Injectivity of relational semantics for (connected) MELL proof-nets via Taylor expansion [extended abstract, cat. 1]

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We show that: (1) the Taylor expansion of a cut-free MELL proof-structure R with atomic axioms is the (most informative part of the) relational semantics of R; (2) every (connected) MELL proof-net is uniquely determined by the element of order 2 of its Taylor expansion; (3) the relational semantics is injective for (connected) MELL proof-nets.

1 Introduction

Starting from investigations on denotational semantics of System F (second order typed λ-calculus), in 1987 Girard [7] introduced linear logic (LL), a refinement of intuitionistic logic. He defines two new modalities, ! and ?, giving a logical status to structural rules and allowing to distinguish between linear resources (i.e. usable exactly once during the cut-elimination process) and resources available at will. One of the main features of LL is the possibility of representing proofs (and λ-terms) geometrically by means of particular graphs: proof-structures. Among proof-structures it is possible to characterize "in a geometric way" the ones corresponding to proofs in LL sequent calculus through the Danos-Regnier correctness criterion (see [2]): a proof-structure corresponds to a proof in LL sequent calculus if and only if it is a proof-net, i.e. it fulfills some conditions about acyclicity and connectedness.

Ehrhard [3] introduced finiteness spaces, a denotational model of LL (and λ-calculus) which interprets formulas by topological vector spaces and proofs by analytical functions: in this model the operations of differentiation and Taylor expansion make sense. Ehrhard and Regnier [4, 5, 6] internalized these operations in the syntax and thus introduced differential linear logic DiLL (and differential λ-calculus), where the promotion rule (the only one in LL which is responsible for introducing the !-modality and hence creating resources available at will) is replaced by three "finitary" rules which are perfectly symmetric to the rules for the ?-modality: this allows a more subtle analysis of the resources consumption during the cut-elimination process. At the syntactic level, Taylor expansion decomposes a LL proof-structure in a (infinite in general) formal sum of DiLL proof-structures (diffnets), each of which contains resources usable only a fixed number of times.

Our contribution aims at looking further into the relationship between Taylor expansion and relational model (a well-known and simple denotational semantics of LL and λ-calculus). More precisely:

1. We show that, given a normal (i.e. cut-free with atomic axioms) proof-structure R of MELL (the multiplicative-exponential fragment of LL, sufficiently expressive to encode the λ-calculus), each

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element of the Taylor expansion of R can be identified with one and only one element of the set of injective points of the interpretation of R in the relational model, quotiented by the equivalence relation induced by atoms renaming (see Thm. 5 below).

2. We show that (see Thm. 9 below) every MELL proof-structure fulfilling the ACC condition (no matter with or without cuts) is uniquely determined by the element of order 2 of its Taylor expansion. Comparing (intuitively) to mathematical analysis, this would correspond to saying that analytical functions fulfilling some condition are uniquely determined by their second derivatives.
3. As a corollary of points 1 and 2, we show that the relational model is injective with respect to MELL proof-nets: given two ACC normal MELL proof-structures, if they have the same relational interpretation then they are identical (see Thm. 10 below). A similar result has already been proven in [1] but following a completely different (and more complicated) approach.

This study also pushes towards a deeper understanding of the Taylor expansion of MELL proof-structures as a bridge between syntax and semantics, which should lead to a more abstract and synthetic representation of this operation (see also [8]).

2 Preliminaries

In order to present our results, one can refer to any notion of MELL proof-structure and DiLL differential net[1] (*diffnet* for short). Actually, in the sequel we refer to the notion of proof-structure presented in [1] and the notion of diffnet presented in [5] (which can be reformulated in the more precise terms of [1]), both using the hypergraph-like syntax of interaction nets (where links are cells, i.e. oriented hyper-edges labeled by MELL connectives, and premises and conclusions of a link are ports, i.e. nodes, see [9, 11]). A proof-structure or diffnet is given with an order on its conclusions and we restrict in this extended abstract to the typed case: every port is labeled by a MELL formula A ($A ::= \alpha \mid \alpha^{\perp} \mid A \parr A \mid A \otimes A \mid ?A \mid !A$).

When drawing a proof-structure or diffnet, we use generalized ?-cells and, for diffnets, generalized !-cells (see [10]) and we order its conclusions from left to right. Also we represent wires (i.e. edges connecting two ports) oriented top-down so that we can speak of cells or wires "above" a given cell/wire.

Definition 1. *For any $n \in \mathbb{N}$, a n-*diffnet *is a diffnet such that every* !*-cell has exactly n premises.*

A MELL proof-structure is then a 1-diffnet with a box-function associating with every !-cell o (whose premise is the *principal door* of the box of o) a set of premises of ?-cell, the *auxiliary doors* of the box of o, in such a way that the nesting condition is fulfilled (see [7, 1]). The (principal or auxiliary) doors of a box B of a !-cell represent the frontier of B: all that is above the doors of B is "inside" B.

Equality between MELL proof-structures is isomorphism of hypergraphs (see [1]).

Definition 2. *Let ρ be a diffnet. A* correctness hypergraph *of ρ is the (undirected) hypergraph obtained from ρ by disconnecting all the premises but one of each* ?*- and* $\parr$*-cell. We say that ρ* is ACC *or ρ* satisfies the correctness criterion *if every correctness hypergraph of ρ is acyclic and connected.*

Let R be a MELL *proof-structure: R is a* proof-net *when R satisfies the* ACC *condition defined as usual by induction on the depth of R (see for example [13, Def. A.6 and Rmk. A.7]).*

A MELL proof-structure is *normal* if it is cut-free and with atomic axioms (i.e. the conclusions of each axiom of R are labeled by dual atomic formulas).

We denote by $\Gamma, \Delta, \dots$ any finite sequence of MELL formulas. Given a finite sequence of MELL formulas $\Gamma = (A_1, \dots, A_n)$ for some $n \in \mathbb{N}^+$, we set $\parr\Gamma = A_1 \parr \cdots \parr A_n$.

[1] For us, diffnets of DiLL are "promotion-free": they may contain multiplicative ($\parr$- and $\otimes$-) links, structural (?-) links and co-structural (!-) links but not boxes.

3 Relational model and Taylor expansion

Let X be an infinite set, whose elements are called *atoms*. In the typed case considered here, the relational model associates the set X with every atomic MELL formula. The interpretation of the other MELL formulas is defined in the well-known way, by induction (for instance, see [13, Def. B.1]). The elements of a set A interpreting a MELL formula are called *points of A*.

Definition 3. *A point x of a set A interpreting a* MELL *formula is* injective*, when every atom occurring in x occurs exactly twice in x.*[2] *For $M \subseteq A$, we denote by M_{inj} the set of injective points of A belonging to M.*

Every bijective function $\sigma : X \to X$ induces in the obvious way a bijective function $\sigma_A : A \to A$, for any set A interpreting a MELL *formula. We denote by $\mathscr{S}_A$ the set of such bijective functions σ_A.*

If A is the interpretation of a MELL *formula, we denote by $\sim_A$ the equivalence relation on A_{inj} defined by: $x \sim_A y \iff \exists \sigma_A \in \mathscr{S}_A \ (x = \sigma_A(y))$.*

Roughly speaking, given a set A interpreting a MELL formula, the equivalence relation $\sim_A$ on A_{inj} identifies any two injective points of A that are equal up to renaming of their atoms.

When R is a MELL proof-structure with conclusion Γ, we denote by $[\![R]\!]$ the interpretation of R in the relational semantics, i.e. the subset of points of the set interpreting $\parr\Gamma$ which are results of the experiments of R; for more details, see [1, Def. 24 and 26]. It is well-known that $[\![R]\!]$ is a morphism from an arbitrary singleton set to the set interpreting $\parr\Gamma$ in the category **Rel** of sets and relations, and it is invariant under cut-elimination and η-expansion (i.e. the substitution of every axiom with conclusions $A, A^\perp$ with the standard proof of $A, A^\perp$ where the conclusions of every axiom are now typed by dual atomic formulas).

Definition 4. *Let R be a* MELL *proof-structure. We denote by $\tau(R)$ the* Taylor expansion *of R.*[3] *Given a $\rho \in \tau(R)$, one can define a function $\tau_{\rho,R}$ associating with every cell of ρ the "corresponding" cell of R. For every $n \in \mathbb{N}$, the n-*diffnet *of R is the (unique) element of $\tau(R)$ which is a n-diffnet.*

Intuitively, given a MELL proof-structure R, an element ρ of $\tau(R)$ is obtained by replacing each box B of R with n_B copies of its content, recursively (for any $n_B \in \mathbb{N}$), so that the function $\tau_{\rho,R}$ establishing the correspondence between the cells of ρ and R can be naturally defined. The n-diffnet of R is then the element of $\tau(R)$ obtained from R by taking n copies of the content of every box of R.

Theorem 5. *For every normal* MELL *proof-structure R with conclusion Γ, let A be the set interpreting the formula $\parr\Gamma$. One has that $[\![R]\!]_{\mathrm{inj}}/\!\sim_A \simeq \tau(R)$.*[4]

Thm. 5 (for the proof, see [12]) says that, for a normal MELL proof-structure R, every element of $\tau(R)$ is a canonical representative of the equivalence class (generated by atoms renaming) of some injective point of $[\![R]\!]$, presented in a geometrical way. In this sense, the Taylor expansion of a normal MELL proof-structure is an object between syntax and semantics.

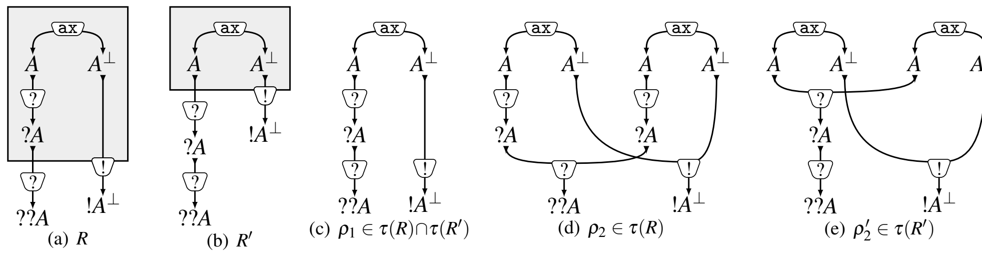
4 Empires for differential nets

The notion of empire is a well-known tool introduced by Girard in [7] in order to prove the sequentialization theorem for ACC proof-structures of the multiplicative fragment of LL. We adapt this notion to diffnets.

[2] In particular, notice that the empty multiset [], which is a point – for example – of the set interpreting the MELL-formula $?\alpha$, is an injective point.

[3] See for example [10, Def. 9] and [11, Def. 5] for details. Notice that the Taylor expansion defined in [6] was given in terms of linear combination of resource λ-terms with scalars in $\mathbb{Q}^{\geq 0}$. With respect to the results achieved in our work, scalars play no role, hence we do not tackle coefficients issue, and we will define Taylor expansions as sets of diffnets, as in [10, 11].

[4] Here $[\![R]\!]_{\mathrm{inj}}/\!\sim_A \simeq \tau(R)$ means that there is a "canonical way" to associate with every element of $[\![R]\!]_{\mathrm{inj}}/\!\sim_A$ an element of $\tau(R)$ and vice-versa.



(a) R (b) R' (c) $\rho_1 \in \tau(R)\cap\tau(R')$ (d) $\rho_2 \in \tau(R)$ (e) $\rho_2' \in \tau(R')$

Figure 1: Two different proof-nets R (Fig. 1(a)) and R' (Fig. 1(b)), having the same 1-diffnet ρ_1 (Fig. 1(c)) but two different 2-diffnets, ρ_2 (Fig. 1(d)) and ρ_2' (Fig. 1(e)) respectively.

Definition 6. *Let ρ be a diffnet and let p be a premise of some cell of ρ. If G is a correctness hypergraph of ρ, the p-*correctness hypergraph of ρ in G *is the hypergraph obtained from G by disconnecting p from its cell. The* empire of p *is the hypergraph obtained from ρ by keeping only ports q (resp. cells l; wires w) such that, for every correctness hypergraph G, q (resp. l; w) is connected to p in the p-correctness hypergraph of ρ in G.*

If ρ is a ACC diffnet and p is a premise of some cell of ρ, then the empire of p is the largest sub-diffnet of ρ having p among its conclusions. Empires play a crucial role in the proof of the following proposition.

Proposition 7. *Let ρ_2 (resp. ρ_1) be a 2-diffnet (resp. 1-diffnet), and let R and R' be two* MELL *proof-nets. If $\{\rho_1, \rho_2\} \subseteq \tau(R) \cap \tau(R')$, then $R = R'$.*

PROOF. Since R and R' have the same 1-diffnet ρ_1, they might only differ in their box-functions. The frontier of the box of a !-cell o of R (and thus the image of the box-function applied to o) coincides with the frontier of the empire of any of the two premises of any !-cell of ρ_2 which is an element of $\tau^{-1}_{\rho_2,R}(o)$. As an example, consider the proof-nets in Fig. 1: as stated, for R (resp. R') one can compute the frontier of the unique box by means of the empire of any of the two premises of the unique !-cell of ρ_2 (resp. ρ_2'). □

Notice that empires do not give the correct information about the frontier of boxes in 1-diffnets: this is the reason why in the statement of Prop. 7 we require that $\tau(R)$ and $\tau(R')$ have not only the same 1-diffnet but also the same 2-diffnet. In Fig. 1, R and R' are two different proof-nets having the same 1-diffnet ρ_1 but different 2-diffnets (ρ_2 and ρ_2' respectively): the empire of the premise of the !-cell in ρ_1 takes over the two ?-cells, but in R (resp. R') the box incorporates only the upper ?-cell (resp. no ?-cells).

In the sequel, for a MELL proof-net R and any $k \in \mathbb{N}$, we denote by $k(R)$ the k-diffnet of R.

Lemma 8. *Let R and R' be two proof-nets. If $2(R) = 2(R')$ then $1(R) = 1(R')$.*

The idea to prove Lemma 8 is that, given a proof-net R and $\rho \in \tau(R)$ such that every !-cell of ρ has more than one premise, if ρ' is the diffnet obtained from ρ by erasing the empire of a premise of a !-cell, then $\rho' \in \tau(R)$. A useful tool to prove that is the proto-Taylor expansion, a notion introduced in [8].

Theorem 9. *Let R and R' be two proof-nets. If $2(R) = 2(R')$, then $R = R'$.*

PROOF. By Lemma 8, R and R' have the same 1-diffnet. Then apply Prop. 7. □

5 Injectivity of relational semantics for MELL proof-nets

Given a syntactic logical system and a denotational model for it, the question of injectivity of the semantics naturally arises: do the equivalence relation on proofs defined by cut-elimination and η-expansion procedures and the one defined by the model coincide? When the answer is positive one says that the model is *injective* (it separates syntactically different proofs). Indeed, two proofs are "syntactically" equivalent when (roughly speaking) they have the same cut-free η-expanded form (in a confluent and weakly normalizing system), and they are "semantically" equivalent in a given denotational model when they have the same interpretation. In the framework of LL, the question of injectivity (of coherent and relational semantics) was first addressed and studied in [13].

Theorem 10 (Injectivity)**.** *Let R and R' be two normal proof-nets. If $[\![R]\!] = [\![R']\!]$ then $R = R'$.*

PROOF. By Thm. 5, $\tau(R) = \tau(R')$ and hence R and R' have the same 2-diffnet. By Thm. 9, $R = R'$. □

The injectivity of the relational model for MELL proof-nets has already recently been proved by de Carvalho and Tortora de Falco in [1]. Our proof, which represents a remarkable simplification, follows a completely different approach based on the notion of Taylor expansion. All these works fit in the general perspective of abolishing the old traditional distinction between syntax and semantics.

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