also head  $\beta_v$ -normal, hence  $M_k$  is head  $\mathbf{v}$ -normal. Certainly,  $M_k = VN_1 \dots N_n$  for some  $n \in \mathbb{N}$ ,  $V \in \Lambda_v$  and  $N_1, \dots, N_n \in \Lambda$ . By confluence of  $\rightarrow_v$  (Proposition 5) and since  $N$  is  $\mathbf{v}$ -normal and  $M_k$  is head  $\mathbf{v}$ -normal, one has  $M_k \xrightarrow{int}_v^* N$  and hence  $N = V'N'_1 \dots N'_n$  for some  $\mathbf{v}$ -normal  $V' \in \Lambda_v$  and some  $\mathbf{v}$ -normal  $N'_1, \dots, N'_n \in \Lambda$  such that  $V \rightarrow_v^* V'$  and  $N_r \rightarrow_v^* N'_r$  for any  $1 \leq r \leq n$ . By induction hypothesis, for every successors path  $(V_i)_{i \in \mathbb{N}}$  of  $V$  and, for any  $1 \leq r \leq n$ , for every successors path  $(L_i^r)_{i \in \mathbb{N}}$  of  $N^r$  there exist  $p, p_1, \dots, p_n \in \mathbb{N}$  such that  $V_p, L_{p_1}^1, \dots, L_{p_n}^n$  are  $\mathbf{v}$ -normal: by confluence of  $\rightarrow_v$  (Proposition 5),  $V_p = V'$  and  $N'_r = L_{p_r}^r$  for any  $1 \leq r \leq n$ .

Let us consider the infinite sequence of terms  $s = (M = M_0, \dots, M_k = VN_1 \dots N_n = V_0N_1 \dots N_n, \dots, V_pN_1 \dots N_n = V'L_0^1N_2 \dots N_n, \dots, V'L_{p_1}^1N_2 \dots N_n = V'N'_1L_0^2 \dots N_n, \dots, V'N'_1N'_2 \dots N'_n = N, N, \dots)$ : this is a successors path of  $M$  and, for an opportune choice of the successors paths  $(V_i)_{i \in \mathbb{N}}$ ,  $(L_i^1)_{i \in \mathbb{N}}$ ,  $\dots$ ,  $(L_i^n)_{i \in \mathbb{N}}$ , one has that  $s = (M_i)_{i \in \mathbb{N}}$ , in particular there exists  $\ell \in \mathbb{N}$  such that  $j \leq k \leq \ell$  and  $M_\ell = N$ .  $\blacktriangleleft$

In ordinary  $\lambda$ -calculus, the well-known theorem corresponding to our Theorem 24 is the Leftmost Reduction Theorem, see [19, Theorem 2.8] or [3, Theorem 13.2.2]. Differently from the leftmost reduction of ordinary  $\lambda$ -calculus, our normalization strategy is not deterministic, i.e., our Theorem 24 provides a family of normalization strategies.

Finally, we have shown at p. 5 that the head  $\sigma$ - and head  $\mathbf{v}$ -reductions are not (locally) confluent and a term may have several head  $\mathbf{v}$ -normal forms. Nevertheless, the characterization of head  $\mathbf{v}$ -normal forms given by Proposition 18 allows us to claim that (see next Proposition 27) in some cases (of interest), more precisely when a term has a head  $\mathbf{v}$ -normal form which is a value or an element of  $\Lambda_a$ , the head  $\mathbf{v}$ -normal form is unique (Proposition 27.1): all terms having several head  $\mathbf{v}$ -normal forms are such that all their head  $\mathbf{v}$ -normal forms are in  $\Lambda_b$ . In particular, every head  $\mathbf{v}$ -normalizable closed term has a unique head  $\mathbf{v}$ -normal form, which is an abstraction and coincides with its head  $\beta_v$ -normal form (Proposition 27.2).

► **Remark 25.** By inspection on the rules of Definition 8, it is easy to check that the head  $\sigma$ -reduction does not reduce to a term in  $\Lambda_a$ , i.e., for any  $M \in \Lambda$  and  $N \in \Lambda_a$ , one has  $M \not\rightarrow_\sigma N$ .

Remark 25 does not hold if we replace  $\xrightarrow{\sigma}$  with  $\xrightarrow{\beta_v}$ : for instance,  $x(II) \xrightarrow{\beta_v} xI \in \Lambda_a$ .

► **Fact 26.** For every  $N \in \Lambda_v \cup \Lambda_a$ , one has  $M \xrightarrow{\beta_v}^* N$  if and only if  $M \xrightarrow{v}^* N$ .

**Proof.** The left-to-right direction follows from  $\xrightarrow{\beta_v} \subseteq \xrightarrow{v}$ . The right-to-left direction is a consequence of Lemma 12.2 and either Fact 10.2 (if  $N \in \Lambda_v$ ) or Remark 25 (if  $N \in \Lambda_a$ ).  $\blacktriangleleft$

Fact 26 means that, given a head  $\mathbf{v}$ -reduction sequence, the head  $\sigma$ -reduction plays no role not only in deciding its termination (as stated in Theorem 21), but also in reaching a particular value or term in  $\Lambda_a$ . Fact 26 will be used in the proof of Proposition 27.

► **Proposition 27** (Uniqueness of “some” head  $\mathbf{v}$ -normal forms). *Let  $M \in \Lambda$  and  $M \xrightarrow{v}^* N$ .*

1. *If  $N \in \Lambda_v \cup \Lambda_a$  then, for every head  $\mathbf{v}$ -normal  $L \in \Lambda$ ,  $M \xrightarrow{v}^* L$  implies  $N = L$ .*
2. *If  $M$  is closed and  $N$  is head  $\mathbf{v}$ -normal, then  $M \xrightarrow{\beta_v}^* N$  and  $N = \lambda x.N'$  for some  $N' \in \Lambda$  such that  $\text{fv}(N') \subseteq \{x\}$ ; moreover, for any head  $\mathbf{v}$ -normal  $L \in \Lambda$ ,  $M \xrightarrow{v}^* L$  implies  $N = L$ .*

**Proof. 1.** Since  $N \in \Lambda_v \cup \Lambda_a$ ,  $M \xrightarrow{h}_v^* N$  implies  $M \xrightarrow{h}_{\beta_v}^* N$  by Fact 26. According to Proposition 18.3,  $N$  is head  $v$ -normal.

Let  $L \in \Lambda$  be head  $v$ -normal and such that  $M \xrightarrow{h}_v^* L$ : by Proposition 18.3,  $L \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ . We claim that  $L \notin \Lambda_b$ . Otherwise,  $L \in \Lambda_b$  and then, by confluence of  $\rightarrow_v$  there would exist  $M' \in \Lambda$  such that  $N \rightarrow_v^* M'$  and  $L \rightarrow_v^* M'$ . According to Proposition 11 and since  $N$  and  $L$  are head  $v$ -normal,  $N \xrightarrow{int}_v^* M'$  and  $L \xrightarrow{int}_v^* M'$ . By Remark 4 (since  $\xrightarrow{int}_v \subseteq \rightarrow_v$ ) and Lemma 19.1,  $M' \in \Lambda_v \cup \Lambda_a$ . By Lemma 19.2,  $M' \in \Lambda_b$ . But  $\Lambda_v \cap \Lambda_b = \emptyset = \Lambda_a \cap \Lambda_b$ : contradiction, therefore  $L \notin \Lambda_b$ .

So,  $L \in \Lambda_v \cup \Lambda_a$  and thus  $M \xrightarrow{h}_{\beta_v}^* L$  by Fact 26, hence  $N = L$  since  $\xrightarrow{h}_{\beta_v}$  is deterministic.

2. Since  $M$  is closed,  $N$  is closed too. Hence, by Proposition 18.3,  $N \in \Lambda_v$  (since the terms in  $\Lambda_a \cup \Lambda_b$  are not closed) and  $N$  is not a variable, therefore  $N = \lambda x.N'$  for some  $N' \in \Lambda$  such that  $\text{fv}(N') \subseteq \{x\}$ . By Fact 26,  $M \xrightarrow{h}_{\beta_v}^* N$ . According to Proposition 27.1, for every head  $v$ -normal  $L \in \Lambda$ ,  $M \xrightarrow{h}_v^* L$  implies  $N = L$ .  $\blacktriangleleft$

Recall that all head  $v$ -normal terms are head  $\beta_v$ -normal, since  $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_v$ .

## 6 Conclusions and future work

In this paper, we have investigated the  $\lambda_v^\sigma$ -calculus introduced in [4], an extension of Plotkin's call-by-value  $\lambda$ -calculus  $\lambda_v$  [15] with the same syntax as  $\lambda_v$  (without constants) and ordinary (i.e. call-by-name)  $\lambda$ -calculus. The peculiarity of  $\lambda_v^\sigma$  is in its reduction rules: the  $v$ -reduction adds to Plotkin's  $\beta_v$ -reduction two commutation rules called  $\sigma_1$  and  $\sigma_3$  which unblock "hidden"  $\beta_v$ -redexes. We have studied the head  $v$ -reduction, a non-confluent sub-reduction of the  $v$ -reduction already introduced in [7]. We now summarize our main contributions:

1. Theorem 21 is about head  $v$ -normalization, it shows that:
  - for the head  $v$ -reduction, normalization coincides with strong normalization;
  - the head  $v$ -reduction is deeply related to Plotkin's deterministic weak evaluation strategy for  $\lambda_v$  (the former terminates if and only if the latter terminates);
  - both head  $v$ -reduction and weak evaluation strategy for  $\lambda_v$  enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary  $\lambda$ -calculus.
2. Theorem 24 is about  $v$ -normalization: it proves that a top-down extension of the head  $v$ -normalization provides a family of normalization strategies for the (full)  $v$ -reduction.
3. Proposition 27 is about the uniqueness of the head  $v$ -normal form: it shows that, even if there are terms having several head  $v$ -normal forms, in some case of interest (for instance, closed terms) the head  $v$ -normal form, if any, is unique.

These results, together with the results proven in [4, 7], show that  $\lambda_v^\sigma$  is a useful tool to study some theoretical and semantic properties of Plotkin's  $\lambda_v$ -calculus, for instance the notions of call-by-value solvability and potential valuability. This is hard (or impossible) to obtain directly in  $\lambda_v$  because of the "weakness" of Plotkin's  $\beta_v$ -reduction. In the case of ordinary (i.e. call-by-name)  $\lambda$ -calculus, head reduction and solvability are the starting point to investigate separability, semi-separability and Böhm's trees. Hence, it may reasonably be supposed that we have all the ingredients for tackling the question of separability, semi-separability and Böhm's trees in a call-by-value setting. In particular, one may reasonably hope to improve in  $\lambda_v^\sigma$  the separability theorem already proven by Paolini [12] for  $\lambda_v$ .

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