# A new point of view on the Taylor expansion of proof-nets and uniformity [8-pages extended abstract]

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We introduce (in the multiplicative and exponential fragment of linear logic) the notion of proto-net and proto-Taylor expansion. We then define a coherence relation on proto-nets and prove the analogue of the result proven for resource lambda-terms, which does not hold for differential nets: a set of proto-nets is included in the proto-Taylor expansion of some proof-structure if and only if it is a clique.

## 1 Introduction

One of the main features of Linear Logic (LL) [7] is the logical status it gives to structural rules, thus stressing the difference between the linear and the non linear use of resources. With the discovery of Differential $\lambda$-calculus [4] and Differential Linear Logic (DiLL) [5], thanks to the notion of Taylor expansion [6] a $\lambda$-term (resp. a proof-structure[1], *ps* for short) is a (usually infinite) set[2] of "linear terms" (resp. "linear nets"), called resource $\lambda$-terms (resp. differential nets[3], *diffnets* for short). One of the aims becomes then to extract informations on the $\lambda$-term (resp. the ps) from its "linear components"; by the way, using elements of the Taylor expansion amounts to use points of the interpretation in the relational model (a denotational semantics based on the category of sets and relations): the link between the two approaches is more or less obvious, and it is precisely stated in the ongoing work [8] (a very first draft can be found in [10]).

Here a remarkable difference between the $\lambda$-calculus and LL appears. In [6] a binary symmetric relation (called *coherence*) on resource $\lambda$-terms is defined, allowing to characterize those sets of resource $\lambda$-terms that are subsets of the Taylor expansion of some (ordinary) $\lambda$-term (see also [1]). This is impossible for diffnets, as shown by the counterexample of [11, p. 244]: there exist three diffnets such that every pair of them belongs to the Taylor expansion of some ps but there is no ps containing in its Taylor expansion the three diffnets. This counterexample can be generalized to show the impossibility of any *n*-ary coherence relation for diffnets. This mismatch between $\lambda$-calculus and LL is clearly due to the fact that while $\lambda$-terms have a natural tree-like structure, this is not the case for ps.

Notice, however, that there exists a tree-like structure underlying a given ps: the nesting of its boxes. A possible way of getting some intuition on the new object introduced in this paper (the *proto-nets* of Definition 1) is to forget everything but the nesting structure of a ps $R$ (thus obtaining the tree of its boxes), and apply then the Taylor expansion (that will be called *proto-Taylor expansion* in Definition 12, denoted

[1] We will speak from now on of the set of $\mathbf{PS}_{\mathsf{MELL}}$, referring to the set of proof-structures of the Multiplicative and Exponential fragment of LL (MELL). All the ps to which we will refer are elements of $\mathbf{PS}_{\mathsf{MELL}}$.

[2] In its original formulation [6] the Taylor expansion of a $\lambda$-term is a (usually infinite) linear combination of resource $\lambda$-terms, but coefficients and sums play no role in our approach, and the linear combination becomes a set (i.e. its support), as in [9].

[3] For us, diffnets of DiLL are "promotion-free": they may contain multiplicative (i.e. $⅋$- and $\otimes$-)links, structural (i.e. ?-)links and co-structural (i.e. !-)links but not boxes.

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by $\mathsf{proto}(R)$). A ps with depth 0 becomes the singleton of an empty sequence, and a box becomes the infinite set of the copies of its content: in case the box contains a ps with depth 0, its Taylor expansion is the set of finite multisets containing empty sequences. In Figures 1, 2(a) and 2(b), we give an example of ps $R$, its ordinary Taylor expansion $\tau(R)$ and its proto-Taylor expansion $\mathsf{proto}(R)$.

With a representative $R'$ (see Definition 11) of a ps $R$ and $\alpha \in \mathsf{proto}(R')$ is associated a unique $\rho \in \tau(R)$: this is not proven in our paper but rather obvious for the reader acquainted with ps and diffnets.[4] Conversely, with a representative $R'$ of a ps $R$ and $\rho \in \tau(R)$ is associated a unique $\alpha \in \mathsf{proto}(R)$: this is not proven either, but while the relationship between diffnets and proto-nets is essential for our intuition, it plays no role in the statements and proofs of our paper. A proto-net is "less" than a diffnet since (for example) the multiplicative structure (which can be preserved in diffnets) is always lost in proto-nets, but it is also "more" because it comes equipped with a tree-like structure (reminiscent of the nesting structure of a ps) which is absent in diffnets.

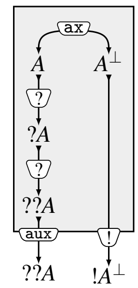


Figure 1: The proof-structure $R$

Once the set of proto-nets (**Proto**, Definition 1) is introduced, one defines a coherence relation on **Proto** (Definition 7): like in [6], the idea is that two proto-nets are coherent when they differ only by the cardinality of their multisets. A clique is a set of proto-nets which are pairwise coherent. One can then prove the expected results that hold for resource $\lambda$-terms (see [6]) and not for diffnets:

1. for every representative $R'$ of any ps $R$ with depth $d$, the set $\mathsf{proto}(R')$ is a maximal clique with depth $d$ (Proposition 14);
2. every clique $\Gamma$ with finite depth $d$ is contained in $\mathsf{proto}(R')$ for some suitable ps $R$ with depth $d$ and some representative $R'$ of $R$ (Corollary 18).

## 2 The proto-nets

When speaking of ps and diffnets, the reader can refer to any presentation of them. In the sequel, for the pictures of ps (resp. diffnets) we will use the syntax presented in [12, Def. A.2] (resp. [5]). See also [9] for a unique formalism for both ps and diffnets.

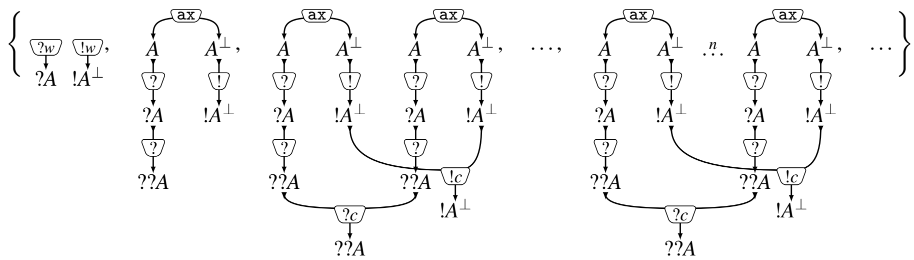
**Notation.** Finite sequences are denoted by $(a_1,\ldots,a_n)$; in particular the empty sequence is denoted by $()$; for any $n \in \mathbb{N}$ we set $a^n = (\overbrace{a,\ldots,a}^{n \text{ times}})$. Concatenation is denoted by $\cdot$: if $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_m)$ then $a \cdot b = (a_1,\ldots,a_n,b_1,\ldots,b_m)$; if $n = 1$ (resp. $m = 1$), then $a_1 \cdot b$ (resp. $a \cdot b_1$) stands for $a \cdot b$. The length of a finite sequence $a = (a_1,\ldots,a_n)$ is $\mathsf{length}(a) = n \in \mathbb{N}$. Finite sequences over $\mathbb{N}$ (used here only for addresses, see Definition 4) are written as words, the empty word is denoted by $\varepsilon$.

Finite multisets are denoted by $[a_1,\ldots,a_n]$, in particular the empty multiset is denoted by $[]$. Multiset union is denoted by $\uplus$: if $a = [a_1,\ldots,a_n]$ and $b = [b_1,\ldots,b_m]$ then $a \uplus b = [a_1,\ldots,a_n,b_1,\ldots,b_m]$.

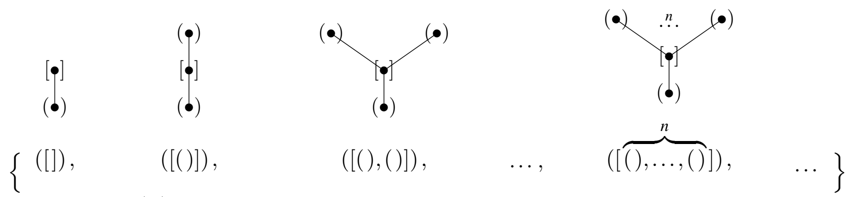
**Definition 1** (Proto-net and depth)**.** *The elements of the set* **Proto***, called* proto-nets *and denoted by* $\alpha,\beta,\gamma,\ldots$*, are defined by induction as follows: given* $n,k_1,\ldots,k_n \in \mathbb{N}$ *and, for any* $1 \leq i \leq n$*, given* $\alpha^i_1,\ldots,\alpha^i_{k_i} \in$ **Proto***, then* $([\alpha^1_1,\ldots,\alpha^1_{k_1}],\ldots,[\alpha^n_1,\ldots,\alpha^n_{k_n}]) \in$ **Proto**.[5]

[4] This means that the Taylor expansion of a ps $R$ can be defined as the set of diffnets associated with the elements of $\mathsf{proto}(R')$, for any representative $R'$ of $R$.

[5] In particular, $() \in$ **Proto** (take $n = 0$): this is the base case of the inductive definition.



(a) $\tau(R)$, the Taylor expansion of $R$.



(b) $\mathsf{proto}(R)$, the proto-Taylor expansion of $R$, also in its tree-like presentation.

Figure 2: The Taylor and proto-Taylor expansion of $R$ (see Figure 1 for $R$).

*For every* $\alpha = ([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]) \in \mathbf{Proto}$ *(for some* $n, k_1, \dots, k_n \in \mathbb{N}$*), the* depth of $\alpha$*, denoted by* $\mathsf{depth}(\alpha)$*, is a natural number defined by induction on* $\alpha$ *as follows:*[6]

$$\mathsf{depth}(\alpha) = \sup\Big\{ \sup_{1 \le j \le k_i} \{\mathsf{depth}(\alpha_j^i)\} + 1 \;\Big|\; 1 \le i \le n \Big\}. \tag{1}$$

*Given* $\Gamma \subseteq \mathbf{Proto}$*, the* depth of $\Gamma$ *is* $\mathsf{depth}(\Gamma) = \sup\{\mathsf{depth}(\alpha) \mid \alpha \in \Gamma\}$.

Intuitively, the depth of a proto-net $\alpha$ is the maximal number of nested multisets occurring in $\alpha$. For $\alpha \in \mathbf{Proto}$ (resp. $\Gamma \subseteq \mathbf{Proto}$), one has $\mathsf{depth}(\alpha) \in \mathbb{N}$ (resp. $\mathsf{depth}(\Gamma) \in \mathbb{N} \cup \{\infty\}$). If $\Gamma \subseteq \mathbf{Proto}$ is such that $\mathsf{depth}(\Gamma) = \infty$ (resp. $\Gamma = \emptyset$), then $\Gamma$ is an infinite set (resp. $\mathsf{depth}(\Gamma) = 0$).

**Example 2.** $()$ is the only proto-net with depth 0; $([()])$ and $([(),()],[])$ and $([])$ are proto-nets with depth 1; $\big([(),()],[],[(),(),([()],[])]\big)$ is a proto-net with depth 2; neither $[]$ nor $[(),()]$ are proto-nets. Other examples of proto-nets are in Figures 2(b), 3 and 4.

**Definition 3.** *Let* $n, k_1, \dots, k_n \in \mathbb{N}$ *and let* $\alpha = ([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n])$ *be a proto-net.*

*The sets of* sub-sequences of $\alpha$ *and* sub-multisets of $\alpha$*, denoted respectively by* $\mathsf{sub_s}(\alpha)$ *and* $\mathsf{sub_m}(\alpha)$*, are defined by induction on* $\mathsf{depth}(\alpha) \in \mathbb{N}$ *as follows:*

$$\mathsf{sub_s}(\alpha) = \{\alpha\} \cup \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le k_i}} \mathsf{sub_s}(\alpha_j^i); \qquad \mathsf{sub_m}(\alpha) = \big\{[\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]\big\} \cup \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le k_i}} \mathsf{sub_m}(\alpha_j^i).$$

**Definition 4** (Addresses in a proto-net)**.** *Let* $\alpha = ([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]) \in \mathbf{Proto}$ *(for some* $n, k_1, \dots, k_n \in \mathbb{N}$*) and let* $x$ *be an occurrence in* $\alpha$ *of an element of* $\mathsf{sub_s}(\alpha) \cup \mathsf{sub_m}(\alpha)$*. The* address of $x$ in $\alpha$ *is a finite sequence* $\mathsf{addr}_\alpha(x)$ *over* $\mathbb{N}^+$ *defined by induction on* $\mathsf{depth}(\alpha) \in \mathbb{N}$ *as follows:*

- *if* $x = \alpha$ *then* $\mathsf{addr}_\alpha(x) = \varepsilon$*;*
- *if* $x = [\alpha_1^i, \dots, \alpha_{k_i}^i]$ *for some* $1 \le i \le n$*, then* $\mathsf{addr}_\alpha(x) = i$*;*
- *if* $x \in \mathsf{sub_s}(\alpha_j^i) \cup \mathsf{sub_m}(\alpha_j^i)$ *and* $\mathsf{addr}_{\alpha_j^i}(x) = \sigma$ *with* $1 \le i \le n$ *and* $1 \le j \le k_i$*, then* $\mathsf{addr}_\alpha(x) = i \cdot \sigma$.

[6] Recall that for $A \subseteq \mathbb{N}$, if $A = \emptyset$ then $\sup(A) = 0$. Thus, Identity (1) takes also into account the cases where $n = 0$ or $n > 0$ with $k_1 = \ldots = k_n = 0$.

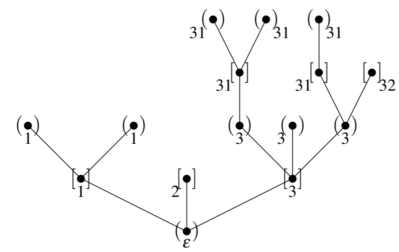
$\big([(),()],[],[([(),()]),(),([()],[])]\big) \rightsquigarrow$ 


Figure 3: Example of a proto-net, also in its tree-like presentation with addresses.

*Let* $\Gamma \subseteq \mathbf{Proto}$ *and let x be an occurrence in some* $\alpha \in \Gamma$ *of an element of* $\mathsf{sub}_{\mathrm{s}}(\alpha) \cup \mathsf{sub}_{\mathrm{m}}(\alpha)$*: we set* $\mathsf{addr}_\Gamma(x) = \mathsf{addr}_\alpha(x)$.[7]

The basic idea is that in a proto-net $\alpha$, different occurrences of elements of $\mathsf{sub}_{\mathrm{s}}(\alpha)$ in the same occurrence of an element of $\mathsf{sub}_{\mathrm{m}}(\alpha)$ represent different copies of the "same thing" and have hence the same address (and indeed with the notations of Definition 4, for every $1 \leq i \leq n$ and every $1 \leq j \leq k_i$ one has $\mathsf{addr}_\alpha(\alpha^i_j) = i$). Definition 4 generalizes this idea.[8]

**Example 5.** The proto-net $\alpha = \big([(),()],[],[([(),()]),(),([()],[])]\big)$ (with $\mathsf{depth}(\alpha) = 2$) is represented as a tree in Figure 3, where we have also specified the addresses in $\alpha$ of all occurrences of elements of $\mathsf{sub}_{\mathrm{s}}(\alpha) \cup \mathsf{sub}_{\mathrm{m}}(\alpha)$ in $\alpha$.

**Remark 6.** Let $\alpha \in \mathbf{Proto}$. One can prove by straightforward induction on $\mathsf{depth}(\alpha) \in \mathbb{N}$:

$$\mathsf{depth}(\alpha) = \sup\{\mathsf{length}(\mathsf{addr}_\alpha(x)) \mid x \text{ is an occurrence in } \alpha \text{ of an element of } \mathsf{sub}_{\mathrm{m}}(\alpha)\}.$$

We define a coherence relation on **Proto**: two proto-nets are coherent when the occurrences of sequences having the same address have the same length. The reader can easily check that this is the case for any two elements of the set represented in Figure 2(b).

**Definition 7** (Coherence, uniformity, clique)**.** *The* coherence relation*, denoted by* $\frown$*, is a binary relation on* **Proto** *defined as follows: given* $\alpha, \beta \in \mathbf{Proto}$*,* $\alpha \frown \beta$ *(say "*$\alpha$ *and* $\beta$ *are coherent") if for any occurrence x (resp. y) in* $\alpha$ *or* $\beta$ *of an element* $s_x$ *(resp.* $s_y$*) of* $\mathsf{sub}_{\mathrm{s}}(\alpha) \cup \mathsf{sub}_{\mathrm{s}}(\beta)$*,* $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$ *implies that* $\mathsf{length}(s_x) = \mathsf{length}(s_y)$*.*

*A proto-net* $\alpha$ *is* uniform *if* $\alpha \frown \alpha$*.*

*A set* $\Gamma \subseteq \mathbf{Proto}$ *is a* clique *(*of **Proto***) when* $\alpha \frown \beta$ *for all* $\alpha, \beta \in \Gamma$*.*

Roughly speaking, two coherent proto-nets can differ recursively only in the cardinality of their occurrences of sub-multisets having the same address. Possibly, an occurrence of a sub-multiset can be empty and another occurrence of a sub-multiset having the same address can be non-empty, therefore in general two coherent proto-nets might have different depth.

**Example 8.** Let $\alpha = ([])$ and $\beta = \big([(),([])]\big)$: $\alpha \frown \alpha$ but $\alpha \not\frown \beta$ and $\beta \not\frown \beta$ because the occurrences in $\beta$ of $()$ and $([])$ have the same address but not the same length.

Note that the coherence relation is symmetric but neither reflexive (the proto-nets $\alpha$ in Example 5 and $\beta$ in Example 8 are not uniform) nor transitive (for instance, take $\alpha = ([()])$, $\beta = ([])$ and $\gamma = ([([])])$: $\alpha \frown \beta$ and $\beta \frown \gamma$ but $\alpha \not\frown \gamma$). In particular, a singleton of a proto-net is not necessarily a clique.

[7] There is no ambiguity in this definition because $x$ is an *occurrence in* $\alpha$ of an element of $\mathsf{sub}_{\mathrm{s}}(\alpha) \cup \mathsf{sub}_{\mathrm{m}}(\alpha)$.

[8] For any multiset $z$, let us denote by $\mathsf{supp}(z)$ its support, i.e. the set of the elements having nonzero multiplicity in $z$. One can prove by induction on $\mathsf{depth}(\alpha) \in \mathbb{N}$ that for every $\alpha \in \mathbf{Proto}$, if $x$ (resp. $y$) is an occurrence in $\alpha$ of some $s_x \in \mathsf{sub}_{\mathrm{s}}(\alpha)$ (resp. $s_y \in \mathsf{sub}_{\mathrm{m}}(\alpha)$) and if $s_x \in \mathsf{supp}(s_y)$, then $\mathsf{addr}_\alpha(x) = \mathsf{addr}_\alpha(y)$.

**Remark 9.** Given $\alpha,\beta \in \mathbf{Proto}$, if $\alpha \frown \beta$ then $\alpha$ and $\beta$ are uniform; in particular, all elements of a clique are uniform. Moreover, $()$ is uniform (since $\mathsf{sub}_\mathrm{s}(()) = \{()\}$) and $\{()\}$ is a maximal[9] clique: for every $\gamma \in \mathbf{Proto}$, either $\gamma = ()$ or $\gamma \not\frown ()$. Following Definition 12, the reader can check that $\mathsf{proto}(R)$ of Figure 2(b) (note that $R$, defined in Figure 1, admits a unique representative) is a maximal clique of **Proto**.

The following lemma gives a nice "inductive" characterization of cliques.

**Lemma 10.** *Let $\Gamma \subseteq \mathbf{Proto}$: $\Gamma$ is a clique iff there exists $n \in \mathbb{N}$ such that for all $\alpha,\beta \in \Gamma$ one has $\alpha = ([\alpha_1^1,\dots,\alpha_{k_1}^1],\dots,[\alpha_1^n,\dots,\alpha_{k_n}^n])$ and $\beta = ([\beta_1^1,\dots,\beta_{h_1}^1],\dots,[\beta_1^n,\dots,\beta_{h_n}^n])$ for some $k_1,h_1,\dots,k_n,h_n \in \mathbb{N}$, and for every $1 \le i \le n$ the set $\{\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i\}$ is a clique.*

PROOF. A preliminary remark: given two proto-nets $\alpha = ([\alpha_1^1,\dots,\alpha_{k_1}^1],\dots,[\alpha_1^n,\dots,\alpha_{k_n}^n])$ and $\beta = ([\beta_1^1,\dots,\beta_{h_1}^1],\dots,[\beta_1^n,\dots,\beta_{h_n}^n])$ with $n,k_1,\dots,k_n,\ h_1,\dots,h_n \in \mathbb{N}$, then for every $1 \le i \le n$, $1 \le j,j' \le k_i$, $1 \le \ell,\ell' \le h_i$ and for all occurrences $x,y$ in $\alpha_j^i$ or $\alpha_{j'}^i$ or $\beta_\ell^i$ or $\beta_{\ell'}^i$ of elements of $\mathsf{sub}_\mathrm{s}(\alpha_j^i) \cup \mathsf{sub}_\mathrm{s}(\alpha_{j'}^i) \cup \mathsf{sub}_\mathrm{s}(\beta_\ell^i) \cup \mathsf{sub}_\mathrm{s}(\beta_{\ell'}^i)$ one has $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y) \iff \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$. Indeed, by Definition 4, $\mathsf{addr}_{\{\alpha,\beta\}}(x) = i{\cdot}\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x)$ and $\mathsf{addr}_{\{\alpha,\beta\}}(y) = i{\cdot}\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$.

Let us now prove the equivalence stated in Lemma 10.

- $\Rightarrow$: Let $\alpha,\beta \in \Gamma$. Since $\alpha \frown \beta$ and $\mathsf{addr}_\alpha(\alpha) = \varepsilon = \mathsf{addr}_\beta(\beta)$, there exists $n \in \mathbb{N}$ such that $\mathsf{length}(\alpha) = n = \mathsf{length}(\beta)$. Thus, $\alpha = ([\alpha_1^1,\dots,\alpha_{k_1}^1],\dots,[\alpha_1^n,\dots,\alpha_{k_n}^n])$ and $\beta = ([\beta_1^1,\dots,\beta_{h_1}^1],\dots,[\beta_1^n,\dots,\beta_{h_n}^n])$ with $k_i,h_i \in \mathbb{N}$ and $\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i \in \mathbf{Proto}$, for any $1 \le i \le n$. Let $1 \le i \le n$, $1 \le j,j' \le k_i$ and $1 \le \ell,\ell' \le h_i$: for any occurrence $x$ (resp. $y$) in $\alpha_j^i$ or $\alpha_{j'}^i$ or $\beta_\ell^i$ or $\beta_{\ell'}^i$ of an element $s_x$ (resp. $s_y$) of $\mathsf{sub}_\mathrm{s}(\alpha_j^i) \cup \mathsf{sub}_\mathrm{s}(\alpha_{j'}^i) \cup \mathsf{sub}_\mathrm{s}(\beta_\ell^i) \cup \mathsf{sub}_\mathrm{s}(\beta_{\ell'}^i)$, $\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$ implies by the previous remark that $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$, so $\mathsf{length}(s_x) = \mathsf{length}(s_y)$ since $\alpha \frown \beta$; thus $\alpha_j^i \frown \beta_\ell^i$, $\alpha_j^i \frown \alpha_{j'}^i$ and $\beta_\ell^i \frown \beta_{\ell'}^i$. Hence, $\{\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i\}$ is a clique, for any $1 \le i \le n$.
- $\Leftarrow$: Let $\alpha,\beta \in \Gamma$: by hypothesis, there exists $n \in \mathbb{N}$ such that $\alpha = ([\alpha_1^1,\dots,\alpha_{k_1}^1],\dots,[\alpha_1^n,\dots,\alpha_{k_n}^n])$ and $\beta = ([\beta_1^1,\dots,\beta_{h_1}^1],\dots,[\beta_1^n,\dots,\beta_{h_n}^n])$ for some $k_i,h_i \in \mathbb{N}$ and $\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i \in \mathbf{Proto}$ such that $\{\alpha_1^i,\dots,\alpha_{k_i}^i,\beta_1^i,\dots,\beta_{h_i}^i\}$ is a clique (for any $1 \le i \le n$). In particular, $\mathsf{length}(\alpha) = n = \mathsf{length}(\beta)$. Let $1 \le i \le n$, $1 \le j,j' \le k_i$ and $1 \le \ell,\ell' \le h_i$: for any occurrence $x$ (resp. $y$) in $\alpha_j^i$ or $\alpha_{j'}^i$ or $\beta_\ell^i$ or $\beta_{\ell'}^i$ of an element $s_x$ (resp. $s_y$) of $\mathsf{sub}_\mathrm{s}(\alpha_j^i) \cup \mathsf{sub}_\mathrm{s} \cup (\alpha_{j'}^i) \cup \mathsf{sub}_\mathrm{s}(\beta_\ell^i) \cup \mathsf{sub}_\mathrm{s}(\beta_{\ell'}^i)$, $\mathsf{addr}_{\{\alpha,\beta\}}(x) = \mathsf{addr}_{\{\alpha,\beta\}}(y)$ implies by the preliminary remark that $\mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(x) = \mathsf{addr}_{\{\alpha_j^i,\alpha_{j'}^i,\beta_\ell^i,\beta_{\ell'}^i\}}(y)$ and hence $\mathsf{length}(s_x) = \mathsf{length}(s_y)$ (because $\alpha_j^i \frown \beta_\ell^i$, $\alpha_j^i \frown \alpha_{j'}^i$ and $\beta_\ell^i \frown \beta_{\ell'}^i$). Since $\mathsf{sub}_\mathrm{s}(\alpha) = \{\alpha\} \cup \bigcup_{\substack{1\le i\le n\\ 1\le j\le k_i}} \mathsf{sub}_\mathrm{s}(\alpha_j^i)$ and $\mathsf{sub}_\mathrm{s}(\beta) = \{\beta\} \cup \bigcup_{\substack{1\le i\le n\\ 1\le \ell\le k_i}} \mathsf{sub}_\mathrm{s}(\alpha_j^i)$, we have shown that for any occurrence $x$ (resp. $y$) in $\alpha$ or $\beta$ of an element $s_x$ (resp. $s_y$) of $\mathsf{sub}_\mathrm{s}(\alpha) \cup \mathsf{sub}_\mathrm{s}(\beta)$, $\mathsf{addr}_\alpha(x) = \mathsf{addr}_\beta(y)$ implies that $\mathsf{length}(s_x) = \mathsf{length}(s_y)$, i.e. $\alpha \frown \beta$ (for all $\alpha,\beta \in \Gamma$). □

# 3 The proto-Taylor expansion

Recall Figures 1, 2(a) and 2(b): we are now going to define the proto-Taylor expansion of a ps.

For any $R \in \mathbf{PS}_{\mathsf{MELL}}$, we recall that: with every !-link $v$ of $R$ is associated a unique sub-graph of $R$, called *box* (*of* $v$), fulfilling some conditions (in particular the *nesting condition*, see for example [12, Def. A.2]); the depth of a link $l$ of $R$ is the number of boxes of $R$ containing $l$; $\mathsf{depth}(R) \in \mathbb{N}$ is the maximal depth of the links of $R$.

[9] The word "maximal" refers here and hereinafter to the order relation $\subseteq$ on the power set of **Proto**.

**Definition 11** (Representative of a MELL proof-structure)**.** *Given* $R \in \mathbf{PS}_{\mathsf{MELL}}$*, a* representative of $R$ *is a triple* $(R, \mathscr{C}_0^{\mathsf{box}}(R), \mathsf{box}_R)$ *defined by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*, where* $\mathscr{C}_0^{\mathsf{box}}(R)$ *is an enumeration of the* $!$*-links of* $R$ *at depth 0, and* $\mathsf{box}_R$ *is a map associating with every* $!$*-link* $v$ *of* $R$ *at depth 0 a representative of the ps associated with* $v$ *in* $R$*.*

**Definition 12** (Proto-Taylor expansion of a representative of a MELL proof-structure)**.** *Let* $R \in \mathbf{PS}_{\mathsf{MELL}}$*. We define, by induction on* $\mathsf{depth}(R) \in \mathbb{N}$*, the* proto-Taylor expansion $\mathsf{proto}(R')$ of any representative $R'$ of $R$*. Let* $R' = (R, \mathscr{C}_0^{\mathsf{box}}(R), \mathsf{box}_R)$ *be a representative of* $R$*, where* $\mathscr{C}_0^{\mathsf{box}}(R) = (v_1, \dots, v_n)$ *and* $v_1, \dots, v_n$ *(with* $n \in \mathbb{N}$*) are the* $!$*-links of* $R$ *at depth* $0$*; for any* $1 \le i \le n$*, let* $R'_i$ *be the representative of the ps* $R_i$ *associated with* $v_i$ *by* $\mathsf{box}_R$*. The proto-Taylor expansion* $\mathsf{proto}(R')$ *of* $R'$ *is the following subset of* **Proto***:*

$$\mathsf{proto}(R') = \big\{([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]) \mid k_i \in \mathbb{N} \text{ and } \alpha_1^i, \dots, \alpha_{k_i}^i \in \mathsf{proto}(R'_i), \forall\, 1 \le i \le n\big\}.$$

Observe that if $n = 0$, i.e. $\mathsf{depth}(R) = 0$, then $\mathsf{proto}(R') = \{()\}$: this is the base case of the induction. The definition of proto-Taylor expansion of a ps depends on the representative of the ps that has been chosen. Intuitively, with reference to notations used in Definition 12, the multiset $[\alpha_1^i, \dots, \alpha_{k_i}^i]$ (for any $1 \le i \le n$) corresponds to $k_i \in \mathbb{N}$ copies of the content $R_i$ of the box associated with the !-link $v_i$ of $R$ at depth 0 (this explains the relationship between the enumeration $\mathscr{C}_0^{\mathsf{box}}(R)$ of the !-links of $R$ at depth 0 and the components of the finite sequence $\alpha$).

**Lemma 13.** *Let* $R'$ *be a representative of some* $R \in \mathbf{PS}_{\mathsf{MELL}}$*. Then, every* $\Gamma \subseteq \mathsf{proto}(R')$ *is a clique. In particular, every* $\alpha \in \mathsf{proto}(R')$ *is a uniform proto-net.*

PROOF. By straightforward induction on $\mathsf{depth}(R) \in \mathbb{N}$. □

The following proposition is the analogue of Lemma 19 in [6].

**Proposition 14.** *For every* $R \in \mathbf{PS}_{\mathsf{MELL}}$ *and every representative* $R'$ *of* $R$*, one has that* $\mathsf{proto}(R')$ *is a non-empty maximal clique with* $\mathsf{depth}(\mathsf{proto}(R')) = \mathsf{depth}(R)$*.*

PROOF. By Lemma 13, $\mathsf{proto}(R')$ is a clique; we prove it is non-empty, maximal and such that $\mathsf{depth}((\mathsf{proto}(R')) = \mathsf{depth}(R)$, by induction on $\mathsf{depth}(R) \in \mathbb{N}$. □

Despite Proposition 14, notice that $\beta \frown \alpha$ and $\alpha \in \mathsf{proto}(R')$ for some representative $R'$ of some $R \in \mathbf{PS}_{\mathsf{MELL}}$ does not imply that $\beta \in \mathsf{proto}(R')$ (because $\frown$ is not transitive). In fact, it is easy to show that there exist $R, S \in \mathbf{PS}_{\mathsf{MELL}}$ such that, for any of their respective representatives $R'$ and $S'$, $\mathsf{proto}(R') \nsubseteq \mathsf{proto}(S')$ and $\mathsf{proto}(S') \nsubseteq \mathsf{proto}(R')$, but $\mathsf{proto}(R') \cap \mathsf{proto}(S') \neq \emptyset$.

Not all maximal cliques of proto-nets are of the shape $\mathsf{proto}(R')$ for some $R \in \mathbf{PS}_{\mathsf{MELL}}$ and some representative $R'$ of $R$. For instance, any maximal extension of the clique $\Gamma = \{([]), ([([])]), ([([([])])]), \dots\}$ cannot be of that shape; notice that $\mathsf{depth}(\Gamma) = \infty$, and indeed this is the only possibility for a maximal clique not to be of the shape $\mathsf{proto}(R')$ for some $R \in \mathbf{PS}_{\mathsf{MELL}}$ and some representative $R'$ of $R$, as we will prove in Corollary 18. In particular, every finite subset of $\Gamma$ is contained in $\mathsf{proto}(R)$ for some $R \in \mathbf{PS}_{\mathsf{MELL}}$ with a unique representative.

**Definition 15** (Merging of proto-nets)**.** *The function* merge *associates with every clique* $A \subseteq$ **Proto** *with finite depth an element of* **Proto***, called the* merging of $A$ *and denoted by* $\mathsf{merge}(A)$*. For a clique* $A = \{\alpha^i \mid i \in I\} \subseteq$ **Proto***, with* $I \subseteq \mathbb{N}$ *and* $\alpha^i = ([\alpha_1^{i,1}, \dots, \alpha_{k_{i,1}}^{i,1}], \dots, [\alpha_1^{i,n}, \dots, \alpha_{k_{i,n}}^{i,n}])$ *for every* $i \in I$ *and some* $n \in \mathbb{N}$*,*[10] $\mathsf{merge}(A)$ *is defined by induction on* $\mathsf{depth}(A) \in \mathbb{N}$ *as follows:*

$$\mathsf{merge}(A) = \begin{cases} () & \text{if } \mathsf{depth}(A) = 0 \\ ([\mathsf{merge}(A_1)], \dots, [\mathsf{merge}(A_n)]) & \text{otherwise} \end{cases}$$

[10] Notice that the coherence hypothesis for the $\alpha^i$'s means that all the $\alpha^i$'s are sequences of the same length $n$.

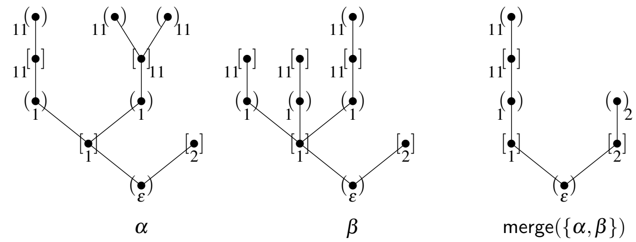



Figure 4: Example of merging of a clique, in its tree-like presentation with addresses.

*where* $A_j = \{\alpha_1^{i,j}, \dots, \alpha_{k_{i,j}}^{i,j} \mid i \in I\}$ *for every* $1 \leq j \leq n$.[11]

The idea is that the elements of a clique $A$ of **Proto** are "compatible" precisely in the sense that there exists a nesting structure compatible with every element of the clique. The proto-net $\mathsf{merge}(A)$ is the "skeleton" of the clique: the elements of $\mathsf{sub}_\mathrm{m}(\mathsf{merge}(A))$ all have cardinality 1, and thus $\mathsf{merge}(A)$ is a tree where the nodes corresponding to multisets all have one son. In other terms, $\mathsf{merge}(A)$ is the prototype of a tree of boxes of some representative of some ps $R$ (the nesting structure of $R$): for every such tree $\mathscr{T}$, it is immediate to find a ps having $\mathscr{T}$ as tree of boxes. Working out this idea, one obtains Theorem 17 and then the converse of Proposition 14: Corollary 18.

**Example 16.** Let $\alpha = \big([([()]),([(),()])],[]\big)$ and $\beta = \big([([]),([]),([()])],[]\big)$: so, $\alpha$ and $\beta$ are coherent proto-nets and $\mathsf{merge}(\{\alpha,\beta\})$ is as follows (see also Figure 4 for their tree-like representation):

$$\begin{aligned}\mathsf{merge}(\{\alpha,\beta\}) &= \big([\mathsf{merge}(\{([()]),([(),()]),([]),([]),([()])\})],[\mathsf{merge}(\emptyset)]\big)\\ &= \big([([\mathsf{merge}(\{(),(),(),()\})])],[()]\big) \;=\; \big([([()])],[()]\big)\,.\end{aligned}$$

The following theorem expresses the fact that $\mathsf{merge}(A)$ is indeed "the skeleton" of every element of the clique $A$, and it is the key-step to prove Corollary 18, the converse (in some sense) of Proposition 14.

**Theorem 17** (1-proto)**.** *Let* $I \subseteq \mathbb{N}$ *and* $\Gamma = \{\alpha^i \mid i \in I\} \subseteq$ **Proto** *be a clique such that* $\mathsf{depth}(\Gamma) \in \mathbb{N}$. *Then* $\mathsf{merge}(\Gamma)$ *is a uniform proto-net such that* $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ *is a clique, every element of* $\mathsf{sub}_\mathrm{m}(\mathsf{merge}(\Gamma))$ *is a multiset of cardinality* 1 *and* $\mathsf{depth}(\mathsf{merge}(\Gamma)) = \mathsf{depth}(\Gamma)$.

PROOF. By induction on $\mathsf{depth}(\Gamma) \in \mathbb{N}$. Observe that the fact that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique implies that $\mathsf{merge}(\Gamma)$ is uniform (see Remark 9). □

We can now conclude with the converse of Proposition 14. We say that a ps $R$ is *ACC* meaning that $R$ is correct in a strong sense (its correctness graphs are acyclic and connected: it is a standard notion dating back to [3], one of the first works on LL, see also [12, Def. A.6 and Rmk. A.7]); but *ACC* has nothing to do with our proof: we could omit it, and it is mentioned only because it strengthens the statement.

**Corollary 18** (Surjectivity)**.** *For every set* $\Gamma$ *of proto-nets with finite depth, if* $\Gamma$ *is a clique, then there exists some ACC and cut-free* $R \in \mathbf{PS}_{\mathsf{MELL}}$ *such that* $\Gamma \subseteq \mathsf{proto}(R')$ *for some representative* $R'$ *of* $R$ *. In particular, for every uniform proto-net* $\alpha$*, there exists some ACC and cut-free* $R \in \mathbf{PS}_{\mathsf{MELL}}$ *such that* $\alpha \in \mathsf{proto}(R')$ *for some representative* $R'$ *of* $R$.

If $\Gamma \subseteq$ **Proto** is a finite set then $\mathsf{depth}(\Gamma) \in \mathbb{N}$, therefore Theorem 17 and Corollary 18 hold in particular for any clique $\Gamma \subseteq$ **Proto** which is a finite set.

## 4 Conclusion and future work

We feel that proto-nets are interesting objects and deserve to be studied, since they offer a new perspective

[11] Notice that $A_j$ is a clique by Lemma 10; moreover, $\mathsf{depth}(A_j) < \mathsf{depth}(A)$.

on Taylor expansion allowing to easily recover the nesting structure of a ps (i.e. the tree of its boxes), as the results proven in this paper witness.

We actually met proto-nets when proving[12] that a ps $R$ fulfilling the *ACC* correctness criterion can be uniquely recovered from the 2-diffnet[13] in its Taylor expansion $\tau(R)$: this yields an alternative (and much simpler) proof of injectivity of the relational semantics in the *ACC* case (a result proven in [2] in a slightly more general case). The main tool for our proof is the extension of the notion of empire (see [7]) to diffnets (which makes sense only in the *ACC* case): given an *ACC* ps $R$, a diffnet $\rho \in \tau(R)$ such that every !-link has at least two premises and a representative $R'$ of $R$, one can build a proto-net $\alpha \in \mathsf{proto}(R')$ such that there is a precise correspondence not only between the !-links of $\rho$ and the multisets occurring in $\alpha$ but also between the empires of the premises of a !-link in $\rho$ and the sequences occurring in $\alpha$.

In an ongoing work we are trying to show that this notion of empire, combined with the one of proto-net, allows to give an alternative criterion to determine whether or not a given finite set of *ACC* diffnets is a subset of the Taylor expansion of some *ACC* ps: for general (*ACC* or not) cut-free ps a (rather complicated) solution to this problem can be found in [9].

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[12] This result too is contained in the mentioned ongoing work [8], a first idea of its proof can be found in [10].

[13] For any $n \in \mathbb{N}$, a $n$-diffnet is a diffnet where every co-contraction link has exactly $n$ premises.

# A Technical appendix

**Proof of Lemma 13.** By straightforward induction on $\mathsf{depth}(R) \in \mathbb{N}$.

If $\mathsf{depth}(R) = 0$ then $\mathsf{proto}(R') = \{()\}$, so $\Gamma \subseteq \mathsf{proto}(R')$ implies that either $\Gamma = \emptyset$ or $\Gamma = \{()\}$: in both cases $\Gamma$ is a clique (see Remark 9).

If $\mathsf{depth}(R) > 0$, then $R' = (R, \mathscr{C}_0^{\mathsf{box}}(R), \mathsf{box}_R)$ where $\mathscr{C}_0^{\mathsf{box}}(R) = (v_1, \dots, v_n)$ and $v_1, \dots, v_n$ (with $n \in \mathbb{N}^+$) are the !-links of $R$ at depth 0, and, for any $1 \leq i \leq n$, $R'_i$ is the representative of the ps associated with $v_i$ by $\mathsf{box}_R$; therefore $\alpha, \beta \in \Gamma \subseteq \mathsf{proto}(R')$ means that

$$\alpha = ([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]) \qquad \text{and} \qquad \beta = ([\beta_1^1, \dots, \beta_{h_1}^1], \dots, [\beta_1^n, \dots, \beta_{h_n}^n])$$

where $k_i, h_i \in \mathbb{N}$ and $\alpha_1^i, \dots, \alpha_{k_i}^i, \beta_1^i, \dots, \beta_{h_i}^i \in \mathsf{proto}(R'_i)$ for any $1 \leq i \leq n$. By induction hypothesis, $\{\alpha_1^i, \dots, \alpha_{k_i}^i, \beta_1^i, \dots, \beta_{h_i}^i\}$ is a clique for any $1 \leq i \leq n$. Hence $\Gamma$ is a clique by Lemma 10. ☐

**Proof of Proposition 14.** By Lemma 13, $\mathsf{proto}(R')$ is a clique; we prove that it is non-empty, maximal and such that $\mathsf{depth}\big(\mathsf{proto}(R')\big) = \mathsf{depth}(R)$, by induction on $\mathsf{depth}(R) \in \mathbb{N}$.

When $\mathsf{depth}(R) = 0$ then $\mathsf{proto}(R') = \{()\}$, which is a non-empty maximal clique (see Remark 9) with $\mathsf{depth}\big(\mathsf{proto}(R')\big) = \mathsf{depth}\big(()\big) = 0 = \mathsf{depth}(R)$.

If $\mathsf{depth}(R) > 0$, then $R' = (R, \mathscr{C}_0^{\mathsf{box}}(R), \mathsf{box}_R)$ where $\mathscr{C}_0^{\mathsf{box}}(R) = (v_1, \dots, v_m)$ and $v_1, \dots, v_m$ (with $m \in \mathbb{N}^+$) are the !-links of $R$ at depth 0, and, for any $1 \leq i \leq m$, $R'_i$ is the representative of the ps associated with $v_i$ by $\mathsf{box}_R$. One has $[]^m \in \mathsf{proto}(R')$ by Definition 12, and thus $\mathsf{proto}(R') \neq \emptyset$.

We prove that the clique $\mathsf{proto}(R')$ is maximal when $\mathsf{depth}(R) > 0$ by showing that for any $\alpha \in \mathbf{Proto} \smallsetminus \mathsf{proto}(R')$ there exists $\beta \in \mathsf{proto}(R')$ such that $\alpha \not\supset\!\!\subset \beta$. Let $\alpha = ([\alpha_1^1, \dots, \alpha_{k_1}^1], \dots, [\alpha_1^n, \dots, \alpha_{k_n}^n]) \in \mathbf{Proto} \smallsetminus \mathsf{proto}(R')$ for some $n \in \mathbb{N}$, where $k_i \in \mathbb{N}$ and $\alpha_1^i, \dots, \alpha_{k_i}^i \in \mathbf{Proto}$ for every $1 \leq i \leq n$. According to Definition 12, the general form of $\beta \in \mathsf{proto}(R')$ is the following: $\beta = ([\beta_1^1, \dots, \beta_{h_1}^1], \dots, [\beta_1^m, \dots, \beta_{h_m}^m])$, where $h_1, \dots, h_m \in \mathbb{N}$ and $\beta_j^i \in \mathsf{proto}(R'_i)$ for any $1 \leq i \leq m$ and any $1 \leq j \leq h_i$. If $n \neq m$ then $\alpha \not\supset\!\!\subset []^m$ because $\mathsf{length}(\alpha) \neq \mathsf{length}([]^m)$, and we are done since $[]^m \in \mathsf{proto}(R')$. When $n = m$, from $\alpha \notin \mathsf{proto}(R')$ we infer that $\alpha_j^i \notin \mathsf{proto}(R'_i)$ for suitable $1 \leq i \leq m$ and $1 \leq j \leq k_i$; since $\mathsf{depth}(R) > \mathsf{depth}(R_i)$, we can apply the induction hypothesis to $R'_i$: the clique $\mathsf{proto}(R'_i)$ is non-empty and maximal, and thus there exists $\beta^i \in \mathsf{proto}(R'_i)$ such that $\alpha_j^i \not\supset\!\!\subset \beta^i$. Let $\beta = []^{i-1} \cdot [\beta^i] \cdot []^{m-i}$: we have $\beta \in \mathsf{proto}(R')$, and by Lemma 10 (since $\alpha_j^i \not\supset\!\!\subset \beta^i$) $\alpha \not\supset\!\!\subset \beta$.

We prove that $\mathsf{depth}(\mathsf{proto}(R')) = \mathsf{depth}(R)$ when $\mathsf{depth}(R) > 0$. For every $1 \leq i \leq m$, one has $\mathsf{depth}(\mathsf{proto}(R'_i)) = \mathsf{depth}(R_i) \in \mathbb{N}$ by induction hypothesis, thus there exists $\beta^i \in \mathsf{proto}(R'_i)$ such that $\mathsf{depth}(\beta^i) = \mathsf{depth}(R_i)$ and every $\gamma^i \in \mathsf{proto}(R'_i)$ is such that $\mathsf{depth}(\gamma^i) \leq \mathsf{depth}(R'_i)$. Therefore, if $\beta = ([\beta^1], \dots, [\beta^m])$ then $\beta \in \mathsf{proto}(R')$ and, according to Definition 1, $\mathsf{depth}(\mathsf{proto}(R')) = \mathsf{depth}(\mathsf{proto}(\beta)) = \sup\{\mathsf{depth}(\beta^i) + 1 \mid 1 \leq i \leq m\} = \sup\{\mathsf{depth}(R_i) + 1 \mid 1 \leq i \leq m\} = \mathsf{depth}(R)$. ☐

**Proof of Theorem 17.** By induction on $\mathsf{depth}(\Gamma) \in \mathbb{N}$. Observe that the fact that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique implies that $\mathsf{merge}(\Gamma)$ is uniform (see Remark 9).

If $\mathsf{depth}(\Gamma) = 0$ then $\Gamma = \emptyset$ or $\Gamma = \{()\}$ and thus $\mathsf{merge}(\Gamma) = () \in \mathbf{Proto}$, so $\Gamma \cup \{\mathsf{merge}(\Gamma)\} = \{()\}$ which is a clique (since $() \supset\!\!\subset ()$), and $\mathsf{depth}(\mathsf{merge}(\Gamma)) = 0 = \mathsf{depth}(\Gamma)$; moreover, $\mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma)) = \emptyset$.

Otherwise $\mathsf{depth}(\Gamma) > 0$: then $I \neq \emptyset$ and there exists[14] $n \in \mathbb{N}^+$ such that, for every $i \in I$, one has $\alpha^i = ([\alpha_1^{i,1}, \dots, \alpha_{k_{i,1}}^{i,1}], \dots, [\alpha_1^{i,n}, \dots, \alpha_{k_{i,n}}^{i,n}])$ for some $k_{i,1}, \dots, k_{i,n} \in \mathbb{N}$ and $\alpha_1^{i,1}, \dots, \alpha_{k_{i,1}}^{i,1}, \dots, \alpha_1^{i,n}, \dots, \alpha_{k_{i,n}}^{i,n} \in \mathbf{Proto}$. For $1 \leq j \leq n$, let $\Gamma_j = \{\alpha_1^{i,j}, \dots, \alpha_{k_{i,j}}^{i,j} \mid i \in I\}$: by definition, $\mathsf{merge}(\Gamma) = ([\mathsf{merge}(\Gamma_1)], \dots, [\mathsf{merge}(\Gamma_n)])$.

[14] See note 10 at p. 6.

Because of Identity (1) of Definition 1 and since $n > 0$, one has, for every $i \in I$:

$$\mathsf{depth}(\alpha^i) = \sup\Big\{\sup_{1\le\ell\le k_{i,j}}\{\mathsf{depth}(\alpha_\ell^{i,j})\} + 1 \;\Big|\; 1 \le j \le n\Big\} = \sup\{\mathsf{depth}(\alpha_\ell^{i,j}) \;\big|\; 1 \le j \le n,\ 1 \le \ell \le k_{i,j}\} + 1$$

whence, since $I \neq \emptyset$:

$$\begin{aligned}\mathsf{depth}(\Gamma) &= \sup\{\mathsf{depth}(\alpha^i) \mid i \in I\} = \sup\{\mathsf{depth}(\alpha_\ell^{i,j}) \mid i \in I, 1 \le j \le n, 1 \le \ell \le k_{i,j}\} + 1 \\ &= \sup\{\mathsf{depth}(\Gamma_j) \mid 1 \le j \le n\} + 1 \end{aligned} \tag{2}$$

thus, for all $1 \le j \le n$, one has $\mathsf{depth}(\Gamma) > \mathsf{depth}(\Gamma_j)$ and hence, by induction hypothesis, $\mathsf{merge}(\Gamma_j) \in$ **Proto**, moreover $\Gamma_j \cup \mathsf{merge}(\Gamma_j)$ is a clique, every element of $\mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma_j))$ is a multiset of cardinality 1 and $\mathsf{depth}(\mathsf{merge}(\Gamma_j)) = \mathsf{depth}(\Gamma_j)$.

Since $\mathsf{merge}(\Gamma_j) \in$ **Proto** for all $1 \le j \le n$, then $\mathsf{merge}(\Gamma) = \big([\mathsf{merge}(\Gamma_1)], \dots, [\mathsf{merge}(\Gamma_n)]\big) \in$ **Proto**, furthermore $\mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma)) = \{[\mathsf{merge}(\Gamma_j)] \mid 1 \le j \le n\} \cup \bigcup_{j=1}^{n} \mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma_j))$ and thus we infer, by induction hypothesis, that every element of $\mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma))$ is a multiset of cardinality 1. Moreover:

$$\begin{aligned}\mathsf{depth}(\mathsf{merge}(\Gamma)) &= \sup\{\mathsf{depth}(\mathsf{merge}(\Gamma_i)) \mid 1 \le j \le n\} + 1 && \text{(by Identity (1) and since } n > 0) \\ &= \sup\{\mathsf{depth}(\Gamma_1), \dots, \mathsf{depth}(\Gamma_n)\} + 1 && \text{(by induction hypothesis)} \\ &= \mathsf{depth}(\Gamma) && \text{(by Identity (2)).}\end{aligned}$$

There is still to prove that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique when $\mathsf{depth}(\Gamma) > 0$. Since $\Gamma$ is a clique, we only have to show that $\alpha^i \frown \mathsf{merge}(\Gamma)$ for every $i \in I$. Let $i \in I$: by Lemma 10, it is sufficient to prove that $\{\alpha_1^{i,j}, \dots, \alpha_{k_{i,j}}^{i,j}\} \cup \{\mathsf{merge}(\Gamma_j)\}$ is a clique for any $1 \le j \le n$, that is true because $\{\alpha_1^{1,j}, \dots, \alpha_{k_{1,j}}^{1,j}, \mathsf{merge}(\Gamma_j)\} \subseteq \Gamma_j \cup \{\mathsf{merge}(\Gamma_j)\}$ which is a clique by induction hypothesis. □

**Proof of Corollary 18.** By Theorem 17, $\mathsf{merge}(\Gamma)$ is a uniform proto-net such that $\Gamma \cup \{\mathsf{merge}(\Gamma)\}$ is a clique, every element of $\mathsf{sub}_{\mathrm{m}}(\mathsf{merge}(\Gamma))$ is a multiset of cardinality 1 and $\mathsf{depth}(\mathsf{merge}(\Gamma)) = \mathsf{depth}(\Gamma)$. Take any *ACC* and cut-free ps $R$ and any representative $R'$ of $R$ having $\mathsf{merge}(\Gamma)$ as nesting box-structure: one has $\mathsf{proto}(R') \subseteq \Gamma$. □