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# Postponement of *raa* and Glivenko's Theorem, Revisited

**Abstract.** We study how to postpone the application of the *reductio ad absurdum* rule (*raa*) in classical natural deduction. This technique is connected with two normalization strategies for classical logic, due to Prawitz and Seldin, respectively. We introduce a variant of Seldin's strategy for the postponement of *raa*, which induces a negative translation (a variant of Kuroda's one) from classical to intuitionistic and minimal logic. Through this translation, Glivenko's theorem from classical to intuitionistic and minimal logic is proven.

*Keywords:* proof theory, natural deduction, negative translation, reductio ad absurdum.

## 1. Introduction

Among the inference rules of classical natural deduction, the *reductio ad absurdum* — denoted by *raa* — formalizes the principle of a “proof by contradiction”: if a contradiction follows from  $\neg A$ , then  $A$  can be asserted and the hypothesis  $\neg A$  can be dropped, i.e. discharged. This principle is rejected by intuitionism and, in general, by constructive accounts of logic. More precisely, *raa* is not an admissible inference rule in intuitionistic natural deduction, even if the latter contains a special case of *raa*, called *ex falso quodlibet* and denoted by *efq*. The rule *efq* formalizes the “principle of explosion”: from a contradiction anything can be asserted, without discharging any hypothesis. In turn, *efq* is not admissible in natural deduction for minimal logic.

For natural deduction of first-order classical logic (with the *raa* rule) there are two general strategies for defining a weak normalization procedure: one due to Prawitz and one due to Seldin (for a first comparison, see [16, pp. 282–283]). For both Prawitz and Seldin a derivation is normal if it has no detour. However, their notions of detour specific to classical logic are not the same.

Prawitz's idea [19] is to restrict to the fragment  $\{\neg, \wedge, \rightarrow, \perp, \forall\}$ , reduce all the applications of *raa* to atomic formulas, and then apply whatever normalization strategy for intuitionistic logic one likes (see [19, pp. 39–41]). Seldin's idea [20], on the other hand, is to restrict to the fragment  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$ , reduce all the discharging applications of *raa* present in a derivation to at most one single application occurring as the last step of the derivation (this

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is the *postponement of raa*), and then apply any normalization strategy for intuitionistic logic (see [20, pp. 638–645]). Prawitz’s and Seldin’s strategies can be seen as “dual”: the former breaks down classical reasoning into a number of atomic steps of *raa*, the latter compactifies classical reasoning into one single (possibly complex) step of *raa*, the final one. A peculiarity of Seldin’s strategy is that Glivenko’s theorem for intuitionistic logic can be obtained as an immediate consequence: it suffices to drop the *raa* rule of a normal (according to Seldin) derivation in classical logic, and replace it with a  $\neg$ -introduction rule discharging the same assumptions (see [20, §3]).

*Glivenko’s theorem*, in its original formulation [8], states that if a formula is provable in classical propositional logic then its double negation is provable in intuitionistic propositional logic (the converse trivially holds). Thus, it allows propositional classical logic to be embedded into propositional intuitionistic logic. Several refinements and generalizations of Glivenko’s theorem are well known: it can be extended to the fragments without  $\forall$  of first-order [10] and second-order [25] logic, to substructural logics [7, 14, 5] and to simple type theory [2]; also, it can embed propositional classical logic without  $\rightarrow$  into minimal logic [3], and full first-order classical logic into intermediate (i.e. between classical and intuitionistic) logics [20, 4]. All these results are obtained using different approaches, both syntactic and semantic.

Starting from a comparison of Prawitz’s and Seldin’s weak normalization strategies for first-order classical natural deduction, we modify Seldin’s reduction steps in order to prove two versions of the postponement of *raa*, one postponing only the discharging instances of *raa* (same result as in [20]), the other postponing all the instances of *raa* (under stronger hypotheses). In this way, we achieve two further goals: on the one hand, we obtain two variants of *Kuroda’s negative translation* [11, 6, 2] of full first-order classical logic into intuitionistic and minimal logic; on the other hand, and consequently, we give a simple proof of Glivenko’s theorem for both intuitionistic logic (from the fragment  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$  of classical logic) and minimal logic (from the fragment  $\{\neg, \wedge, \vee, \perp, \exists\}$  of classical logic).

We obtain these results using only proof-theoretic methods. The three main reasons for the interest of our approach are:

1. We point out that the postponement of *raa* not only is an interesting result in itself, with remarkable consequences such as weak normalization and Glivenko theorems (as first observed by Seldin [20]), but also it allows them to be proved in a *uniform* way in, at least, a triple sense:
  - (a) we prove the postponement of *raa* and its consequences for first-order classical logic without  $\forall$ , but our methods and techniques can also be

- applied in *other systems*, such as second-order classical logic without  $\forall$  (see point 2 below), and the modal logics S4 and S5 without  $\forall$ ;
- (b) the postponement of *raa* allows us to derive Glivenko’s theorem for *both* intuitionistic and minimal logic using the same proof-theoretic approach based on our reduction steps; and
  - (c) the reduction steps we have defined are essentially variants of the ones used by Seldin, Prawitz and others (see §2) to prove weak normalization for classical natural deduction.
2. Our proof of the postponement of *raa* is proof-theoretic in a “*geometric*” way, in the sense that it relies on a notion of size for a derivation based only on the distance of the instances of *raa* from the conclusion of the derivation; the complexity of formulas plays no role in this definition of size. Two remarks about this “*geometric*” approach:
    - (a) we prove the postponement of *raa* in a *weak* form, in that if our reduction steps are applied by selecting *suitable* instances of *raa*, then the size of the derivation decreases; we conjecture that, by refining this notion of size, our reduction steps allow the postponement of *raa* in a strong sense, i.e. they can be applied following whatever strategy one likes;
    - (b) we could generalize the postponement of *raa* and its corollaries to second-order classical logic, since the substitution of formulas for propositional variables does not impact the size we have defined.
  3. We show that, in classical natural deduction, the postponement of *raa* *induces* (two variants of) Kuroda’s negative translation, just as, dually, the atomization of *raa* proposed by Prawitz’s weak normalization strategy is deeply related to the Gödel–Gentzen negative translation (see [15]).

## 2. Normalization of Classical logic: an Overview

Looking closer at both Prawitz’s and Seldin’s weak normalization strategies for first-order classical natural deduction, we note that their differences are not as sharp as it might appear at a first glance. In particular, each of these strategies can be exploited to eliminate classical detours by postponing the use of *raa*. Let us clarify this point.

### 2.1. Prawitz’s (Weak) Normalization Strategy and its Legacy

Prawitz’s original weak normalization strategy for classical natural deduction [19, pp. 39-41] was conceived only for the fragment  $\{\neg, \wedge, \rightarrow, \perp, \forall\}$ , which is

adequate for the full first-order language of classical logic.<sup>1</sup> The first step in his normalization strategy consists in atomizing all the instances of  $\text{raa}$  in a derivation. Let us give an example for the case of conjunction. Prawitz observes that the derivation (1) below, with a complex instance of  $\text{raa}$  (its conclusion is  $A \wedge B$ ), can be reduced to the derivation (2), which has less complex instances of  $\text{raa}$  (their conclusions are  $A$  and  $B$ ).

$$\begin{array}{c}
 \Gamma \neg(A \wedge B)^{\neg 1} \\
 \vdots \pi \\
 \frac{\perp}{A \wedge B} \text{raa}^1 \quad (1) \\
 \vdots
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\Gamma \neg A^{\neg 2} \quad \frac{\Gamma A \wedge B^{\neg 1}}{A} \wedge_{e1}}{\perp} \neg_e \\
 \frac{\perp}{\neg(A \wedge B)} \neg_i^1 \\
 \vdots \pi \\
 \frac{\perp}{A} \text{raa}^2 \\
 \hline
 A \wedge B \\
 \vdots
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{\Gamma \neg B^{\neg 4} \quad \frac{\Gamma A \wedge B^{\neg 3}}{B} \wedge_{e2}}{\perp} \neg_e \\
 \frac{\perp}{\neg(A \wedge B)} \neg_i^3 \\
 \vdots \pi \\
 \frac{\perp}{B} \text{raa}^4 \\
 \hline
 B \\
 \wedge_i
 \end{array}
 \quad (2)$$

But is it necessary to reduce all complex instances of  $\text{raa}$  (i.e. the instances of  $\text{raa}$  whose conclusions are non-atomic formulas)? Maybe it would be enough to focus only on a particular subset of them: the instances of  $\text{raa}$  that introduce a formula occurrence  $A$  and are immediately followed by an instance of an elimination rule having  $A$  as major premiss. The reason is that only in this situation we are creating a rules' configuration which is similar to the standard (intuitionistic) detours of the form  $\circ$ -introduction/ $\circ$ -elimination (where  $\circ$  is any connective):  $\text{raa}$  plays the role of an introduction rule. This configuration can be called *classical detour à la Prawitz*, since it corresponds to Prawitz's definition of maximum formula given in [19, p. 34].

Thus, instead of (1), one could consider the derivation (3) below and reduce it to the derivation (4), which has a less complex instance of  $\text{raa}$ .

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<sup>1</sup>Actually, Prawitz [19] and Seldin [20] do not consider  $\neg$  as primitive, but define it by  $\neg A = A \rightarrow \perp$ , so its introduction and elimination rules are special cases of the introduction and elimination rules for  $\rightarrow$ . As we will see in §3, our approach is different.

$$\begin{array}{c}
 \Gamma \neg(A \wedge B)^{\neg 1} \\
 \vdots \pi \\
 \frac{\perp}{A \wedge B} \text{raa}^1 \\
 \frac{\quad}{A} \wedge_{e1} \\
 \vdots
 \end{array} \quad (3)
 \qquad
 \begin{array}{c}
 \frac{\Gamma \neg A^{\neg 2} \quad \frac{\Gamma A \wedge B^{\neg 1}}{A} \wedge_{e1}}{\neg(A \wedge B)} \neg_e \\
 \frac{\perp}{\neg(A \wedge B)} \neg_i^1 \\
 \vdots \pi \\
 \frac{\perp}{A} \text{raa}^2 \\
 \vdots
 \end{array} \quad (4)$$

It is worth noticing that (4) is nothing but a subderivation of (2). However, adopting this second kind of reduction does not mean that we are simply applying a special case of Prawitz's original strategy. There is indeed a crucial difference between the two reductions. In the reduced derivation (2), the distance from the conclusion of the two instances of *raa* is one unit greater than the distance from the conclusion of the instance of *raa* which is present in the original derivation (1). In contrast, in the reduced derivation (4), the distance from the conclusion of the instance of *raa* is one unit smaller than the distance from the conclusion of the instance of *raa* in the original derivation (3). In other words, where Prawitz's original strategy (1)-(2) brings the application of *raa* forward, the second strategy (3)-(4) *postpones* it.

Several works used this notion of classical detour *à la* Prawitz in order to solve the (weak) normalization problem for the full language of first-order classical natural deduction. In particular, Statman [23], Stålmårck [22], von Plato and Siders [18] developed normalization techniques based on the following general reduction scheme for classical detour *à la* Prawitz:

$$\begin{array}{c}
 \Gamma \neg A^{\neg 1} \\
 \vdots \pi \\
 \frac{\perp}{A} \text{raa}^1 \\
 \frac{\quad}{C} \circ_e \\
 \vdots
 \end{array} \quad \text{reduces to} \quad
 \begin{array}{c}
 \frac{\Gamma \neg C^{\neg 2} \quad \frac{\Gamma A^{\neg 1}}{C} \circ_e}{\neg A} \neg_e \\
 \frac{\perp}{\neg A} \neg_i^1 \\
 \vdots \pi \\
 \frac{\perp}{C} \text{raa}^2 \\
 \vdots
 \end{array} = II' \quad (5)$$

where  $\circ_e$  is an elimination rule for any connective  $\circ$  of the full language of first-order logic and  $A$  is the major premiss of  $\circ_e$  (cf. [23, pp. 78–79]). This means that all undischarged assumptions of  $II'$  are undischarged assumptions of  $II$ , since  $\circ_e$  cannot discharge any assumption of the subderivation  $\pi$  of  $II$ .



Seldin's solution consists in adopting two alternative schemata ( $C \neq \perp$ ):

$$\begin{array}{c}
 \begin{array}{c}
 \Gamma \neg A^{\neg 1} \\
 \vdots \pi \\
 \frac{\perp}{A} \text{raa}^1 \quad \vdots \pi_1 \\
 \hline
 C \\
 \vdots
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{\Gamma \neg C^{\neg 2} \quad \frac{\Gamma A^{\neg 1} \quad \vdots \pi_1}{C} \text{s}}{\perp} \neg_e \\
 \frac{\perp}{\neg A} \neg_i^1 \\
 \vdots \pi \\
 \frac{\perp}{A} \text{efq} \quad \vdots \pi_1 \\
 \hline
 C \\
 \hline
 \frac{\Gamma \neg C^{\neg 2} \quad C}{\perp} \neg_e \\
 \frac{\perp}{C} \text{raa}^2 \\
 \vdots
 \end{array}
 \end{array}
 \quad (6)$$

$$\begin{array}{c}
 \begin{array}{c}
 \Gamma \neg A^{\neg 1} \\
 \vdots \pi \\
 \frac{\perp}{A} \text{raa}^1 \quad \vdots \pi_1 \\
 \hline
 \perp \\
 \vdots
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{\Gamma A^{\neg 1} \quad \vdots \pi_1}{\perp} \text{s} \\
 \frac{\perp}{\neg A} \neg_i^1 \\
 \vdots \pi \\
 \frac{\perp}{A} \text{efq} \quad \vdots \pi_1 \\
 \hline
 \perp \\
 \vdots
 \end{array}
 \end{array}
 \quad (7)$$

Note that these reductions make a crucial use of a **efq** rule (we will come back on this point in §2.3). As it concerns the normalization strategy, Seldin follows the same pattern as Prawitz. First, all classical detours are eliminated, and secondly, all intuitionistic detours are eliminated. The difference to Prawitz is the way in which classical detours are defined and eliminated. In Seldin, the elimination of classical detour consists in pushing down all discharging instances of **raa** with respect to all the other rules, and then contract these instances of **raa** into one. In this way, what tells him when the elimination of classical detours has to stop is the position of **raa** in the derivation tree, and not, like in Prawitz, the complexity of the formula to which **raa** is applied. By borrowing a terminology from Girard's jargon, we could say that for Seldin the termination of classical detours elimination can be characterized in a *geometrical* way rather than in a *syntactical* one.

Even Seldin's strategy needs some restrictions in order to work. The  $\forall$  has to be dropped (i.e. Seldin's approach works for the  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$

fragment, which is as expressive as the full language of first-order classical logic), since the reduction schemata (5) and (6)-(7) cannot be applied when  $\mathbf{s}$  is a  $\forall_i$  rule. Indeed if  $\mathbf{s} = \forall_i$ , according to (5) or (6) we would get, respectively,

$$\begin{array}{c}
 \frac{\frac{\frac{\vdots}{\vdash A} \text{raa}^1}{\forall x A} \forall_i}{\vdash A} \neg^1 \\
 \vdots \pi \\
 \frac{\vdash A}{} \neg^1
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{\frac{\frac{\frac{\vdash A}{} \neg^1}{\forall x A} \forall_i}{\vdash A} \neg^1}{\vdash A} \neg^1 \\
 \vdots \pi \\
 \frac{\vdash A}{} \text{raa}^2 \\
 \vdots
 \end{array}
 \text{ or }
 \begin{array}{c}
 \frac{\frac{\frac{\frac{\frac{\frac{\vdash A}{} \neg^1}{\forall x A} \forall_i}{\vdash A} \neg^1}{\vdash A} \neg^1}{\vdash A} \neg^1}{\vdash A} \neg^1 \\
 \vdots \pi \\
 \frac{\vdash A}{} \text{efq} \\
 \frac{\vdash A}{\forall x A} \forall_i \\
 \frac{\vdash A}{\forall x A} \text{raa}^2 \\
 \vdots
 \end{array}$$

but in the right-hand side we do not have derivations in natural deduction because the rule  $\forall_i$  is not correctly instantiated there: indeed, the variable  $x$  may occur free in  $A$  and  $A$  is an undischarged assumption when the rule  $\forall_i$  is applied. There is no (reasonable) way to treat the  $\forall_i$  case without adding any rule of some intermediate logic such as MH, see [20, notably pp. 639-640].

Nevertheless, as already anticipated in §1, the problem concerning  $\forall$  is a small limitation with respect to the great advantage of Seldin's strategy, consisting in obtaining Glivenko's theorem for intuitionistic logic as an immediate corollary (see [20, pp. 637-638]; the same result can be obtained using von Plato's reduction rules [17, pp. 87-88; pp. 142-143]): in a derivation where discharging  $\text{raa}$  is postponed (possibly it contains several instances of  $\text{efq}$  that are not its last rule), it is sufficient to replace its last rule — an instance of  $\text{raa}$  — by a  $\neg$ -introduction rule discharging the same assumptions.

### 2.3. Towards a unified approach

In this article we will focus on the postponement of  $\text{raa}$ : we aim to show that Seldin's result [20] about the postponement of  $\text{raa}$  can be refined in such a way that a Glivenko's theorem can be obtained not only for intuitionistic but also for minimal logic. In order to do this, we will proceed in two steps.

First, we observe that to postpone  $\text{raa}$  with respect to classical detours *à la* Seldin, the use of  $\text{efq}$  in the reduction steps (6)-(7) can be limited to just the case where  $\mathbf{s}$  is a  $\rightarrow_i$  rule. However, the reduction steps that we will define do not follow a uniform reduction scheme — as in Seldin's original formulation — but they will come out of a mixing of techniques. More precisely, we can divide the definition of our reduction steps in §4 according to two main cases:

1. when the conclusion  $A$  of an *raa* is a premiss of an elimination rule  $s$ ,
  - (a) if  $A$  is the major premiss of  $s$ , we will follow the general scheme (5);
  - (b) if  $A$  is one of the minor premisses of  $s$ , we will introduce new specific reduction steps;
2. when the conclusion  $A$  of an *raa* is a premiss of an introduction rule, we will follow (with some emendations for the case of the implication) the reduction steps proposed by von Plato [17, p. 85–86].

Thus, we can postpone not only the discharging instances of *raa* in a derivation without the  $\forall_i$  rule (as in [20]), but also all the (discharging and non-discharging) instances of *raa* in a derivation without the  $\forall_i$  and  $\rightarrow_i$  rules.

As it will become clear in §5 (see also Definition 1), our main concern is the “geometrical” character of the postponement of *raa*, while normalization of classical logic is only an indirect target. In this sense, we will see that it is not necessary to take into account the complexity of formulas introduced by the *raa* rule. This will lead us to consider — and reduce — at the same time multiple occurrences of *raa* when they are immediately above the same instance of a rule. Even if our reduction steps create maximum formulas of greater complexity, we can eventually prove a (weak) normalization theorem for the —adequate— fragment  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$  of first-order classical logic.

Secondly, we shed light on a issue raised by Pereira [15] about the relation between normalization strategies for classical logic and negative translations. As remarked in [15], the atomization of *raa* in Prawitz's original normalization strategy [19, pp. 39-40] for the —adequate— fragment  $\{\neg, \wedge, \rightarrow, \perp, \forall\}$  of classical natural deduction induces a negative translation of full first-order classical logic into the fragment  $\{\neg, \wedge, \rightarrow, \perp, \forall\}$  of intuitionistic logic. Indeed, if we set  $A \vee B = \neg(\neg A \wedge \neg B)$  and  $\exists x A = \neg \forall x \neg A$  and then replace each atomic instance of *raa* in a normal classical proof  $\pi$  with an instance of  $\neg_i$  discharging the same assumptions (modifying the rest of  $\pi$  consequently), then everything is ready for a modular definition of a variant  $(\cdot)^{\mathfrak{g}}$  of Gentzen's translation:

$$\begin{aligned}
 (P(t_1, \dots, t_n))^{\mathfrak{g}} &= \neg \neg P(t_1, \dots, t_n) & (\perp)^{\mathfrak{g}} &= \perp & (A \rightarrow B)^{\mathfrak{g}} &= A^{\mathfrak{g}} \rightarrow B^{\mathfrak{g}} \\
 (A \vee B)^{\mathfrak{g}} &= \neg(\neg A^{\mathfrak{g}} \wedge \neg B^{\mathfrak{g}}) & (\neg A)^{\mathfrak{g}} &= \neg A^{\mathfrak{g}} & (A \wedge B)^{\mathfrak{g}} &= A^{\mathfrak{g}} \wedge B^{\mathfrak{g}} \\
 (\exists x A)^{\mathfrak{g}} &= \neg \forall x \neg A^{\mathfrak{g}} & (\forall x A)^{\mathfrak{g}} &= \forall x A^{\mathfrak{g}}.
 \end{aligned}$$

Note that, since  $B^{\mathfrak{g}}$  is minimally equivalent to  $\neg \neg B^{\mathfrak{g}}$ ,  $A^{\mathfrak{g}} \rightarrow B^{\mathfrak{g}}$  is minimally equivalent to  $A^{\mathfrak{g}} \rightarrow \neg \neg B^{\mathfrak{g}}$ ; but  $A^{\mathfrak{g}} \rightarrow \neg \neg B^{\mathfrak{g}}$  is also minimally equivalent to

$\neg(A^{\mathcal{E}} \wedge \neg B^{\mathcal{E}})$ . Hence, just by using minimal logic steps,  $(\cdot)^{\mathcal{E}}$  is equivalent to a variant of Gödel’s negative translation (cf. [15, p. 22] and [6, p. 228]).<sup>2</sup>

In a similar way, we will show in §6 that in Seldin’s normalization strategy for the —adequate— fragment  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$ , the postponement of *raa* induces another negative translation, namely (a variant of) Kuroda’s one embedding full first-order classical logic into the fragment  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$  of intuitionistic logic. However, a crucial difference exists between the Gödel–Gentzen translation and Kuroda’s original translation: the former can embed classical logic into minimal logic, while the latter cannot (see [6, pp. 228–229]). The parallel between Prawitz’s and Seldin’s strategies would then work only partially. Actually, this is too harsh a conclusion. Indeed, by a slight modification of Kuroda’s translation induced by *our* reduction steps, we will show how to obtain an embedding of full first-order classical logic into the fragment  $\{\neg, \wedge, \vee, \perp, \exists\}$  of minimal logic (this would be impossible starting from Seldin’s original reduction steps). Also, the second (resp. first) variant of Kuroda’s translation embeds the —adequate— fragment  $\{\neg, \wedge, \vee, \perp, \exists\}$  (resp.  $\{\neg, \wedge, \vee, \rightarrow, \perp, \exists\}$ ) of first-order classical logic into minimal (resp. intuitionistic) logic simply by adding a double negation in front of formulas: we get in this way a Glivenko theorem for minimal (resp. intuitionistic) logic.

In [21, pp. 203, 216], Seldin already proved a form of Glivenko’s theorem consisting of embedding the system  $TD^*$  into minimal logic. But  $TD^*$  is weaker than first-order classical logic, it corresponds to first-order minimal logic plus the rule of *consequentia mirabilis*, and in this system neither *raa* nor *efq* are admissible (see [1] for details). Our result is thus more general.

It is worth noting that, unlike the algebraic proof given in [3], our demonstration of Glivenko’s theorem for minimal logic makes use of purely proof-theoretic tools and is not restricted to the propositional fragment. Also Tennant [24] gives a proof-theoretic demonstration of Glivenko’s theorem for first-order minimal logic without  $\forall$  and  $\rightarrow$ . However, unlike our approach, he does not appeal to the postponement of *raa* for classical logic, but he translates each classical inference rule into a corresponding derivable rule in the fragment  $\{\neg, \wedge, \vee, \perp, \exists\}$  of minimal logic. In this way, by induction on the length of a derivation, he can then transform a classical derivation into a derivation in minimal logic (for more details, see Appendix A.2 of [9]).

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<sup>2</sup>Even when the full language fragment is considered, like in Stålmarch’s [22] or in von Plato and Siders’ [18] approaches to normalization, it is possible to detect the use of some kind of negative translations. See Appendix A.1 of [9] for more details.



that can be obtained from a given set of inference rules. In other words, a derivation system is identified with the set of its inference rules. The complete list of inference rules that we will consider for any derivation system in first-order natural deduction is in Figure 1. Given two derivation systems  $D$  and  $D'$ ,  $D$  is a *subsystem* of  $D'$  if  $D \subseteq D'$ ; hence, any derivation in  $D$  is also a derivation in  $D'$ . The notions of derivation, conclusion and (major, minor) premisses of an instance of rule are taken for granted (see for example [19]).

In Figure 1, observe that **efq** is nothing but the special case of **raa** where no assumption is discharged. This means that, in a derivation  $\pi$ , every instance of the rule **efq** is just an instance of the rule **raa** discharging no assumption. We can thus say that, in a derivation  $\pi$ , an instance of **raa** is *discharging* if it is not an instance of **efq** (i.e. it discharges at least one assumption).

Negation  $\neg$  is here considered as primitive:  $\neg A$  will not be treated as a shorthand for  $A \rightarrow \perp$ , and the inferences rules  $\neg_i$  and  $\neg_e$  (see Figure 1) will be not special cases of  $\rightarrow_i$  and  $\rightarrow_e$ , respectively. Indeed, for our purposes (see in particular §5-6), it turns out that the rules  $\rightarrow_i$  and  $\neg_i$  have different behavior.

We say that the *first-order minimal natural deduction* is the derivation system  $NM = \{\top_i, \neg_i, \neg_e, \wedge_i, \wedge_{e_1}, \wedge_{e_2}, \vee_i, \vee_{i_2}, \vee_e, \rightarrow_i, \rightarrow_e, \forall_i, \forall_e, \exists_i, \exists_e\}$  (i.e. in  $NM$  there are all the inference rules in Figure 1 except **raa** and **efq**); the *first-order intuitionistic natural deduction* is the derivation system  $NJ = NM \cup \{\text{efq}\}$ ; and the *first-order classical natural deduction* is the derivation system  $NK = NM \cup \{\text{raa}\}$ . This means that  $NM$  is a subsystem of  $NJ$ , and that  $NM$  and  $NJ$  are both subsystems of  $NK$ . All the derivation systems we will consider are subsystems of  $NK$ .

NOTATION. Let  $D \subseteq NK$  be a derivation system.

- (i) Derivations in  $D$  are denoted by  $\Pi, \Pi', \dots$ , or also  $\pi, \pi', \dots$ .
- (ii) Given a derivation  $\pi$  in  $D$ , let  $\text{raa}_\pi$  (resp.  $\text{raa}_\pi^+$ ) denote the set of instances (resp. discharging instances) of the rule **raa** in  $\pi$ .
- (iii) Given a formula  $B$ , a set of formulas  $\Gamma$ , and a derivation  $\pi$  in  $D$ , we write  $\pi: \Gamma \vdash_D B$  (or simply  $\pi: \Gamma \vdash B$  when no ambiguity arises) to indicate that the conclusion of  $\pi$  is  $B$  and all the undischarged assumptions of  $\pi$  are occurrences of some formulas in  $\Gamma$ ; possibly, not all formulas in  $\Gamma$  occur as undischarged assumptions in  $\pi$ .
- (iv) If there is a derivation  $\pi: \Gamma \vdash_D B$ , we write  $\Gamma \vdash_D B$  and say that  $B$  is *derivable from  $\Gamma$*  (or  $\Gamma \vdash B$  is *derivable*) in  $D$ ; otherwise we write  $\Gamma \not\vdash_D B$ .

Clearly,  $\text{raa}_\pi^+ \subseteq \text{raa}_\pi$  for every derivation  $\pi$  in  $D \subseteq NK$ , and  $\text{raa}_\pi \setminus \text{raa}_\pi^+$  is the set of instances of **efq** in  $\pi$ .

Given an instance  $r$  of a rule in a derivation  $\pi$ , it is natural to define the notion of distance of  $r$  from the conclusion of  $\pi$  as the number of instances of rules in  $\pi$  between  $r$  and the last rule of  $\pi$ . More formally, we have that:

**DEFINITION 1** (Thread; distance of a rule; *raa*-size of a proof; standard derivation). *Let  $\pi$  be a derivation in  $D \subseteq \text{NK}$ .*

*Given two formula occurrences  $A$  and  $B$  in  $\pi$ , a thread from  $A$  to  $B$  in  $\pi$  is a sequence  $\mathbf{t} = (A_i)_{0 \leq i \leq n}$  (with  $n \in \mathbb{N}$ ) of formula occurrences in  $\pi$  such that  $A_0 = A$ ,  $A_n = B$  and, for all  $0 \leq i < n$ , there is an instance of rule in  $\pi$  having  $A_i$  as a premise and  $A_{i+1}$  as its conclusion; the length of  $\mathbf{t}$  is  $n$ .<sup>4</sup>*

*For every instance  $r$  of a rule in  $\pi$ , the distance of  $r$ , denoted by  $\text{dist}_\pi(r)$ , is the length of the thread from the conclusion of  $r$  to the conclusion of  $\pi$ .<sup>5</sup>*

*Let  $r \in \text{raa}_\pi$ :  $r$  is *raa* $_\pi$ -maximal if  $\text{dist}_\pi(r) \geq \text{dist}_\pi(r')$  for every  $r' \in \text{raa}_\pi$ ;  $r$  is *raa* $_\pi^+$ -maximal if  $\text{dist}_\pi(r) \geq \text{dist}_\pi(r')$  for every  $r' \in \text{raa}_\pi^+$ .*

*The *raa*-size of  $\pi$  is  $\text{size}_{\text{raa}}(\pi) = \sum_{r \in \text{raa}_\pi} \text{dist}_\pi(r)$ .*

*The *raa* $^+$ -size of  $\pi$  is  $\text{size}_{\text{raa}^+}(\pi) = \sum_{r \in \text{raa}_\pi^+} \text{dist}_\pi(r)$ .*

*We say that  $\pi$  is *m*-standard (resp. *j*-standard) if it contains at most one instance (resp. one discharging instance) of the rule *raa*, and this instance, if any, is the last rule of  $\pi$ , the rest of  $\pi$  being a derivation in *NM* (resp. *NJ*).*

The notion of *m*-standard (resp. *j*-standard) derivation characterizes exactly the derivations in *NK* where the *raa* (resp. discharging *raa*) is postponed. These notions will be used in Theorem 8 and Corollary 9. A *j*-standard derivation might contain several instances of *efq* that are not its last rule.

Clearly, for any derivation  $\pi$  and any instance  $r$  of a rule in  $\pi$ ,  $\text{dist}_\pi(r) \in \mathbb{N}$  and  $\text{size}_{\text{raa}}(\pi) \geq \text{size}_{\text{raa}^+}(\pi) \in \mathbb{N}$ . Also,  $\text{dist}_\pi(r) = 0$  iff  $r$  is the last rule in  $\pi$ .

**REMARK 2.** Let  $\pi$  be a derivation in  $D \subseteq \text{NK}$ :

1.  $\text{size}_{\text{raa}^+}(\pi) = 0$  if and only if  $\pi$  is *j*-standard;
2.  $\text{size}_{\text{raa}}(\pi) = 0$  if and only if  $\pi$  is *m*-standard.

Since we are mainly interested in the postponement of *raa* instead of normalization, differently from the approaches discussed in §2, our definitions of *raa* $_\pi$ -maximal and *raa* $_\pi^+$ -maximal, as well as of  $\text{size}_{\text{raa}}(\pi)$  and  $\text{size}_{\text{raa}^+}(\pi)$ , depend only on the distances of the instances of *raa* from the conclusion of the derivation  $\pi$ , without taking into account the complexity of the formulas

<sup>4</sup>Note that if  $A = B$  (as formula occurrences in  $\pi$ ) then the length of  $\mathbf{t}$  is 0.

<sup>5</sup>This notion is well-defined since a derivation is a tree (i.e. a rooted acyclic connected graph) whose nodes are formula occurrences. Hence, for every formula occurrence  $A$  in  $\pi$ , there exists exactly one thread from  $A$  to the conclusion of  $\pi$ .

occurring in the conclusions of these instances. In a way, as we will see in §5, our approach to prove the postponement of  $\mathbf{raa}$  is purely “geometrical”.

Since a derivation  $\pi$  is a finite tree, if  $\mathbf{raa}_\pi \neq \emptyset$  (resp.  $\mathbf{raa}_\pi^+ \neq \emptyset$ ) — i.e. if there is at least one instance (resp. discharging instance) of the rule  $\mathbf{raa}$  in  $\pi$  — then there is an  $\mathbf{raa}_\pi$ -maximal instance (resp.  $\mathbf{raa}_\pi^+$ -maximal discharging instance) of the rule  $\mathbf{raa}$  in  $\pi$ . Intuitively, an instance  $r$  of  $\mathbf{raa}$  is  $\mathbf{raa}_\pi$ -maximal (resp.  $\mathbf{raa}_\pi^+$ -maximal) when there are no instances (resp. discharging instances) of  $\mathbf{raa}$  farther from the conclusion of  $\pi$  than  $r$  is.<sup>6</sup>

#### 4. Reduction Steps for the Postponement of $\mathbf{raa}$

We define reduction steps on derivations so as to push an instance of  $\mathbf{raa}$  downward or erase it. Their case-by-case definition depends on the inference rule  $r$  instantiated immediately below the instance of  $\mathbf{raa}$  under focus (thus there is no case with a 0-ary inference rule). For the lack of space, the cases where  $r = \forall_i$  and  $r = \vee_e$  are omitted here, the reader can find them in [9, §4]; all the results related to our reduction steps hold also in those cases. There is no reduction step when  $r = \forall_i$  (see Remark 4 on p. 21).

$\neg$  **introduction:**

$$\Pi = \frac{\frac{\frac{\frac{\frac{\Gamma A^{\neg 2}, \Gamma \neg \perp^{\neg 1}}{\vdots \pi_1}}{\perp} \mathbf{raa}^1}{\neg A} \neg_i^2}{\vdots \pi}}{\perp} \neg_i^1}{\vdots \pi} \rightsquigarrow \frac{\frac{\frac{\frac{\Gamma \perp^{\neg 1}}{\neg \perp} \neg_i^1}{\vdots \pi_1}}{\neg A} \neg_i^2}{\vdots \pi}}{\Gamma A^{\neg 2}, \frac{\Gamma \perp^{\neg 1}}{\neg \perp} \neg_i^1} \neg_i^1 = \Pi' \quad (8)$$

$\neg$  **elimination:**

$$\Pi = \frac{\frac{\frac{\frac{\frac{\Gamma \neg \neg A^{\neg 1}}{\vdots \pi_1}}{\perp} \mathbf{raa}^1}{\neg A} \neg_i^1}{\vdots \pi_2} A \neg_e}{\perp} \neg_e}{\vdots \pi} \rightsquigarrow \frac{\frac{\frac{\frac{\Gamma \neg A^{\neg 1}}{\neg \neg A} \neg_i^1}{\vdots \pi_1}}{A} \neg_e}{\vdots \pi_2}}{\perp} \neg_e = \Pi' \quad (9a)$$

<sup>6</sup>By definition,  $\mathbf{raa}_\pi$ -maximality implies  $\mathbf{raa}_\pi^+$ -maximality. Given  $r_1 \in \mathbf{raa}_\pi$  and  $r_2 \in \mathbf{raa}_\pi \setminus \mathbf{raa}_\pi^+$ , both  $r_1$  and  $r_2$  might be  $\mathbf{raa}_\pi^+$ -maximal even if  $\text{dist}_\pi(r_1) \neq \text{dist}_\pi(r_2)$ .

where the last rule of the derivation  $\pi_2$  is not an instance of the rule *raa*;

$$\Pi = \frac{\frac{\frac{\vdots \pi_1}{\neg A} \text{raa}^1 \quad \frac{\frac{\vdots \pi_2}{A} \text{raa}^1}{\perp} \neg_e}{\perp} \neg_e}{\vdots \pi} \rightsquigarrow \frac{\frac{\vdots \pi_1}{\neg A} \text{raa}^1 \quad \frac{\vdots \pi_2}{A} \text{raa}^1}{\perp} \neg_e = \Pi' \quad (9b)$$

where the last rule of the derivation  $\pi_1$  is not an instance of the rule *raa*;

$$\Pi = \frac{\frac{\frac{\vdots \pi_1}{\neg \neg A} \text{raa}^1 \quad \frac{\frac{\vdots \pi_2}{\neg A} \text{raa}^2}{\perp} \neg_e}{\perp} \neg_e}{\vdots \pi} \rightsquigarrow \frac{\frac{\frac{\frac{\frac{\vdots \pi_2}{A} \text{raa}^2}{\perp} \neg_e}{\neg A} \neg_i^1}{\perp} \neg_e}{\neg \neg A} \neg_i^2}{\vdots \pi_1} = \Pi' \quad (9c)$$

**$\wedge$  introduction:**

$$\Pi = \frac{\frac{\frac{\frac{\vdots \pi_1}{A} \text{raa}^1 \quad \frac{\vdots \pi_2}{B} \wedge_i}{A \wedge B} \wedge_i}{\perp} \neg_e}{\vdots \pi} \rightsquigarrow \frac{\frac{\frac{\frac{\frac{\vdots \pi_2}{A \wedge B} \wedge_i}{\perp} \neg_e}{\neg A} \neg_i^1}{\perp} \neg_e}{A \wedge B} \text{raa}^2}{\vdots \pi} = \Pi' \quad (10a)$$

where the last rule of the derivation  $\pi_2$  is not an instance of the rule *raa*;

$$\begin{array}{c}
\vdots \pi_1 \\
\vdots \pi_2 \\
\vdots \pi_1 \\
\frac{\perp}{A} \text{raa}^1 \\
\frac{\perp}{A \wedge B} \wedge_i \\
\vdots \pi
\end{array}
\rightsquigarrow
\frac{\frac{\frac{\frac{\perp}{\neg B} \neg_i^1}{A \wedge B} \text{raa}^2}{\neg(A \wedge B) \neg^2} \wedge_i}{\neg(A \wedge B) \neg^2} \neg_e
= \Pi' \quad (10b)$$

where the last rule of the derivation  $\pi_1$  is not an instance of the rule  $\text{raa}$ ;

$$\begin{array}{c}
\frac{\frac{\frac{\perp}{A} \text{raa}^1}{A} \wedge_i \quad \frac{\frac{\perp}{B} \text{raa}^2}{B} \wedge_i}{A \wedge B} \wedge_i \\
\vdots \pi_1 \quad \vdots \pi_2 \\
\vdots \pi_1 \quad \vdots \pi_2 \\
\frac{\perp}{A} \text{raa}^1 \quad \frac{\perp}{B} \text{raa}^2 \\
\vdots \pi
\end{array}
\rightsquigarrow
\frac{\frac{\frac{\frac{\perp}{\neg A} \neg_i^2}{A \wedge B} \text{raa}^3}{\neg(A \wedge B) \neg^3} \wedge_i}{\neg(A \wedge B) \neg^3} \neg_e
= \Pi' \quad (10c)$$

**$\wedge$  elimination:**

$$\begin{array}{c}
\frac{\frac{\perp}{A \wedge B} \text{raa}^1}{A} \wedge_{e_1} \\
\vdots \pi_1 \\
\vdots \pi
\end{array}
\rightsquigarrow
\frac{\frac{\frac{\perp}{\neg(A \wedge B)} \neg_i^1}{A} \wedge_{e_1}}{\neg(A \wedge B) \neg^1} \neg_e
= \Pi' \quad (11a)$$

$$\begin{array}{c}
 \Gamma \neg(A \wedge B)^{\neg 1} \\
 \vdots \pi_1 \\
 \frac{\perp}{A \wedge B} \text{raa}^1 \\
 \frac{\quad}{B} \wedge_{e_2} \\
 \vdots \pi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{\Gamma \neg B^{\neg 2} \quad \frac{\Gamma A \wedge B^{\neg 1}}{B} \wedge_{e_2}}{\neg(A \wedge B)} \neg_e \\
 \frac{\perp}{\neg(A \wedge B)} \neg_i^1 \\
 \vdots \pi_1 \\
 \frac{\perp}{B} \text{raa}^2 \\
 \vdots \pi
 \end{array}
 = \Pi' \quad (11b)$$

→ **introduction:**

$$\begin{array}{c}
 \Gamma A^{\neg 2}, \Gamma \neg B^{\neg 1} \\
 \vdots \pi_1 \\
 \frac{\perp}{B} \text{raa}^1 \\
 \frac{\quad}{A \rightarrow B} \rightarrow_i^2 \\
 \vdots \pi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \Gamma \neg(A \rightarrow B)^{\neg 3} \quad \frac{\Gamma B^{\neg 1}}{A \rightarrow B} \rightarrow_i^0 \\
 \frac{\quad}{\neg(A \rightarrow B)} \neg_e \\
 \Gamma A^{\neg 2}, \frac{\perp}{\neg B} \neg_i^1 \\
 \vdots \pi_1 \\
 \frac{\perp}{B} \text{efq} \\
 \frac{\quad}{A \rightarrow B} \rightarrow_i^2 \\
 \frac{\Gamma \neg(A \rightarrow B)^{\neg 3} \quad \frac{\quad}{A \rightarrow B} \rightarrow_i^2}{\frac{\perp}{A \rightarrow B} \text{raa}^3} \neg_e \\
 \vdots \pi
 \end{array}
 = \Pi' \quad (12)$$

→ **elimination:**

$$\begin{array}{c}
 \Gamma \neg(A \rightarrow B)^{\neg 1} \\
 \vdots \pi_1 \\
 \frac{\perp}{A \rightarrow B} \text{raa}^1 \\
 \frac{\quad}{B} \rightarrow_e \\
 \vdots \pi
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \Gamma \neg B^{\neg 2} \quad \frac{\Gamma A \rightarrow B^{\neg 1} \quad A}{B} \rightarrow_e \\
 \frac{\quad}{\neg(A \rightarrow B)} \neg_e \\
 \frac{\perp}{\neg(A \rightarrow B)} \neg_i^1 \\
 \vdots \pi_1 \\
 \frac{\perp}{B} \text{raa}^2 \\
 \vdots \pi
 \end{array}
 = \Pi' \quad (13a)$$

where the last rule of the derivation  $\pi_2$  is not an instance of the rule *raa*;



$\exists$  **elimination:**

$$\begin{array}{c}
 \Gamma \neg \exists x A^{\neg 1} \\
 \vdots \pi_1 \\
 \frac{\perp}{\exists x A} \text{raa}^1 \\
 \frac{\quad}{C} \exists_e^2 \\
 \vdots \pi \\
 \Pi =
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma A^{\neg 2} \\
 \vdots \pi_2 \\
 \frac{\quad}{C} \text{raa}^1 \\
 \frac{\quad}{\exists_e^2} \\
 \vdots \pi \\
 \sim\rightsquigarrow
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma A^{\neg 1} \\
 \vdots \pi_2 \\
 \frac{\Gamma \neg C^{\neg 3}}{C} \neg_e \\
 \frac{\Gamma \exists x A^{\neg 2}}{\perp} \exists_e^1 \\
 \frac{\perp}{\neg \exists x A} \neg_i^2 \\
 \vdots \pi_1 \\
 \frac{\perp}{C} \text{raa}^3 \\
 \vdots \pi \\
 = \Pi'
 \end{array}
 \quad (15a)$$

where the last rule of the derivation  $\pi_2$  is not an instance of the rule *raa* (notice that the variable  $x$  does not occur free in  $C$  and hence even in  $\neg C$ , thus the rule  $\exists_e$  is correctly instantiated in  $\Pi'$ );

$$\begin{array}{c}
 \Gamma \neg C^{\neg 1}, \Gamma A^{\neg 2} \\
 \vdots \pi_2 \\
 \frac{\quad}{\exists x A} \text{raa}^1 \\
 \frac{\quad}{C} \exists_e^2 \\
 \vdots \pi \\
 \Pi =
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \neg C^{\neg 2}, \Gamma A^{\neg 1} \\
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{\exists x A}{\perp} \exists_e^1 \\
 \frac{\perp}{C} \text{raa}^2 \\
 \vdots \pi \\
 = \Pi'
 \end{array}
 \quad (15b)$$

where the last rule of the derivation  $\pi_1$  is not an instance of the rule *raa* (notice that the variable  $x$  does not occur free in  $C$  and hence even in  $\neg C$ , thus the rule  $\exists_e$  is correctly instantiated in  $\Pi'$ );

$$\begin{array}{c}
 \Gamma \neg \exists x A^{\neg 1} \\
 \vdots \pi_1 \\
 \frac{\perp}{\exists x A} \text{raa}^1 \\
 \frac{\quad}{C} \exists_e^2 \\
 \vdots \pi \\
 \Pi =
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \neg C^{\neg 2}, \Gamma A^{\neg 3} \\
 \vdots \pi_2 \\
 \frac{\perp}{C} \text{raa}^2 \\
 \frac{\quad}{\exists_e^2} \\
 \vdots \pi \\
 \sim\rightsquigarrow
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \neg C^{\neg 3}, \Gamma A^{\neg 1} \\
 \vdots \pi_2 \\
 \frac{\Gamma \exists x A^{\neg 2}}{\perp} \exists_e^1 \\
 \frac{\perp}{\neg \exists x A} \neg_i^2 \\
 \vdots \pi_1 \\
 \frac{\perp}{C} \text{raa}^3 \\
 \vdots \pi \\
 = \Pi'
 \end{array}
 \quad (15c)$$



subderivation of  $\Pi$  might be erased or duplicated in  $\Pi'$ , depending on the number of assumptions that are discharged by the active instances of *raa* in  $\Pi \rightsquigarrow \Pi'$ . Moreover, some subderivations of  $\Pi$  can be moved in  $\Pi'$  above some other subderivations of  $\Pi$  (this corresponds to composition of proofs).

When  $\Pi \rightsquigarrow \Pi'$ , there is no reduction step that introduces in  $\Pi'$  a “new” discharging instance of *raa*: any discharging instance of *raa* in  $\Pi'$  can thus be seen as a “residual” of an instance of *raa* in  $\Pi$  (possibly non-discharging or applied to another formula). However, it is not true that any instance of *raa* in  $\Pi$  has a residual in  $\Pi'$ ; for example, in the reduction steps (8), (9a), (9b), (9c) and (17) the active instances of *raa* in  $\Pi$  vanish in  $\Pi'$ .

REMARK 3. The case where  $\Pi \rightsquigarrow \Pi'$  by applying the reduction step (12) (i.e. the  $\rightarrow_i$  case) is the only one introducing in  $\Pi'$  a new instance of *efq*: the instance of *efq* in  $\Pi'$  explicitly represented in the right-hand side of the reduction step (12) is not a residual of any instance of *efq* in  $\Pi$ . In a way, it is impossible to avoid adding an instance of *efq* in the  $\rightarrow_i$  case: this is deeply related to the fact that  $(A \rightarrow \neg\neg B) \rightarrow \neg\neg(A \rightarrow B)$  is provable in NJ but not in NM. Indeed, if it were possible to define the following reduction step (where in the subderivations  $\pi$  and  $\pi'$  there is no instance of *raa*, and the formulas occurring in the undischarged assumptions of  $\Pi'$  are a subset of the formulas occurring in the undischarged assumptions of  $\Pi$ )

$$\Pi = \frac{\frac{\frac{\frac{\vdots \pi}{\perp} \text{raa}^1}{B} \rightarrow_i^2}{A \rightarrow B} \rightarrow_i^1}{\vdots \pi', \vdots \pi'} \rightsquigarrow \frac{\frac{\frac{\frac{\vdots \pi'}{B} \rightarrow_i^1}{A \rightarrow B} \rightarrow_e}{\perp} \text{raa}^2}{\neg\neg(A \rightarrow B) \rightarrow_i^1} = \Pi'$$

then, by replacing the instances of *raa* in  $\Pi$  and  $\Pi'$  by instances of  $\neg_i$ , the conclusions of  $\Pi$  and  $\Pi'$  would be  $A \rightarrow \neg\neg B$  and  $\neg\neg(A \rightarrow B)$ , respectively: this would mean that the derivability of  $A \rightarrow \neg\neg B$  in NM would imply the derivability of  $\neg\neg(A \rightarrow B)$  in NM, which is impossible.

REMARK 4. It is easy to check that if  $\Pi$  is a derivation in NK and  $\Pi \rightsquigarrow \Pi'$ , then  $\Pi'$  is a derivation in NK and  $\Pi$  is not in NM ( $\Pi$  has at least one instance of *raa*, discharging or not). In the reduction steps listed above, the  $\forall_i$  case (where in  $\Pi$  an instance of the rule  $\forall_i$  is immediately below the instance of *raa* under focus) is absent, otherwise  $\Pi'$  could not be a derivation in NK, as we have pointed out in §2.2. In other words, if  $\Pi \overset{r}{\rightsquigarrow} \Pi'$  then  $r$  is not an instance of *raa* in  $\Pi$  whose conclusion is the premise of an instance of  $\forall_i$ .

REMARK 5. The reduction steps (9c), (10c) and (13c) possess a certain degree of arbitrariness since they could also be defined so that, in  $\Pi'$ , the subderivation  $\pi_1$  would be put above  $\pi_2$ , and not vice-versa.

Observe the similarities between the reduction steps in the  $\wedge_i$  and  $\rightarrow_e$  cases, or in the  $\wedge_e$  and  $\forall_e$  cases. In contrast, the reduction steps for the  $\neg_i$  and  $\rightarrow_i$  cases (resp.  $\neg_e$  and  $\rightarrow_e$  cases) are rather different: the former — reduction step (8) (resp. (9)) — erases an instance of **raa**, whereas the latter — reduction step (12) (resp. (13)) — postpones an instance of **raa** after an instance of  $\rightarrow_i$  (resp.  $\rightarrow_e$ ), moreover in the  $\rightarrow_i$  case the reduction step introduces a new instance of **efq**, unlike the  $\neg_i$  case. This differing behavior justifies our choice to consider  $\neg$  as primitive and not to treat  $\neg A$  as a shorthand for  $A \rightarrow \perp$ .

## 5. Postponement of **raa**

In this section we prove the first main result of this paper: the postponement of **raa** (Theorem 8, Corollary 9), i.e. the fact that in a derivation in **NK** it is possible to push all its instances of **raa** downward until they vanish or occur only in the last rule, preserving the *same conclusion* and *without adding any new undischarged assumptions*. More precisely, we show that, by repeated applications of the reduction steps of §4 following a suitable strategy:

- a derivation  $\pi$  in **NK** without the rule  $\forall_i$  reduces all its discharging instances of **raa** to at most one discharging instance of **raa**, and this instance (if any) occurs as the last rule (the rest of the reduced derivation is in **NJ** — Theorem 8.1);
- a derivation  $\pi$  in **NK** without the rules  $\forall_i$  and  $\rightarrow_i$  reduces all its instances of **raa** to at most one instance of **raa**, and this instance (if any) occurs as the last rule (the rest of the reduced derivation is in **NM** — Theorem 8.2).

These results will be then reformulated by focusing on the form of the conclusion and of the undischarged assumptions of the starting derivation  $\pi$ , rather than on the kind of inference rules used in  $\pi$  (Corollary 9).

Two lemmas are used in the proof of Theorem 8. Lemma 6 says that when a reduction step of §4 is applied, no new undischarged assumptions are added in the reduced derivation, and the conclusion of the original derivation is preserved. Moreover, the reduction steps do not introduce any new instance of the rules  $\rightarrow_i$  and  $\forall_i$ . Lemma 7 says that the size  $\text{size}_{\text{raa}}$  (resp.  $\text{size}_{\text{raa}}^+$ ) of a derivation  $\pi$  strictly decreases after a reduction step with an active instance of **raa** that is **raa** $_{\pi}$ -maximal (resp. **raa** $_{\pi}^+$ -maximal and discharging).

LEMMA 6 (Preservations). *Let  $\pi$  and  $\pi'$  be derivations in NK with  $\pi \rightsquigarrow \pi'$ .*

1. *If  $\pi : \Gamma \vdash A$ , then  $\pi' : \Gamma \vdash A$ ;*
2. *If  $\pi$  has no instance of the rule  $\rightarrow_i$  (resp.  $\forall_i$ ), then  $\pi'$  has no instance of the rule  $\rightarrow_i$  (resp.  $\forall_i$ ).*

PROOF. By straightforward inspection of the reduction steps listed in §4 (and, more in general, in [9, §4]).  $\blacksquare$

The converses of Lemmas 6.1-2 do not hold, as shown by the following counterexample (take  $\Gamma = \{\neg P, P\}$  and  $A = Q \wedge (R_1 \rightarrow R_2)$ , apply the reduction step (10a), and notice that in  $\pi'$  there is no instance of  $\rightarrow_i$ ):

$$\pi = \frac{\frac{\frac{\neg P \quad P}{\perp} \neg_e}{Q} \text{efq}}{Q \wedge (R_1 \rightarrow R_2)} \wedge_i \quad \frac{R_2}{R_1 \rightarrow R_2} \rightarrow_i^0}{\wedge_i} \rightsquigarrow \frac{\frac{\frac{\neg P \quad P}{\perp} \neg_e}{Q \wedge (R_1 \rightarrow R_2)} \text{efq}}{Q \wedge (R_1 \rightarrow R_2)} = \pi'. \quad (18)$$

In (18) we have  $\pi' : \Gamma \vdash A$ , but  $\pi : \Gamma \not\vdash A$  because the set of undischarged assumptions of  $\pi$  also contains an occurrence of the formula  $R_2 \notin \Gamma$  (see point (iii) about notations on p. 12). In other words, Lemma 6.1 says that if  $\pi \rightsquigarrow \pi'$  then the formulas occurring among the undischarged assumptions of  $\pi'$  are a *subset* of the formulas occurring among the undischarged assumptions of  $\pi$ .

Derivability in NJ is not preserved by the reduction steps of §4: the fact that  $\pi$  is a derivation in NJ and  $\pi \rightsquigarrow \pi'$  does not imply that the derivation  $\pi'$  is in NJ, because an instance of *efq* in  $\pi$  can be transformed into a discharging instance of *raa* in  $\pi'$ . For example, by applying the reduction step (12),

$$\pi = \frac{\frac{\perp}{Q} \text{efq}}{P \rightarrow Q} \rightarrow_i^0 \rightsquigarrow \frac{\frac{\frac{\perp}{Q} \text{efq}}{P \rightarrow Q} \rightarrow_i^0}{\perp} \text{raa}^1}{\frac{\Gamma \neg(P \rightarrow Q) \neg^1}{P \rightarrow Q} \rightarrow_i^0} \neg_e = \pi'. \quad (19)$$

LEMMA 7 (Size decreasing). *Let  $\pi$  and  $\pi'$  be derivations in NK such that  $\pi \xrightarrow{r_1, \dots, r_n} \pi'$ , where  $r_1, \dots, r_n$  are the active instances of *raa* in  $\pi$  and  $n \in \mathbb{N}^+$ .*

1. *If  $r_i$  is  $\text{raa}_\pi$ -maximal (resp.  $\text{raa}_\pi^+$ -maximal) for some  $1 \leq i \leq n$ , then  $r_j$  is  $\text{raa}_\pi$ -maximal (resp.  $\text{raa}_\pi^+$ -maximal) for all  $1 \leq j \leq n$ .*
2. *If  $r_i \in \text{raa}_\pi^+$  is  $\text{raa}_\pi^+$ -maximal for some  $1 \leq i \leq n$ ,  $\text{size}_{\text{raa}^+}(\pi') < \text{size}_{\text{raa}^+}(\pi)$ .*
3. *If  $\pi$  contains no instance of the rule  $\rightarrow_i$ , and if  $r_j$  is  $\text{raa}_\pi$ -maximal for some  $1 \leq j \leq n$ , then  $\text{size}_{\text{raa}}(\pi') < \text{size}_{\text{raa}}(\pi)$ .*

PROOF. By inspection of all the reduction steps of §4 (and, more in general, of [9, §4]). Lemma 7.1 is crucial to prove Lemmas 7.2-3: the maximality of *all* the active instances of *raa* ensures that when a subderivation is moved above another subderivation according to the reduction step, no instance (or discharging instance) of *raa* is moved away from the conclusion of the derivation. ■

Note that, in the proof of Lemma 7, the complexity of the formulas occurring in the conclusions of the instances of *raa* plays no role (see Definition 1): the fact that the sizes  $\text{size}_{\text{raa}}$  and  $\text{size}_{\text{raa}^+}$  strictly decrease when applying the reduction steps listed in §4 is purely “geometrical”, due to the strict decrease of the distance of a maximal *raa* from the conclusion of the derivation.

Lemma 7.3 becomes false if  $\pi$  has an instance of the rule  $\rightarrow_i$ , as shown by the counterexample (19) on p. 23 where  $\text{size}_{\text{raa}}(\pi) = 1 < 3 = \text{size}_{\text{raa}}(\pi')$ .

THEOREM 8 (Postponement of *raa*, version 1).

1. If  $\Pi: \Gamma \vdash A$  is a derivation in  $\text{NK} \setminus \{\forall_i\}$ , then there exists a *j*-standard derivation  $\Pi': \Gamma \vdash A$  in  $\text{NK} \setminus \{\forall_i\}$  such that  $\Pi \rightsquigarrow^* \Pi'$ .
2. If  $\Pi: \Gamma \vdash A$  is a derivation in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ , then there is a *m*-standard derivation  $\Pi': \Gamma \vdash A$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$  such that  $\Pi \rightsquigarrow^* \Pi'$ .

PROOF.

1. By induction on  $\text{size}_{\text{raa}^+}(\Pi) \in \mathbb{N}$ .  
 If  $\text{size}_{\text{raa}^+}(\Pi) = 0$ , then just take  $\Pi' = \Pi$ , according to Remark 2.1.  
 Otherwise,  $\text{size}_{\text{raa}^+}(\Pi) > 0$  and there is  $r \in \text{raa}^+_{\Pi}$  which is  $\text{raa}^+_{\Pi}$ -maximal and which is not the last rule of  $\Pi$ . Since there is no instance of the rule  $\forall_i$  in  $\Pi$ , there necessarily exists a  $\Pi'$  such that  $\Pi \overset{r_1, \dots, r_n}{\rightsquigarrow} \Pi'$  where  $n \in \mathbb{N}^+$  and  $r = r_j$  for some  $1 \leq j \leq n$ . We have  $\Pi': \Gamma \vdash A$  by Lemma 6.1, and  $\Pi'$  has no instance of the rule  $\forall_i$  by Lemma 6.2. According to Lemma 7.2,  $\text{size}_{\text{raa}^+}(\Pi') < \text{size}_{\text{raa}^+}(\Pi)$ . Hence, by the induction hypothesis, there is a *j*-standard derivation  $\Pi'': \Gamma \vdash A$  in  $\text{NK} \setminus \{\forall_i\}$  such that  $\Pi \rightsquigarrow \Pi' \rightsquigarrow^* \Pi''$ .
2. By induction on  $\text{size}_{\text{raa}}(\Pi) \in \mathbb{N}$ .  
 If  $\text{size}_{\text{raa}}(\Pi) = 0$ , then just take  $\Pi' = \Pi$ , according to Remark 2.2.  
 Otherwise,  $\text{size}_{\text{raa}}(\Pi) > 0$  and there is an  $\text{raa}_{\Pi}$ -maximal  $r \in \text{raa}_{\Pi}$  which is not the last rule of  $\Pi$ . Since there is no instance of the rule  $\forall_i$  in  $\Pi$ , there is a  $\Pi'$  such that  $\Pi \overset{r_1, \dots, r_n}{\rightsquigarrow} \Pi'$  with  $n \in \mathbb{N}^+$  and  $r = r_i$  for some  $1 \leq i \leq n$ . Thus,  $\Pi': \Gamma \vdash A$  by Lemma 6.1, and  $\Pi'$  has no instance of the rules  $\forall_i$  and  $\rightarrow_i$  by Lemma 6.2. According to Lemma 7.3,  $\text{size}_{\text{raa}}(\Pi') < \text{size}_{\text{raa}}(\Pi)$ . So, by the induction hypothesis, there is a *m*-standard derivation  $\Pi'': \Gamma \vdash A$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$  such that  $\Pi \rightsquigarrow \Pi' \rightsquigarrow^* \Pi''$ . ■

Theorem 8 can be seen as a *weak* standardization theorem: for every derivation  $\Pi$  in NK (fulfilling suitable conditions), we have shown that there exists a particular strategy for the application of the reduction steps of §4 (we fire only maximal instances of *raa*) transforming  $\Pi$  into a “standard” derivation in NK, where “standard” is here understood in the sense of Definition 1. We conjecture that Theorem 8 can be strengthened to a *strong* standardization theorem: whatever strategy in the application of the reduction steps of §4 terminates in a “standard” derivation in NK. To prove that, one should refine the notion of size of a derivation and proceed by a more complex induction.

Thanks to the normalization theorem and subformula property for (the *full* first-order language of) NK proved by Stålmårck in [22, pp. 130, 135] (see also [18, p. 208]), we can reformulate Theorem 8.1 (resp. Theorem 8.2) into Corollary 9 below with a more satisfactory hypothesis: instead of supposing that the derivation  $\Pi: \Gamma \vdash A$  in NK is without any instance of the rule  $\forall_i$  (resp. rules  $\forall_i$  and  $\rightarrow_i$ ), it is sufficient to suppose that  $A$  and the formulas in  $\Gamma$  do not contain any occurrence of  $\forall$  (resp.  $\forall$  and  $\rightarrow$ ). Note that, since in Stålmårck's normalization strategy *raa* is pushed downward only with respect to elimination rules, a normal (in his sense) derivation can still contain instances of *raa* that can be reduced via our reduction steps. In this respect, Corollary 9 does not follow trivially from Stålmårck's normalization for NK.

COROLLARY 9 (Postponement of *raa*, version 2). *Suppose  $\Gamma \vdash_{\text{NK}} A$ .*

1. *If  $A$  and the formulas in  $\Gamma$  do not contain any occurrence of  $\forall$ , then there exists a  $j$ -standard derivation  $\Pi': \Gamma \vdash A$  in NK.*
2. *If  $A$  and the formulas in  $\Gamma$  do not contain any occurrence of  $\forall$  or  $\rightarrow$ , then there exists a  $m$ -standard derivation  $\Pi': \Gamma \vdash A$  in NK.*

PROOF. As  $\Gamma \vdash A$  is derivable in NK, there exists a *normal* (in the sense of [22, p. 130]) derivation  $\Pi: \Gamma \vdash A$  in NK and hence, in conformity to the aforementioned *subformula principle* for NK (see [22, p. 130]), each formula occurrence  $B$  in  $\Pi: \Gamma \vdash A$  satisfies one of the clauses (i)-(iii) below:

- (i)  $B$  is an occurrence of a subformula of  $A$  or of some formula in  $\Gamma$ ;
- (ii)  $B$  is an assumption discharged by some instance of the rule *raa*,  $B$  has the form  $\neg C$ , and  $C$  is a subformula of  $A$  or of some formula in  $\Gamma$ ;
- (iii)  $B$  has the form  $\perp$  and stands immediately below an assumption which satisfies (ii) above.

Using this subformula principle, we can now prove Corollaries 9.1-2.

1. According to the subformula principle, there are no instances of the rule  $\forall_i$  in  $\Pi$ . By Theorem 8.1, there is a  $j$ -standard derivation  $\Pi': \Gamma \vdash_{\text{NK}} A$ .
2. By the subformula principle, there are no instances of the rules  $\rightarrow_i$  and  $\forall_i$  in  $\Pi$ . By Theorem 8.2, there is a  $m$ -standard derivation  $\Pi': \Gamma \vdash_{\text{NK}} A$ . ■

Thus, according to Corollary 9.1 (resp. Corollary 9.2), when we look for a derivation of  $A$  from  $\Gamma$  in  $\text{NK}$ , where  $\forall$  does (resp.  $\forall$  and  $\rightarrow$  do) not occur in  $A$  or in  $\Gamma$ , we can consider the use of classical (resp. classical or intuitionistic) reasoning — i.e. of a discharging (resp. either discharging or non-discharging) instance of **raa** — only in the last rule of the derivation, if this use is required; the rest of the derivation is based on intuitionistic (resp. minimal) reasoning.

## 6. Generalized Glivenko's Theorem

As already mentioned in §1-2, an immediate consequence of the postponement of **raa** is the *weak normalization of*  $\text{NK} \setminus \{\forall_i\}$ . Indeed, a normalization strategy is the following: by Theorem 8.1, any derivation  $\pi: \Gamma \vdash A$  in  $\text{NK} \setminus \{\forall_i\}$  reduces to a  $j$ -standard derivation  $\pi': \Gamma \vdash A$  in  $\text{NK} \setminus \{\forall_i\}$  (which is a derivation in  $\text{NJ} \setminus \{\forall_i\}$ , possibly except for its last rule which could be a discharging instance of **raa**), then one can apply Prawitz's original weak normalization theorem for  $\text{NJ}$  [19, p. 50] to  $\pi'$  (or to  $\pi'$  without its last rule), so as to obtain a normal derivation  $\pi'': \Gamma \vdash A$  in  $\text{NK} \setminus \{\forall_i\}$ .

Other consequences of the postponement of **raa** are two strengthened forms of *Glivenko's theorem*, one from the fragment  $\{\perp, \top, \neg, \wedge, \vee, \rightarrow, \exists\}$  of classical logic into intuitionistic logic (Theorem 14.2), the other from the fragment  $\{\perp, \top, \neg, \wedge, \vee, \exists\}$  of classical logic into minimal logic (Theorem 13.2). The idea is that, if  $A$  is derivable from  $\Gamma$  in  $\text{NK}$  where  $A$  and  $\Gamma$  do not contain any occurrence of  $\forall$  (resp.  $\forall$  or  $\rightarrow$ ), then by Corollary 9 there is a  $j$ -standard (resp.  $m$ -standard) derivation of  $A$  from  $\Gamma$  in  $\text{NK}$ ; the only instance of **raa** — if any — can be replaced by an instance of  $\neg_i$  discharging the same assumptions, so as to derive  $\neg\neg A$  from  $\Gamma$  in  $\text{NJ}$  (resp.  $\text{NM}$ ). Actually, the postponement of **raa** induces two more general embeddings of *full* first-order classical logic into the fragments  $\{\perp, \top, \neg, \wedge, \vee, \rightarrow, \exists\}$  of intuitionistic logic (Theorem 14.1) and  $\{\perp, \top, \neg, \wedge, \vee, \exists\}$  of minimal logic (Theorem 13.1), via two translations that get rid of  $\forall$ , and of  $\forall$  and  $\rightarrow$ , respectively (Definition 10). Thus, Glivenko's theorems cited above follow immediately. Let us see this in detail (a variant of this approach is discussed in Appendix A.3 of [9]).

We define a translation  $(\cdot)^m$  (resp.  $(\cdot)^j$ ) on formulas that just redefines the implication and the universal quantifier (resp. only the universal quantifier) in

a classical way, using the negation, the disjunction and the existential quantifier (resp. the negation and the existential quantifier). All other connectives and the existential quantifier are left alone.

DEFINITION 10 (Minimal and intuitionistic translations). *The minimal translation is a function  $(\cdot)^m$  associating with every formula  $A$  a formula  $A^m$  defined by induction on  $A$  as follows:*

$$\begin{aligned} (P(t_1, \dots, t_n))^m &= P(t_1, \dots, t_n) & \top^m &= \top & \perp^m &= \perp \\ (A \wedge B)^m &= A^m \wedge B^m & (A \vee B)^m &= A^m \vee B^m & (\neg A)^m &= \neg A^m \\ (A \rightarrow B)^m &= \neg A^m \vee B^m & (\forall x A)^m &= \neg \exists x \neg A^m & (\exists x A)^m &= \exists x A^m. \end{aligned}$$

*The intuitionistic translation is a function  $(\cdot)^j$  associating with every formula  $A$  a formula  $A^j$  defined by induction on  $A$  as follows:*

$$\begin{aligned} (P(t_1, \dots, t_n))^j &= P(t_1, \dots, t_n) & \top^j &= \top & \perp^j &= \perp \\ (A \wedge B)^j &= A^j \wedge B^j & (A \vee B)^j &= A^j \vee B^j & (\neg A)^j &= \neg A^j \\ (A \rightarrow B)^j &= A^j \rightarrow B^j & (\forall x A)^j &= \neg \exists x \neg A^j & (\exists x A)^j &= \exists x A^j. \end{aligned}$$

Given a set of formulas  $\Gamma$ , we set  $\Gamma^m = \{A^m \mid A \in \Gamma\}$  and  $\Gamma^j = \{A^j \mid A \in \Gamma\}$ .

The difference between  $(\cdot)^m$  and  $(\cdot)^j$  is only in the translation of  $A \rightarrow B$ . Our minimal and intuitionistic translations are deeply related to *Kuroda's negative translation*: if  $(\cdot)^{m'}$  and  $(\cdot)^{j'}$  are translations defined as in Definition 10, except for

$$(\forall x A)^{m'} = \forall x \neg \neg A^{m'} \qquad (\forall x A)^{j'} = \forall x \neg \neg A^{j'},$$

then the negative translation  $A \mapsto \neg \neg A^{j'}$  is the one defined by Kuroda in [11], while  $A \mapsto \neg \neg A^{m'}$  is a variant of Kuroda's negative translation introduced in [6, p. 229]. Note that  $\neg \exists x \neg A$  and  $\forall x \neg \neg A$  are provably equivalent in NM, for every formula  $A$ .

Using the terminology of [6],  $(\cdot)^m$  and  $(\cdot)^j$  are *modular* translations in the sense that the translation of a formula is based on the translation of its immediate subformulas. The names “minimal” and “intuitionistic” associated with  $(\cdot)^m$  and  $(\cdot)^j$ , respectively, are due to the derivability relation they preserve: this will be clarified in Theorems 13.1-14.1 below (also their converses hold, see [9, §6]), which imply that  $\vdash_{\text{NK}} A$  iff  $\vdash_{\text{NJ}} \neg \neg A^j$  iff  $\vdash_{\text{NM}} \neg \neg A^m$ . Consequently, via Remark 11.1 below, the translations  $A \mapsto \neg \neg A^m$  and  $A \mapsto \neg \neg A^j$  are *modular negative* according to [6, Definition 1]. Besides, the modular negative translations  $A \mapsto \neg \neg A^m$  and  $A \mapsto \neg \neg A^{m'}$  (resp.  $A \mapsto \neg \neg A^j$

and  $A \mapsto \neg\neg A^j$ ) are the same according to [6, Definition 2], in the sense that  $A^m$  and  $A^{m'}$  (resp.  $A^j$  and  $A^{j'}$ ) are interderivable in minimal logic; however, quite interestingly, they have a very different behavior with respect to the postponement of  $\text{raa}$ : only the negative translation  $A \mapsto \neg\neg A^m$  (resp.  $A \mapsto \neg\neg A^j$ ) allows one to use Theorem 8.2 (resp. Theorem 8.1), since in  $A^{m'}$  (resp.  $A^{j'}$ ) the universal quantifier might occur, with the disturbing effect pointed out in §2.2 and Remark 4.

REMARK 11. For every formula  $A$ , by induction on  $A$  we can prove that:

1.  $A^m \dashv\vdash_{\text{NK}} A \dashv\vdash_{\text{NK}} A^j$  (where  $B \dashv\vdash_{\text{D}} C$  stands for  $B \vdash_{\text{D}} C$  and  $C \vdash_{\text{D}} B$ );
2.  $A^m$  contains no occurrences of  $\rightarrow$  or  $\forall$ ;  $A^j$  contains no occurrences of  $\forall$ ;
3. the free variables in  $A^m$  and  $A^j$  are the same as in  $A$ , and  $(A\{t/x\})^m = A^m\{t/x\}$  and  $(A\{t/x\})^j = A^j\{t/x\}$  for any term  $t$ ;
4.  $A^m = A$  if  $\rightarrow$  and  $\forall$  do not occur in  $A$ ;  $A^j = A$  if  $\forall$  does not occur in  $A$ .

Actually, one direction of the equivalences in Remark 11.1 can be reformulated in a more informative and constructive way from a proof-theoretic viewpoint, thanks to (the proof of) the following lemma.

LEMMA 12 (Preservation of derivability in NK with respect to translations). *For every derivation  $\Pi: \Gamma \vdash A$  in NK there exist a derivation  $\Pi': \Gamma^m \vdash A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$  and a derivation  $\Pi'': \Gamma^j \vdash A^j$  in  $\text{NK} \setminus \{\forall_i\}$ .*

PROOF. By induction on the derivation  $\Pi$ . Let us consider its last rule  $r$ . Due to Definition 10, to construct a derivation  $\Pi': \Gamma^m \vdash A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$  the only interesting cases are when  $r$  is an instance of  $\rightarrow_i$  or  $\rightarrow_e$  or  $\forall_i$  or  $\forall_e$ .

If  $r$  is an instance of  $\rightarrow_i$ , then  $\Pi: \Gamma \vdash A \rightarrow B$  and there is a derivation  $\pi: \Gamma, A \vdash B$  in NK such that

$$\Pi = \frac{\frac{\Gamma A^{\neg 1} \vdots \pi}{B}}{A \rightarrow B} \rightarrow_i^1.$$

By the induction hypothesis, there is a derivation  $\pi': \Gamma^m, A^m \vdash B^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ . Let  $\Pi'$  be the following derivation:



Thus,  $\Pi': \Gamma^m \vdash (\forall xA)^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ , since  $(\forall xA)^m = \neg\exists x\neg A^m$ .

If  $r$  is an instance of  $\forall_e$ , then  $\Pi: \Gamma \vdash A\{t/x\}$  and there is a derivation  $\pi: \Gamma \vdash \forall xA$  in  $\text{NK}$  such that

$$\Pi = \frac{\vdots \pi}{\forall xA} \forall_e \cdot \frac{\quad}{A\{t/x\}}$$

By the induction hypothesis, there is a derivation  $\pi': \Gamma^m \vdash \neg\exists x\neg A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ , since  $(\forall xA)^m = \neg\exists x\neg A^m$ . Let

$$\Pi' = \frac{\vdots \pi' \quad \frac{\Gamma \neg A^m\{t/x\}^{\neg 1}}{\exists x\neg A^m} \exists_i}{\frac{\perp}{A^m\{t/x\}} \text{raa}^1} \neg_e \quad \text{with } (A\{t/x\})^m = A^m\{t/x\} \text{ by Remark 11.3.}$$

Therefore,  $\Pi': \Gamma^m \vdash (A\{t/x\})^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ .

The proof of the existence of a derivation  $\Pi'': \Gamma^j \vdash A^j$  in  $\text{NK} \setminus \{\forall_i\}$  is analogous to the proof of the existence of a derivation  $\Pi': \Gamma^m \vdash A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ , but the only interesting cases are when  $r$  is an instance of  $\forall_i$  or  $\forall_e$ . ■

In general, derivability in  $\text{NM}$  is not preserved via the translation  $(\cdot)^m$ : e.g.  $\vdash_{\text{NM}} P \rightarrow P$  but  $\not\vdash_{\text{NM}} \neg P \vee P$ , where  $(P \rightarrow P)^m = \neg P \vee P$ . Also, a formula  $A$  in general is not provably equivalent to  $A^j$  in  $\text{NJ}$  (resp.  $A^m$  in  $\text{NM}$ ), since  $\forall xA$  is not equivalent to  $\neg\exists x\neg A$  in intuitionistic (resp. minimal) logic.

**THEOREM 13** (Generalized Glivenko's theorem, minimal version).

1. If  $\Gamma \vdash_{\text{NK}} A$ , then  $\Gamma^m \vdash_{\text{D}} \neg\neg A^m$  and  $\Gamma^m, \neg A^m \vdash_{\text{D}} \perp$ , where  $\text{D} = \text{NM} \setminus \{\rightarrow_i, \rightarrow_e, \forall_i, \forall_e\}$ .
2. If  $\rightarrow$  and  $\forall$  occur neither in  $A$  nor in any formula of  $\Gamma$ , then
  - (a)  $\Gamma \vdash_{\text{NK}} A$ ,    iff    (b)  $\Gamma \vdash_{\text{NM}} \neg\neg A$ ,    iff    (c)  $\Gamma, \neg A \vdash_{\text{NM}} \perp$ .

If moreover  $A = \neg B$ , then:  $\Gamma \vdash_{\text{NK}} \neg B$  if and only if  $\Gamma \vdash_{\text{NM}} \neg B$ .

**PROOF.**

1. Since  $\Gamma \vdash A$  is derivable in  $\text{NK}$ , according to Lemma 12, there exists a derivation  $\Pi: \Gamma^m \vdash A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$ , and by Theorem 8.2, there exists a derivation  $\Pi': \Gamma^m \vdash A^m$  in  $\text{NK} \setminus \{\rightarrow_i, \forall_i\}$  with at most one instance of the rule  $\text{raa}$ : this instance, if any, is the last rule of  $\Pi'$ , the rest of  $\Pi'$  being a derivation in  $\text{NM}$ . Only two cases are possible:

- either the last rule of  $\Pi'$  is not an instance of *raa*, and thus  $\Pi'$  is a derivation in NM, so that  $\Pi'' : \Gamma^m \vdash \neg\neg A^m$  and  $\Pi''' : \Gamma^m, \neg A^m \vdash \perp$  are derivations in NM, where:

$$\Pi'' = \frac{\frac{\vdots \Pi'}{A^m \quad \Gamma\neg A^{m\neg 1}} \neg_e}{\frac{\perp}{\neg\neg A^m} \neg_i^1} \neg_e \quad \text{and} \quad \Pi''' = \frac{\frac{\vdots \Pi'}{A^m \quad \neg A^m}}{\perp} \neg_e ;$$

- or the last rule of  $\Pi'$  is an instance of *raa*, i.e.  $\Pi'$  is the derivation (20) below, where  $\pi : \Gamma^m, \neg A^m \vdash \perp$  is a derivation in NM. Therefore, the derivation  $\Pi'' : \Gamma^m \vdash \neg\neg A^m$  (see (21) below) is in NM since  $\Pi''$  is obtained from  $\Pi'$  by replacing the instance of *raa* with an instance of  $\neg_i$  discharging the same assumptions.

$$\Pi' = \frac{\frac{\frac{\Gamma\neg A^{m\neg 1}}{\vdots \pi}}{\perp} \text{raa}^1}{A^m} \quad (20) \quad \Pi'' = \frac{\frac{\frac{\Gamma\neg A^{m\neg 1}}{\vdots \pi}}{\perp} \neg_i^1}{\neg\neg A^m} \quad (21)$$

We have thus proved that  $\Gamma^m \vdash \neg\neg A^m$  and  $\Gamma^m, \neg A^m \vdash \perp$  are derivable in NM. According to Remark 11.2, neither  $A^m$  nor any formula in  $\Gamma^m$  contain occurrences of  $\rightarrow$  and  $\forall$ ; hence, according to the normalization theorem and the subformula property for NM [19, p. 53],  $\Gamma^m \vdash \neg\neg A^m$  and  $\Gamma^m, \neg A^m \vdash \perp$  are derivable in  $\text{NM} \setminus \{\rightarrow_i, \rightarrow_e, \forall_i, \forall_e\}$ .

- (a) **implies** (b): By Theorem 13.1, since  $\Gamma \vdash A$  is derivable in NK, there is a derivation  $\Pi : \Gamma^m \vdash \neg\neg A^m$  in NM. According to Remark 11.4,  $\Gamma^m = \Gamma$  and  $A^m = A$ . So,  $\Pi : \Gamma \vdash \neg\neg A$  (in NM).
- (b) **implies** (c): If  $\Pi : \Gamma \vdash_{\text{NM}} \neg\neg A$ , then  $\Pi' : \Gamma, \neg A \vdash_{\text{NM}} \perp$  where

$$\Pi' = \frac{\frac{\vdots \Pi}{\neg\neg A \quad \neg A} \neg_e}{\perp} \neg_e .$$

- (c) **implies** (a): Since  $\text{NM} \subseteq \text{NK}$ , if  $\Pi : \Gamma, \neg A \vdash \perp$  is a derivation in NM, then  $\Pi$  is a derivation in NK. Therefore,  $\Pi' : \Gamma \vdash_{\text{NK}} A$  where

$$\Pi' = \frac{\frac{\frac{\Gamma\neg A^{\neg 1}}{\vdots \Pi}}{\perp} \text{raa}^1}{A} .$$

This proves the equivalences: (a) iff (b) iff (c). Suppose now that moreover  $A = \neg B$ . We show that  $\Gamma \vdash_{\text{NK}} \neg B$  if and only if  $\Gamma \vdash_{\text{NM}} \neg B$ .

**if:** Any derivation  $\pi: \Gamma \vdash \neg B$  in NM is also in NK because  $\text{NM} \subseteq \text{NK}$ .

**only if:** Since  $\Gamma \vdash_{\text{NK}} \neg B$ , there exists  $\pi: \Gamma \vdash_{\text{NM}} \neg\neg\neg B$  according to the implication (a) $\Rightarrow$ (b) we have just proved. Thus,  $\pi': \Gamma \vdash_{\text{NM}} \neg B$  where

$$\pi' = \frac{\frac{\frac{\vdots \pi_B}{\neg B}}{\neg\neg\neg B \rightarrow \neg B} \rightarrow_i^3}{\neg B} \rightarrow_e \quad \vdots \pi \quad \text{with} \quad \pi_B = \frac{\frac{\frac{\frac{\vdots \perp}{\neg\neg\neg B} \neg_i^1}{\neg\neg\neg B} \neg_e}{\perp} \neg_i^2}{\neg B} \neg_e \quad \blacksquare$$

**THEOREM 14** (Generalized Glivenko's theorem, intuitionistic version).

1. If  $\Gamma \vdash_{\text{NK}} A$ , then  $\Gamma^j \vdash_{\text{D}} \neg\neg A^j$  and  $\Gamma^j, \neg A^j \vdash_{\text{D}} \perp$  where  $\text{D} = \text{NJ} \setminus \{\forall_i, \forall_e\}$ .
2. If  $\forall$  occurs neither in  $A$  nor in any formula of  $\Gamma$ , then

$$(a) \Gamma \vdash_{\text{NK}} A, \quad \text{iff} \quad (b) \Gamma \vdash_{\text{NJ}} \neg\neg A, \quad \text{iff} \quad (c) \Gamma, \neg A \vdash_{\text{NJ}} \perp.$$

If moreover  $A = \neg B$ , then:  $\Gamma \vdash_{\text{NK}} \neg B$  if and only if  $\Gamma \vdash_{\text{NJ}} \neg B$ .

**PROOF.** The proof of each part of Theorem 14 is analogous to the proof of the respective part of Theorem 13 above, replacing  $(\cdot)^m$  with  $(\cdot)^j$ , NM with NJ,  $\text{NM} \setminus \{\rightarrow_i, \forall_i\}$  with  $\text{NJ} \setminus \{\forall_i\}$ ,  $\text{NM} \setminus \{\rightarrow_i, \rightarrow_e, \forall_i, \forall_e\}$  with  $\text{NJ} \setminus \{\forall_i, \forall_e\}$ . To prove Theorem 14.1 we use Theorem 8.1 instead of Theorem 8.2, and the normalization theorem and the subformula property for NJ instead of the normalization theorem and the subformula property for NM (see [19, p. 53]). To prove Theorem 14.2 we use Theorem 14.1 instead of Theorem 13.1.  $\blacksquare$

An immediate consequence of Theorem 14.2 (resp. Theorem 13.2) is the next corollary: in the fragment  $\{\perp, \top, \neg, \vee, \wedge, \rightarrow, \exists\}$  (resp.  $\{\perp, \top, \neg, \vee, \wedge, \exists\}$ ) of first-order logic, the consistency of a set of formulas in classical logic is equivalent to its consistency in intuitionistic (resp. minimal) logic.

**COROLLARY 15** (Relative consistency). *Let  $\Gamma$  be a set of formulas.*

1. If  $\forall$  does not occur in any formula of  $\Gamma$ , then:  $\Gamma \vdash_{\text{NK}} \perp$  iff  $\Gamma \vdash_{\text{NJ}} \perp$ .
2. If  $\rightarrow$  and  $\forall$  do not occur in any formula of  $\Gamma$ , then:  $\Gamma \vdash_{\text{NK}} \perp$  iff  $\Gamma \vdash_{\text{NM}} \perp$ .

**PROOF.** The proof of Corollary 15.1 is analogous to the proof of Corollary 15.2: it is sufficient to replace NM with NJ, and use Theorem 14.2 instead of Theorem 13.2. We now prove Corollary 15.2.

**only if:** Since  $\Gamma \vdash \perp$  is a derivable in NK, then by Theorem 13.2 there is a derivation  $\pi: \Gamma, \neg\perp \vdash_{\text{NM}} \perp$ . So,  $\Pi: \Gamma \vdash \perp$  is a derivation in NM where

$$\Pi = \frac{\frac{\frac{\vdots \perp}{\neg\perp} \neg_i^1}{\perp} \neg_i^1}{\perp} \neg_i^1 \quad \blacksquare$$

**if:** Since  $\text{NM} \subseteq \text{NK}$ , every derivation  $\pi: \Gamma \vdash \perp$  in  $\text{NM}$  is also in  $\text{NK}$ . ■

The fact that Theorem 13.2 and Corollary 15.2 (resp. Theorem 14.2 and Corollary 15.1) are restricted to the fragment  $\{\perp, \top, \neg, \wedge, \vee, \exists\}$  (resp.  $\{\perp, \top, \neg, \wedge, \vee, \rightarrow, \exists\}$ ) of the first-order language of classical logic is not a limit because this fragment is adequate for full first-order classical logic.

## 7. Conclusion

The literature on the connections between classical and constructive logic is extremely rich. Our aim in this paper was to give a unifying view of some of these results, by adopting a proof-theoretic perspective, and, in particular, by focusing on a very specific technique: that of postponing the application of the rule of *reductio ad absurdum* (*raa*) in the proofs of classical logic.

After having sketched the evolution of this technique starting from the seminal work of Prawitz in his monograph on natural deduction [19], we have focused our attention on a particular strategy of postponement: the one adopted by Seldin [20]. The interest of this strategy is that it can be characterized in a sort of geometrical way. In this sense, we proposed a modified version of it by reasoning only on the distance from the conclusion of the instances of *raa* present in a given derivation, and we left aside any consideration on the syntactic structure of the formulas introduced by the instances of *raa*. This insensitivity to syntactic considerations makes the technique extensible to logic systems going beyond first-order classical logic, like modal classical logic and, especially, second-order classical logic. We also conjecture that, even if our postponing strategy is a weak one, it is possible to transform it into a strong one, in the sense that the order of application of our reduction steps is not essential: any order of application should allow one to push *raa* downward with respect to all the other rules.

A further research direction is to investigate the link between the postponement of *raa* in classical natural deduction and the notion of uniform provability (a device for structuring proof search [13]) in classical sequent calculus. A sequent calculus proof is uniform if the last rule in the derivation of a complex formula at any stage in the proof is always the introduction of the top-level logical symbol of that formula.

The other aspect on which we focused our attention is the possibility of extracting some constructive content from the postponing strategy that we presented. In particular, we have been able to obtain Glivenko's theorem in a uniform way, that is, working for both intuitionistic and minimal logic. As for the postponement of *raa*, it should not be difficult to extend it to systems

that go beyond first-order logic, such as modal logic and second-order logic.

Finally, since the proof of Glivenko’s theorem rests on the use of a negative translation, and since negative translation is closely related to the *continuation-passing style* (CPS) transformations in functional programming, it would be interesting to investigate which is the proper computational interpretation (in terms of  $\lambda$ -calculus) that can be assigned to the negative translation induced by our postponing strategy. In particular, since our translation of classical logic into intuitionistic logic is just a variant of the Kuroda translation, it is reasonable to expect that our translation simulates a call-by-value evaluation strategy in a call-by-name interpreter (see [6, p. 255], [12, p. 158 ff.]). And since we can define also a translation of classical logic into minimal logic, it would be interesting to understand whether this second translation generates a different CPS transformation or not.

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