A new viewpoint on the Taylor expansion of proof-structures and uniformity

Giulio Guerrieri
Laboratoire PPS, UMR 7126
Université Paris Diderot, Sorbonne Paris Cité
F-75205 Paris, France
giulio.guerrieri@pps.univ-paris-diderot.fr

Lorenzo Tortora de Falco
Dipartimento di Matematica e Fisica
Università Roma Tre
Rome, Italy
tortora@uniroma3.it

We introduce (in the differential extension of linear logic) the notion of proto-net and proto-Taylor expansion. This approach suggests a slightly different viewpoint on the Taylor expansion of proof-structures. We then define a coherence relation on proto-nets and prove the analogue of the result proven for resource lambda-terms, which does not hold for differential nets: a set of proto-nets is included in the proto-Taylor expansion of some proof-structure if and only if it is a clique.

1 Introduction

One of the main features of Linear Logic (LL) [9] is the logical status given to structural rules, thus stressing the difference between the linear and the non linear use of resources. With the discovery of Differential λ-calculus [6] and Differential Linear Logic (DiLL)[1] [7], thanks to the notion of Taylor expansion [5, 8] a λ-term (resp. a proof-structure, *ps* for short)[2] is decomposed into a (usually infinite) set[3] of "(products of) linear terms" (resp. "(products of) linear nets"), called resource λ-terms (resp. *diffnets* or DiLL_0-*ps*, i.e. the proof-structures of DiLL_0). One of the aims becomes then to extract informations on the λ-term (resp. the ps) from its "linear components"; by the way, using elements of the Taylor expansion amounts to use points of the interpretation in the relational model (a denotational semantics based on the category of sets and relations): the link between the two approaches is more or less obvious, and it is precisely stated in an ongoing work with Luc Pellissier (a very first draft can be found in [15]).

Here a remarkable difference between the λ-calculus and LL appears. In [8] a binary symmetric relation (called *coherence*) on resource λ-terms is defined, allowing to characterize those sets of resource λ-terms that are subsets of the Taylor expansion of some (ordinary) λ-term (see also [1]). This is impossible for diffnets, as shown by the counterexample of [16, p. 244]: there exist three diffnets such that every pair of them belongs to the Taylor expansion of some ps but there is no ps containing in its Taylor expansion the three diffnets. This counterexample can be generalized to show the impossibility of any *n*-ary coherence relation for diffnets. This mismatch between λ-calculus and LL is clearly due to the fact that while λ-terms have a natural tree-like structure, this is not the case for ps.

[1] DiLL is the extension of MELL (the multiplicative-exponential fragment of LL) with the same language as MELL and provided with both promotion rule (i.e. boxes) and co-structural rules (i.e. the duals of the structural rules dealing with ? modality) to handle !-modality. MELL (resp. DiLL_0) can be seen as the fragment of DiLL without co-structural rules (resp. without boxes).

[2] In this Introduction, when we talk about ps, the reader can refer to MELL-ps (proof-structures of MELL) or more generally DiLL-ps (proof-structures of DiLL). For a formal definition of ps, see Definition 12. We use the expression "proof-structure" like in [9, 3], instead of "proof-net" or "net", to stress that correctness criterion plays no role in this paper.

[3] In its original formulation [8] the Taylor expansion of a λ-term is a (usually infinite) linear combination of resource λ-terms with scalars in some semiring. With respect to the results achieved in our work, scalars play no role, so we do not tackle coefficients issue, and we will define the Taylor expansion of a DiLL-ps as a set (and not a linear combination) of DiLL_0-ps, like in [12, 14].

Submitted to:
Linearity 2014

Notice, however, that there exists a tree-like structure underlying a given ps: the nesting of its boxes. A possible way of getting some intuition on the new object introduced in this paper (the *proto-nets* of Definition 1) is to forget everything but the nesting structure of a ps R (thus obtaining the tree of its boxes), and apply then the Taylor expansion (that will be called *proto-Taylor expansion* in Definition 22, denoted by $\mathscr{T}_R^{\mathsf{proto}}$). A ps with depth 0 becomes the singleton of an empty sequence, and a box becomes the infinite set of the copies of its content: in case the box contains a ps with depth 0, its proto-Taylor expansion can be seen as the set of finite multisets containing empty sequences. In Figures 1, 2(a) and 2(b), we give an example of ps R, its ordinary Taylor expansion $\mathscr{T}_R$ and its proto-Taylor expansion $\mathscr{T}_R^{\mathsf{proto}}$.

With a representative R' (Definition 21) of a ps R and $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$ is associated a unique $\rho \in \mathscr{T}_R$: the Taylor expansion can thus be computed pointwise (Definition 23). Our approach can be compared to the computation of the interpretation of a ps pointwise using the notion of *experiment* [9, 17, 2, 3]. With the usual definitions (such as the ones given in [12, 14]), in order to compute the Taylor expansion of the ps S obtained by applying a $\otimes$ rule to two occurrences R_1 and R_2 of the ps R of Figure 1 (thus obtaining a ps with conclusion $??A, !A^\perp \otimes !A^\perp, ??A$), one had to compute the whole Taylor expansion $\mathscr{T}_{R_1}$ of the box R_1, the whole Taylor expansion $\mathscr{T}_{R_2}$ of the box R_2, and then apply a $\otimes$-cell to every pair of elements of $\mathscr{T}_{R_1}$ and $\mathscr{T}_{R_2}$. The passage through the proto-Taylor expansion $\mathscr{T}_S^{\mathsf{proto}}$ of S allows instead to compute separately every element of $\mathscr{T}_S$ from S and an element of $\mathscr{T}_S^{\mathsf{proto}}$ ($\mathscr{T}_S^{\mathsf{proto}}$ can itself be computed pointwise, see Defininition 22).

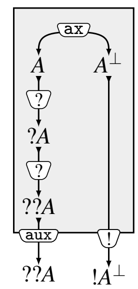


Figure 1: The proof-structure R

Since one could also show that with a representative R' of a ps R and $\rho \in \mathscr{T}_R$ is associated a unique $\alpha \in \mathscr{T}_R^{\mathsf{proto}}$, it is natural to wonder what is the relationship between diffnets and proto-nets: a proto-net is "less" than a diffnet since (for example) the multiplicative structure (which can be preserved in diffnets) is always lost in proto-nets, but it is also "more" because it comes equipped with a tree-like structure (reminiscent of the nesting box-structure of a ps) which is absent in diffnets.

Once the set of proto-nets (**Proto**, Definition 1) is introduced, one defines a coherence relation on **Proto** (Definition 7): like in [8], the idea is that two proto-nets are coherent when they differ only by the cardinality of their "corresponding finite multisets" (actually, they are finite sequences of the shape $\langle \cdot \rangle$ with the same address, see Definition 4). A clique is a set of proto-nets which are pairwise coherent. One can then prove the expected results that hold for resource λ-terms (see [8]) and not for diffnets:

1. for every representative R' of any ps R with depth d, the set $\mathscr{T}_{R'}^{\mathsf{proto}}$ is a maximal clique with depth d (Proposition 31);
2. any clique Γ with finite depth d is contained in $\mathscr{T}_{R'}^{\mathsf{proto}}$ for some representative R' of some ps R with depth d and some (Corollary 35); for this purpose, we define the merging of a clique (Definition 32).

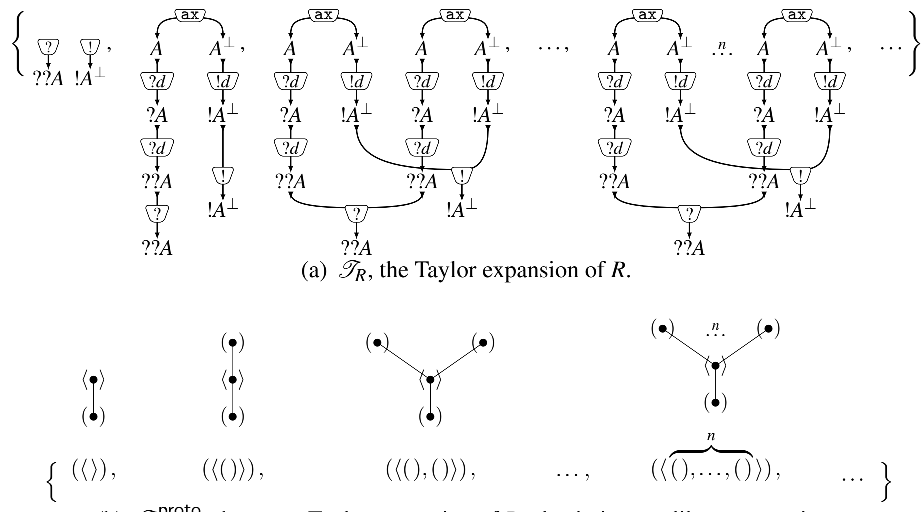
Preliminaries and notations

We set $\mathscr{L}_{\mathsf{MELL}} = \{1, \perp, \otimes, \parr, box, ?, ?d, ax, cut\}$ and $\mathscr{L}_{\mathsf{DiLL}} = \mathscr{L}_{\mathsf{MELL}} \cup \{!, !d\}$.

Let $\mathscr{V}_{\mathsf{MELL}}$ be a countably infinite set whose elements, denoted by $X, Y, Z, \ldots$, are called *propositional variables*. The set $\mathscr{F}_{\mathsf{MELL}}$ of MELL *formulas*, denoted by $A, B, C, \ldots$, is generated by the grammar:

$$A, B, C ::= X \mid X^\perp \mid 1 \mid \perp \mid A \otimes B \mid A \parr B \mid !A \mid ?A\,.$$

For every $A \in \mathscr{F}_{\mathsf{MELL}}$, the *dual*, or *negation*, *of* A, denoted by $(A)^\perp$ or $A^\perp$, is defined by induction as follows: $(X)^\perp = X^\perp$, $(X^\perp)^\perp = X$, $(1)^\perp = \perp$, $(\perp)^\perp = 1$, $(A \otimes B)^\perp = (A)^\perp \parr (B)^\perp$, $(A \parr B)^\perp = (A)^\perp \otimes (B)^\perp$, $(!A)^\perp = ?(A)^\perp$ and $(?A)^\perp = !(A)^\perp$. Therefore, $A^{\perp\perp} = A$ for any $A \in \mathscr{F}_{\mathsf{MELL}}$.



(a) $\mathscr{T}_R$, the Taylor expansion of R.

(b) $\mathscr{T}_R^{\mathsf{proto}}$, the proto-Taylor expansion of R, also in its tree-like presentation.

Figure 2: The Taylor and proto-Taylor expansion of R (see Figure 1 for R).

Notation. Finite sequences are denoted by $(a_1,\dots,a_n)$ or $\langle a_1,\dots,a_n\rangle$ (and we say that $a_1,\dots,a_n$ are the *components* of the finite sequence); in particular, the empty sequence is denoted by $(\,)$ or $\langle\,\rangle$; for any $n \in \mathbb{N}$ we set $a^n = (a,\dots,a)$ (n times a). Concatenation is denoted by $\cdot$: if $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_m)$ then $a{\cdot}b = (a_1,\dots,a_n,b_1,\dots,b_m)$; if $n = 1$ (resp. $m = 1$), then $a_1{\cdot}b$ (resp. $a{\cdot}b_1$) stands for $a{\cdot}b$. The length of a finite sequence $a = (a_1,\dots,a_n)$ is $\mathsf{length}(a) = n \in \mathbb{N}$. Finite sequences over $\mathbb{N}$ (used here only for addresses, see Definition 4) are written as words, the empty word is denoted by ε.

For every set $\mathscr{A}$, $\mathfrak{P}(\mathscr{A})$ is the power set of $\mathscr{A}$ and, if $\mathscr{A}$ is finite, $\mathsf{card}(\mathscr{A})$ is the cardinality of $\mathscr{A}$.

Let $\mathscr{A},\mathscr{B}$ be sets and $\mathsf{f}\colon \mathscr{A} \to \mathscr{B}$ be a function. We identify f with its graph; in particular, $\emptyset$ denotes also the empty function. We set $\mathsf{im}(\mathsf{f}) = \{\mathsf{f}(a) \mid a \in \mathscr{A}\}$. The function $\bar{\mathsf{f}}\colon \mathfrak{P}(\mathscr{A}) \to \mathfrak{P}(\mathscr{B})$ is defined by $\bar{\mathsf{f}}(\mathscr{A}') = \{\mathsf{f}(a) \mid a \in \mathscr{A}'\}$ for any $\mathscr{A}' \subseteq \mathscr{A}$. Given $\mathscr{A}' \subseteq \mathscr{A}$, the function $\mathsf{f}\!\upharpoonright_{\mathscr{A}'}\colon \mathscr{A}' \to \mathscr{B}$ is defined by $\mathsf{f}\!\upharpoonright_{\mathscr{A}'}(a) = \mathsf{f}(a)$ for any $a \in \mathscr{A}'$.

If f is an enumeration of a finite set $\mathscr{A}$, i.e. a bijection from $\{1,\dots,\mathsf{card}(\mathscr{A})\}$ into $\mathscr{A}$, then f is also denoted like a finite sequence $(\mathsf{f}(1),\dots,\mathsf{f}(\mathsf{card}(\mathscr{A})))$. For every set $\mathscr{A}$, we denote by $\mathsf{id}_{\mathscr{A}}$ the identity function on $\mathscr{A}$. For any $n \in \mathbb{N}$, we denote by $\mathfrak{S}_n$ the set of permutations of $\{1,\dots,n\}$.

2 Proto-nets

Roughly speaking, a proto-net is a finite sequence of the shape $(\cdot)$ (called $(\cdot)$-*sequence*) whose components are finite sequences of the shape $\langle\cdot\rangle$ (called $\langle\cdot\rangle$-*sequences*) whose components are proto-nets (the definition is by induction, with a natural notion of depth). Even if they are the same mathematical object, $(\cdot)$-sequences play a completely different role from $\langle\cdot\rangle$-sequences, as we will see in Section 3.2. Actually, a $\langle\cdot\rangle$-sequence can be seen as a finite multiset (intuitively it says how many copies to take of the content a box), but we preferred to define them as sequences in order to facilitate establishing the relationship between a proto-net and a (representative of a) proof-structure (see Definitions 22 and 23).

Definition 1 (Proto-net and depth)**.** *The elements of the set* **Proto***, called* proto-nets *and denoted by* $\alpha,\beta,\gamma,\dots$*, are defined by induction as follows: given* $n,k_1,\dots,k_n \in \mathbb{N}$ *and, for any* $1 \le i \le n$*, given* $\alpha^i_1,\dots,\alpha^i_{k_i} \in$ **Proto***, one has that* $(\langle\alpha^1_1,\dots,\alpha^1_{k_1}\rangle,\dots,\langle\alpha^n_1,\dots,\alpha^n_{k_n}\rangle) \in$ **Proto**.[4]

For every $\alpha = (\langle\alpha^1_1,\dots,\alpha^1_{k_1}\rangle,\dots,\langle\alpha^n_1,\dots,\alpha^n_{k_n}\rangle) \in$ **Proto** *(for some* $n,k_1,\dots,k_n \in \mathbb{N}$*), the* depth of α*, denoted by* $\mathsf{depth}(\alpha)$*, is a natural number defined by induction on* α *as follows:*[5]

$$\mathsf{depth}(\alpha) = \sup\Big\{ \sup_{1\le j\le k_i} \{\mathsf{depth}(\alpha^i_j)\} + 1 \;\Big|\; 1 \le i \le n \Big\}. \tag{1}$$

Given $\Gamma \subseteq$ **Proto***, the* depth of Γ *is* $\mathsf{depth}(\Gamma) = \sup\{\mathsf{depth}(\alpha) \mid \alpha \in \Gamma\}$.

Intuitively, the depth of a proto-net α is the maximal number of nested $\langle\cdot\rangle$-sequences occurring in α. For $\alpha \in$ **Proto** (resp. $\Gamma \subseteq$ **Proto**), one has $\mathsf{depth}(\alpha) \in \mathbb{N}$ (resp. $\mathsf{depth}(\Gamma) \in \mathbb{N}\cup\{\infty\}$). If $\Gamma \subseteq$ **Proto** is such that $\mathsf{depth}(\Gamma) = \infty$ (resp. $\Gamma = \emptyset$), then Γ is an infinite set (resp. $\mathsf{depth}(\Gamma) = 0$).

Example 2. The only proto-net with depth 0 is $(\,)$; on the other hand, $(\langle(\,)\rangle)$ and $(\langle(\,),(\,)\rangle,\langle\,\rangle)$ and $(\langle\,\rangle)$ are proto-nets with depth 1; $\big(\langle(\,),(\,)\rangle,\langle\,\rangle,\langle(\,),(\,),(\langle(\,)\rangle,\langle\,\rangle)\rangle\big)$ is a proto-net with depth 2; neither $\langle\,\rangle$ nor $\langle(\,),(\,)\rangle$ are proto-nets. Other examples of proto-nets are in Figures 2(b), 3 and 4.

Definition 3. *Let* $n,k_1,\dots,k_n \in \mathbb{N}$ *and let* $\alpha = (\langle\alpha^1_1,\dots,\alpha^1_{k_1}\rangle,\dots,\langle\alpha^n_1,\dots,\alpha^n_{k_n}\rangle)$ *be a proto-net.*

The sets $\mathsf{sub}_{(\cdot)}(\alpha)$ *and* $\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$ *are defined by induction on* $\mathsf{depth}(\alpha) \in \mathbb{N}$ *as follows:*

$$\mathsf{sub}_{(\cdot)}(\alpha) = \{\alpha\} \cup \bigcup_{\substack{1\le i\le n\\ 1\le j\le k_i}} \mathsf{sub}_{(\cdot)}(\alpha^i_j)\,; \qquad \mathsf{sub}_{\langle\cdot\rangle}(\alpha) = \big\{\langle\alpha^1_1,\dots,\alpha^1_{k_1}\rangle,\dots,\langle\alpha^n_1,\dots,\alpha^n_{k_n}\rangle\big\} \cup \bigcup_{\substack{1\le i\le n\\ 1\le j\le k_i}} \mathsf{sub}_{\langle\cdot\rangle}(\alpha^i_j).$$

For any $\alpha \in$ **Proto**, $\mathsf{sub}_{(\cdot)}(\alpha)$ (resp. $\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$) is the set of the $(\cdot)$-(resp. $\langle\cdot\rangle$-)subsequences of α.

Definition 4 (Addresses in a proto-net)**.** *Let* $\alpha = (\langle\alpha^1_1,\dots,\alpha^1_{k_1}\rangle,\dots,\langle\alpha^n_1,\dots,\alpha^n_{k_n}\rangle) \in$ **Proto** *(for some* $n,k_1,\dots,k_n \in \mathbb{N}$*) and let* x *be an occurrence in* α *of an element of* $\mathsf{sub}_{(\cdot)}(\alpha)\cup\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$*. The* address of x in α *is a finite sequence* $\mathsf{addr}_\alpha(x)$ *over* $\mathbb{N}^+$ *defined by induction on* $\mathsf{depth}(\alpha) \in \mathbb{N}$ *as follows:*

- *if* $x = \alpha$ *then* $\mathsf{addr}_\alpha(x) = \varepsilon$*;*
- *if* $x = \langle\alpha^i_1,\dots,\alpha^i_{k_i}\rangle$ *for some* $1 \le i \le n$*, then* $\mathsf{addr}_\alpha(x) = i$*;*
- *if* $x \in \mathsf{sub}_{(\cdot)}(\alpha^i_j)\cup\mathsf{sub}_{\langle\cdot\rangle}(\alpha^i_j)$ *and* $\mathsf{addr}_{\alpha^i_j}(x) = \sigma$ *with* $1 \le i \le n$ *and* $1 \le j \le k_i$*, then* $\mathsf{addr}_\alpha(x) = i\cdot\sigma$.

Let $\Gamma \subseteq$ **Proto** *and let* x *be an occurrence in some* $\alpha \in \Gamma$ *of an element of* $\mathsf{sub}_{(\cdot)}(\alpha)\cup\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$*: we set* $\mathsf{addr}_\Gamma(x) = \mathsf{addr}_\alpha(x)$.[6]

The basic idea is that in a proto-net α, different occurrences of elements of $\mathsf{sub}_{(\cdot)}(\alpha)$ in the same occurrence of an element of $\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$ represent different copies of the "same thing" and have hence the same address (and indeed with the notations of Definition 4, for every $1 \le i \le n$ and every $1 \le j \le k_i$ one has $\mathsf{addr}_\alpha(\alpha^i_j) = i$). Definition 4 generalizes this idea.[7]

Example 5. The proto-net $\alpha = \big(\langle(\,),(\,)\rangle,\langle\,\rangle,\langle(\langle(\,),(\,)\rangle),(\,),(\langle(\,)\rangle,\langle\,\rangle)\rangle\big)$ (with $\mathsf{depth}(\alpha) = 2$) is represented as a tree in Figure 3, where we have also specified the addresses in α of all occurrences of elements of $\mathsf{sub}_{(\cdot)}(\alpha)\cup\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$ in α.

[4] In particular, $(\,) \in$ **Proto** (take $n = 0$). More generally, $\langle\,\rangle^n \in$ **Proto** for every $n \in \mathbb{N}$ (take $k_1 = \cdots = k_n = 0$; recall that $\langle\,\rangle^n = (\langle\,\rangle,\dots,\langle\,\rangle)$ with n times $\langle\,\rangle$): these are the base cases of the inductive definition.

[5] Recall that for $A \subseteq \mathbb{N}$, if $A = \emptyset$ then $\sup(A) = 0$. Thus, Identity (1) takes also into account the cases where $n = 0$ or $n > 0$ with $k_1 = \ldots = k_n = 0$.

[6] There is no ambiguity in this definition because x is an *occurrence in* α of an element of $\mathsf{sub}_{(\cdot)}(\alpha)\cup\mathsf{sub}_{\langle\cdot\rangle}(\alpha)$.

[7] One can prove by induction on $\mathsf{depth}(\alpha) \in \mathbb{N}$ that for every $\alpha \in$ **Proto**, if x (resp. y) is an occurrence in α of some $s_x \in \mathsf{sub}_{(\cdot)}(\alpha)$ (resp. $s_y \in \mathsf{sub}_{\langle\cdot\rangle}(\alpha)$) and if x is a component of the sequence y, then $\mathsf{addr}_\alpha(x) = \mathsf{addr}_\alpha(y)$.

$$(\langle(),()\rangle,\langle\rangle,\langle((\langle(),()\rangle),(),(\langle()\rangle,\langle\rangle))\rangle) \quad \rightsquigarrow$$

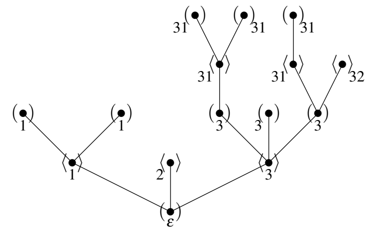


Figure 3: Example of a proto-net, also in its tree-like presentation with addresses.

Remark 6. Let $\alpha \in \mathbf{Proto}$. One can prove by straightforward induction on $\mathsf{depth}(\alpha) \in \mathbb{N}$ that $\mathsf{depth}(\alpha) = \sup\{\mathsf{length}(\mathsf{addr}_\alpha(x)) \mid x$ is an occurrence in α of an element of $\mathsf{sub}_{\langle\cdot\rangle}(\alpha)\}$.

We define a coherence relation on **Proto**: two proto-nets are coherent when all occurrences of $(\cdot)$-sequences with the same address have the same length. The reader can easily check that this is the case for any two elements of the set represented in Figure 2(b).

Definition 7 (Coherence, uniformity, clique)**.** *The* coherence relation*, denoted by* $\frown$*, is a binary relation on* **Proto** *defined as follows: given* $\alpha,\beta \in \mathbf{Proto}$*,* $\alpha \frown \beta$ *(say "*α *and* β *are coherent") if for any occurrence* x *(resp.* y*) in* α *or* β *of an element* s_x *(resp.* s_y*) of* $\mathsf{sub}_{(\cdot)}(\alpha) \cup \mathsf{sub}_{(\cdot)}(\beta)$*,* $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$ *implies that* $\mathsf{length}(s_x) = \mathsf{length}(s_y)$.

A proto-net α *is* uniform *if* $\alpha \frown \alpha$*. A set* $\Gamma \subseteq$ **Proto** *is a* clique *(*of **Proto***) if* $\alpha \frown \beta$ *for all* $\alpha,\beta \in \Gamma$*.*

Roughly speaking, two coherent proto-nets can differ recursively only in the length (possibly 0) of the occurrences of their $\langle\cdot\rangle$-subsequences having the same address. In general, two coherent proto-nets might have different depth, see for instance α and β in Example 8.

Example 8. Let $\alpha = (\langle(\langle\rangle)\rangle)$, $\beta = (\langle\rangle)$ and $\gamma = \big(\langle(),(\langle\rangle)\rangle\big)$: $\alpha \frown \alpha$ and $\beta \frown \beta$ but $\alpha \not\frown \gamma \not\frown \beta$ and $\gamma \not\frown \gamma$ because the occurrences in γ of $()$ and $(\langle\rangle)$ have the same address but not the same length; moreover $\alpha \frown \beta$ with $\mathsf{depth}(\alpha) = 2 \neq 1 = \mathsf{depth}(\beta)$.

Note that the coherence relation is symmetric but neither reflexive (the proto-nets α in Example 5 and γ in Example 8 are not uniform) nor transitive (for instance, take $\alpha = (\langle(\langle\rangle)\rangle)$, $\beta = (\langle\rangle)$ and $\gamma = (\langle()\rangle)$: $\alpha \frown \beta$ and $\beta \frown \gamma$ but $\alpha \not\frown \gamma$). In particular, a singleton of a proto-net is not necessarily a clique.

Remark 9. Given $\alpha,\beta \in \mathbf{Proto}$, if $\alpha \frown \beta$ then α and β are uniform; in particular, all elements of a clique are uniform. Moreover, $()$ is uniform (since $\mathsf{sub}_{(\cdot)}(()) = \{()\}$) and $\{()\}$ is a maximal[8] clique: for every $\gamma \in \mathbf{Proto}$, either $\gamma = ()$ or $\gamma \not\frown ()$. Following Definition 22, $\mathscr{T}_R^{\mathsf{proto}}$ of Figure 2(b) (note that R, defined in Figure 1, admits a unique representative, see Definition 21) is a maximal clique of **Proto**.

The following lemma gives a nice "inductive" characterization of cliques.

Lemma 10. *Let* $\Gamma \subseteq$ **Proto***:* Γ *is a clique iff there exists* $n \in \mathbb{N}$ *such that for all* $\alpha,\beta \in \Gamma$ *one has* $\alpha = (\langle\alpha_1^1,\ldots,\alpha_{k_1}^1\rangle,\ldots,\langle\alpha_1^n,\ldots,\alpha_{k_n}^n\rangle)$ *and* $\beta = (\langle\beta_1^1,\ldots,\beta_{h_1}^1\rangle,\ldots,\langle\beta_1^n,\ldots,\beta_{h_n}^n\rangle)$ *for some* $k_1,h_1,\ldots,k_n,h_n \in \mathbb{N}$*, and for every* $1 \leq i \leq n$ *the set* $\{\alpha_1^i,\ldots,\alpha_{k_i}^i,\beta_1^i,\ldots,\beta_{h_i}^i\}$ *is a clique.* Proof in Appendix

3 DiLL-proof structures and their proto-Taylor and Taylor expansion

The syntax that we present here is inspired by the ones used in [12, 13, 14, 19, 2, 3] and above all [10]. We use the terminology of interactions nets [11, 7], even if properly speaking our objects are not interaction nets. The main novelty in our syntax is that there are no wires (the same port may be auxiliary for some cell and principal for another cell). Unlike [10], our syntax is inductive and boxes have an explicit constructor.

[8] The word "maximal" refers here and hereinafter to the order relation $\subseteq$ on the power set of **Proto**.

3.1 DiLL-, DiLL_0- and MELL-proof structures

We define here our basic syntactical object: DiLL-*proof structure* (DiLL-*ps*, for short). The other syntactical objects corresponding to some framework or extension of LL (MELL- and DiLL_0-*proof structures*), will be some special cases of DiLL-ps. As in [12, 13, 14] and unlike [2, 3], our DiLL-ps are typed by MELL formulas, but they can be easily defined in the untyped case too. We use !- and ?-cells having n premises for any $n \in \mathbb{N}$, as in [17, 19], for generalized (co-)contractions (the type does not change passing from the premises to the conclusion); dereliction (resp. co-dereliction) cells have the special type $?d$ (resp. $!d$).

Unlike [14, 3], our syntactical objects are not necessarily cut-free (nor η-expansed). Moreover it is possible to define the cut-elimination and η-expansion[9] for DiLL-proof structures (and in particular for DiLL_0- and MELL-proof structures), but their rewriting rules will be presented in a further work.

Definition 11 (Ground-structure, ports, cells)**.** *A* ground-structure *(*g-structure *for short) is a* 7*-tuple* $\Phi = (\mathscr{P}_\Phi, \mathscr{C}_\Phi, \mathsf{tc}_\Phi, \mathsf{P}^{\mathsf{pri}}_\Phi, \mathsf{P}^{\mathsf{aux}}_\Phi, \mathsf{P}^{\mathsf{left}}_\Phi, \mathsf{tp}_\Phi)$ *such that:*

- $\mathscr{P}_\Phi$ *(resp.* $\mathscr{C}_\Phi$*) is a finite set, whose elements are the* ports *(resp.* cells *or* links*) of* Φ*;*
- tc_Φ *is a function from* $\mathscr{C}_\Phi$ *to* $\mathscr{L}_{\mathsf{DiLL}}$*; for every* $l \in \mathscr{C}_\Phi$, $\mathsf{tc}_\Phi(l)$ *is the* label*, or* type*, of* l*; for every* $t, t' \in \mathscr{L}_{\mathsf{DiLL}}$*, we set* $\mathscr{C}^t_\Phi = \{l \in \mathscr{C}_\Phi \mid \mathsf{tc}_\Phi(l) = t\}$ *(whose elements are the* t-cells*, or* t-links*, of* Φ*) and* $\mathscr{C}^{t,t'}_\Phi = \mathscr{C}^t_\Phi \cup \mathscr{C}^{t'}_\Phi$*;*
- $\mathsf{P}^{\mathsf{pri}}_\Phi$ *is a function from* $\mathscr{C}_\Phi$ *to* $\mathfrak{P}(\mathscr{P}_\Phi)$ *such that* $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\Phi) = \mathscr{P}_\Phi$*, and moreover, for all* $l, l' \in \mathscr{C}_\Phi$*,*
  - *if* $l \neq l'$ *then* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{pri}}_\Phi(l') = \emptyset$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 1$ *if* $\mathsf{tc}_\Phi(l) \in \{1, \bot, \otimes, \parr, !, !d, ?d, ?\}$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) \geq 1$ *if* $\mathsf{tc}_\Phi(l) = box$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 2$ *(resp.* $\mathsf{card}(\mathsf{P}^{\mathsf{pri}}_\Phi(l)) = 0$*) if* $\mathsf{tc}_\Phi(l) = ax$ *(resp.* $\mathsf{tc}_\Phi(l) = cut$*);*

  for any $l \in \mathscr{C}_\Phi$*, the elements of* $\mathsf{P}^{\mathsf{pri}}_\Phi(l)$ *are the* principal ports*, or* conclusions*, of* l in Φ*;*
- $\mathsf{P}^{\mathsf{aux}}_\Phi$ *is a function from* $\mathscr{C}_\Phi$ *to* $\mathfrak{P}(\mathscr{P}_\Phi)$ *such that, for all* $l, l' \in \mathscr{C}_\Phi$*,*
  - $\mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l') = \emptyset$ *if* $l \neq l'$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 0$ *if* $\mathsf{tc}_\Phi(l) \in \{1, \bot, ax, box\}$*,*
  - $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 2$ *(resp.* $\mathsf{card}(\mathsf{P}^{\mathsf{aux}}_\Phi(l)) = 1$*) if* $\mathsf{tc}_\Phi(l) \in \{\otimes, \parr, cut\}$ *(resp. if* $\mathsf{tc}_\Phi(l) \in \{!d, ?d\}$*);*

  for any $l \in \mathscr{C}_\Phi$*, the elements of* $\mathsf{P}^{\mathsf{aux}}_\Phi(l)$ *are the* auxiliary ports*, or* premises*, of* l in Φ*; we set* $\mathscr{P}^{\mathsf{aux}}_\Phi = \bigcup \mathsf{im}(\mathsf{P}^{\mathsf{aux}}_\Phi)$ *whose elements are the* auxiliary ports of Φ*,* $\mathscr{P}^{\mathsf{free}}_\Phi = \mathscr{P}_\Phi \smallsetminus \mathscr{P}^{\mathsf{aux}}_\Phi$ *whose elements are the* free ports*, or* conclusions, of Φ*, and* $\mathscr{C}^{\mathsf{free}}_\Phi = \{l \in \mathscr{C}_\Phi \mid \mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathscr{P}^{\mathsf{free}}_\Phi \neq \emptyset\}$ *whose elements are the* free*, or* terminal*,* cells of Φ*;*
- $\mathsf{P}^{\mathsf{left}}_\Phi$ *is a function from* $\mathscr{C}^{\otimes,\parr}_\Phi$ *to* $\mathscr{P}^{\mathsf{aux}}_\Phi$ *such that* $\mathsf{P}^{\mathsf{left}}_\Phi(l) \in \mathsf{P}^{\mathsf{aux}}_\Phi(l)$ *for any* $l \in \mathscr{C}^{\otimes,\parr}_\Phi$*;*
- tp_Φ *is a function from* $\mathscr{P}_\Phi$ *to* $\mathscr{F}_{\mathsf{MELL}}$ *such that, for any* $l \in \mathscr{C}_\Phi$*, one has*
  - $\mathsf{tp}_\Phi(p_1) = A$ *and* $\mathsf{tp}_\Phi(p_2) = A^\bot$*, if* $\mathsf{tc}_\Phi(l) = ax$ *(resp.* $\mathsf{tc}_\Phi(l) = cut$*),* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p_1, p_2\}$ *(resp.* $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$*) and* $A \in \mathscr{F}_{\mathsf{MELL}}$*,*
  - $\mathsf{tp}_\Phi(p) = A$*, if* $\mathsf{tc}_\Phi(l) = A \in \{1, \bot\}$ *and* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$*,*
  - $\mathsf{tp}_\Phi(p) = \mathsf{tp}_\Phi(p_1) \odot \mathsf{tp}_\Phi(p_2)$ *if* $\mathsf{tc}_\Phi(l) = \odot \in \{\otimes, \parr\}$*,* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$*,* $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, p_2\}$ *and* $\mathsf{P}^{\mathsf{left}}_\Phi(l) = p_1$*,*

[9] The η-expansion of a DiLL-ps is the substitution of every ax-cell whose conclusions are typed by $A, A^\bot$ with the "standard" DiLL-ps with conclusions typed by $A, A^\bot$ and where the conclusions of every ax-cell are now typed by dual atomic formulas.

- $\mathsf{tp}_\Phi(p) = \mathsf{tp}_\Phi(p_i) = \lozenge A$ *for any* $1 \le i \le n$*, if* $\mathsf{tc}_\Phi(l) = \lozenge \in \{!, ?\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$ *and* $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{p_1, \dots, p_n\}$ *for some* $n \in \mathbb{N}$*,*
- $\mathsf{tp}_\Phi(p) = \lozenge \mathsf{tp}_\Phi(q)$*, if* $\mathsf{tc}_\Phi(l) = \lozenge d$ *with* $\lozenge \in \{!, ?\}$, $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p\}$ *and* $\mathsf{P}^{\mathsf{aux}}_\Phi(l) = \{q\}$*,*
- $\mathsf{tp}_\Phi(p) = !A$ *and* $\mathsf{tp}_\Phi(p_i) = ?A_i$ *for any* $1 \le i \le n$*, if* $\mathsf{tc}_\Phi(l) = box$*,* $A, A_1, \dots, A_n \in \mathscr{F}_{\mathsf{MELL}}$ *and* $\mathsf{P}^{\mathsf{pri}}_\Phi(l) = \{p, p_1, \dots, p_n\}$ *for some* $n \in \mathbb{N}$*; p is the* principal door of l in Φ *and* $p_1, \dots, p_n$ *are the* auxiliary doors of l in Φ*;*

for every $p \in \mathscr{P}_\Phi$*,* $\mathsf{tp}_\Phi(p)$ *is the* label*, or* type*, of* p*.*

In a g-structure Φ, the function $\mathsf{P}^{\mathsf{left}}_\Phi$ fixes an order on the two premises of any $\otimes$- and $⅋$-cell of Φ; the premises of the other types of cells are unordered. The conditions $\bigcup \mathsf{im}(\mathsf{P}^{\mathsf{pri}}_\Phi) = \mathscr{P}_\Phi$ and, for all $l, l' \in \mathscr{C}_\Phi$, if $l \neq l'$ then $\mathsf{P}^{\mathsf{pri}}_\Phi(l) \cap \mathsf{P}^{\mathsf{pri}}_\Phi(l') = \emptyset = \mathsf{P}^{\mathsf{aux}}_\Phi(l) \cap \mathsf{P}^{\mathsf{aux}}_\Phi(l')$, mean that every port is conclusion of exactly one cell and premise of at most one cell; the ports that are not premises of any cell are the conclusions of Φ.

A g-structure Φ can be seen as a directed labeled hypergraph whose nodes are the ports of Φ labeled by MELL formulas, and whose hyperedges are the cells of Φ, labeled by the elements of $\mathscr{L}_{\mathsf{DiLL}}$ and oriented from their premises to their conclusions.

A cell is graphically depicted as a trapezoid with its label inside it, its principal port being on the shorter base and its auxiliary ports on the longer base (in such a way that when the principal port is downwards the left auxiliary ports of a $\otimes$- or $⅋$-cell is placed on the left). A port which is principal for one cell l and auxiliary for another cell l' is depicted as an oriented wire from l to l'. In such graphical representations the names of ports and cells are omitted, unless indicated to the contrary.

Definition 12 (DiLL-, DiLL$_0$-, MELL-proof structure)**.** *We define the set* $\mathbf{PS}_{\mathsf{DiLL}_d}$ *by induction on* $d \in \mathbb{N}$*.*

The set $\mathbf{PS}_{\mathsf{DiLL}_0}$ *is defined by:* $R \in \mathbf{PS}_{\mathsf{DiLL}_0}$ *iff* $R = (\mathsf{ground}(R), \mathsf{b}^0_R)$ *where* $\mathsf{ground}(R)$ *is a g-structure such that* $\mathscr{C}^{box}_R = \emptyset$*, and* $\mathsf{b}^0_R = \emptyset$ *(the empty function). The elements of* $\mathbf{PS}_{\mathsf{DiLL}_0}$ *are called* DiLL$_0$-proof structures *(*DiLL$_0$-ps *for short).*

For any $d \in \mathbb{N}$*, the set* $\mathbf{PS}_{\mathsf{DiLL}_{d+1}}$ *is defined by:* $R \in \mathbf{PS}_{\mathsf{DiLL}_{d+1}}$ *iff* $R = (\mathsf{ground}(R), \mathsf{b}^0_R)$ *where* $\mathsf{ground}(R)$ *is a g-structure (and we set* $\mathscr{C}^{box_0}_R = \mathscr{C}^{box}_{\mathsf{ground}(R)}$*,* $\mathscr{P}^{\mathsf{free}}_R = \mathscr{P}^{\mathsf{free}}_{\mathsf{ground}(R)}$ *and* $\mathscr{C}^{\mathsf{free}}_R = \mathscr{C}^{\mathsf{free}}_{\mathsf{ground}(R)}$*) and* b^0_R *is a function associating with any* $l \in \mathscr{C}^{box_0}_R$ *a pair* $\mathsf{b}^0_R(l) = (R_l, \mathsf{concl}_l)$ *where* $R_l \in \mathbf{PS}_{\mathsf{DiLL}_{d'}}$ *with* $d' \le d$ *(*R_l *is called* the content of l*) and* $\mathsf{concl}_l \colon \mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(R)}(l) \to \mathscr{P}^{\mathsf{free}}_{R_l}$ *is a bijection such that if p is the principal (resp. an auxiliary) door of l in* $\mathsf{ground}(R)$ *then* $\mathsf{tp}_{\mathsf{ground}(R)}(p) = !\mathsf{tp}_{\mathsf{ground}(R_l)}(\mathsf{concl}_l(p))$ *(resp.* $\mathsf{tp}_{\mathsf{ground}(R)}(p) = \mathsf{tp}_{\mathsf{ground}(R_l)}(\mathsf{concl}_l(p))$*) and we say that* $\mathsf{concl}_l(p)$ *is the* principal *(resp. an* auxiliary*)* door of R_l*.*

We set $\mathbf{PS}_{\mathsf{DiLL}} = \bigcup_{d \in \mathbb{N}} \mathbf{PS}_{\mathsf{DiLL}_d}$*, whose elements are called* DiLL-proof structures *(*DiLL-ps *for short). For any* $R \in \mathbf{PS}_{\mathsf{DiLL}}$*, the* depth of R*, denoted by* $\mathsf{depth}(R)$*, is the minimal* $d \in \mathbb{N}$ *such that* $R \in \mathbf{PS}_{\mathsf{DiLL}_d}$*.*[10]

A MELL-proof structure *(*MELL*-ps for short) is a* DiLL*-ps R such that* $\mathscr{C}^{!,!d}_{\mathsf{ground}(R)} = \emptyset$ *and, for every* $l \in \mathscr{C}^{box_0}_R$*, the content of l is a* MELL*-ps (the definition is by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*).*

We set $\mathbf{PS}_{\mathsf{MELL}} = \{R \in \mathbf{PS}_{\mathsf{DiLL}} \mid R$ *is a* MELL*-ps*$\}$*.*

Observe that if R is a DiLL-ps and $l \in \mathscr{C}^{box_0}_R$ where R_l is the content of l and p is an auxiliary door of l in $\mathsf{ground}(R)$, then $\mathsf{tp}_{\mathsf{ground}(R)}(p) = \mathsf{tp}_{\mathsf{ground}(R_l)}(\mathsf{concl}_l(p)) = ?A$ for some $A \in \mathscr{F}_{\mathsf{MELL}}$.

As usual in literature, given a DiLL-ps R and $l \in \mathscr{C}^{box_0}_R$, the cell l is graphically depicted by a rectangular frame where the principal door (resp. any auxiliary door) of l is under a trapezoid in the border of the frame labeled by ! (resp. by *aux*) and inside the rectangular frame there is the content R_l of l. In order to make visual the correspondence defined by concl_l between the conclusions of l and the conclusions of R_l, two corresponding conclusions of l and R_l are placed beneath each other.

---

[10] In [13, 18, 19] DiLL-ps are defined in such a way that inside a box there is a sum (i.e. a set or a multiset) of DiLL-ps, not necessarily exactly one DiLL-ps as in our syntax. Anyway, our definition is not limiting if one considers that a sum of two DiLL-ps inside a box is equivalent to two boxes side by side linked by a co-contraction cell (see [18, p. 86]).

Definition 13 (Isomorphism on ground- and DiLL-proof structures)**.** *Let Φ and Ψ be some g-structures. An* isomorphism from Φ to Ψ *is a pair $\varphi = (\varphi_{\mathscr{P}}, \varphi_{\mathscr{C}})$ of bijections $\varphi_{\mathscr{P}} \colon \mathscr{P}_\Phi \to \mathscr{P}_\Psi$ and $\varphi_{\mathscr{C}} \colon \mathscr{C}_\Phi \to \mathscr{C}_\Psi$ such that all diagrams (2) commute. We write then $\varphi \colon \Phi \simeq \Psi$.*

$$
\begin{array}{ccccc}
\mathfrak{P}(\mathscr{P}_\Phi) & \xleftarrow{\mathsf{P}^{\mathsf{aux}}_{\Phi}} & \mathscr{C}_\Phi & \xrightarrow{\mathsf{P}^{\mathsf{pri}}_{\Phi}} & \mathfrak{P}(\mathscr{P}_\Phi) \\
\downarrow{\scriptstyle \overline{\varphi_{\mathscr{P}}}} & & \downarrow{\scriptstyle \varphi_{\mathscr{C}}} & & \downarrow{\scriptstyle \overline{\varphi_{\mathscr{P}}}} \\
\mathfrak{P}(\mathscr{P}_\Psi) & \xleftarrow{\mathsf{P}^{\mathsf{aux}}_{\Psi}} & \mathscr{C}_\Psi & \xrightarrow{\mathsf{P}^{\mathsf{pri}}_{\Psi}} & \mathfrak{P}(\mathscr{P}_\Psi)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}_\Phi & \xrightarrow{\mathsf{tc}_\Phi} & \mathscr{L}_{\mathsf{DiLL}} \\
\downarrow{\scriptstyle \varphi_{\mathscr{C}}} & \nearrow{\scriptstyle \mathsf{tc}_\Psi} & \\
\mathscr{C}_\Psi & &
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{P}_\Phi & \xrightarrow{\mathsf{tp}_\Phi} & \mathscr{F}_{\mathsf{MELL}} \\
\downarrow{\scriptstyle \varphi_{\mathscr{P}}} & \nearrow{\scriptstyle \mathsf{tp}_\Psi} & \\
\mathscr{P}_\Psi & &
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{C}_\Phi^{\otimes,\parr} & \xrightarrow{\mathsf{P}^{\mathsf{left}}_{\Phi}} & \mathscr{P}_\Phi \\
\downarrow{\scriptstyle \varphi_{\mathscr{C}}} & & \downarrow{\scriptstyle \varphi_{\mathscr{P}}} \\
\mathscr{C}_\Psi^{\otimes,\parr} & \xrightarrow{\mathsf{P}^{\mathsf{left}}_{\Psi}} & \mathscr{P}_\Psi
\end{array}
\tag{2}
$$

Let $R = (\mathsf{ground}(R), \mathsf{b}^0_R)$ and $S = (\mathsf{ground}(S), \mathsf{b}^0_S)$ be some DiLL*-ps. An* isomorphism from R to S *is a pair $\varphi = (\varphi^{\mathsf{ground}}, (\varphi_l)_{l \in \mathscr{C}_R^{box_0}})$ such that $\varphi^{\mathsf{ground}} = (\varphi^{\mathsf{ground}}_{\mathscr{P}}, \varphi^{\mathsf{ground}}_{\mathscr{C}}) \colon \mathsf{ground}(R) \simeq \mathsf{ground}(S)$ and, for every $l \in \mathscr{C}_R^{box_0}$ with $\mathsf{b}^0_R(l) = (R_l, \mathsf{concl}_l)$ and $\mathsf{b}^0_S(l') = (S_{l'}, \mathsf{concl}_{l'})$ where $l' = \varphi^{\mathsf{ground}}_{\mathscr{C}}(l)$,[11] one has $\varphi_l \colon R_l \simeq S_{l'}$ and diagram (3) commutes (the definition is by induction on $\mathsf{depth}(R) \in \mathbb{N}$).*

$$
\begin{array}{ccc}
\mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(R)}(l) & \xrightarrow{\mathsf{concl}_l} & \mathscr{P}^{\mathsf{free}}_{R_l} \\
\downarrow{\scriptstyle \varphi^{\mathsf{ground}}_{\mathscr{P}}} & & \downarrow{\scriptstyle \varphi^{\mathsf{ground}}_{l\,\mathscr{P}}} \\
\mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(S)}(l') & \xrightarrow{\mathsf{concl}_{l'}} & \mathscr{P}^{\mathsf{free}}_{S_{l'}}
\end{array}
\tag{3}
$$

If, given two DiLL*-ps R and S, there is an isomorphism from R to S, we say that R* and *S* are isomorphic *and we write $R \simeq S$.*

An isomorphism between g-structures is nothing but an isomorphism between hypergraphs. The idea is that two g-structures are isomorphic iff they are identical up to the names of their cells and ports: two isomorphic g-structures are "morally" the same object (bijections $\varphi_{\mathscr{P}}$ and $\varphi_{\mathscr{C}}$ of an isomorphism φ on g-structures are a renaming of ports and cells, ports and cells being nothing but their names) and indeed they have the same graphical representation up to the order of the premises of their !- and ?-cells. The relation $\simeq$ is an equivalence on the set of g-structures and a canonical representative of an equivalence class with respect to $\simeq$ on the set of g-structures is the graphical representation of any g-structure in this class up to the order of the premises of its !- and ?-cells.

An isomorphism between two DiLL-ps generalizes this idea recursively on the depth of the DiLL-ps, being careful that the correspondence between the conclusions of a *box*-cell l and the conclusions of the content of l commutes with the isomorphism (see diagram (3) in Definition 13).

The following Definitions 14–20 will be used to define the Taylor expansion (Definitions 23 and 27).

Definition 14 (Ports and cells of a DiLL-proof structure, good-name)**.** *Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$; for every $l \in \mathscr{C}_R^{box_0}$ we denote by R_l the content of l. The following definitions are by induction on $\mathsf{depth}(R) \in \mathbb{N}$.*

We set: $\mathscr{P}_R = \mathscr{P}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathscr{P}_{R_l}$ and $\mathscr{C}_R = \mathscr{C}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathscr{C}_{R_l}$. The elements of $\mathscr{P}_R$ (resp. $\mathscr{C}_R$) are the ports *(resp.* cells*) of R.*

We say that R has good-names *if, for all $l, l' \in \mathscr{C}_R^{box_0}$, one has: $\mathscr{P}_{\mathsf{ground}(R)} \cap \mathscr{P}_{R_l} = \emptyset = \mathscr{P}_{R_l} \cap \mathscr{P}_{R_{l'}}$, $\mathscr{C}_{\mathsf{ground}(R)} \cap \mathscr{C}_{R_l} = \emptyset = \mathscr{C}_{R_l} \cap \mathscr{C}_{R_{l'}}$, R_l has good-names, $((l,l),()) \notin \mathscr{P}_{\mathsf{ground}(R)} \cup \mathscr{P}_{R_{l'}}$ and, for every $p \in \mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(R)}(l)$, $((l,p),()) \notin \mathscr{C}_{\mathsf{ground}(R)} \cup \mathscr{C}_{R_{l'}}$.*

[11] Notice that diagrams (2) and $\varphi^{\mathsf{ground}} \colon \mathsf{ground}(R) \to \mathsf{ground}(S)$ imply that $\overline{\varphi^{\mathsf{ground}}_{\mathscr{C}}}(\mathscr{C}_R^{box_0}) = \mathscr{C}_S^{box_0}$.

Let $R = (\mathsf{ground}(R), \mathsf{b}^0_R)$ be a DiLL*-ps having good-names. We set:* $\mathsf{tc}_R = \mathsf{tc}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{tc}_{R_l}$, $\mathsf{P}^{\mathsf{pri}}_R = \mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{P}^{\mathsf{pri}}_{R_l}$, $\mathsf{P}^{\mathsf{aux}}_R = \mathsf{P}^{\mathsf{aux}}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{P}^{\mathsf{aux}}_{R_l}$, $\mathsf{P}^{\mathsf{left}}_R = \mathsf{P}^{\mathsf{left}}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{P}^{\mathsf{left}}_{R_l}$, $\mathsf{tp}_R = \mathsf{tp}_{\mathsf{ground}(R)} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{tp}_{R_l}$, *and* $\mathsf{b}_R = \mathsf{b}^0_R \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \mathsf{b}_{R_l}$. *For any $t, t' \in \mathscr{L}_{\mathsf{DiLL}}$, we set* $\mathscr{C}^t_R = \{l \in \mathscr{C}_R \mid \mathsf{tc}_R(l) = t\}$ *(whose elements are the t-*cells *of R) and* $\mathscr{C}^{t,t'}_R = \mathscr{C}^t_R \cup \mathscr{C}^{t'}_R$. *The elements of $\mathscr{P}^{\mathsf{free}}_R$ (resp. $\mathscr{C}^{\mathsf{free}}_R$) are the* free ports*, or* conclusions *(resp.* free cells*, or* terminal cells*), of R. The elements of $\mathscr{C}_R^{box_0}$ are the* box-cells with depth 0 of *R.*

Let R and S be some DiLL*-ps having good-names, let $\varphi = (\varphi^{\mathsf{ground}}, (\varphi_l)_{l \in \mathscr{C}_R^{box_0}})$ be an isomorphism from R to S. We set $\varphi_{\mathscr{P}} = \varphi^{\mathsf{ground}}_{\mathscr{P}} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \varphi_{l\,\mathscr{P}}$ and $\varphi_{\mathscr{C}} = \varphi^{\mathsf{ground}}_{\mathscr{C}} \cup \bigcup_{l \in \mathscr{C}_R^{box_0}} \varphi_{l\,\mathscr{C}}$.*

A DiLL-ps R has good-names if all its ports and all its cells (not only the ones of $\mathsf{ground}(R)$) have different names. All DiLL_0-ps have good-names. For every DiLL-ps R there is a DiLL-ps S having good-names and such that $R \simeq S$. In the sequel, we will always suppose implicitly that *all* DiLL*-ps have good-names*. This hypothesis is crucial to ensure that the Taylor expansion of a (representative of a) DiLL-ps (Definitions 23 and 27) is a set of DiLL_0-ps, each of them having different names for different ports or cells. Furthermore, good-naming allows one to consider a DiLL-ps not only as an inductive structure but also as a "flatten hypergraph", as we have seen in Definition 14 when we suppose that a DiLL-ps has good-names.

Definition 15 (Well- and empty-naming, interface)**.** *Let R and S be some* DiLL*-ps. We say that:*

- *R is* well-named *if for every $d \in (\mathscr{P}_R \cup \mathscr{C}_R) \smallsetminus (\mathscr{P}^{\mathsf{free}}_R \cup \mathscr{C}^{\mathsf{free}}_R)$ there exists a finite sequence a of ordered pairs such that $d = (d', a)$ for some d', and every $c \in \mathscr{P}^{\mathsf{free}}_R \cup \mathscr{C}^{\mathsf{free}}_R$ is such that $c = (c', ())$ for some c'; R is* empty-named *if every $c \in \mathscr{P}_R \cup \mathscr{C}_R$ is such that $c = (c', ())$ for some c';*
- *R and S are* weakly interfaced *if $\mathscr{P}^{\mathsf{free}}_R = \mathscr{P}^{\mathsf{free}}_S$ and $\mathsf{tp}_R \restriction_{\mathscr{P}^{\mathsf{free}}_R} = \mathsf{tp}_S \restriction_{\mathscr{P}^{\mathsf{free}}_S}$; R and S are* interfaced *if they are weakly interfaced and $\mathscr{C}^{\mathsf{free}}_R = \mathscr{C}^{\mathsf{free}}_S$, $\mathsf{P}^{\mathsf{pri}}_R \restriction_{\mathscr{C}^{\mathsf{free}}_R} = \mathsf{P}^{\mathsf{pri}}_S \restriction_{\mathscr{C}^{\mathsf{free}}_S}$ and $\mathsf{tc}_R \restriction_{\mathscr{C}^{\mathsf{free}}_R} = \mathsf{tc}_S \restriction_{\mathscr{C}^{\mathsf{free}}_S}$; R and S are* strongly interfaced *if they are interfaced and $\mathscr{P}_R \cap \mathscr{P}_S = \mathscr{P}^{\mathsf{free}}_R$ and $\mathscr{C}_R \cap \mathscr{C}_S = \mathscr{C}^{\mathsf{free}}_R$.*

Definition 16 (Opening and border of a *box*-cell)**.** *Let $R = (\mathsf{ground}(R), \mathsf{b}^0_R) \in \mathbf{PS}_{\mathsf{DiLL}}$ and $l \in \mathscr{C}_R^{box_0}$, where R_l is the content of l and concl_l is the bijection between the doors of l in $\mathsf{ground}(R)$ and the doors of R_l, i.e. $\mathsf{b}^0_R(l) = (R_l, \mathsf{concl}_l)$. Let p be the principal door of l in $\mathsf{ground}(R)$.*

The opening *of l is the* DiLL*-ps $R^{\mathsf{op}}_l = (\mathsf{ground}(R^{\mathsf{op}}_l), \mathsf{b}^0_{R^{\mathsf{op}}_l})$ defined by:*

- *$\mathscr{P}_{R^{\mathsf{op}}_l} = \mathscr{P}_{R_l} \cup \{((l,l),())\} \cup \mathsf{P}^{\mathsf{pri}}_R(l)$, $\mathscr{C}_{R^{\mathsf{op}}_l} = \mathscr{C}_{R_l} \cup \{l\} \cup \{((l,q),()) \mid q \in \mathsf{P}^{\mathsf{pri}}_R(l)\}$ and $\mathsf{tc}_{R^{\mathsf{op}}_l}$ is such that $\mathsf{tc}_{R^{\mathsf{op}}_l} \restriction_{\mathscr{C}_{R_l}} = \mathsf{tc}_{R_l}$, $\mathsf{tc}_{R^{\mathsf{op}}_l}(l) = \,!$, $\mathsf{tc}_{R^{\mathsf{op}}_l}\big(((l,p),())\big) = \,!d$, and $\mathsf{tc}_{R^{\mathsf{op}}_l}\big(((l,q),())\big) = \,?$ for any auxiliary door q of l in $\mathsf{ground}(R)$; $\mathsf{b}^0_{R^{\mathsf{op}}_l} = \mathsf{b}_R \restriction_{\mathscr{C}_{R_l}^{box_0}}$;*
- *$\mathsf{P}^{\mathsf{pri}}_{R^{\mathsf{op}}_l}$ is such that $\mathsf{P}^{\mathsf{pri}}_{R^{\mathsf{op}}_l} \restriction_{\mathscr{P}_{R_l}} = \mathsf{P}^{\mathsf{pri}}_{R_l}$ and $\mathsf{P}^{\mathsf{pri}}_{R^{\mathsf{op}}_l}(l) = \{p\}$ and $\mathsf{P}^{\mathsf{pri}}_{R^{\mathsf{op}}_l}\big(((l,p),())\big) = \{((l,l),())\}$ and $\mathsf{P}^{\mathsf{pri}}_{R^{\mathsf{op}}_l}\big(((l,q),())\big) = \{q\}$ for any auxiliary door q of l in $\mathsf{ground}(R)$; $\mathsf{P}^{\mathsf{aux}}_{R^{\mathsf{op}}_l}$ is such that $\mathsf{P}^{\mathsf{aux}}_{R^{\mathsf{op}}_l} \restriction_{\mathscr{P}_{R_l}} = \mathsf{P}^{\mathsf{aux}}_{R_l}$, $\mathsf{P}^{\mathsf{aux}}_{R^{\mathsf{op}}_l}(l) = \{((l,l),())\}$, $\mathsf{P}^{\mathsf{aux}}_{R^{\mathsf{op}}_l}\big(((l,p),())\big) = \{\mathsf{concl}_l(p)\}$ and $\mathsf{P}^{\mathsf{aux}}_{R^{\mathsf{op}}_l}\big(((l,q),())\big) = \{\mathsf{concl}_l(q)\}$ for any auxiliary door q of l in $\mathsf{ground}(R)$;*
- *$\mathsf{P}^{\mathsf{left}}_{R^{\mathsf{op}}_l} = \mathsf{P}^{\mathsf{left}}_{R_l}$; $\mathsf{tp}_{R^{\mathsf{op}}_l}$ is defined by $\mathsf{tp}_{R^{\mathsf{op}}_l} \restriction_{\mathscr{P}_{R_l} \cup \mathsf{P}^{\mathsf{pri}}_R(l)} = \mathsf{tp}_R \restriction_{\mathscr{P}_{R_l} \cup \mathsf{P}^{\mathsf{pri}}_R(l)}$ and $\mathsf{tp}_{R^{\mathsf{op}}_l}\big(((l,l),())\big) = \mathsf{tp}_R(p)$.*

The border *of l is the DiLL_0-ps R^0_l defined by:*

- *$\mathscr{P}_{R^0_l} = \mathsf{P}^{\mathsf{pri}}_R(l)$, $\mathscr{C}_{R^0_l} = \{l\} \cup \{((l,q),()) \mid q \in \mathsf{P}^{\mathsf{pri}}_R(l) \smallsetminus \{p\}\}$ and $\mathsf{tc}_{R^0_l}$ is such that $\mathsf{tc}_{R^0_l}(l) = \,!$ and $\mathsf{tc}_{R^0_l}\big(((l,q),())\big) = \,?$ for any auxiliary door q of l in $\mathsf{ground}(R)$;*

- $\mathsf{P}^{\mathsf{pri}}_{R^0_l}$ *is such that* $\mathsf{P}^{\mathsf{pri}}_{R^0_l}(l) = p$ *and* $\mathsf{P}^{\mathsf{pri}}_{R^0_l}\big(((l,q),())\big) = q$ *for any auxiliary door* q *of* l *in* $\mathsf{ground}(R)$, $\mathsf{P}^{\mathsf{aux}}_{R^0_l}(l') = \emptyset$ *for any* $l' \in \mathscr{C}_{R^0_l}$, $\mathsf{tp}_{R^0_l} = \mathsf{tp}_R\restriction_{\mathsf{P}^{\mathsf{pri}}_R(l)}$, $\mathsf{P}^{\mathsf{left}}_{R^0_l} = \emptyset$ *and* $\mathsf{b}^0_{R^0_l} = \emptyset$.

If $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and $l \in \mathscr{C}^{box_0}_R$, then $\mathscr{P}^{\mathsf{free}}_{R^{\mathsf{op}}_l} = \mathsf{P}^{\mathsf{pri}}_R(l) = \mathscr{P}^{\mathsf{free}}_{R^0_l}$ and $\mathsf{tp}_{R^{\mathsf{op}}_l}(q) = \mathsf{tp}_R(q) = \mathsf{tp}_{R^0_l}(q)$ for any $q \in \mathsf{P}^{\mathsf{pri}}_R(l)$, moreover R^0_l and R^{op}_l are interfaced and if R is empty-named, then R^{op}_l and R^0_l are empty-named. The (implicit) hypothesis that $R \in \mathbf{PS}_{\mathsf{DiLL}}$ has good-names is fundamental in order to ensure that, for example, for any $q \in \mathsf{P}^{\mathsf{pri}}_R(l)$ the new cell $((l,q),())$ added in R^{op}_l does not already exist. See Figure 5 at p. 16 for an example of the opening, the border and the content of a *box*-cell. The opening of a *box*-cell is very similar to the content of a *box*-cell, the only difference being in their free cells and free ports.

Notice that, for any $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and $l \in \mathscr{C}^{box_0}_R$, one has $l \notin \mathscr{C}^{box}_{R^{\mathsf{op}}_l}$ and $\mathsf{depth}(R) > \mathsf{depth}(R^{\mathsf{op}}_l)$.

Definition 17 (Product of DiLL_0-proof structures)**.** *Let* $\mathscr{A} = \{R_1,\dots,R_n\}$ *(with* $n \in \mathbb{N}$*) be a finite set of pairwise strongly interfaced* DiLL_0*-ps such that* $\mathscr{C}^{\mathsf{free}}_{R_i} \subseteq \mathscr{C}^{!,?}_{R_i}$ *for any* $1 \le i \le n$. *The* product of $\mathscr{A}$ *is the* DiLL_0*-ps* $S = \prod_{i=1}^n R_i = \prod \mathscr{A}$ *defined by:*

- $\mathscr{P}_S = \bigcup_{i=1}^n \mathscr{P}_{R_i}$, $\mathscr{C}_S = \bigcup_{i=1}^n \mathscr{C}_{R_i}$ *and* $\mathsf{tc}_S = \bigcup_{i=1}^n \mathsf{tc}_{R_i}$*;*
- $\mathsf{P}^{\mathsf{pri}}_S = \bigcup_{i=1}^n \mathsf{P}^{\mathsf{pri}}_{R_i}$ *and* $\mathsf{P}^{\mathsf{aux}}_S$ *is such that* $\mathsf{P}^{\mathsf{aux}}_S\restriction_{\mathscr{C}_S \smallsetminus \mathscr{C}^{\mathsf{free}}_{\mathscr{A}}} = \bigcup_{i=1}^n \mathsf{P}^{\mathsf{aux}}_{R_i}\restriction_{\mathscr{C}_{R_i} \smallsetminus \mathscr{C}^{\mathsf{free}}_{R_i}}$ *and* $\mathsf{P}^{\mathsf{aux}}_S(l) = \bigcup_{i=1}^n \mathsf{P}^{\mathsf{aux}}_{R_i}(l)$ *for any* $l \in \mathscr{C}^{\mathsf{free}}_{\mathscr{A}}$*, where*[12] $\mathscr{C}^{\mathsf{free}}_{\mathscr{A}} = \bigcup_{i=1}^n \mathscr{C}^{\mathsf{free}}_{R_i}$*;* $\mathsf{P}^{\mathsf{left}}_S = \bigcup_{i=1}^n \mathsf{P}^{\mathsf{left}}_{R_i}$*,* $\mathsf{tp}_S = \bigcup_{i=1}^n \mathsf{tp}_{R_i}$ *and* $\mathsf{b}^0_S = \emptyset$.

Notice that if $\mathscr{A}$ is a finite set of pairwise strongly interfaced DiLL_0-ps such that $\mathscr{C}^{\mathsf{free}}_R \subseteq \mathscr{C}^{!,?}_R$ for any $R \in \mathscr{A}$, then $\mathscr{C}^{\mathsf{free}}_{\prod \mathscr{A}} \subseteq \mathscr{C}^{!,?}_{\prod \mathscr{A}}$, and R and $\prod \mathscr{A}$ are interfaced for any $R \in \mathscr{A}$. Since the premises of !- and ?-cells are unordered, it is not necessary to fix an order on the elements of $\mathscr{A}$ to compute $\prod \mathscr{A}$.

Definition 18 (Sub-DiLL-proof structure, box of a *box*-cell)**.** *Let* $R = (\mathsf{ground}(R), \mathsf{b}^0_R)$ *be a* DiLL*-ps.*

A sub-DiLL-proof structure *(*sub-DiLL-ps *for short)* of R *is a* DiLL*-ps* $S = (\mathsf{ground}(S), \mathsf{b}^0_S)$ *such that (definition by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*) either* S *is a sub-*DiLL*-ps of the content of some* $l \in \mathscr{C}^{box_0}_R$ *or:*

- $\mathscr{P}_{\mathsf{ground}(S)} \subseteq \mathscr{P}_{\mathsf{ground}(R)}$, $\mathscr{C}_{\mathsf{ground}(S)} \subseteq \mathscr{C}_{\mathsf{ground}(R)}$ *and* $\mathsf{tc}_{\mathsf{ground}(S)} = \mathsf{tc}_R\restriction_{\mathscr{C}_{\mathsf{ground}(S)}}$*;*
- $\mathsf{P}^{\mathsf{pri}}_{\mathsf{ground}(S)} = \mathsf{P}^{\mathsf{pri}}_R\restriction_{\mathscr{C}_{\mathsf{ground}(S)}}$, $\mathsf{P}^{\mathsf{aux}}_{\mathsf{ground}(S)}$ *is such that* $\mathsf{P}^{\mathsf{aux}}_{\mathsf{ground}(S)}\restriction_{\mathscr{C}_{\mathsf{ground}(S)} \smallsetminus \mathscr{C}^{!,?}_{\mathsf{ground}(S)}} = \mathsf{P}^{\mathsf{aux}}_R\restriction_{\mathscr{C}_{\mathsf{ground}(S)} \smallsetminus \mathscr{C}^{!,?}_{\mathsf{ground}(S)}}$ *and* $\mathsf{P}^{\mathsf{aux}}_{\mathsf{ground}(S)}(l) \subseteq \mathsf{P}^{\mathsf{aux}}_R(l)$ *for any* $l \in \mathscr{C}^{!,?}_{\mathsf{ground}(S)}$, $\mathsf{P}^{\mathsf{left}}_{\mathsf{ground}(S)} = \mathsf{P}^{\mathsf{left}}_R\restriction_{\mathscr{C}^{\otimes,\parr}_{\mathsf{ground}(S)}}$*;*
- $\mathsf{tp}_{\mathsf{ground}(S)} = \mathsf{tp}_R\restriction_{\mathscr{P}_{\mathsf{ground}(S)}}$ *and* $\mathsf{b}^0_S = \mathsf{b}^0_R\restriction_{\mathscr{C}^{box_0}_S}$.[13]

Let $l \in \mathscr{C}^{box_0}_R$ *where* R_l *is the content of* l. *The* box of l *is the sub-*DiLL*-ps* R^{box}_l *of* R *such that* $\mathscr{P}_{R^{\mathsf{box}}_l} = \mathscr{P}_{R_l} \cup \mathsf{P}^{\mathsf{pri}}_R(l)$ *and* $\mathscr{C}_{R^{\mathsf{box}}_l} = \mathscr{C}_{R_l} \cup \{l\}$.

Given $R \in \mathbf{PS}_{\mathsf{DiLL}}$, every sub-DiLL-ps of R which is a DiLL_0-ps is just a sub-hypergraph of $\mathsf{ground}(R)$. A sub-DiLL-ps S of R generalizes this notion of sub-hypergraph, recursively on the depth of R, being careful that if l is a *box*-cell of R inside S and R_l is the content of l, then the whole R_l is in S.

See Figure 5 at p. 16 for an example of the box of some *box*-cell of a DiLL-ps.

Definition 19 (Substitution of sub-DiLL-proof structures)**.** *Let* $n \in \mathbb{N}$, *let* $R, S_1, \dots, S_n \in \mathbf{PS}_{\mathsf{DiLL}}$ *and, for every* $1 \le i \le n$, *let* R_i *be a sub-*DiLL*-ps of* R *such that* R_i *and* S_i *are weakly interfaced. If for every* $1 \le i \ne j \le n$ *one has* $\mathscr{P}_{R_i} \cap \mathscr{P}_{R_j} = \emptyset = \mathscr{C}_{R_i} \cap \mathscr{C}_{R_j}$, $\mathscr{P}_{S_i} \cap \mathscr{P}_{S_j} = \emptyset = \mathscr{C}_{S_i} \cap \mathscr{C}_{S_j}$ *and* $\mathscr{P}_{S_i} \cap (\mathscr{P}_R \smallsetminus \bigcup_{k=1}^n \mathscr{P}_{R_k}) = \emptyset = \mathscr{C}_{S_i} \cap (\mathscr{C}_R \smallsetminus \bigcup_{k=1}^n \mathscr{C}_{R_k})$, *then the* substitution of $S_1,\dots,S_n$ for $R_1,\dots,R_n$ in R *is the* DiLL*-ps* $S = R[S_1/R_1,\dots,S_n/R_n]$ *defined by (we set* $\mathscr{C}_{R \smallsetminus \bigcup_i R_i} = \mathscr{C}_R \smallsetminus \bigcup_{i=1}^n \mathscr{C}_{R_i}$ *and* $\mathscr{C}^{\otimes,\parr}_{R \smallsetminus \bigcup_i R_i} = \mathscr{C}^{\otimes,\parr}_R \smallsetminus \bigcup_{i=1}^n \mathscr{C}^{\otimes,\parr}_{R_i}$*):*

[12] If $n = 0$ then $\mathscr{C}^{\mathsf{free}}_{\mathscr{A}} = \emptyset$, otherwise $\mathscr{C}^{\mathsf{free}}_{\mathscr{A}} = \mathscr{C}^{\mathsf{free}}_{R_i}$ for any $1 \le i \le n$, because $\mathscr{A}$ is a set of pairwise interfaced DiLL-ps.

[13] Observe that a sub-DiLL-ps S of a DiLL-ps R such that $\mathscr{C}_{\mathsf{ground}(S)} \cap \mathscr{C}_{\mathsf{ground}(R)} \neq \emptyset$ is univocally defined by fixing $\mathscr{P}_{\mathsf{ground}(S)}$, $\mathscr{C}_{\mathsf{ground}(S)}$ and $\mathsf{P}^{\mathsf{aux}}_{\mathsf{ground}(S)}(l)$ for any $l \in \mathscr{C}_{\mathsf{ground}(S)} \cap \mathscr{C}^{!,?}_R$.

- $\mathscr{P}_S = (\mathscr{P}_R \smallsetminus \bigcup_{i=1}^n \mathscr{P}_{R_i}) \cup \bigcup_{i=1}^n \mathscr{P}_{S_i}$, $\mathscr{C}_S = \mathscr{C}_{R \smallsetminus \bigcup_i R_i} \cup \bigcup_{i=1}^n \mathscr{C}_{S_i}$ *and* $\mathsf{tc}_S = \mathsf{tc}_R\restriction_{\mathscr{C}_{R \smallsetminus \bigcup_i R_i}} \cup \bigcup_{i=1}^n \mathsf{tc}_{S_i}$;
- $\mathsf{P}_S^{\mathsf{pri}} = \mathsf{P}_R^{\mathsf{pri}}\restriction_{\mathscr{C}_{R \smallsetminus \bigcup_i R_i}} \cup \bigcup_{i=1}^n \mathsf{P}_{S_i}^{\mathsf{pri}}$, $\mathsf{P}_S^{\mathsf{aux}} = \mathsf{P}_R^{\mathsf{aux}}\restriction_{\mathscr{C}_{R \smallsetminus \bigcup_i R_i}} \cup \bigcup_{i=1}^n \mathsf{P}_{S_i}^{\mathsf{aux}}$, $\mathsf{P}_S^{\mathsf{left}} = \mathsf{P}_R^{\mathsf{left}}\restriction_{\mathscr{C}^{\otimes,\parr}_{R \smallsetminus \bigcup_i R_i}} \cup \bigcup_{i=1}^n \mathsf{P}_{S_i}^{\mathsf{left}}$, $\mathsf{tp}_S = \mathsf{tp}_R\restriction_{\mathscr{C}_{R \smallsetminus \bigcup_i R_i}} \cup \bigcup_{i=1}^n \mathsf{tp}_{S_i}$ *and* $\mathsf{b}_S = \mathsf{b}_R\restriction_{\mathscr{C}_{R \smallsetminus \bigcup_i R_i} \cap \mathscr{C}_R^{box}} \cup \bigcup_{i=1}^n \mathsf{b}_{S_i}$.

Definition 20 (Empty-renaming, n-th copy of a DiLL-proof structure)**.** *Let* $R \in \mathbf{PS}_{\mathsf{DiLL}}$. *The* empty-renaming of *R is the empty-named* DiLL*-ps* R_ε *such that* $\mathscr{P}_{R_\varepsilon} = \{(p,()) \mid p \in \mathscr{P}_R\}$, $\mathscr{C}_{R_\varepsilon} = \{(l,()) \mid l \in \mathscr{C}_R\}$ *and* $\varphi \colon R \simeq R_\varepsilon$ *where* φ *is such that* $\varphi_{\mathscr{P}}(p) = (p,())$ *and* $\varphi_{\mathscr{C}}(l) = (l,())$ *for any* $p \in \mathscr{P}_R$ *and* $l \in \mathscr{C}_R$.

Let $n \in \mathbb{N}^+$, *let R be a well-named* DiLL*-ps with* $(l,()) \in \mathscr{C}_R^{\mathsf{free}}$. *The* n-th l-copy of *R is the* DiLL*-ps* R_n *such that* $\mathscr{P}_{R_n} = \mathscr{P}_R^{\mathsf{free}} \cup \{(p,(l,n)\cdot a) \mid (p,a) \in \mathscr{P}_R \smallsetminus \mathscr{P}_R^{\mathsf{free}}\}$, $\mathscr{C}_{R_n} = \mathscr{C}_R^{\mathsf{free}} \cup \{(l',(l,n)\cdot a) \mid (l',a) \in \mathscr{C}_R \smallsetminus \mathscr{C}_R^{\mathsf{free}}\}$ *and* $\varphi \colon R \simeq R_n$ *where* φ *is such that* $\varphi_{\mathscr{P}}((p,a)) = (p,(l,n)\cdot a)$ *and* $\varphi_{\mathscr{C}}((l',a)) = (l',(l,n)\cdot a)$ *for any* $p \in \mathscr{P}_R \smallsetminus \mathscr{P}_R^{\mathsf{free}}$ *and* $l' \in \mathscr{C}_R \smallsetminus \mathscr{C}_R^{\mathsf{free}}$, *and* $\varphi_{\mathscr{P}}\restriction_{\mathscr{P}_R^{\mathsf{free}}} = \mathsf{id}_{\mathscr{P}_R^{\mathsf{free}}}$ *and* $\varphi_{\mathscr{C}}\restriction_{\mathscr{C}_R^{\mathsf{free}}} = \mathsf{id}_{\mathscr{C}_R^{\mathsf{free}}}$.

The intuition is: given $R \in \mathbf{PS}_{\mathsf{DiLL}}$ well-named and $(l,()) \in \mathscr{C}_R^{\mathsf{free}}$, if R_n is the n-th l-copy of R and $c = (p,(l,n)\cdot a) \in \mathscr{P}_{R_n}$ (resp. $c = (l',(l,n)\cdot a) \in \mathscr{C}_{R_n}$), then c is the n-th copy of $(p,a) \in \mathscr{P}_R$ (resp. $(l',a) \in \mathscr{C}_R$) associated with l. In this way, for any DiLL-ps R, taking its empty renaming R_ε, it is possible to build n different copies of R (associated with some $l \in \mathscr{C}_R^{\mathsf{free}}$) and make their product, like in Definition 23.

3.2 Proto-Taylor and Taylor expansion of a DiLL-proof structure

Recall Figures 1, 2(a) and 2(b): we are now going to define the proto-Taylor expansion (Definition 22) of a representative (Definition 21) of a DiLL-ps and the Taylor expansion of a DiLL-ps (Definition 27).

Roughly speaking, an element of the Taylor expansion of a MELL-ps or more generally DiLL-ps R is obtained from R by replacing each box B in R with n_B copies of its content (for any $n_B \in \mathbb{N}$), recursively on the depth of R. Proto-nets are abstract objects able to say recursively how many copies to take for each box, so we can build the Taylor expansion of R starting from the proto-Taylor expansion of R, i.e. the set of proto-nets associated with R: one of the advantages of our pointwise approach with respect to other ones such as [12, 14] is that it makes easier to define the correspondence between ports and cells of any element ρ of the Taylor expansion of a DiLL-ps R and ports and cells of R, to recognize in which box (if any) the correspondents in R of ports and cells of ρ are, and to differentiate the various copies in ρ of the content of a same box in R. For this purpose ρ is a well-named DiLL_0-ps and a port or cell of ρ is of the shape (c,a), c is the corresponding port of R and the finite sequence a has to be intended as a list of indexes which keeps track of all boxes containing c and says in which copy of the content of each box (c,a) is. These indexes are a syntactic counterpart of the ones used in the definition of k-experiment of *PLPS* in [3, Def. 35]. In [10] we show that the informations encoded by indexes are useful to compute a DiLL-ps from a certain element of its Taylor expansion. Here, we make use of these indexes only to ensure that in each element of the Taylor expansion of a DiLL-ps different ports or cells have different names.

Definition 21 (Representative of a DiLL-proof structure)**.** *Let* $R \in \mathbf{PS}_{\mathsf{DiLL}}$. *A* representative of *R is a pair* $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ *defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$, *where* $\mathsf{box}^0_{R'}$ *is an enumeration of* $\mathscr{C}_R^{box_0}$*and* $\mathsf{rep}^0_{R'}$ *is a map associating with any* $l \in \mathscr{C}_R^{box_0}$ *a representative of the opening* R_l^{op} *of* l.

Intuitively, taking a representative of a DiLL-ps R means fixing an order on the *box*-cells of R, recursively on the depth of R. This order has nothing to do with the left-right order of the *box*-cells of R when R is depicted or that might be inherited by an order in the conclusions of R. This approach makes sense by considering that if R is a DiLL-ps consisting of two *box*-cells l_1, l_2 and a terminal ?-cell having two premises, which are the unique conclusion of l_1 and l_2 respectively, then the choice of the first *box*-cell is arbitrary (since the premises of a ?-cell are unordered) and two distinct representatives of R are possible.

Definition 22 (Proto-Taylor expansion of a representative of a DiLL-proof structure). *Let $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ be a representative of a* DiLL*-ps R, where $\mathsf{box}^0_{R'} = (l_1, \dots, l_n)$ (with $n \in \mathbb{N}$). The* proto-Taylor expansion $\mathscr{T}^{\mathsf{proto}}_{R'}$ *of R' is the following subset of* **Proto**, *defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*:*

$$\mathscr{T}^{\mathsf{proto}}_{R'} = \big\{ (\langle \alpha^1_1, \dots, \alpha^1_{k_1} \rangle, \dots, \langle \alpha^n_1, \dots, \alpha^n_{k_n} \rangle) \mid k_i \in \mathbb{N} \text{ and } \alpha^i_1, \dots, \alpha^i_{k_i} \in \mathscr{T}^{\mathsf{proto}}_{\mathsf{rep}^0_{R'}(l_i)}, \text{ for any } 1 \le i \le n \big\}.$$

Observe that if $n = 0$, i.e. $\mathsf{depth}(R) = 0$, then $\mathscr{T}^{\mathsf{proto}}_{R'} = \{()\}$: this is the base case of the induction.

Notice that the proto-Taylor expansion of some representative of a DiLL-ps R depends on the representative of R chosen: this is the price to pay if we wish proto-nets can say something about R.

An element of the proto-Taylor expansion of a representative $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ of a (empty-named) DiLL-ps R allows one to build an element of the Taylor expansion of R', as stated precisely in Definition 23, in such a way that $(\cdot)$- and $\langle\cdot\rangle$-sequences play a completely different role: if $\alpha = (a_1, \dots, a_n) \in \mathscr{T}^{\mathsf{proto}}_{R'}$ with $n \in \mathbb{N}$ and, for any $1 \le i \le n$, $a_i = \langle \alpha^i_1, \dots, \alpha^i_{k_i} \rangle$ for some $k_i \in \mathbb{N}$, then R has exactly n *box*-cells with depth 0 and for the i-th *box*-cell l_i with depth 0, according to the order fixed by $\mathsf{box}^0_{R'}$, a_i asks for taking k_i copies of (the representative according to $\mathsf{rep}^0_{R'}$ of) the content (actually, the opening) of l_i.

Definition 23 (Taylor expansion of a representative of a empty-named DiLL-proof structure). *Let R be an empty-named* DiLL*-ps where $\mathscr{C}^{box_0}_R = \{(l_1, ()), \dots, (l_n, ())\}$ for some $n \in \mathbb{N}$ and some pairwise distinct $l_1, \dots, l_n$; for any $1 \le i \le n$, let R_i (resp. R^0_i) be the box (resp. the border of the box) of l_i in R. Let $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ be a representative of R, where $\mathsf{box}^0_{R'} = \big((l_1, ()), \dots, (l_n, ())\big)$.*

With every $\alpha = (\langle \alpha^1_1, \dots, \alpha^1_{k_1} \rangle, \dots, \langle \alpha^n_1, \dots, \alpha^n_{k_n} \rangle) \in \mathscr{T}^{\mathsf{proto}}_{R'}$ (with $k_1, \dots, k_n \in \mathbb{N}$) is associated a well-named DiLL_0*-ps $\tau_{R'}(\alpha)$, defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*, such that $\tau_{R'}(\alpha)$ and R are weakly interfaced. For every $1 \le i \le n$ and $1 \le j \le k_i$, let ρ^j_i be the j-th l_i-copy of $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$ and let*[14]

$$\rho_i = \begin{cases} \prod_{j=1}^{k_i} \rho^j_i & \text{if } k_i > 0 \\ R^0_i & \text{otherwise.} \end{cases} \qquad \text{Then, we set } \tau_{R'}(\alpha) = R[\rho_1/R_1, \dots, \rho_n/R_n].$$

The Taylor expansion of R' *is* $\mathscr{T}_{R'} = \{\tau_{R'}(\alpha) \mid \alpha \in \mathscr{T}^{\mathsf{proto}}_{R'}\}$.

We now show that the definition of the Taylor expansion of a DiLL-ps R does not depend on the representative of R chosen (see Proposition 26 and Definition 27). To prove this, given two representatives R' and R'' of R, we define a "recursive permutation" p (Definition 24) inducing two functions (both denoted by p) such that: $\mathsf{p}(R') = R''$ and, for any $\alpha \in \mathscr{T}^{\mathsf{proto}}_{R'}$, $\mathsf{p}(\alpha) \in \mathscr{T}^{\mathsf{proto}}_{R''}$ and $\tau_{R'}(\alpha) = \tau_{R''}(\mathsf{p}(\alpha))$.

Definition 24 (Permutation of a representative of a DiLL-proof structure). *Let $R' = (\mathsf{box}^0_{R'}, \mathsf{rep}^0_{R'})$ be a representative of some* DiLL*-ps R with $\mathsf{box}^0_{R'} = (l_1, \dots, l_n)$ for some $n \in \mathbb{N}$.*

A permutation of R' *is a $n+1$-tuple $\mathsf{p} = (\sigma, \mathsf{p}_1, \dots, \mathsf{p}_n)$ defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*, where $\sigma \in \mathfrak{S}_n$ and p_i is a permutation of $\mathsf{rep}^0_{R'}(l_i)$ for any $1 \le i \le n$.*

Let $\mathsf{p} = (\sigma, \mathsf{p}_1, \dots, \mathsf{p}_n)$ be a permutation of R'. The representative of R induced by p *is $\mathsf{p}(R') = (\mathsf{box}^0_{\mathsf{p}(R')}, \mathsf{rep}^0_{\mathsf{p}(R')})$ where $\mathsf{box}^0_{\mathsf{p}(R')} = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$ and $\mathsf{rep}^0_{\mathsf{p}(R')}$ is defined (by induction on $\mathsf{depth}(R) \in \mathbb{N}$) as follows: $\mathsf{rep}^0_{\mathsf{p}(R')}(l_i) = \mathsf{p}_i(\mathsf{rep}^0_{R'}(l_i))$ for any $1 \le i \le n$.*

Let $\mathsf{p} = (\sigma, \mathsf{p}_1, \dots, \mathsf{p}_n)$ be a permutation of R'. If $\alpha = (\langle \alpha^1_1, \dots, \alpha^1_{k_1} \rangle, \dots, \langle \alpha^n_1, \dots, \alpha^n_{k_n} \rangle) \in \mathscr{T}^{\mathsf{proto}}_{R'}$ with $k_1, \dots, k_n \in \mathbb{N}$, then we set (again, the definition is by induction on $\mathsf{depth}(R) \in \mathbb{N}$): $\mathsf{p}(\alpha) = (\langle \mathsf{p}_{\sigma(1)}(\alpha^{\sigma(1)}_1), \dots, \mathsf{p}_{\sigma(1)}(\alpha^{\sigma(1)}_{k_{\sigma(1)}}) \rangle, \dots, \langle \mathsf{p}_{\sigma(n)}(\alpha^{\sigma(n)}_1), \dots, \mathsf{p}_{\sigma(n)}(\alpha^{\sigma(n)}_{k_{\sigma(n)}}) \rangle)$.[15]

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[14] Notice that, for any $1 \le i \le n$ and $1 \le j \ne h \le k_i$, one has that: $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$ is well-named by induction hypothesis; ρ^j_i and ρ^h_i are strongly interfaced; R^0_i, $\tau_{\mathsf{rep}^0_{R'}((l_i,()))}(\alpha^i_j)$ and $\prod_{j=1}^{k_i} \rho^j_i$ are interfaced, and each of them is weakly interfaced with R_i.

[15] In the base case, i.e. when $\mathsf{depth}(R) = 0$, one has that $\mathscr{C}^{box_0}_R = \emptyset$, thus $\mathsf{box}^0_{R'} = ()$ and $\mathsf{rep}^0_{R'} = \emptyset$, therefore $\mathsf{p} = (\emptyset)$ and $\mathsf{p}(R') = ((), \emptyset) = R'$; moreover, $\alpha \in \mathscr{T}^{\mathsf{proto}}_{R'}$ entails $\alpha = ()$ and hence $\mathsf{p}(\alpha) = ()$.

Fact 25. *Let R' be a representatives of some* DiLL*-ps R.* Proof in Appendix

1. *If* p *is a permutation of R' and $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$, then $\mathsf{p}(\alpha) \in \mathscr{T}_{\mathsf{p}(R')}^{\mathsf{proto}}$ and $\tau_{\mathsf{p}(R')}(\mathsf{p}(\alpha)) = \tau_{R'}(\alpha)$.*
2. *For any representative R'' of R, there exists a permutation* p *of R' such that $R'' = \mathsf{p}(R')$.*

Proposition 26. *Let R' and R'' be some representatives of a empty-named* DiLL*-ps R. Then $\mathscr{T}_{R'} = \mathscr{T}_{R''}$.* Proof in Appendix

Definition 27 (Taylor expansion of a DiLL-proof structure)**.** *Let R be a* DiLL*-ps and let R_ε be the empty-renaming of R. The* Taylor expansion of *R is $\mathscr{T}_R = \mathscr{T}_{R'_\varepsilon}$ where R'_ε is any representative of R_ε.*

Remark 28. If R is a DiLL-ps and $\rho \in \mathscr{T}_R$ then R and ρ are not (weakly) interfaced strictly speaking, anyway there is a one-to-one correspondence between $\mathscr{P}_R^{\mathsf{free}}$ and $\mathscr{P}_\rho^{\mathsf{free}}$, since $p \in \mathscr{P}_R^{\mathsf{free}}$ iff $(p,())\in \mathscr{P}_\rho^{\mathsf{free}}$; moreover $\mathsf{tp}_R(p) = \mathsf{tp}_\rho\big((p,())\big)$ for any $p \in \mathscr{P}_R^{\mathsf{free}}$. If $\mathscr{C}_R^{box_0} \cap \mathscr{C}_R^{\mathsf{free}} = \emptyset$ then: $l \in \mathscr{C}_R^{\mathsf{free}}$ iff $(l,()) \in \mathscr{C}_\rho^{\mathsf{free}}$; moreover, $\mathsf{tc}_R(l) = \mathsf{tc}_\rho\big((l,())\big)$ for any $l \in \mathscr{C}_R^{\mathsf{free}}$. One can say that R and ρ are "morally interfaced".

From $\rho \in \mathscr{T}_R$ for some DiLL-ps R and $\sigma \simeq \rho$, one cannot conclude that $\sigma \in \mathscr{T}_R$ (and there may exist a DiLL-ps $S \not\simeq R$ such that $\sigma \in \mathscr{T}_S$). This means that all the informations about R available in ρ thanks to the names of its ports and cells (see [10]) are lost in σ, though ρ and σ "morally" represent the same object.

The definitions of Taylor expansion of a MELL-ps in [12, Def. 9] and [14, Def. 5] forget all the information which we have encoded in our formalism, so they correspond to the following definition.

Definition 29 (Quotiented Taylor expansion)**.** *Let R be a* DiLL*-ps. The* quotiented Taylor expansion of *R is $\mathscr{T}_R^{\simeq} = \big\{\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\} \mid \rho \in \mathscr{T}_R\big\}$.*

Let $R \in \mathbf{PS}_{\mathsf{DiLL}}$: the binary relation $\approx_R$ on $\mathbf{PS}_{\mathsf{DiLL}_0}$ defined by "$\tau \approx_R \tau'$ iff there is $\rho \in \mathscr{T}_R$ such that $\tau \simeq \rho \simeq \tau'$" is a partial equivalence, and, for any $\rho \in \mathscr{T}_R$, $\{\tau \in \mathbf{PS}_{\mathsf{DiLL}_0} \mid \tau \simeq \rho\}$ is a partial equivalence class.

4 Uniformity for proto-nets

We conclude the paper by proving for proto-nets the analogue of the result proven for resource λ-terms in [8], which does not hold for diffnets [16, p. 244]: a set of proto-nets is included in the proto-Taylor expansion of some representative of a DiLL-ps if and only if it is a clique (Proposition 31, Corollary 35).

Lemma 30. *Let R' be a representative of some $R \in \mathbf{PS}_{\mathsf{DiLL}}$. Then, every $\Gamma \subseteq \mathscr{T}_{R'}^{\mathsf{proto}}$ is a clique. In particular, every $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$ is a uniform proto-net.* Proof in Appendix

The following proposition is the analogue of Lemma 19 in [8].

Proposition 31. *For every $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and every representative R' of R, one has that $\mathscr{T}_{R'}^{\mathsf{proto}}$ is a non-empty maximal clique with $\mathsf{depth}(\mathscr{T}_{R'}^{\mathsf{proto}}) = \mathsf{depth}(R)$.* Proof in Appendix

Despite Proposition 31, notice that $\beta \frown \alpha$ and $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$ for some representative R' of some $R \in \mathbf{PS}_{\mathsf{MELL}}$ does not imply that $\beta \in \mathscr{T}_{R'}^{\mathsf{proto}}$ (because $\frown$ is not transitive). In fact, it is easy to show that there exist $R, S \in \mathbf{PS}_{\mathsf{MELL}}$ such that, for any of their respective representatives R' and S', $\mathscr{T}_{R'}^{\mathsf{proto}} \nsubseteq \mathscr{T}_{S'}^{\mathsf{proto}}$ and $\mathscr{T}_{S'}^{\mathsf{proto}} \nsubseteq \mathscr{T}_{R'}^{\mathsf{proto}}$, but $\mathscr{T}_{R'}^{\mathsf{proto}} \cap \mathscr{T}_{S'}^{\mathsf{proto}} \neq \emptyset$.

Similarly to λ-calculus, not all maximal cliques of proto-nets are of the shape $\mathscr{T}_{R'}^{\mathsf{proto}}$ for some representative R' of some DiLL-ps R: any maximal extension of the clique $\Gamma = \{(\langle\rangle), (\langle(\langle\rangle)\rangle), (\langle(\langle(\langle\rangle)\rangle)\rangle), \dots\}$ cannot be of that shape; notice that $\mathsf{depth}(\Gamma) = \infty$, and indeed this is the only possibility for a maximal clique not to be of the shape $\mathscr{T}_{R'}^{\mathsf{proto}}$ for some $R \in \mathbf{PS}_{\mathsf{DiLL}}$ and some representative R' of R, as we will prove in Corollary 35. In particular, every finite subset of Γ is contained in $\mathscr{T}_{R}^{\mathsf{proto}}$ for some $R \in \mathbf{PS}_{\mathsf{MELL}}$ (we did not speak of the representatives of R since such an R has a unique representative).

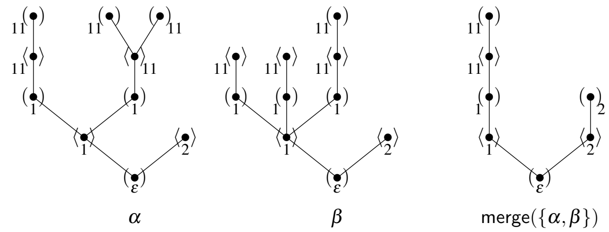


Figure 4: Example of merging of a clique, in its tree-like presentation with addresses.

Definition 32 (Merging of proto-nets)**.** *The function* merge *associates with every clique $A \subseteq$* **Proto** *with finite depth an element of* **Proto**, *called the* merging of *A and denoted by* merge(A). *For a clique $A = \{\alpha^i \mid i \in I\} \subseteq$* **Proto**, *with $I \subseteq \mathbb{N}$ and $\alpha^i = (\langle \alpha_1^{i,1}, \dots, \alpha_{k_{i,1}}^{i,1} \rangle, \dots, \langle \alpha_1^{i,n}, \dots, \alpha_{k_{i,n}}^{i,n} \rangle)$ for every $i \in I$ and some $n \in \mathbb{N}$,*[16] merge(A) *is defined by induction on* depth$(A) \in \mathbb{N}$ *as follows:*

$$\mathsf{merge}(A) = \begin{cases} () & \text{if } \mathsf{depth}(A) = 0 \\ (\langle \mathsf{merge}(A_1) \rangle, \dots, \langle \mathsf{merge}(A_n) \rangle) & \text{otherwise,} \end{cases}$$

where $A_j = \{\alpha_1^{i,j}, \dots, \alpha_{k_{i,j}}^{i,j} \mid i \in I\}$ for every $1 \leq j \leq n$.[17]

The idea is that the elements of a clique A of **Proto** are "compatible" precisely in the sense that there exists a nesting box-structure compatible with every element of the clique. The proto-net merge(A) is the "skeleton" of the clique: the elements of $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(A))$ all have length 1, and thus merge(A) is a tree where the nodes corresponding to $\langle\cdot\rangle$-sequences all have exactly one son. In other terms, merge(A) is the prototype of a tree of boxes of some representative of some DiLL- or more specifically MELL-ps R (the nesting box-structure of R): for every such tree $\mathfrak{T}$, it is immediate to find a MELL-ps having $\mathfrak{T}$ as its tree of boxes. Working out this idea, one gets Theorem 34 and then the converse of Proposition 31: Corollary 35.

Example 33. Let $\alpha = (\langle ((\langle () \rangle), (\langle (), () \rangle)\rangle, \langle \rangle)$ and $\beta = (\langle (\langle \rangle), (\langle \rangle), (\langle () \rangle)\rangle, \langle \rangle)$: so, α and β are coherent proto-nets and merge$(\{\alpha, \beta\})$ is as follows (see also Figure 4 for their tree-like representation):

$$\begin{aligned} \mathsf{merge}(\{\alpha,\beta\}) &= (\langle \mathsf{merge}(\{(\langle () \rangle), (\langle (), () \rangle), (\langle \rangle), (\langle \rangle), (\langle () \rangle)\}) \rangle, \langle \mathsf{merge}(\emptyset) \rangle) \\ &= (\langle (\langle \mathsf{merge}(\{(), (), (), ()\}) \rangle) \rangle, \langle () \rangle) \;=\; (\langle (\langle () \rangle) \rangle, \langle () \rangle)\,. \end{aligned}$$

The next theorem expresses the fact that merge(Γ) is indeed "the skeleton" of every element of the clique Γ, and it is the key-step to prove Corollary 35, the converse (in some sense) of Proposition 31.

Proof in Appendix

Theorem 34 (1-proto)**.** *Let $I \subseteq \mathbb{N}$ and $\Gamma = \{\alpha^i \mid i \in I\} \subseteq$* **Proto** *be a clique such that* depth$(\Gamma) \in \mathbb{N}$. *Then* merge(Γ) *is a uniform proto-net such that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique, every element of $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma))$ is a $\langle\cdot\rangle$-sequence of length* 1 *and* depth$(\mathsf{merge}(\Gamma)) =$ depth(Γ).

We can now conclude with the converse of Proposition 31. We say that a DiLL-ps R is ACC meaning that R is correct in a strong sense (its correctness graphs are acyclic and connected: it is a standard notion dating back to [4]; see also [17, Def. A.6 and Rmk. A.7]); but ACC has nothing to do with our proof: we could omit it, and it is mentioned only because it strengthens the statement.

Proof in Appendix

Corollary 35 (Surjectivity)**.** *For every set Γ of proto-nets with finite depth, if Γ is a clique, then there exists some* ACC *and cut-free $R \in \mathbf{PS}_{\mathsf{MELL}}$ such that $\Gamma \subseteq \mathscr{T}_{R'}^{\mathsf{proto}}$ for some representative R' of R . In particular, for every uniform proto-net α, there exists some* ACC *and cut-free $R \in \mathbf{PS}_{\mathsf{MELL}}$ such that $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$ for some representative R' of R.*

If $\Gamma \subseteq$ **Proto** is a finite set then depth$(\Gamma) \in \mathbb{N}$, therefore Theorem 34 and Corollary 35 hold in particular for any clique $\Gamma \subseteq$ **Proto** which is a finite set.

[16] Notice that the coherence hypothesis for the α^i's means that all the α^i's are sequences of the same length n.

[17] Notice that A_j is a clique by Lemma 10; moreover, depth$(A_j) <$ depth(A).

References

[1] Pierre Boudes, Fanny He & Michele Pagani (2013): *A characterization of the Taylor expansion of lambda-terms*. In: *Proceedings of the 22nd EACSL Annual Conference Computer Science Logic CSL 2013*.

[2] Daniel de Carvalho, Michele Pagani & Lorenzo Tortora de Falco (2011): *A semantic measure of the execution time in Linear Logic*. *Theoretical Computer Science, Special issue Girard's Festschrift* 412(20), pp. 1884–1902.

[3] Daniel de Carvalho & Lorenzo Tortora de Falco (2012): *The relational model is injective for Multiplicative Exponential Linear Logic (without weakenings)*. *Annals of Pure and Applied Logic* 163(9), pp. 1210–1236.

[4] Vincent Danos & Laurent Regnier (1989): *The structure of multiplicatives*. *Archive for Mathematical logic* 28(3), pp. 181–203.

[5] Thomas Ehrhard (2005): *Finiteness spaces*. *Mathematical Structures in Computer Science* 15(4), pp. 615–646.

[6] Thomas Ehrhard & Laurent Regnier (2003): *The differential lambda-calculus*. *Theoretical Computer Science* 309(1-3), pp. 1–41.

[7] Thomas Ehrhard & Laurent Regnier (2006): *Differential interaction nets*. *Theoretical Computer Science* 364(2), pp. 166–195.

[8] Thomas Ehrhard & Laurent Regnier (2008): *Uniformity and the Taylor expansion of ordinary lambda-terms*. *Theoretical Computer Science* 403(2-3), pp. 347–372.

[9] Jean-Yves Girard (1987): *Linear logic*. *Theoretical Computer Science* 50(1), pp. 1–102.

[10] Giulio Guerrieri, Luc Pellissier & Lorenzo Tortora de Falco (2014): *Computing connected proof(-structure)s from their Taylor expansion*. Technical Report, presented at the workshop Termgraph 2014. Available at `http://www.pps.univ-paris-diderot.fr/~giuliog/connectedtaylor.pdf`.

[11] Yves Lafont (1995): *From Proof-Nets to Interaction Nets*. In Jean-Yves Girard, Yves Lafont & Laurent Regnier, editors: *Advances in Linear Logic*, *London Mathematical Society Lecture Notes Series* 222, Cambridge University Press, pp. 225–247.

[12] Damiano Mazza & Michele Pagani (2007): *The Separation Theorem for Differential Interaction Nets*. In Nachum Dershowitz, editor: *Proceedings of the 14th International Conference on Logic for Programming Artificial Intelligence and Reasoning (LPAR 2007)*, *Lecture Notes in Artificial Intelligence* 4790, Springer, pp. 393–407.

[13] Michele Pagani (2009): *The Cut-Elimination Thereom for Differential Nets with Boxes*. In Pierre-Louis Curien, editor: *Proceedings of the Ninth International Conference on Typed Lambda Calculi and Applications (TLCA 2009)*, Lecture Notes in Computer Science, Springer, pp. 219–233.

[14] Michele Pagani & Christine Tasson (2009): *The Taylor Expansion Inverse Problem in Linear Logic*. In Andrew Pitts, editor: *Proceedings of the Twenty-Fourth Annual IEEE Symposium on Logic in Computer Science (LICS 2009)*, IEEE Computer Society Press, pp. 222–231.

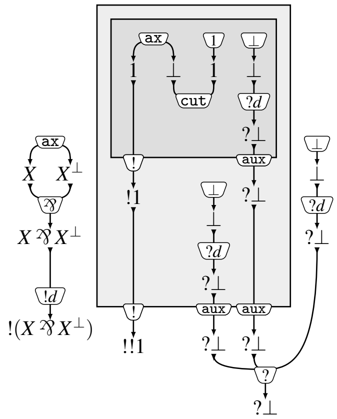
[15] Luc Pellissier (2012): *The differential nets seen as elements of a relational model*. Rapport de stage master 1, Ecole Normale Supérieure de Cachan.

[16] Christine Tasson (2009): *Sémantiques et syntaxes vectorielles de la logique linéaire*. Ph.D. thesis, Université Paris 7. Available at `http://www.pps.univ-paris-diderot.fr/~tasson/doc/recherche/these_tasson.pdf`.

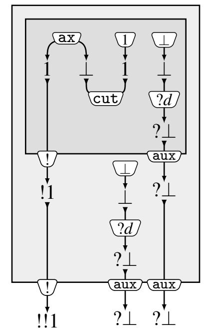
[17] Lorenzo Tortora de Falco (2003): *Obsessional Experiments For Linear Logic Proof-Nets*. *Mathematical Structures in Computer Science* 13(6), pp. 799–855.

[18] Paolo Tranquilli (2009): *Nets between Determinism and Nondeterminism*. Ph.D. thesis, Università Roma Tre / Université Paris 7. Available at `http://www.cs.unibo.it/~tranquil/content/docs/phd.pdf`.

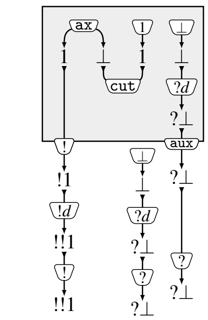
[19] Paolo Tranquilli (2011): *Intuitionistic differential nets and lambda-calculus*. *Theoretical Computer Science* 412(20), pp. 1979–1997.



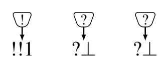
(a) The DiLL-ps S with $\mathscr{C}_S^{box_0} = \{l\}$.



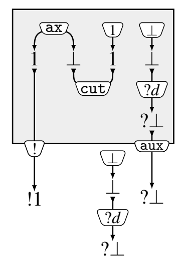
(b) The box of l.



(c) The opening of l.



(d) The border of l.



(e) The content of l.

Figure 5: A DiLL-ps S (where $\mathscr{C}_S^{box_0} = \{l\}$), with the box of l, the opening of l, the border of l and the content of l.

A Technical appendix

Proof of Lemma 10. A preliminary remark: given two proto-nets $\alpha = (\langle\alpha_1^1,\dots,\alpha_{k_1}^1\rangle,\dots,\langle\alpha_1^n,\dots,\alpha_{k_n}^n\rangle)$ and $\beta = (\langle\beta_1^1,\dots,\beta_{h_1}^1\rangle,\dots,\langle\beta_1^n,\dots,\beta_{h_n}^n\rangle)$ with $n,k_1,\dots,k_n, h_1,\dots,h_n \in \mathbb{N}$, then for any $1 \leq i \leq n$, $1 \leq j, j' \leq k_i$, $1 \leq \ell,\ell' \leq h_i$ and for all occurrences x,y in α_j^i or $\alpha_{j'}^i$ or β_ℓ^i or $\beta_{\ell'}^i$ of elements of $\mathsf{sub}_{(\cdot)}(\alpha_j^i) \cup \mathsf{sub}_{(\cdot)}(\alpha_{j'}^i) \cup \mathsf{sub}_{(\cdot)}(\beta_\ell^i) \cup \mathsf{sub}_{(\cdot)}(\beta_{\ell'}^i)$ one has that: $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$ iff $\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$. Indeed, by Definition 4, $\mathsf{addr}_{\{\alpha,\beta\}}(x) = i{\cdot}\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x)$ and $\mathsf{addr}_{\{\alpha,\beta\}}(y) = i{\cdot}\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$.

Let us now prove the equivalence stated in Lemma 10.

- $\Rightarrow$: Let $\alpha,\beta \in \Gamma$. Since $\alpha \frown \beta$ and $\mathsf{addr}_\alpha(\alpha) = \varepsilon = \mathsf{addr}_\beta(\beta)$, there exists $n \in \mathbb{N}$ such that $\mathsf{length}(\alpha) = n = \mathsf{length}(\beta)$. So, $\alpha = (\langle\alpha_1^1,\dots,\alpha_{k_1}^1\rangle,\dots,\langle\alpha_1^n,\dots,\alpha_{k_n}^n\rangle)$ and $\beta = (\langle\beta_1^1,\dots,\beta_{h_1}^1\rangle,\dots,\langle\beta_1^n,\dots,\beta_{h_n}^n\rangle)$ with $k_i,h_i \in \mathbb{N}$ and $\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i \in \mathbf{Proto}$, for any $1 \leq i \leq n$. Let $1 \leq i \leq n$, $1 \leq j,j' \leq k_i$ and $1 \leq \ell,\ell' \leq h_i$: for any occurrence x (resp. y) in α_j^i or $\alpha_{j'}^i$ or β_ℓ^i or $\beta_{\ell'}^i$ of an element s_x (resp. s_y) of $\mathsf{sub}_{(\cdot)}(\alpha_j^i) \cup \mathsf{sub}_{(\cdot)}(\alpha_{j'}^i) \cup \mathsf{sub}_{(\cdot)}(\beta_\ell^i) \cup \mathsf{sub}_{(\cdot)}(\beta_{\ell'}^i)$, $\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$ implies by the previous remark that $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$, so $\mathsf{length}(s_x) = \mathsf{length}(s_y)$ since $\alpha \frown \beta$; thus $\alpha_j^i \frown \beta_\ell^i$, $\alpha_j^i \frown \beta_{\ell'}^i$, $\alpha_{j'}^i \frown \beta_\ell^i$, $\alpha_{j'}^i \frown \beta_{\ell'}^i$, $\alpha_j^i \frown \alpha_{j'}^i$ and $\beta_\ell^i \frown \beta_{\ell'}^i$. Hence, $\{\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i\}$ is a clique, for any $1 \leq i \leq n$.

$\Leftarrow$: Let $\alpha, \beta \in \Gamma$: by hypothesis, there exists $n \in \mathbb{N}$ such that $\alpha = (\langle \alpha_1^1, \dots, \alpha_{k_1}^1 \rangle, \dots, \langle \alpha_1^n, \dots, \alpha_{k_n}^n \rangle)$ and $\beta = (\langle \beta_1^1, \dots, \beta_{h_1}^1 \rangle, \dots, \langle \beta_1^n, \dots, \beta_{h_n}^n \rangle)$ for some $k_i, h_i \in \mathbb{N}$ and $\alpha_1^i, \dots, \alpha_{k_i}^i, \beta_1^i, \dots, \beta_{h_i}^i \in \mathbf{Proto}$ such that $\{\alpha_1^i, \dots, \alpha_{k_i}^i, \beta_1^i, \dots, \beta_{h_i}^i\}$ is a clique (for any $1 \leq i \leq n$). In particular, $\mathsf{length}(\alpha) = n = \mathsf{length}(\beta)$. Let $1 \leq i \leq n$, $1 \leq j, j' \leq k_i$ and $1 \leq \ell, \ell' \leq h_i$: for any occurrence x (resp. y) in α_j^i or $\alpha_{j'}^i$ or β_ℓ^i or $\beta_{\ell'}^i$ of an element s_x (resp. s_y) of $\mathsf{sub}_{(\cdot)}(\alpha_j^i) \cup \mathsf{sub}_{(\cdot)} \cup (\alpha_{j'}^i) \cup \mathsf{sub}_{(\cdot)}(\beta_\ell^i) \cup \mathsf{sub}_{(\cdot)}(\beta_{\ell'}^i)$, $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$ implies $\mathsf{addr}_{\{\alpha_j^i, \alpha_{j'}^i, \beta_\ell^i, \beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i, \alpha_{j'}^i, \beta_\ell^i, \beta_{\ell'}^i\}}(y)$ by the preliminary remark, so $\mathsf{length}(s_x) = \mathsf{length}(s_y)$ (because $\alpha_j^i \frown \beta_\ell^i$, $\alpha_j^i \frown \beta_{\ell'}^i$, $\alpha_{j'}^i \frown \beta_\ell^i$, $\alpha_{j'}^i \frown \beta_{\ell'}^i$, $\alpha_j^i \frown \alpha_{j'}^i$ and $\beta_\ell^i \frown \beta_{\ell'}^i$). Since $\mathsf{sub}_{(\cdot)}(\alpha) = \{\alpha\} \cup \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k_i}} \mathsf{sub}_{(\cdot)}(\alpha_j^i)$ and $\mathsf{sub}_{(\cdot)}(\beta) = \{\beta\} \cup \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq \ell \leq k_i}} \mathsf{sub}_{(\cdot)}(\beta_\ell^i)$, we have shown that for any occurrence x (resp. y) in α or β of an element s_x (resp. s_y) of $\mathsf{sub}_{(\cdot)}(\alpha) \cup \mathsf{sub}_{(\cdot)}(\beta)$, $\mathsf{addr}_\alpha(x) = \mathsf{addr}_\beta(y)$ implies $\mathsf{length}(s_x) = \mathsf{length}(s_y)$, i.e. $\alpha \frown \beta$ (for all $\alpha, \beta \in \Gamma$). □

Proof of Fact 25. Both proofs are by induction on $\mathsf{depth}(R) \in \mathbb{N}$. Let $R' = (\mathsf{box}_{R'}^0, \mathsf{rep}_{R'}^0)$, with $\mathsf{box}_{R'}^0 = (l_1, \dots, l_n)$ for some $n \in \mathbb{N}$ such that $\mathsf{card}(\mathscr{C}_R^{box_0}) = n$.

1. The proof that $\tau_{\mathsf{p}(R')}(\mathsf{p}(\alpha)) = \tau_{R'}(\alpha)$ is left to the reader: it is based on the remark that, given a representative R' of a DiLL-ps R, the names of ports and cells of each element of $\mathscr{T}_{R'}$ does not depend on the enumeration $\mathsf{box}_{R'}^0$ and the other enumerations fixed recursively by $\mathsf{rep}_{R'}^0$. Let us prove that $\mathsf{p}(\alpha) \in \mathscr{T}_{\mathsf{p}(R')}^{\mathsf{proto}}$. Let $\mathsf{p} = (\sigma, \mathsf{p}_1, \dots, \mathsf{p}_n)$, where $\sigma \in \mathfrak{S}_n$ and p_i is a permutation of $\mathsf{rep}_{R'}^0(l_i)$ for any $1 \leq i \leq n$.

   If $\mathsf{depth}(R) = 0$ then $\mathscr{C}_R^{box_0} = \emptyset$ and $n = 0$, thus $\mathsf{box}_{R'}^0 = (\,)$ and $\mathsf{rep}_{R'}^0 = \emptyset$, so $R' = ((\,), \emptyset)$. Hence, $\mathsf{p} = (\emptyset)$ because $\emptyset$ is the only element of $\mathfrak{S}_0$, and $\mathsf{p}(R') = ((\,), \emptyset) = R'$ by Definition 24. Moreover, $\mathscr{T}_{R'}^{\mathsf{proto}} = \{(\,)\}$ and thus $\alpha = (\,) = \mathsf{p}(\alpha)$ (by Definition 24); therefore $\mathsf{p}(\alpha) = \alpha \in \mathscr{T}_{R'}^{\mathsf{proto}} = \mathscr{T}_{\mathsf{p}(R')}^{\mathsf{proto}}$.

   If $\mathsf{depth}(R) > 0$ then $n \in \mathbb{N}^+$ and $\alpha = (\langle \alpha_1^1, \dots, \alpha_{k_1}^1 \rangle, \dots, \langle \alpha_1^n, \dots, \alpha_{k_n}^n \rangle)$ with $k_i \in \mathbb{N}$ and $\alpha_1^i, \dots, \alpha_{k_i}^i \in \mathscr{T}_{\mathsf{rep}_{R'}^0(l_i)}^{\mathsf{proto}}$ for any $1 \leq i \leq n$. By Definition 24, $\mathsf{p}(R') = (\mathsf{box}_{\mathsf{p}(R')}^0, \mathsf{rep}_{\mathsf{p}(R')}^0)$ where $\mathsf{rep}_{\mathsf{p}(R')}^0(l_i) = \mathsf{p_i}(\mathsf{rep}_{R'}^0(l_i))$ for any $1 \leq i \leq n$, and $\mathsf{box}_{\mathsf{p}(R')}^0 = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$; by Definition 24 again, $\mathsf{p}(\alpha) = (\langle \mathsf{p}_{\sigma(1)}(\alpha_1^{\sigma(1)}), \dots, \mathsf{p}_{\sigma(1)}(\alpha_{k_{\sigma(1)}}^{\sigma(1)}) \rangle, \dots, \langle \mathsf{p}_{\sigma(n)}(\alpha_1^{\sigma(n)}), \dots, \mathsf{p}_{\sigma(n)}(\alpha_{k_{\sigma(n)}}^{\sigma(n)}) \rangle)$. By induction hypothesis, $\mathsf{p}_{\sigma(i)}(\alpha_1^{\sigma(i)}), \dots, \mathsf{p}_{\sigma(i)}(\alpha_{k_{\sigma(i)}}^{\sigma(i)}) \in \mathscr{T}_{\mathsf{p}_{\sigma(i)}(\mathsf{rep}_{R'}^0(l_{\sigma(i)}))}^{\mathsf{proto}} = \mathscr{T}_{\mathsf{rep}_{\mathsf{p}(R')}^0(l_{\sigma(i)})}^{\mathsf{proto}}$ for any $1 \leq i \leq n$. According to Definition 22, $\mathsf{p}(\alpha) \in \mathscr{T}_{\mathsf{p}(R')}^{\mathsf{proto}}$ since $\mathsf{box}_{\mathsf{p}(R')}^0 = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$.

2. Let $R'' = (\mathsf{box}_{R''}^0, \mathsf{rep}_{R''}^0)$.

   If $\mathsf{depth}(R) = 0$ then $\mathscr{C}_R^{box_0} = \emptyset$, thus $\mathsf{box}_{R'}^0 = (\,) = \mathsf{box}_{R''}^0$ and $\mathsf{rep}_{R'}^0 = \emptyset = \mathsf{rep}_{R''}^0$, so $R' = ((\,), \emptyset) = R''$. Hence, $\mathsf{p} = (\emptyset)$ is a permutation of R (since $\emptyset \in \mathfrak{S}_0$) and $\mathsf{p}(R') = ((\,), \emptyset) = R' = R''$.

   If $\mathsf{depth}(R) > 0$ then $n \in \mathbb{N}^+$ and there exists $\sigma \in \mathfrak{S}_n$ such that $\mathsf{box}_{R''}^0 = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$, since both $\mathsf{box}_{R'}^0$ and $\mathsf{box}_{R''}^0$ are enumerations of the same set $\mathscr{C}_R^{box_0}$. By induction hypothesis, for any $1 \leq i \leq n$, there exists a permutation p_i of $\mathsf{rep}_{R'}^0(l_i)$ such that $\mathsf{rep}_{R''}^0(l_i) = \mathsf{p}_i(\mathsf{rep}_{R'}^0(l_i))$. Hence, $\mathsf{p} = (\sigma, \mathsf{p}_1, \dots, \mathsf{p}_n)$ is a permutation of R and $\mathsf{p}(R') = R''$. □

Proof of Proposition 26. If $\rho \in \mathscr{T}_{R'}$, then there exists $\alpha \in \mathscr{T}_{R'}^{\mathsf{proto}}$ such that $\tau_{R'}(\alpha) = \rho$. By Facts 25.1-2, there exists a permutation p of R' such that $R'' = \mathsf{p}(R')$ and $\mathsf{p}(\alpha) \in \mathscr{T}_{R''}^{\mathsf{proto}}$ with $\tau_{R''}(\mathsf{p}(\alpha)) = \rho$, hence $\rho \in \mathscr{T}_{R''}$. Therefore, $\mathscr{T}_{R'} \subseteq \mathscr{T}_{R''}$. Similarly, one can prove that $\mathscr{T}_{R''} \subseteq \mathscr{T}_{R'}$. □

Proof of Lemma 30. By straightforward induction on $\mathsf{depth}(R) \in \mathbb{N}$.

If $\mathsf{depth}(R) = 0$ then $\mathscr{T}_{R'}^{\mathsf{proto}} = \{(\,)\}$, so $\Gamma \subseteq \mathscr{T}_{R'}^{\mathsf{proto}}$ implies that either $\Gamma = \emptyset$ or $\Gamma = \{(\,)\}$: in both cases Γ is a clique (see Remark 9).

If $\mathsf{depth}(R) > 0$, then $R' = (R, \mathscr{C}_R^{box_0}, \mathsf{box}_R)$ where $\mathscr{C}_R^{box_0} = (v_1, \ldots, v_n)$ and $v_1, \ldots, v_n$ (with $n \in \mathbb{N}^+$) are the !-links of R at depth 0, and, for any $1 \leq i \leq n$, R'_i is the representative of the ps associated with v_i by box_R; therefore $\alpha, \beta \in \Gamma \subseteq \mathscr{T}_{R'}^{\mathsf{proto}}$ means that

$$\alpha = (\langle \alpha_1^1, \ldots, \alpha_{k_1}^1 \rangle, \ldots, \langle \alpha_1^n, \ldots, \alpha_{k_n}^n \rangle) \qquad \text{and} \qquad \beta = (\langle \beta_1^1, \ldots, \beta_{h_1}^1 \rangle, \ldots, \langle \beta_1^n, \ldots, \beta_{h_n}^n \rangle)$$

where $k_i, h_i \in \mathbb{N}$ and $\alpha_1^i, \ldots, \alpha_{k_i}^i, \beta_1^i, \ldots, \beta_{h_i}^i \in \mathscr{T}_{R'_i}^{\mathsf{proto}}$ for any $1 \leq i \leq n$. By induction hypothesis, for any $1 \leq i \leq n$, $\{\alpha_1^i, \ldots, \alpha_{k_i}^i, \beta_1^i, \ldots, \beta_{h_i}^i\}$ is a clique. Hence Γ is a clique by Lemma 10. □

Proof of Proposition 31. By Lemma 30, $\mathscr{T}_{R'}^{\mathsf{proto}}$ is a clique; we prove that it is non-empty, maximal and such that $\mathsf{depth}\big(\mathscr{T}_{R'}^{\mathsf{proto}}\big) = \mathsf{depth}(R)$, by induction on $\mathsf{depth}(R) \in \mathbb{N}$.

When $\mathsf{depth}(R) = 0$ then $\mathscr{T}_{R'}^{\mathsf{proto}} = \{()\}$, which is a non-empty maximal clique (see Remark 9) with $\mathsf{depth}\big(\mathscr{T}_{R'}^{\mathsf{proto}}\big) = \mathsf{depth}\big(()\big) = 0 = \mathsf{depth}(R)$.

If $\mathsf{depth}(R) > 0$, then $R' = (\mathsf{box}_{R'}^0, \mathsf{rep}_{R'}^0)$ where $\mathsf{box}_{R'}^0 = (v_1, \ldots, v_m)$ and $v_1, \ldots, v_m$ (with $m \in \mathbb{N}^+$) are the !-links of R at depth 0, and, for any $1 \leq i \leq m$, R'_i is the representative of the ps associated with v_i by box_R. One has $\langle\rangle^m \in \mathscr{T}_{R'}^{\mathsf{proto}}$ by Definition 22, and thus $\mathscr{T}_{R'}^{\mathsf{proto}} \neq \emptyset$.

We prove that the clique $\mathscr{T}_{R'}^{\mathsf{proto}}$ is maximal when $\mathsf{depth}(R) > 0$ by showing that for any $\alpha \in \mathbf{Proto} \smallsetminus \mathscr{T}_{R'}^{\mathsf{proto}}$ there exists $\beta \in \mathscr{T}_{R'}^{\mathsf{proto}}$ such that $\alpha \not\frown \beta$. Let $\alpha = (\langle \alpha_1^1, \ldots, \alpha_{k_1}^1 \rangle, \ldots, \langle \alpha_1^n, \ldots, \alpha_{k_n}^n \rangle) \in \mathbf{Proto} \smallsetminus \mathscr{T}_{R'}^{\mathsf{proto}}$ for some $n \in \mathbb{N}$, where $k_i \in \mathbb{N}$ and $\alpha_1^i, \ldots, \alpha_{k_i}^i \in \mathbf{Proto}$ for every $1 \leq i \leq n$. According to Definition 22, the general form of $\beta \in \mathscr{T}_{R'}^{\mathsf{proto}}$ is the following: $\beta = (\langle \beta_1^1, \ldots, \beta_{h_1}^1 \rangle, \ldots, \langle \beta_1^m, \ldots, \beta_{h_m}^m \rangle)$, where $h_1, \ldots, h_m \in \mathbb{N}$ and $\beta_j^i \in \mathscr{T}_{R'_i}^{\mathsf{proto}}$ for any $1 \leq i \leq m$ and any $1 \leq j \leq h_i$. If $n \neq m$ then $\alpha \not\frown \langle\rangle^m$ because $\mathsf{length}(\alpha) \neq \mathsf{length}(\langle\rangle^m)$, and we are done since $\langle\rangle^m \in \mathscr{T}_{R'}^{\mathsf{proto}}$. When $n = m$, from $\alpha \notin \mathscr{T}_{R'}^{\mathsf{proto}}$ we infer that $\alpha_j^i \notin \mathscr{T}_{R'_i}^{\mathsf{proto}}$ for suitable $1 \leq i \leq m$ and $1 \leq j \leq k_i$; since $\mathsf{depth}(R) > \mathsf{depth}(R_i)$, we can apply the induction hypothesis to R'_i: the clique $\mathscr{T}_{R'_i}^{\mathsf{proto}}$ is non-empty and maximal, and thus there exists $\beta^i \in \mathscr{T}_{R'_i}^{\mathsf{proto}}$ such that $\alpha_j^i \not\frown \beta^i$. Let $\beta = \langle\rangle^{i-1} \cdot \langle \beta^i \rangle \cdot \langle\rangle^{m-i}$: we have $\beta \in \mathscr{T}_{R'}^{\mathsf{proto}}$, and by Lemma 10 (since $\alpha_j^i \not\frown \beta^i$) $\alpha \not\frown \beta$.

We prove that $\mathsf{depth}(\mathscr{T}_{R'}^{\mathsf{proto}}) = \mathsf{depth}(R)$ when $\mathsf{depth}(R) > 0$. For every $1 \leq i \leq m$, one has $\mathsf{depth}(\mathscr{T}_{R'_i}^{\mathsf{proto}}) = \mathsf{depth}(R_i) \in \mathbb{N}$ by induction hypothesis, thus there is $\beta^i \in \mathscr{T}_{R'_i}^{\mathsf{proto}}$ such that $\mathsf{depth}(\beta^i) = \mathsf{depth}(R_i)$ and every $\gamma^i \in \mathscr{T}_{R'_i}^{\mathsf{proto}}$ is such that $\mathsf{depth}(\gamma^i) \leq \mathsf{depth}(R'_i)$. Therefore, if $\beta = (\langle \beta^1 \rangle, \ldots, \langle \beta^m \rangle)$ then $\beta \in \mathscr{T}_{R'}^{\mathsf{proto}}$ and, according to Definition 1, $\mathsf{depth}(\mathscr{T}_{R'}^{\mathsf{proto}}) = \mathsf{depth}(\beta) = \sup\{\mathsf{depth}(\beta^i) + 1 \mid 1 \leq i \leq m\} = \sup\{\mathsf{depth}(R_i) + 1 \mid 1 \leq i \leq m\} = \mathsf{depth}(R)$. □

Proof of Theorem 34. By induction on $\mathsf{depth}(\Gamma) \in \mathbb{N}$. Observe that the fact that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique implies that $\mathsf{merge}(\Gamma)$ is uniform (see Remark 9).

If $\mathsf{depth}(\Gamma) = 0$ then $\Gamma = \emptyset$ or $\Gamma = \{()\}$ and thus $\mathsf{merge}(\Gamma) = () \in \mathbf{Proto}$, so $\Gamma \cup \{\mathsf{merge}(\Gamma)\} = \{()\}$ which is a clique (since $() \frown ()$), and $\mathsf{depth}(\mathsf{merge}(\Gamma)) = 0 = \mathsf{depth}(\Gamma)$; moreover, $\mathsf{sub}_{\langle\rangle}(\mathsf{merge}(\Gamma)) = \emptyset$.

Otherwise $\mathsf{depth}(\Gamma) > 0$: then $I \neq \emptyset$ and there is[18] $n \in \mathbb{N}^+$ such that, for any $i \in I$, one has $\alpha^i = (\langle \alpha_1^{i,1}, \ldots, \alpha_{k_{i,1}}^{i,1} \rangle, \ldots, \langle \alpha_1^{i,n}, \ldots, \alpha_{k_{i,n}}^{i,n} \rangle)$ for some $k_{i,1}, \ldots, k_{i,n} \in \mathbb{N}$ and $\alpha_1^{i,1}, \ldots, \alpha_{k_{i,1}}^{i,1}, \ldots, \alpha_1^{i,n}, \ldots, \alpha_{k_{i,n}}^{i,n} \in \mathbf{Proto}$. For $1 \leq j \leq n$, let $\Gamma_j = \{\alpha_1^{i,j}, \ldots, \alpha_{k_{i,j}}^{i,j} \mid i \in I\}$: by definition, $\mathsf{merge}(\Gamma) = (\langle \mathsf{merge}(\Gamma_1) \rangle, \ldots, \langle \mathsf{merge}(\Gamma_n) \rangle)$.

Because of Identity (1) of Definition 1 and since $n > 0$, one has, for every $i \in I$:

$$\mathsf{depth}(\alpha^i) = \sup\Big\{ \sup_{1 \leq \ell \leq k_{i,j}} \{\mathsf{depth}(\alpha_\ell^{i,j})\} + 1 \;\Big|\; 1 \leq j \leq n \Big\} = \sup\{\mathsf{depth}(\alpha_\ell^{i,j}) \;\big|\; 1 \leq j \leq n,\ 1 \leq \ell \leq k_{i,j}\} + 1$$

[18]See note 16 at p. 14.

whence, since $I \neq \emptyset$:

$$\begin{aligned}\mathsf{depth}(\Gamma) &= \sup\{\mathsf{depth}(\alpha^i) \mid i \in I\} = \sup\{\mathsf{depth}(\alpha^{i,j}_\ell) \mid i \in I, 1 \leq j \leq n, 1 \leq \ell \leq k_{i,j}\} + 1 \\ &= \sup\{\mathsf{depth}(\Gamma_j) \mid 1 \leq j \leq n\} + 1 \end{aligned} \tag{4}$$

thus, for all $1 \leq j \leq n$, one has $\mathsf{depth}(\Gamma) > \mathsf{depth}(\Gamma_j)$ and hence, by induction hypothesis, $\mathsf{merge}(\Gamma_j) \in$ **Proto**, moreover $\Gamma_j \cup \mathsf{merge}(\Gamma_j)$ is a clique, every element of $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma_j))$ is a $\langle\cdot\rangle$-sequence of length 1 and $\mathsf{depth}(\mathsf{merge}(\Gamma_j)) = \mathsf{depth}(\Gamma_j)$.

Since $\mathsf{merge}(\Gamma_j) \in$ **Proto** for all $1 \leq j \leq n$, then $\mathsf{merge}(\Gamma) = \big(\langle \mathsf{merge}(\Gamma_1)\rangle, \dots, \langle \mathsf{merge}(\Gamma_n)\rangle\big) \in$ **Proto**, furthermore $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma)) = \{\langle \mathsf{merge}(\Gamma_j)\rangle \mid 1 \leq j \leq n\} \cup \bigcup_{j=1}^n \mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma_j))$ and thus we infer, by induction hypothesis, that every element of $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma))$ is a $\langle\cdot\rangle$-sequence of length 1. Moreover:

$$\begin{aligned}\mathsf{depth}(\mathsf{merge}(\Gamma)) &= \sup\{\mathsf{depth}(\mathsf{merge}(\Gamma_i)) \mid 1 \leq j \leq n\} + 1 && \text{(by Identity (1) and since } n > 0) \\ &= \sup\{\mathsf{depth}(\Gamma_1), \dots, \mathsf{depth}(\Gamma_n)\} + 1 && \text{(by induction hypothesis)} \\ &= \mathsf{depth}(\Gamma) && \text{(by Identity (4)).}\end{aligned}$$

There is still to prove that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique when $\mathsf{depth}(\Gamma) > 0$. Since Γ is a clique, we only have to show that $\alpha^i \frown \mathsf{merge}(\Gamma)$ for every $i \in I$. Let $i \in I$: by Lemma 10, it is sufficient to prove that $\{\alpha^{i,j}_1, \dots, \alpha^{i,j}_{k_{i,j}}\} \cup \{\mathsf{merge}(\Gamma_j)\}$ is a clique for any $1 \leq j \leq n$, that is true because $\{\alpha^{1,j}_1, \dots, \alpha^{1,j}_{k_{1,j}}, \mathsf{merge}(\Gamma_j)\} \subseteq \Gamma_j \cup \{\mathsf{merge}(\Gamma_j)\}$ which is a clique by induction hypothesis. □

Proof of Corollary 35. By Theorem 34, $\mathsf{merge}(\Gamma)$ is a uniform proto-net such that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique, every element of $\mathsf{sub}_{\langle\cdot\rangle}(\mathsf{merge}(\Gamma))$ is a $\langle\cdot\rangle$-sequence of length 1 and $\mathsf{depth}(\mathsf{merge}(\Gamma)) = \mathsf{depth}(\Gamma)$. Take any ACC and cut-free MELL-ps R and any representative R' of R having $\mathsf{merge}(\Gamma)$ as nesting box-structure: one has $\mathscr{T}^{\mathsf{proto}}_{R'} \subseteq \Gamma$. □