Postponement of raa and Glivenko's theorem, revisited

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Abstract

This article focuses on the technique of postponing the application of the *reduction ad absurdum* rule (raa) in classical natural deduction. First, it is shown how this technique is connected with two normalization strategies for classical logic: one given by Prawitz, and the other by Seldin. Secondly, a variant of Seldin's strategy for the postponement of raa is proposed, and the similarities with Prawitz's approach are investigated. In particular, it is shown that, as for Prawitz, it is possible to use this variant of Seldin's strategy in order to induce a negative translation from classical to intuitionistic and minimal logic, which is nothing but a variant of the Kuroda's translation. Through this translation, the Glivenko's theorem for intuitionistic and minimal logic is proved.

1 Introduction

Among the inference rules of classical natural deduction, the *reductio ad absurdum* – denoted by raa – formalizes the principle of a "proof by contradiction": if a contradiction is obtained from $\neg A$, then A can asserted and the hypothesis $\neg A$ can be dropped, i.e. discharged. This principle is rejected by intuitionism and, in general, by constructive accounts of logic. More precisely, raa is not an admissible inference rule in intuitionistic natural deduction, even if the latter contains a special case of raa, called *ex falso quodlibet* and denoted by efq. This rule formalizes the "principle of explosion": from a contradiction anything can be asserted, without discharging any hypothesis. The rule efq, in turn, is not admissible in a more restrictive system of constructive logic, like minimal natural deduction.

For natural deduction of first-order classical logic (with the raa rule) there exist two general ways of defining a weak normalization procedure: either by adopting Prawitz's strategy or by adopting Seldin's one (see [15, pp. 282–283] for a first comparison).

Prawitz's idea [22] consists in restricting to the fragment $\{\neg, \wedge, \rightarrow, \bot, \forall\}$, reducing all the applications of raa to atomic formulas, and then applying whatever normalization strategy for intuitionistic logic one likes (see [22, pp. 39–41]). Seldin's idea [25] consists, instead, in restricting to the fragment $\{\neg, \wedge, \vee, \rightarrow, \bot, \exists\}$, reducing all the applications of raa present in a derivation tree to at most one single application occurring as the last step of the tree (this is the *postponement of* raa), and then applying whatever normalization strategy for intuitionistic logic one likes (see [25, pp. 638–645]).[1]

Prawitz's and Seldin's strategies can be seen as one the "dual" of the other: the former breaks down classical reasoning into a number of atomic steps of raa, the latter compactifies classical reasoning into one single (possibly complex) step of raa, the final one. Moreover, a peculiarity of Seldin's strategy is that Glivenko's theorem (for intuitionistic logic) can be obtained as an immediate consequence: it is

[1]Both Prawitz and Seldin consider that a normal derivation is a derivation in which there are no detours. However, the notion of detour specific to classical logic is not the same for them. We will analyze this point in details in §2.

sufficient to drop the raa rule of a normal (according to Seldin) derivation in classical logic, and replace it with a ¬-introduction rule discharging the same assumptions (see [25, §3]).

The Glivenko theorem, in its original formulation [8], states that if a formula is provable in classical propositional logic then its double negation is provable in propositional intuitionistic logic (the converse trivially holds). Thus, it allows to embed propositional classical logic into propositional intuitionistic logic. Several refinements and generalizations of Glivenko's theorem are well-known (see [4] for a partial survey), they extend the result to first-order [10, 25, 4], second-order [31] and substructural logics [7, 13, 5], or they embed classical logic into minimal logic [3]. All these results are obtained using different approaches, both syntactic and semantic.

Starting from a comparison between Prawitz's and Seldin's weak normalization strategies, we will modify Seldin's reduction steps for the postponement of raa in order to achieve two purposes: on the one hand, to induce two variants of Kuroda's negative translation [11, 6] of first-order classical logic into intuitionistic and minimal logic; on the other hand, to give an elegant proof of Glivenko's theorem for both intuitionistic and minimal logic. These results are obtained by adopting a completely proof-theoretic approach. The interest of this approach rests on three main reasons:

1. We point out that the postponement of raa not only is an interesting result in itself, with remarkable consequences such as weak normalization and Glivenko theorems (as first observed by Seldin [25]), but it allows to prove them in a *uniform* way in, at least, a triple sense:
   (a) we prove the postponement of raa and its consequences for first-order classical logic, but our methods and techniques can be applied also in other systems, such as second-order classical logic (see point 2 below), and modal logic S4 and S5 with quantifiers;
   (b) the postponement of raa allows to derive Glivenko's theorem for both intuitionistic and minimal logic using the same proof-theoretic approach based on our reduction steps;
   (c) the reduction steps we have defined are essentially some variants of the ones used by Seldin, Prawitz and others (see §2) to prove weak normalization for classical natural deduction.
2. Our proof of the postponement of raa is proof-theoretic in a "geometric" way, in the sense that it relies on a notion of size for a derivation based only on the distance of the instances of raa from the conclusion of the derivation; the complexity of formulas play no role in this definition of size. This approach has two immediate consequences:
   (a) it allows to generalize the postponement of raa and its corollaries to second-order classical logic, since the substitution of formulas for propositional variables does not impact on the size we have defined;
   (b) we prove the postponement of raa in a weak form, in the sense that if we apply our reduction steps to *suitable* instances of raa, then our size of the derivation decreases; but we strongly conjecture that, by refining this notion of size, our reduction steps allow to postpone raa in a strong sense, i.e. they can be applied following whatever strategy one likes.
3. As well as the atomization of raa proposed by Prawitz's weak normalization strategy for classical natural deduction is deeply related to Gödel-Gentzen's negative translation, we show that the postponement of raa induces (a variant of) Kuroda's negative translation.

Outline The article will be structured into two parts. In the first part, namely §2, we will present a survey of the relevant literature concerning the (weak) normalization strategies for classical natural deduction. The second part presents our technical contributions. More precisely, §3 is devoted to basic definitions for first-order language and natural deduction. In §4 we will introduce our reduction steps, and in §5 we will use them to prove the postponement of raa. Finally, in §6 we will deduce from that the relationship with Kuroda's translation, and the Glivenko theorems for intuitionistic and minimal logic.

2 Normalization of classical logic: an overview

Looking closer at the distinction between Prawitz's and Seldin's weak normalization strategies, it can be noted that, in fact, this distinction is not so sharp as it could seem at a first glance. In particular, both of these strategies can be exploited for eliminating classical detours by postponing the use of raa. Let us clarify this point.

2.1 Prawitz's (weak) normalization strategy: its legacy (1965-2012)

As already mentioned, Prawitz's original weak normalization strategy for classical natural deduction [22] was conceived only for the fragment $\{\neg, \wedge, \rightarrow, \bot, \forall\}$, which is adequate for the full first-order language $\mathcal{L}$ of classical logic.[2] One of the challenges that arose from Prawitz's work was to find a weak normalization strategy for the full $\mathcal{L}$. The idea was to prove this result by somehow relaxing Prawitz's strategy: instead of proving the atomization of all the raa occurrences present in a given derivation, it would be sufficient to eliminate all classical detours, where a *classical detour* is defined as an instance of the raa rule introducing a formula occurrence A, which is immediately followed by an instance of an elimination rule having A as major premiss. In this paper, such detours will be called classical detours *à la* Prawitz, since they correspond to the definition of maximum formula given by Prawitz in [22, p. 34].

In order to better appreciate this departure from Prawitz's original strategy we proceed by giving an example. According to Prawitz, the following derivation contains a complex instance of raa

$$\begin{array}{c} \ulcorner\neg(A\wedge B)\urcorner^{1} \\ \vdots\,\pi \\ \dfrac{\bot}{A\wedge B}\,\mathsf{raa}^{1} \\ \vdots\,\pi_0 \end{array} \tag{1}$$

and thus, it has to be reduced on a derivation using less complex instances of raa, i.e.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\ulcorner\neg A\urcorner^{2} \quad \dfrac{\ulcorner A\wedge B\urcorner^{1}}{A}\,\wedge_{e_1}}{\dfrac{\bot}{\neg(A\wedge B)}\,\neg_{i}^{1}}\,\neg_{e} \\ \vdots\,\pi \\ \bot\end{array}}{A}\,\mathsf{raa}^{2} \qquad \dfrac{\begin{array}{c}\dfrac{\ulcorner\neg B\urcorner^{4} \quad \dfrac{\ulcorner A\wedge B\urcorner^{3}}{B}\,\wedge_{e_2}}{\dfrac{\bot}{\neg(A\wedge B)}\,\neg_{i}^{3}}\,\neg_{e} \\ \vdots\,\pi \\ \bot\end{array}}{B}\,\mathsf{raa}^{4}}{\begin{array}{c}A\wedge B \\ \vdots\,\pi_0\end{array}}\,\wedge_{i} \tag{2}$$

However, it can be claimed that it is not necessary to reduce all complex instances of raa (i.e. the instances of raa whose conclusion is a non-atomic formula); it would be already sufficient to focus only on a particular subset of them: the ones forming classical detours *à la* Prawitz. The reason is that only in this situation we are creating a rules' configuration which is similar to the standard (intuitionistic) detours of the form $\circ$-introduction/$\circ$-elimination (for a certain connective $\circ$): the rule raa plays the role of an introduction rule. Thus, instead of (1), we could consider

[2] In Prawitz [22], as well as in the other approaches that will be analyzed in this section, the negation $\neg$ is not treated as a primitive operator, but it is defined by $\rightarrow$ and $\bot$, i.e. $\neg A := A \rightarrow \bot$, and its introduction and elimination rules are special cases of the introduction and elimination rules for the implication. As we will see in §3, our approach is different.

$$\begin{array}{c}\ulcorner\neg(A\wedge B)\urcorner^1\\ \vdots\ \pi\\ \dfrac{\dfrac{\bot}{A\wedge B}\ \mathsf{raa}^1}{A}\ \wedge_{\mathsf{e}_1}\\ \vdots\ \pi_0\end{array} \tag{3}$$

and reduce it to

$$\begin{array}{c}\dfrac{\dfrac{\ulcorner\neg A\urcorner^2 \qquad \dfrac{\ulcorner A\wedge B\urcorner^1}{A}\ \wedge_{\mathsf{e}_1}}{\bot}\ \neg_{\mathsf{e}}}{\neg(A\wedge B)}\ \neg_{\mathsf{i}}^1\\ \vdots\ \pi\\ \dfrac{\bot}{A}\ \mathsf{raa}^2\\ \vdots\ \pi_0\end{array} \tag{4}$$

It is worth noticing that (4) is nothing but a sub-derivation of (2). However, adopting this second kind of reduction does not mean that we are simply applying a special case of Prawitz's original strategy. There is indeed a crucial difference between the two reductions presented here. In the reduced derivation (2), the distance from its conclusion of the two instances of raa is one unit greater than the distance from the conclusion of the instance of raa which is present in the original derivation (1). On the contrary, in the reduced derivation (4), the distance from its conclusion of the instance of raa is one unit smaller that the distance from the conclusion of the instance of raa which is present in the original derivation (3). In other words, while Prawitz's original strategy (1)-(2) makes the application of raa to be anticipated, the second strategy (3)-(4) makes the application of raa to be postponed.

2.1.1 Statman's approach (1974)

As far as we know, the first who took into consideration this postponing strategy for solving the normalization problem of classical natural deduction for the *full* first-order language was Statman [28]. He considered, in particular, the following general reduction scheme for classical detours *à la* Prawitz:

$$\begin{array}{c}\dfrac{\begin{array}{c}\ulcorner\neg A\urcorner^1\\ \vdots\ \pi\\ \dfrac{\bot}{A}\ \mathsf{raa}^1\end{array} \qquad \begin{array}{c}\vdots\ \pi'\end{array}}{C}\ \circ_{\mathsf{e}}\\ \vdots\ \pi_0\end{array} \quad \text{reduces to} \quad \begin{array}{c}\dfrac{\dfrac{\ulcorner\neg C\urcorner^2 \qquad \dfrac{\ulcorner A\urcorner^1 \quad \vdots\ \pi'}{C}\ \circ_{\mathsf{e}}}{\bot}\ \neg_{\mathsf{e}}}{\neg A}\ \neg_{\mathsf{i}}^1\\ \vdots\ \pi\\ \dfrac{\bot}{C}\ \mathsf{raa}^2\\ \vdots\ \pi_0\end{array} \tag{5}$$

where $\circ_{\mathsf{e}}$ is an elimination rule for any connective $\circ$ of the full language of first-order logic (see [28, pp. 78–79]) and A is the major premiss of $\circ_{\mathsf{e}}$.[3] This means that, differently from Prawitz, Statman

[3] The fact that, in the derivation Π on the left-hand side of (5), A is the major premiss of $\circ_{\mathsf{e}}$ ensures that $\circ_{\mathsf{e}}$ does not

has not to drop the disjunction and the existential quantifier in order to apply his reduction steps. However, instead of reasoning in a pure combinatorial way on the application of the reduction steps – as previously done by Prawitz – Statman adopts a different approach, which basically consists of two phases. First, second order predicate classical logic is considered, and it is embedded into a system of second order intuitionistic propositional logic by using a homomorphism which preserves the reduction relations. Secondly, a (strong) normalization theorem for this intuitionistic system is proved. To prove this theorem, impredicative methods are used.

A part the use of impredicative methods, there is another problem which could make Statman's approach not completely satisfactory: the $\lor$- and $\exists$-elimination rules can be applied only in the restricted case in which their conclusion is $\bot$ (see [28, p. 90]), i.e.

$$\dfrac{A \lor B \qquad \begin{array}{c}\ulcorner A \urcorner^1\\ \vdots \\ \bot\end{array} \qquad \begin{array}{c}\ulcorner B \urcorner^1\\ \vdots\\ \bot\end{array}}{\bot}\,{\lor'}_{\mathsf{e}}^{1} \qquad \text{and} \qquad \dfrac{\exists x A \qquad \begin{array}{c}\ulcorner A \urcorner^1\\ \vdots \\ \bot\end{array}}{\bot}\,{\exists'}_{\mathsf{e}}^{1}$$

Statman's solution seems then to lack of generality and uniformity.

Some arguments can be invoked in order to justify Statman's choice to work with these restricted versions of the $\lor$- and $\exists$-elimination rules. For simplicity, let us restrict to the case of disjunction. First, it should be noted that, within the framework of classical logic, the usual $\lor$-elimination rule – i.e. $\lor_{\mathsf{e}}$ – and the restricted version used by Statman – i.e. ${\lor'}_{\mathsf{e}}$ – are equivalent from the point of view of derivability. Indeed, ${\lor'}_{\mathsf{e}}$ is just a special case of $\lor_{\mathsf{e}}$, and conversely, $\lor_{\mathsf{e}}$ is classically derivable from ${\lor'}_{\mathsf{e}}$:

$$\dfrac{\dfrac{A \lor B \qquad \dfrac{\ulcorner \neg C \urcorner^2 \qquad \begin{array}{c}\ulcorner A \urcorner^1\\ \vdots \\ C\end{array}}{\bot}\,\neg_{\mathsf{e}} \qquad \dfrac{\ulcorner \neg C \urcorner^2 \qquad \begin{array}{c}\ulcorner B \urcorner^1\\ \vdots \\ C\end{array}}{\bot}\,\neg_{\mathsf{e}}}{\bot}\,{\lor'}_{\mathsf{e}}^{1}}{C}\,\mathsf{raa}^2$$

The restriction imposed on the $\lor$-elimination rule can also be explained by a second argument. In his proof of (strong) normalization for classical natural deduction, Statman passes through an intermediate step: he shows that the set of classical derivations can be embedded into the subset of classical derivations not containing $\lor$ and $\exists$, and that this embedding preserves the reduction relations between derivations. This means that, differently from Prawitz, Statman is not replacing from the beginning the full first-order language with the fragment $\{\neg, \land, \to, \bot, \forall\}$ (which is still adequate for classical logic). In other words, Statman is not reasoning at the level of the derivability relation between sentences, but he is reasoning at the level of the reduction relations between derivations. Thus, strictly speaking, he maintains the full language of classical logic and uses an operation m_0 – corresponding to Gentzen's negative translation – just to narrow down the set of derivations which have to be analyzed with respect to the reduction relations.

In such a context, disjunction can be defined in terms of negation and conjunction, so that $m_0(A \lor B) \coloneqq \neg(\neg m_0(A) \land \neg m_0(B))$.[4] And if we have two derivations of the form

$$\begin{array}{c}m_0(A)\\ \vdots \\ \bot\end{array} \qquad \text{and} \qquad \begin{array}{c}m_0(B)\\ \vdots \\ \bot\end{array}$$

discharge any assumptions of the sub-derivation π, hence all assumptions in the derivation on the right-hand side of (5) are already assumptions of Π. See also §2.2 below.

[4] Note that Statman composes then the operation m_0 with another operation m_1 – corresponding to the so-called Russell-Prawitz's translation [22, p. 67] – in order to further narrow down his analysis to the set of derivations and reductions which use only implication and second order universal quantifier (see [28, p. 91–92]).

we could obtain $\neg m_0(A) \wedge \neg m_0(B)$ (namely, by discharging $m_0(A)$ and $m_0(B)$ through $\neg$-introduction rules, and applying then a $\wedge$-introduction), so that in presence of $\neg(\neg m_0(A) \wedge \neg m_0(B))$ – which, as we said, is equal to $m_0(A \vee B)$ – we can conclude $\bot$. This shows that the rule

$$\dfrac{m_0(A \vee B) \qquad \begin{matrix}\ulcorner m_0(A) \urcorner^1 \\ \vdots \\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner m_0(B)\urcorner^1 \\ \vdots \\ \bot\end{matrix}}{\bot}\, S^1$$

is derivable. And this is nothing but a way to say that working with a restricted version of the $\vee$-elimination rule – where $\bot$ is the conclusion – does not represent a real limitation if one works modulo m_0: the rule S and the rule $\vee'$ share the same inferential structure. In other words, the operation m_0 is an homomorphism which preserves the inferential structure.

2.1.2 Stålmarck's approach (1991)

Being it conceived for second-order logic, Statman's approach rests on an impredicative analysis of derivations and of their reduction relations. An alternative approach to the (weak) normalization of classical natural deduction for the full first-order language by using only pure combinatorial means was proposed by Stålmarck [27].

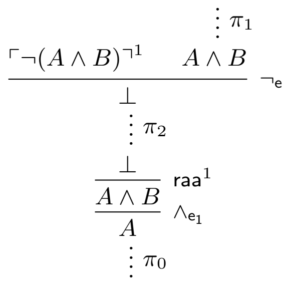
Like Statman, Stålmarck's fundamental idea consists in removing the maximum formulas which appear in classical detours *à la* Prawitz by pushing raa downwards with respect to the elimination rules for the connectives of the full first-order language. But, differently from Statman, Stålmarck has no restrictions on $\vee$- and $\exists$-elimination rules.[5] The problem for Stålmarck is that, by using the reduction scheme (5), when $\circ$ is a disjunction or an existential quantifier, the formula C introduced by raa in the reduced derivation could be more complex than the formula A introduced by raa in the original derivation. Thus, *a priori* there is nothing which tells him when to stop the reduction of the classical detours. In particular, in order to prove the weak normalization of classical natural deduction for the full first-order language, Stålmarck cannot proceed like Prawitz in two phases, that is, first, by atomizing all the occurrences of raa, and secondly, by applying any normalization strategy for intuitionistic logic. On the contrary, Stålmarck has to treat classical and intuitionistic detours on a par, and he has to eliminate them together by (a main) induction on the complexity of the maximum formulas of the detours.

The problem is that this method could conflict with certain applications of the reduction scheme (5). Consider the case of conjunction. In (4), the formula $\neg(A \wedge B)$ introduced by a $\neg_i$ rule could be the major premiss of a $\neg_e$ rule; it could thus create a new detour with a maximum formula more complex than the original one (namely $A \wedge B$, associated with the classical detour). Indeed, in this case the derivation (4) has the form

$$\begin{matrix} \dfrac{\dfrac{\dfrac{\ulcorner\neg A\urcorner^2 \quad \dfrac{\ulcorner A\wedge B\urcorner^1}{A}\wedge_{e_1}}{\bot}\neg_e}{\neg(A\wedge B)}\neg_i^1 \qquad \begin{matrix}\vdots\,\pi_1\\ A\wedge B\end{matrix}}{\bot}\neg_e \\ \vdots\,\pi_2 \\ \dfrac{\bot}{A}\,\mathsf{raa}^2 \\ \vdots\,\pi_0 \end{matrix}$$

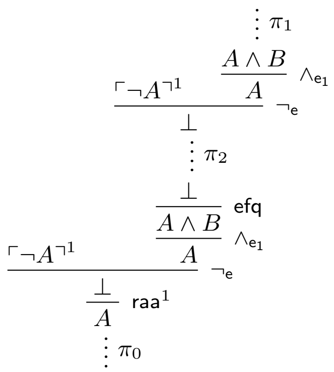
[5] Like Statman, Stålmarck's main interest is a strong normalization theorem for classical logic. However, since this article mainly focuses on the problem of the downward postponement of raa, it is already sufficient to consider here only Stålmarck's weak normalization strategy (see [27, pp. 133–135]).

and, according to the reduction scheme (5), the derivation we have before the application of the reduction step is the following, where the maximum formula of the classical detour is $A \wedge B$:



(6)

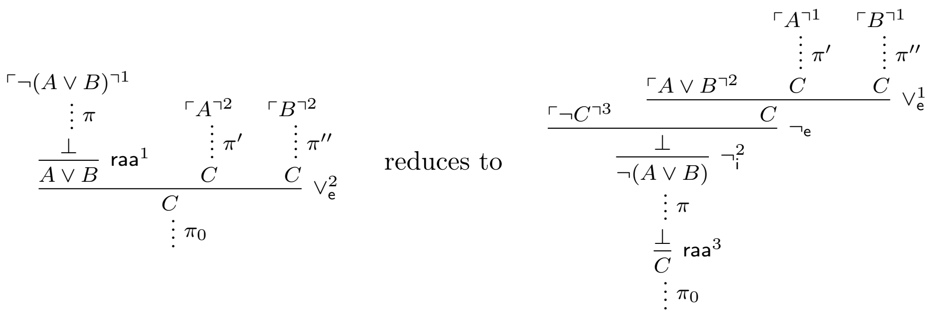
Stålmarck's solution consists in adding to (5) a new reduction scheme which transforms (6) into



In this way, the occurrence of raa is not only pushed downward, but also applied to a formula of lower complexity than the original one, respecting in this sense Prawitz's original idea. However, differently from Prawitz's atomization procedure, in order to define this reduction, a step of efq has to be added. This represents an important point on which we will return later (see §2.2).

The other difference with respect to Prawitz's procedure is that, by considering the full first-order language, Stålmarck has to define also reduction steps for classical detours *à la* Prawitz created by a raa immediately followed by a $\vee$- or $\exists$-elimination rule. Let us focus on the $\vee$ case.

As we have seen for conjunction, if we define a straightforward postponement of raa with respect to $\vee_{\mathsf{e}}$ following the reduction scheme (5), we would have that



(7)

and thus, when in the sub-derivation π the assumption $\neg(A \vee B)$ is the major premiss of a $\neg_{\mathsf{e}}$, we could create a new detour of the type $\neg_{\mathsf{i}}/\neg_{\mathsf{e}}$, having $\neg(A \vee B)$ as maximum formula, which is more complex than the maximum formula eliminated by the reduction step itself, namely $A \vee B$. However, when we are dealing with disjunction, the solution proposed by Stålmarck for conjunction is no more sufficient in order to circumvent this problem. More precisely, the derivation

$$\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\ulcorner\neg(A\vee B)\urcorner^1 \quad \begin{array}{c}\vdots\,\pi_1\\ A\vee B\end{array}}{\bot}\neg_{\mathsf{e}}\\ \vdots\,\pi_2\\ \dfrac{\bot}{A\vee B}\mathsf{raa}^1\end{array} \qquad \begin{array}{c}\ulcorner A\urcorner^2\\ \vdots\,\pi'\\ C\end{array} \quad \begin{array}{c}\ulcorner B\urcorner^2\\ \vdots\,\pi''\\ C\end{array}}{C}\vee_{\mathsf{e}}^2\\ \vdots\,\pi_0\end{array} \quad \text{would reduce to} \quad \begin{array}{c}\dfrac{\ulcorner\neg C\urcorner^2 \quad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ A\vee B\end{array} \quad \begin{array}{c}\ulcorner A\urcorner^1\\ \vdots\,\pi'\\ C\end{array} \quad \begin{array}{c}\ulcorner B\urcorner^1\\ \vdots\,\pi''\\ C\end{array}}{C}\vee_{\mathsf{e}}^1}{\bot}\neg_{\mathsf{e}}\\ \vdots\,\pi_2\\ \dfrac{\bot}{C}\mathsf{raa}^2\\ \vdots\,\pi_0\end{array} \tag{8}$$

but since C could be any compound formula, it could create a new classical detour (when C in π_0 is the major premiss of an elimination rule) more complex than the one just eliminated.[6]

In order to solve this problem, Stålmarck's idea is to decompose the reduction for the $\vee$ case into two steps. A first part consists in operating a sort of expansion of the $\vee_{\mathsf{e}}$ rules present in a given derivation, where $\bot$ plays the role of a minimum formula in the two threads starting from the assumptions A and B, respectively, and the end-formula C (when $C \neq \bot$) is introduced by raa,[7] i.e. the derivation

$$\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\ulcorner\neg(A\vee B)\urcorner^1 \quad \begin{array}{c}\vdots\,\pi_1\\ A\vee B\end{array}}{\bot}\neg_{\mathsf{e}}\\ \vdots\,\pi_2\\ \dfrac{\bot}{A\vee B}\mathsf{raa}^1\end{array} \qquad \begin{array}{c}\ulcorner A\urcorner^2\\ \vdots\,\pi'\\ C\end{array} \quad \begin{array}{c}\ulcorner B\urcorner^2\\ \vdots\,\pi''\\ C\end{array}}{C}\vee_{\mathsf{e}}^2\\ \vdots\,\pi_0\end{array}$$

is transformed into

$$\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\ulcorner\neg(A\vee B)\urcorner^1 \quad \begin{array}{c}\vdots\,\pi_1\\ A\vee B\end{array}}{\bot}\neg_{\mathsf{e}}\\ \vdots\,\pi_2\\ \dfrac{\bot}{A\vee B}\mathsf{raa}^1\end{array} \qquad \dfrac{\ulcorner\neg C\urcorner^3 \quad \begin{array}{c}\ulcorner A\urcorner^2\\ \vdots\,\pi'\\ C\end{array}}{\bot}\neg_{\mathsf{e}} \quad \dfrac{\ulcorner\neg C\urcorner^3 \quad \begin{array}{c}\ulcorner B\urcorner^2\\ \vdots\,\pi''\\ C\end{array}}{\bot}\neg_{\mathsf{e}}}{\bot}\vee_{\mathsf{e}}^2\\ \dfrac{\bot}{C}\mathsf{raa}^3\\ \vdots\,\pi_0\end{array}$$

The second part consists in applying the original reduction procedure, following the complexity of the maximum formulas of the detours (namely, starting with C). This is possible since the reduction of a $\vee$ classical detour is now operated only when the conclusion of the $\vee_{\mathsf{e}}$ is $\bot$, so no new (possibly) more complex detours are created.[8] When it comes the turn of the $\vee$ classical detour, we have that

[6] Note also that in this reduction for $\vee_{\mathsf{e}}$, differently from the $\wedge$ case, the appeal to the efq rule would be pointless. If efq was used after the $\bot$ of the π derivation in order to obtain $A \vee B$, then another step of $\vee_{\mathsf{e}}$ would be needed. This would force one to conclude C, and thus to use $\neg_{\mathsf{e}}$ again – with $\neg C$ as major premiss – followed by the π derivation. Hence, the result would be nothing but a repeated duplication of the original reduction pattern.

[7] We use here the term 'thread' in the sense of Prawitz [22, p. 25] (see also Definition 1, *infra*). For the definition of expansion and minimum formula see Prawitz ([23, part II.3]).

[8] The reason is that the burden of the creation of new (possibly) more complex detours rests now on the preliminary step corresponding to the "expansion" of the $\vee_{\mathsf{e}}$ rule described above.

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_2\\ \dfrac{\ulcorner\neg(A\vee B)\urcorner^1 \qquad \begin{matrix}\vdots\,\pi_1\\ A\vee B\end{matrix}}{\bot}\,\neg_{\mathsf{e}}\end{matrix}}{\bot}}{A\vee B}\,\mathsf{raa}^1 \qquad \begin{matrix}\ulcorner A\urcorner^2\\ \vdots\,\pi'\\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner B\urcorner^2\\ \vdots\,\pi''\\ \bot\end{matrix}}{\begin{matrix}\bot\\ \vdots\,\pi_0\end{matrix}}\,\vee^2_{\mathsf{e}_1}
\quad \text{reduces to} \quad
\dfrac{\dfrac{\begin{matrix}\dfrac{\begin{matrix}\vdots\,\pi_1\\ A\vee B\end{matrix} \qquad \begin{matrix}\ulcorner A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner B\urcorner^1\\ \vdots\,\pi''\\ \bot\end{matrix}}{\bot}\,\vee^1_{\mathsf{e}_1}\\ \vdots\,\pi_2\\ \bot\end{matrix}}{A\vee B}\,\mathsf{efq} \qquad \begin{matrix}\ulcorner A\urcorner^2\\ \vdots\,\pi'\\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner B\urcorner^2\\ \vdots\,\pi''\\ \bot\end{matrix}}{\begin{matrix}\bot\\ \vdots\,\pi_0\end{matrix}}\,\wedge^2_{\mathsf{e}_1}
\qquad (9)
$$

Note that Stålmarck's first step for the reduction of the $\vee$ case can also be described as the application of *permutative conversion* between the $\vee_{\mathsf{e}}$ and the $\neg_{\mathsf{e}}$ rules in the reduced derivation of (8), as remarked by de Groote ([9, p. 184]).[9] It is for this reason that we can say that Stålmarck works with the standard set of inference rules for classical logic. More precisely, differently from Statman – who works with a system of classical logic where the rule $\vee_{\mathsf{e}}$ is replaced by the rule $\vee'_{\mathsf{e}}$ – Stålmarck keeps defining the system of classical rules using the standard $\vee_{\mathsf{e}}$. In particular, since the permutative conversions are taken into account when a $\vee_{\mathsf{e}}$ is involved in a classical detours, this rule is used in a restricted way only with respect to the reduction procedure for classical detours, but not with respect to the simple construction of a derivation. What is lost then is the uniformity of the reduction procedure for classical detours, even if there is no loss from the point of view of the derivability.

2.1.3 von Plato and Siders' approach (2012)

A uniform solution to the weak normalization problem of classical natural deduction for the full first-order language – i.e. a solution imposing no sort of restriction to the application of the inference rules of the full first-order classical natural deduction – was finally given by von Plato and Siders [21].[10]

They consider the same notion of classical detour *à la* Prawitz as Stålmarck, but they use a system of natural deduction for classical logic where all the elimination rules are presented in a general form as done in [16]. For example, the elimination rules for conjunction and implication become respectively

$$
\dfrac{A\wedge B \qquad \begin{matrix}\ulcorner A, B\urcorner^1\\ \vdots\,\pi\\ C\end{matrix}}{C}\,\wedge^1_{\mathsf{e}}
\qquad \text{and} \qquad
\dfrac{A\to B \qquad A \qquad \begin{matrix}\ulcorner B\urcorner^1\\ \vdots\,\pi\\ C\end{matrix}}{C}\,\to^1_{\mathsf{e}}
$$

Using these general elimination rules von Plato and Siders are able to define the postponement of raa (for classical detours *à la* Prawitz) in an uniform way, following the general reduction scheme (5).

In this way, differently from Stålmarck's reductions, no use of efq is needed. Moreover, Stålmarck's treatment of $\vee_{\mathsf{e}}$ through a permutative conversion with a $\neg_{\mathsf{e}}$ can be extended to all the other elimination rules, thanks to their general form. Besides, differently from Stålmarck, the appeal to permutative conversions does not represent a necessary step, since there is no need for reasoning over the complexity of the maximum formulas of classical detours. The reason is that, by using general elimination rules, a derivation in classical logic can be considered to be normal when all major premisses of the elimination rules are assumptions. In order to obtain this normal form it is sufficient to apply, first, the reductions associated to the classical detours – as described in (5) – and then the reductions associated to the intuitionistic detours. In this way Prawitz's original strategy is respected, even if the reductions of classical detours are not necessarily associated with the atomization of the applications of raa rules.

[9] See also [27, p. 136]. The definition of permutative conversions (or reductions) is given in [22, p. 51] and [23, p. 253].

[10] In [21], von Plato and Siders propose only a weak normalization theory for classical logic. A strong normalization strategy is suggested in [18].

Thus, as in Prawitz (see [22, p. 41] and [23, part. II.3.2]), a classical proof in normal form becomes a derivation essentially composed by two parts:[11]

> [. . .] from the endformula upward, there will be a sequence of I-rules and their nested premisses [. . .] until a conclusion of a rule DN [*i.e.* raa] is reached. Its premiss is $\bot$. Looking from the other direction, from top formulas downward, we find a nested sequence of major premisses of E-rules. [. . .] the presence of rule DN can force conclusions of E-rules in normal derivations to be equal to the premiss $\bot$, without any *a priori* requirement that this should be so. ([21, p. 208])

This means, in particular, that in a normal derivation the application of the raa rule is what separates one of the parts containing the elimination rules from the part containing the introduction rules. Schematically, this takes the form:

$$
\begin{array}{ccc}
\dfrac{\ulcorner\neg A_1\urcorner^1 \quad \begin{array}{c}\vdots\,\pi_1\\ A_1\end{array}}{\bot}\,\neg_{\mathsf{e}} & & \dfrac{\ulcorner\neg A_n\urcorner^n \quad \begin{array}{c}\vdots\,\pi_n\\ A_n\end{array}}{\bot}\,\neg_{\mathsf{e}} \\
\vdots\ E\text{-rules} & & \vdots\ E\text{-rules}\\
\dfrac{\bot}{A_1}\,\mathsf{raa}^1 & & \dfrac{\bot}{A_n}\,\mathsf{raa}^n\\
\ddots & I\text{-rules} & \cdot^{\cdot^{\cdot}} \\
 & \vdots & \\
 & C &
\end{array}
$$

This makes clear that in order to obtain the normal form for classical natural deduction (with respect to classical detours *à la* Prawitz), it is sufficient to push raa downward with respect to the elimination rules whose major premiss is the conclusion of raa, while nothing is said about the possibility of pushing raa downward with respect to introduction rules, or to elimination rules whose major premiss is not a conclusion of raa. Nevertheless, in [21, p. 210] von Plato and Siders also mention the possibility of defining the reduction steps for pushing raa downward with respect to the introduction rules of the propositional fragment. These reduction steps are then explicitly given by von Plato in [17, pp. 86–87].

2.2 Seldin's normalization strategy (1986)

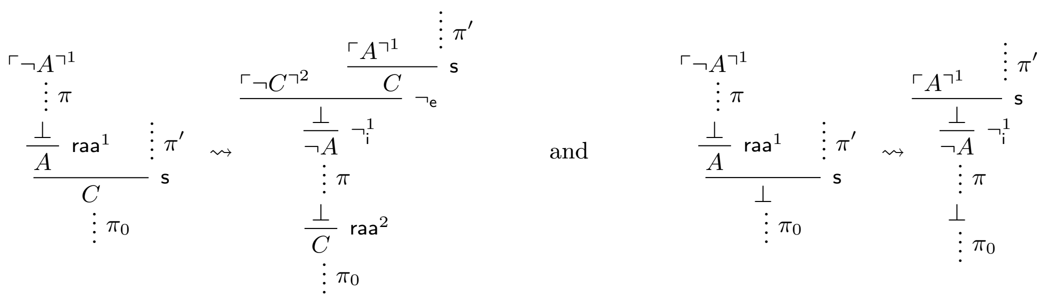
Turning now to Seldin's approach [25] for the weak normalization of first-order classical natural deduction, we can see that the principal difference with Prawitz rests on the notion of classical detour which is adopted. Seldin considers that a classical detour consists in a raa rule introducing a formula occurrence A, which is immediately followed by another rule having A as one of its premisses (see [25, p. 638]).

This characterization is more general than Prawitz's one: in order to eliminate a classical detour it is asked to push raa downward not only with respect to the major premiss of the elimination rules, but also with respect to any (introduction or elimination) inference rule immediately below raa.

It would be tempting, for such a purpose, to appeal to the reduction scheme (5) and to generalize it in order to define a reduction procedure for Seldin's notion of classical detour. The idea would be to

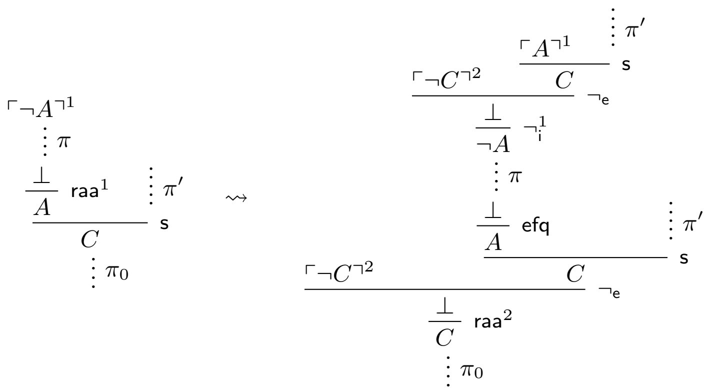
[11] Note that, by restricting to the fragment $\{\neg, \wedge, \to, \bot, \forall\}$, Prawitz's notion of classical normal derivation can be defined with respect to the notion of branch (see [22, p. 52]). On the other hand, working with the full language and using general elimination rules obliges Siders and von Plato to define the notion of classical normal derivation with respect to another notion, that of thread (see [21, p. 208]). The reader has to pay attention that von Plato and Siders' definition of thread is different from Prawitz's one (already mentioned here at p. 8). More precisely, von Plato and Siders' notion of thread is conceived as a sort of generalization of Prawitz's notion of path, allowing one to go through a derivation by jumping from the major premiss of a general elimination rule to the assumptions of the minor premisses of the same rule. For more detail see Negri and von Plato ([19, p. 196 et sqq.] and [20, p. 26–27]).

replace the elimination rule $\circ_{\mathsf{e}}$ in (5) by a rule s whatsoever. More precisely, given a derivation with an instance r of raa which is not its last rule, a reduction step $\rightsquigarrow$ either pushes r downward or erases r, i.e. if s is an instance of a (1-, 2- or 3-ary) rule immediately below r, one has (for $C \neq \bot$):

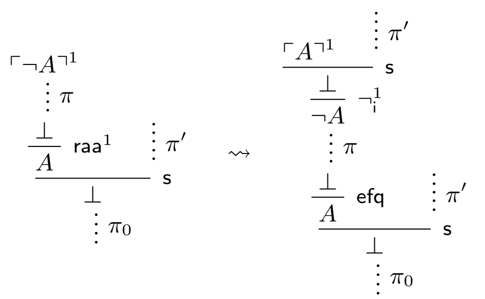


However, as already noticed by Seldin [25, pp. 642, 645], these schemata work only if no assumption of π is discharged at s in the derivations on the left-hand side of $\rightsquigarrow$; otherwise the transformation $\rightsquigarrow$ would change the set of non-discharged assumptions, adding new formulas in it (think for example of the case in which s is a $\rightarrow_{\mathsf{i}}$ rule, or a $\exists_{\mathsf{e}}$ rule with A as its minor premiss).

Seldin's solution consists in adopting the two following alternative schemata (where $C \neq \bot$):



(10)



(11)

Note that these reductions make use of a efq rule. We have already seen the same occurring in Stålmarck's reductions for the $\{\neg, \wedge, \rightarrow, \bot, \forall\}$ fragment. In fact, Stålmarck's reductions are just a particular case of Seldin's ones, as they only work when s is an elimination rule with A as major premiss.

It should also be noted that in Seldin's case no restrictions have to be made on the applications of $\vee_{\mathsf{e}}$ and $\exists_{\mathsf{e}}$ rules during the process of normalization. The reason is that his normalization strategy is not like Stålmarck's one, but follows the same pattern as Prawitz. First, all classical detours are eliminated, and secondly, all intuitionistic detours are eliminated. What differs with Prawitz is the way in which classical detours are eliminated. In Seldin, the elimination of classical detour consists in pushing down all instances of raa with respect to all the other rules, and then contract these instances of raa into one. In this way, what tells him when the elimination of classical detours has to stop is the position of raa in the derivation tree, and not, like in Prawitz, the complexity of the formula to which raa is applied. By borrowing a terminology from Girard's jargon, we could say that for Seldin the termination of classical detours elimination can be characterized in a *geometrical* way rather than in a *syntactical* one.[12]

However, even Seldin's strategy is not immune from some restrictions in order to work. The $\forall$ has to be dropped (i.e. Seldin's approach works for the $\{\neg, \wedge, \vee, \rightarrow, \bot, \exists\}$ fragment, which is as expressive as the full language of first-order classical logic), since the reduction schemata (10)-(11) cannot be applied when s is a $\forall_\mathsf{i}$ rule. Indeed, the natural way to treat the $\forall_\mathsf{i}$ case would be the following reduction step:

$$\Pi = \begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\, \pi' \\ \dfrac{\dfrac{\bot}{A}\,\mathsf{raa}^1}{\forall x A}\,\forall_\mathsf{i} \\ \vdots\, \pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\ulcorner \neg \forall x A \urcorner^2 \qquad \dfrac{\ulcorner A \urcorner^1}{\forall x A}\,\forall_\mathsf{i}}{\bot}\,\neg_\mathsf{e}}{\neg A}\,\neg_i^1 \\ \vdots\, \pi' \\ \dfrac{\bot}{\forall x A}\,\mathsf{raa}^2 \\ \vdots\, \pi \end{array} = \Pi'$$

but Π' is not a derivation in classical natural deduction (nor in its subsystems) because in Π' the rule $\forall_\mathsf{i}$ is not correctly instantiated, indeed the variable x may occur free in A and A is a non-discharged assumption when the rule $\forall_\mathsf{i}$ is applied in Π'. There is no (reasonable) way to treat the $\forall_\mathsf{i}$ case without adding any rule of some intermediate logic such as MH, see [25, in particular pp. 639-640].

Nevertheless, as already anticipated in the introduction, the problem concerning $\forall$ is a small limitation with respect to the great advantage of Seldin's strategy, consisting in obtaining Glivenko's theorem for intuitionistic logic as an immediate corollary (see [25, pp. 637-638])[13]: in a derivation where raa is postponed (possibly it contains several instances of efq that are not its last rule), it is sufficient to replace its last rule – a raa rule – by a $\neg$-introduction rule discharging the same assumptions.

2.3 Towards a unified approach

In this article we will focus on the postponement of raa: we aim to show that Seldin's result about the postponement of raa can be generalized in such a way that a Glivenko's theorem not only for intuitionistic but also for minimal logic can be obtained. In order to do this, we will proceed in two steps.

First, we will show that, with respect to Seldin's definition of classical detour, the use of the efq rule in the reduction steps can be limited only to the case where s corresponds to the $\rightarrow_\mathsf{i}$ rule. However, the reduction steps that we define for classical detours will not come out from a uniform reduction scheme – as in Seldin's original formulation – but they will come out from a mixing of techniques. More precisely, we can divide the definition of our reduction steps according to two main cases (see §4):

1. when the maximum formula A is obtained from a raa followed by an elimination rule s,

[12] Note that also von Plato and Siders' normalization strategy can be characterized in geometric rather than in syntactic terms: in order to establish when a proof is in normal form it is sufficient to look at the position of the major premisses of the elimination rules – namely, the fact of being in the position of assumptions – and not at their syntactical form.

[13] Note that the same result can be obtained using the reduction rules proposed by von Plato, that we mentioned at the end of §2.1.3 (see [17, pp. 87–88; pp. 142–143]).

(a) if A is the major premiss of s, we will follow the general scheme (5);

(b) if A is one of the minor premisses of s, we will introduce new specific reduction steps.

2. when the maximum formula A is obtained from a raa followed by an introduction rule, we will follow (with some emendations for the case of the implication) the reduction steps proposed by von Plato ([17, p. 85–86]; cf. also the end of §2.1.3, *supra*).

As it will become clear in §5 (see also Definition 1), our main concern is the "geometrical" character of the postponement of raa, while normalization of classical logic is only an indirect target. In this sense, we will see that it is not necessary to take into consideration the complexity of the formulas introduced by the raa rule. This will lead us to consider also other kinds of reduction steps involving instances of rules in which more than one of their premisses is obtained from raa. Multiple occurrences of raa has then to be considered – and reduced – at the same time, even if they create maximum formulas of greater complexity. Nevertheless, even by working with this new kind of reduction steps, we can eventually show that a (weak) normalization theorem for classical logic can be recovered.

Secondly, we will show how the reductions that we propose can shed light on a problem raised by Pereira [14] and consisting in determining the exact relations between normalization strategies for classical logic and negative translations. It is not difficult to see that Prawitz's original normalization strategy [22, pp.39-40] for the fragment $\{\neg, \wedge, \rightarrow, \bot, \forall\}$ of classical natural deduction induces a negative translation: it is sufficient to replace every atomic instance of raa present in a normal classical proof with a $\neg_{\mathsf{i}}$ in order to obtain a variant $(\cdot)^{\mathsf{g}}$ of Gentzen's translation, such that

$$
\begin{aligned}
(P(t_1, \ldots, t_n))^{\mathsf{g}} &= \neg\neg P(t_1, \ldots, t_n) & \qquad (\bot)^{\mathsf{g}} &= \bot \\
(A \wedge B)^{\mathsf{g}} &= A^{\mathsf{g}} \wedge B^{\mathsf{g}} & (A \rightarrow B)^{\mathsf{g}} &= A \rightarrow B^{\mathsf{g}} \\
(\neg A)^{\mathsf{g}} &= \neg A^{\mathsf{g}} & (\forall x A)^{\mathsf{g}} &= \forall x A^{\mathsf{g}}
\end{aligned}
$$

and where $\vee$ and $\exists$ are already translated, since they are defined via a combinations of the primitive connectives, i.e. $A \vee B := \neg(\neg A \wedge \neg B)$ and $\exists x A := \neg\forall x \neg A$. Note also that since A^{g} is minimally equivalent to $\neg\neg A^{\mathsf{g}}$, then $A \rightarrow B^{\mathsf{g}}$ is minimally equivalent to $A \rightarrow \neg\neg B^{\mathsf{g}}$; but $A \rightarrow \neg\neg B^{\mathsf{g}}$ is also minimally equivalent to $\neg(A \wedge B^{\mathsf{g}})$. Hence, just by using minimal logic steps this variant of Gentzen's negative translation can be transformed into a variant of Gödel's negative translation (cf. [14, p. 22]).[14]

In a similar way, we will show that Seldin's normalization strategy induces another kind of negative translation, namely (a variant of) Kuroda's one. However, a crucial difference exists between Gödel-Gentzen's translation and original Kuroda's translation: the former can embed classical logic into minimal logic, while the latter can not (see [6]). The parallel between Prawitz's and Seldin's strategies would then work only partially. Actually, we will show that this is a too harsh conclusion. Indeed, by a slight modification of the Kuroda's translation induced by our reduction steps, we obtain an embedding of full first-order classical logic into the fragment $\{\neg, \wedge, \vee, \bot, \exists\}$ of minimal logic. In particular, the (adequate) fragment $\{\neg, \wedge, \vee, \bot, \exists\}$ of first-order classical logic (where $\rightarrow$ and $\forall$ are defined using the other connectives) is embedded into minimal logic via this variant of Kuroda's translation simply by adding a double negation in front of formulas: we get in this way a Glivenko theorem for minimal logic.

In fact, in [26, pp. 203,216] Seldin already proved a form of Glivenko's theorem consisting in embedding the system TD^* into minimal logic. But TD^* is a weaker system than first-order classical logic, since it corresponds to the first-order minimal logic plus the rule of *consequentia mirabilis*

$$
\dfrac{\begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\, \pi \\ A \end{array}}{A}\ \mathsf{cm}^1
$$

[14] Even when the full language-fragment is considered, like in Stålmarck's [27] or in von Plato and Siders' [21] approaches, it is possible to detect the use of some kind of negative translations. See Appendix A.1 for more details.

$$\frac{\begin{array}{c}\vdots\\ \bot\end{array}}{A}\ \mathsf{efq} \qquad \frac{\begin{array}{c}\ulcorner\neg A\urcorner^{*}\\ \vdots\\ \bot\end{array}}{A}\ \mathsf{raa}^{*} \qquad \frac{}{\top}\ \top_{\mathsf{i}} \qquad \frac{\begin{array}{c}\ulcorner A\urcorner^{*}\\ \vdots\\ \bot\end{array}}{\neg A}\ \neg_{\mathsf{i}}^{*} \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ \neg A & A\end{array}}{\bot}\ \neg_{\mathsf{e}} \qquad \frac{\begin{array}{c}\ulcorner A\urcorner^{*}\\ \vdots\\ B\end{array}}{A \to B}\ \to_{\mathsf{i}}^{*} \qquad \frac{\begin{array}{cc}\vdots & \vdots\\ A \to B & A\end{array}}{B}\ \to_{\mathsf{e}}$$

$$\frac{\begin{array}{cc}\vdots & \vdots\\ A & B\end{array}}{A \wedge B}\ \wedge_{\mathsf{i}} \qquad \frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A}\ \wedge_{\mathsf{e}_1} \qquad \frac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{B}\ \wedge_{\mathsf{e}_2} \qquad \frac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\ \vee_{\mathsf{i}_1} \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B}\ \vee_{\mathsf{i}_2} \qquad \frac{\begin{array}{ccc} & \ulcorner A\urcorner^{*} & \ulcorner B\urcorner^{*}\\ \vdots & \vdots & \vdots\\ A \vee B & C & C\end{array}}{C}\ \vee_{\mathsf{e}}^{*}$$

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{\forall x A}\ \forall_{\mathsf{i}} \qquad \frac{\begin{array}{c}\vdots\\ \forall x A\end{array}}{A\{t/x\}}\ \forall_{\mathsf{e}} \qquad \frac{\begin{array}{c}\vdots\\ A\{t/x\}\end{array}}{\exists x A}\ \exists_{\mathsf{i}} \qquad \frac{\begin{array}{cc} & \ulcorner A\urcorner^{*}\\ \vdots & \vdots\ \pi\\ \exists x A & C\end{array}}{C}\ \exists_{\mathsf{e}}^{*}$$

Every formula occurrence in a derivation π that is not the conclusion of some instance of rule in π is an *assumption of* π: it may be *discharged* or *non-discharged* (in π). A discharged assumption A of a derivation π is denoted by $\ulcorner A\urcorner$ with a marker – here noted with a place-hoder $*$ for a numeral – for indicating the instance of rule in π that has discharged it. The rules entitled to discharge assumptions are raa, $\to_{\mathsf{i}}$, $\vee_{\mathsf{e}}$ and $\exists_{\mathsf{e}}$: any instance of these rules may discharge an arbitrary number (possibly 0) of assumptions. In the rule $\forall_{\mathsf{i}}$ the variable x is not free in the non-discharged assumptions. In the rule $\exists_{\mathsf{e}}$ the variable x is not free in C and in the non-discharged assumptions of π different from A.

Figure 1: Inference rules for first-order natural deduction (i stands for intro, e for elim).

and in this system neither raa nor efq are derivable (see [2] for details). Our result is thus more general.

It is worth noting that, differently from the algebraic demonstration given in [3], our demonstration makes use of purely proof-theoretic tools and is not restricted to the propositional fragment. But this does not mean that our result is the only proof-theoretic demonstration of the Glivenko's theorem for first-order minimal logic. A proof-theoretic demonstration of this theorem is also given by Tennant [29]. However, differently from our approach, he does not appeal to the postponement of raa for classical logic, but he translates each classical inference rule into a corresponding derivable rule in the fragment $\{\neg, \wedge, \vee, \bot, \exists\}$ of minimal logic. In this way, by induction on the length of a derivation, he can then transform a classical derivation into a derivation in minimal logic (for more details see Appendix A.2).

3 The syntax of first-order natural deduction

Let us first recall (quite informally) the language of first-order logic that we will use for our presentation.

Formulas are generated by the propositional connectives $\top$ (*truth*), $\bot$ (*falsehood*), $\neg$ (*negation*), $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\to$ (*implication*), and the quantifiers $\forall$ (*universal*), $\exists$ (*existential*), starting from an infinite set of individual variables (denoted by x, y, z, etc.) and, for any $n \in \mathbb{N}$, a set of n-ary function symbols and a set of n-ary predicate symbols (denoted by P, Q, R, etc.).[15] Terms are denoted by s, t, etc.; formulas are denoted by A, B, C, etc., in particular atomic formulas different from $\bot, \top$ are denoted by $P(t_1, \dots, t_n)$ where P is a n-ary predicate symbol ($n \in \mathbb{N}$). Sets of formulas are denoted by Γ, Δ, etc.

Formulas are identified up to renaming of bound variables. The capture-avoiding substitution of a term t for all the free occurrences of an individual variable x in a formula A is denoted by $A\{t/x\}$: it is implicitly assumed that all individual variables occurring in t are not bound in A (this condition can always be fulfilled by renaming the bound variables of A).

[15] In particular, for $n = 0$, we get a set of individual constants and a set of proposition symbols: therefore, we will consider propositional natural deduction as a subsystem of first-order natural deduction.

A *derivation system* in first-order natural deduction is the set of derivations that can be obtained from a given set of inference rules. In other words, a derivation system is identified with the set of its inference rules. The complete list of inference rules that we will consider for any derivation system in first-order natural deduction is in Figure 1. Given two derivation systems D and D', D is a *subsystem* of D' if $\mathsf{D} \subseteq \mathsf{D}'$; hence, any derivation in D is also a derivation in D'. The notions of derivation, conclusion and (major, minor) premisses of an instance of rule are taken for granted (see for example [22]).

Looking at the inference rules in Figure 1, observe that efq is nothing but the special case of raa where no assumption is discharged. This means that, in a derivation π, every instance of the rule efq is just an instance of the rule raa discharging no assumption. We can thus say that, in a derivation π, an instance of raa is *discharging* if it is not an instance of efq (i.e. it discharges at least one assumption).

Note that negation $\neg$ is here considered as primitive:[16] $\neg A$ will not be treated as a shorthand for $A \to \bot$, and the inferences rules $\neg_{\mathsf{i}}$ and $\neg_{\mathsf{e}}$ (see Figure 1) will be not special cases of $\to_{\mathsf{i}}$ and $\to_{\mathsf{e}}$, respectively. The reason is that, for our purposes (see in particular §6), it it turns out that the rules $\to_{\mathsf{i}}$ and $\neg_{\mathsf{i}}$ have different behavior.[17]

We say that the *first-order minimal natural deduction* is the derivation system $\mathsf{NM} = \{\top_{\mathsf{i}}, \neg_{\mathsf{i}}, \neg_{\mathsf{e}}, \wedge_{\mathsf{i}}, \wedge_{\mathsf{e}_1}, \wedge_{\mathsf{e}_2}, \vee_{\mathsf{i}_1}, \vee_{\mathsf{i}_2}, \vee_{\mathsf{e}}, \to_{\mathsf{i}}, \to_{\mathsf{e}}, \forall_{\mathsf{i}}, \forall_{\mathsf{e}}, \exists_{\mathsf{i}}, \exists_{\mathsf{e}}\}$ (i.e. in NM there are all the inference rules in Figure 1 except raa and efq); the *first-order intuitionistic natural deduction* is the derivation system $\mathsf{NJ} = \mathsf{NM} \cup \{\mathsf{efq}\}$; and the *first-order classical natural deduction* is the derivation system $\mathsf{NK} = \mathsf{NM} \cup \{\mathsf{raa}\}$. This means that NM is a subsystem of NJ, and that NM and NJ are both subsystems of NK. All derivation systems we will consider are subsystems of NK.

Notation. Let $\mathsf{D} \subseteq \mathsf{NK}$ be a derivation system.

(i) Derivations in D are denoted by $\Pi, \Pi', \dots$, or also $\pi, \pi', \dots$, possibly with subscripts.

(ii) Given a derivation π in D, we denote by RAA_π (resp. RAA_π^+) the set of instances (resp. discharging instances) of the rule raa in π.

(iii) Given a formula B, a set of formulas Γ, and a derivation π in D, we write $\pi\colon \Gamma \vdash_{\mathsf{D}} B$ (or simply $\pi\colon \Gamma \vdash B$ when no ambiguity arises) for indicating that the conclusion of π is B and the non-discharged assumptions of π are occurrences of some formulas in Γ; possibly, not all formulas in Γ occur as non-discharged assumptions in π.

(iv) If there exists a derivation $\pi\colon \Gamma \vdash_{\mathsf{D}} B$, we write $\Gamma \vdash_{\mathsf{D}} B$ and we say that *B is derivable from Γ in* D (or *$\Gamma \vdash B$ is derivable in* D); otherwise we write $\Gamma \nvdash_{\mathsf{D}} B$.

Clearly, $\mathsf{RAA}_\pi^+ \subseteq \mathsf{RAA}_\pi$ for every derivation π in $\mathsf{D} \subseteq \mathsf{NK}$.

Given an instance r of a rule in a derivation π, it is natural to define the notion of distance of r in π as the number of instances of rules in π between r and the last rule of π. More formally, we have that:

Definition 1 (Thread; distance of a rule; RAA-size of a proof; standard derivation)**.** *Let π be a derivation in $\mathsf{D} \subseteq \mathsf{NK}$.*

Given two formula occurrences A and B in π, a thread from A to B in π *is a sequence* $\mathsf{t} = (A_i)_{0 \leq i \leq n}$ *(with $n \in \mathbb{N}$) of formula occurrences in π such that $A_0 = A$, $A_n = B$ and, for any $0 \leq i < n$ there is an instance of rule in π having A_i as a premise and A_{i+1} as its conclusion; the* length of t *is n.*[18]

[16] This is also what is done by Andou [1] in his proof of (weak) normalization for classical logic (cf. [1, p. 152]). Taking $\neg$ as primitive is essential for him in order to define what he calls a *regular proof*, that is, a proof in which all the discharged assumptions of raa are major premisses of a $\neg_{\mathsf{e}}$ [1, p. 154]. We have not analyzed Andou's proposal in §2 since we should have considered not only the detour reductions, but also other kinds of transformations necessary for putting non-normal proofs in the regular form.

[17] In fact, taking $\neg$ as primitive is just a matter of convenience: all our results can be proved in a setting where $\neg A := A \to \bot$ and the rules $\neg_{\mathsf{i}}$ and $\neg_{\mathsf{e}}$ are special cases of $\to_{\mathsf{i}}$ and $\to_{\mathsf{e}}$. However, this requires to distinguish, for all implicative sentences $A \to B$, whether B is equal or different from $\bot$.

[18] Note that if $A = B$ (as formula occurrences in π) then the length of t is 0.

For every instance r *of a rule in* π*, the* distance of r *(*from the conclusion of π*), denoted by* $\mathsf{dist}_\pi(\mathsf{r})$*, is the length of the thread from the conclusion of* r *to the conclusion of* π*.*[19]

For every $\mathsf{r} \in \mathsf{RAA}_\pi$*, we say that* r *is* RAA_π-maximal *if* $\mathsf{dist}_\pi(\mathsf{r}) \geq \mathsf{dist}_\pi(\mathsf{r}')$ *for any* $\mathsf{r}' \in \mathsf{RAA}_\pi$*.*

For every $\mathsf{r} \in \mathsf{RAA}^+_\pi$*, we say that* r *is* RAA^+_π-maximal *if* $\mathsf{dist}_\pi(\mathsf{r}) \geq \mathsf{dist}_\pi(\mathsf{r}')$ *for any* $\mathsf{r}' \in \mathsf{RAA}^+_\pi$*.*

The RAA-size of π *is* $\mathsf{size}_{\mathsf{RAA}}(\pi) = \sum_{\mathsf{r}\in\mathsf{RAA}_\pi} \mathsf{dist}_\pi(\mathsf{r})$*.*

The RAA^+-size of π *is* $\mathsf{size}_{\mathsf{RAA}^+}(\pi) = \sum_{\mathsf{r}\in\mathsf{RAA}^+_\pi} \mathsf{dist}_\pi(\mathsf{r})$*.*

We say that π *is* m-standard *if in* π *there is at most one instance of the rule* raa*, and this instance, if any, is the last rule of* π*, the rest of* π *being a derivation in* NM*.*

We say that π *is* j-standard *if in* π *there is at most one discharging instance of the rule* raa*, and this instance, if any, is the last rule of* π*, the rest of* π *being a derivation in* NJ*.*

The notion of m-standard (resp. j-standard) derivations characterizes exactly the derivations in NK where the raa (resp. discharging raa) is postponed. These notions will be used in Theorem 8 and Corollary 9. A j-standard derivation might contains several instances of efq, which are not its last rule.

Note that, for any derivation π and any instance r of a rule in π, one has that $\mathsf{dist}_\pi(\mathsf{r}) \in \mathbb{N}$ and $\mathsf{size}_{\mathsf{RAA}}(\pi) \geq \mathsf{size}_{\mathsf{RAA}^+}(\pi) \in \mathbb{N}$. Moreover, $\mathsf{dist}_\pi(\mathsf{r}) = 0$ if and only if r is the last rule in π.

Since we are mainly interested in the postponement of raa instead of normalization, differently from the approaches discussed in §2, our definitions of RAA_π-maximal and RAA^+_π-maximal in a derivation π depend only on the distances of the instances of raa from the conclusion of π, without taking into account the complexity of the formulas occurring in the conclusions of these instances. In a way, as we will see in §5, our approach to prove the postponement of raa is purely "geometrical".

Remark 2. Let π be a derivation in $\mathsf{D} \subseteq \mathsf{NK}$:

1. $\mathsf{size}_{\mathsf{RAA}^+}(\pi) = 0$ if and only if π is j-standard;
2. $\mathsf{size}_{\mathsf{RAA}}(\pi) = 0$ if and only if π is m-standard.

Intuitively, in a derivation π in $\mathsf{D} \subseteq \mathsf{NK}$, an instance (resp. a discharging instance) r of the rule raa is RAA_π-maximal (resp. RAA^+_π-maximal) when there are no other instances (resp. discharging instances) of raa above r.

Since a derivation π in $\mathsf{D} \subseteq \mathsf{NK}$ can be seen as a finite tree, if $\mathsf{RAA}_\pi \neq \emptyset$ (resp. $\mathsf{RAA}^+_\pi \neq \emptyset$), i.e. if there is at least one instance (resp. discharging instance) of the rule raa in π, then there is a RAA_π-maximal (resp. RAA^+_π-maximal discharging) instance of the rule raa in π.

4 Reduction steps for the postponement of raa

We define reduction steps case by case, depending on the inference rule instantiated immediately below the instance of raa under focus (hence, there is no case with a 0-ary inference rule).

¬ introduction:

$$
\Pi = \begin{array}{c}
\ulcorner A\urcorner^{2}, \ulcorner\neg\bot\urcorner^{1}\\
\vdots\,\pi'\\
\dfrac{\dfrac{\bot}{\bot}\,\mathsf{raa}^{1}}{\neg A}\,\neg_{\mathsf{i}}^{2}\\
\vdots\,\pi
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}
\ulcorner A\urcorner^{2},\ \dfrac{\ulcorner\bot\urcorner^{1}}{\neg\bot}\,\neg_{\mathsf{i}}^{1}\\
\vdots\,\pi'\\
\dfrac{\bot}{\neg A}\,\neg_{\mathsf{i}}^{2}\\
\vdots\,\pi
\end{array}
= \Pi' \tag{12}
$$

[19]This notion is well-defined since a derivation is a tree (i.e. a rooted acyclic connected graph) whose nodes are formula occurrences. Hence, for every formula occurrence A in π, there exists exactly one thread from A to the conclusion of π.

¬ elimination:

$$\Pi = \dfrac{\dfrac{\begin{matrix}\ulcorner\neg\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{\neg A}\,\mathsf{raa}^1 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{\begin{matrix}\bot\\ \vdots\,\pi\end{matrix}}\,\neg_{\mathsf{e}} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner\neg A\urcorner^1 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{\bot}\,\neg_{\mathsf{e}}}{\neg\neg A}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi'\\ \bot\\ \vdots\,\pi\end{matrix} = \Pi' \tag{13a}$$

where the last rule of the derivation π'' is not an instance of the rule raa;

$$\Pi = \dfrac{\begin{matrix}\vdots\,\pi'\\ \neg A\end{matrix} \qquad \dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi''\\ \bot\end{matrix}}{A}\,\mathsf{raa}^1}{\begin{matrix}\bot\\ \vdots\,\pi\end{matrix}}\,\neg_{\mathsf{e}} \quad\rightsquigarrow\quad \begin{matrix}\vdots\,\pi'\\ \neg A\\ \vdots\,\pi'' = \Pi'\\ \bot\\ \vdots\,\pi\end{matrix} \tag{13b}$$

where the last rule of the derivation π' is not an instance of the rule raa;

$$\Pi = \dfrac{\dfrac{\begin{matrix}\ulcorner\neg\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{\neg A}\,\mathsf{raa}^1 \qquad \dfrac{\begin{matrix}\ulcorner\neg A\urcorner^2\\ \vdots\,\pi''\\ \bot\end{matrix}}{A}\,\mathsf{raa}^2}{\begin{matrix}\bot\\ \vdots\,\pi\end{matrix}}\,\neg_{\mathsf{e}} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\begin{matrix}\dfrac{\dfrac{\ulcorner\neg A\urcorner^2 \qquad \ulcorner A\urcorner^1}{\bot}\,\neg_{\mathsf{e}}}{\neg A}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi''\\ \bot\end{matrix}}{\neg\neg A}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \bot\\ \vdots\,\pi\end{matrix} = \Pi' \tag{13c}$$

∧ introduction:

$$\Pi = \dfrac{\dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A}\,\mathsf{raa}^1 \qquad \begin{matrix}\vdots\,\pi''\\ B\end{matrix}}{\begin{matrix}A\wedge B\\ \vdots\,\pi\end{matrix}}\,\wedge_{\mathsf{i}} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\begin{matrix}\dfrac{\dfrac{\ulcorner\neg(A\wedge B)\urcorner^2 \qquad \dfrac{\ulcorner A\urcorner^1 \qquad \begin{matrix}\vdots\,\pi''\\ B\end{matrix}}{A\wedge B}\,\wedge_{\mathsf{i}}}{\bot}\,\neg_{\mathsf{e}}}{\neg A}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A\wedge B}\,\mathsf{raa}^2\\ \vdots\,\pi\end{matrix} = \Pi' \tag{14a}$$

where the last rule of the derivation π'' is not an instance of the rule raa;

$$\Pi = \begin{array}{c}\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\end{array} \qquad \dfrac{\begin{array}{c}\ulcorner\neg B\urcorner^1\\ \vdots\,\pi''\\ \bot\end{array}}{B}\,\mathsf{raa}^1}{A\wedge B}\,\wedge_{\mathsf{i}}\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\dfrac{\ulcorner\neg(A\wedge B)\urcorner^2 \qquad \dfrac{\begin{array}{c}\vdots\,\pi'\\ A\end{array}\quad \ulcorner B\urcorner^1}{A\wedge B}\,\wedge_{\mathsf{i}}}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi''\\ \bot\end{array}}{A\wedge B}\,\mathsf{raa}^2\\ \vdots\,\pi\end{array} = \Pi' \tag{14b}$$

where the last rule of the derivation π' is not an instance of the rule raa;

$$\Pi = \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi'\\ \bot\end{array}}{A}\,\mathsf{raa}^1 \qquad \dfrac{\begin{array}{c}\ulcorner\neg B\urcorner^2\\ \vdots\,\pi''\\ \bot\end{array}}{B}\,\mathsf{raa}^2}{A\wedge B}\,\wedge_{\mathsf{i}}\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\dfrac{\dfrac{\ulcorner\neg(A\wedge B)\urcorner^3 \qquad \dfrac{\ulcorner A\urcorner^2\quad \ulcorner B\urcorner^1}{A\wedge B}\,\wedge_{\mathsf{i}}}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi''\\ \bot\end{array}}{\neg A}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \bot\end{array}}{A\wedge B}\,\mathsf{raa}^3\\ \vdots\,\pi\end{array} = \Pi' \tag{14c}$$

$\wedge$ **elimination:**

$$\Pi = \begin{array}{c}\ulcorner\neg(A\wedge B)\urcorner^1\\ \vdots\,\pi'\\ \dfrac{\dfrac{\bot}{A\wedge B}\,\mathsf{raa}^1}{A}\,\wedge_{\mathsf{e}_1}\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\dfrac{\ulcorner\neg A\urcorner^2 \qquad \dfrac{\ulcorner A\wedge B\urcorner^1}{A}\,\wedge_{\mathsf{e}_1}}{\bot}\,\neg_{\mathsf{e}}}{\neg(A\wedge B)}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi'\\ \dfrac{\bot}{A}\,\mathsf{raa}^2\\ \vdots\,\pi\end{array} = \Pi' \tag{15a}$$

$$\Pi = \begin{array}{c}\ulcorner\neg(A\wedge B)\urcorner^1\\ \vdots\,\pi'\\ \dfrac{\dfrac{\bot}{A\wedge B}\,\mathsf{raa}^1}{B}\,\wedge_{\mathsf{e}_2}\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\dfrac{\ulcorner\neg B\urcorner^2 \qquad \dfrac{\ulcorner A\wedge B\urcorner^1}{B}\,\wedge_{\mathsf{e}_2}}{\bot}\,\neg_{\mathsf{e}}}{\neg(A\wedge B)}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi'\\ \dfrac{\bot}{B}\,\mathsf{raa}^2\\ \vdots\,\pi\end{array} = \Pi' \tag{15b}$$

$\vee$ **introduction:**

$$\Pi = \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\ulcorner \neg A \urcorner^1\\ \vdots\,\pi' \\ \bot\end{array}}{A}\,\mathsf{raa}^1}{A\vee B}\,\vee_{\mathsf{i}_1} \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}\dfrac{\dfrac{\ulcorner\neg(A\vee B)\urcorner^2 \qquad \dfrac{\ulcorner A\urcorner^1}{A\vee B}\,\vee_{\mathsf{i}_1}}{\bot}\,\neg_{\mathsf{e}}}{\neg A}\,\neg_{\mathsf{i}}^1 \\ \vdots\,\pi' \\ \bot\end{array}}{A\vee B}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} = \Pi' \tag{16a}$$

$$\Pi = \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\ulcorner \neg B \urcorner^1\\ \vdots\,\pi' \\ \bot\end{array}}{B}\,\mathsf{raa}^1}{A\vee B}\,\vee_{\mathsf{i}_2} \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}\dfrac{\dfrac{\ulcorner\neg(A\vee B)\urcorner^2 \qquad \dfrac{\ulcorner B\urcorner^1}{A\vee B}\,\vee_{\mathsf{i}_2}}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^1 \\ \vdots\,\pi' \\ \bot\end{array}}{A\vee B}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} = \Pi' \tag{16b}$$

$\vee$ **elimination:**

$$\Pi = \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\ulcorner\neg(A\vee B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{array}}{A\vee B}\,\mathsf{raa}^1 \quad \begin{array}{c}\ulcorner A\urcorner^2\\ \vdots\,\pi''\\ C\end{array} \quad \begin{array}{c}\ulcorner B\urcorner^2\\ \vdots\,\pi'''\\ C\end{array}}{C}\,\vee_{\mathsf{e}}^2 \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\begin{array}{c}\dfrac{\dfrac{\ulcorner A\vee B\urcorner^2 \quad \dfrac{\ulcorner\neg C\urcorner^3 \quad \begin{array}{c}\ulcorner A\urcorner^1\\ \vdots\,\pi''\\ C\end{array}}{\bot}\,\neg_{\mathsf{e}} \quad \dfrac{\ulcorner\neg C\urcorner^3 \quad \begin{array}{c}\ulcorner B\urcorner^1\\ \vdots\,\pi'''\\ C\end{array}}{\bot}\,\neg_{\mathsf{e}}}{\bot}\,\vee_{\mathsf{e}}^1}{\neg(A\vee B)}\,\neg_{\mathsf{i}}^2 \\ \vdots\,\pi' \\ \bot\end{array}}{C}\,\mathsf{raa}^3 \\ \vdots\,\pi \end{array} = \Pi' \tag{17a}$$

where the last rule of the derivations π'' and π''' is not an instance of the rule raa;

$$\Pi = \begin{array}{c} \dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array} \quad \dfrac{\begin{array}{c}\ulcorner A\urcorner^2,\ulcorner\neg C\urcorner^1\\ \vdots\,\pi''\\ \bot\end{array}}{C}\,\mathsf{raa}^1 \quad \begin{array}{c}\ulcorner B\urcorner^2\\ \vdots\,\pi'''\\ C\end{array}}{C}\,\vee_{\mathsf{e}}^2 \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array} \quad \begin{array}{c}\ulcorner A\urcorner^1,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi''\\ \bot\end{array} \quad \dfrac{\ulcorner\neg C\urcorner^2 \quad \begin{array}{c}\ulcorner B\urcorner^1\\ \vdots\,\pi'''\\ C\end{array}}{\bot}\,\neg_{\mathsf{e}}}{\bot}\,\vee_{\mathsf{e}}^1}{C}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} = \Pi' \tag{17b}$$

where the last rule of the derivations π' and π''' is not an instance of the rule raa;

$$\Pi = \begin{array}{c}\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array}\quad \begin{array}{c}\ulcorner A\urcorner^2\\ \vdots\,\pi''\\ C\end{array}\quad \dfrac{\begin{array}{c}\ulcorner B\urcorner^2,\ulcorner\neg C\urcorner^1\\ \vdots\,\pi'''\\ \bot\end{array}}{C}\,\mathsf{raa}^1}{C}\,\vee_{\mathsf{e}}^2\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array}\quad \dfrac{\ulcorner\neg C\urcorner^2\quad \begin{array}{c}\ulcorner A\urcorner^1\\ \vdots\,\pi''\\ C\end{array}}{\bot}\,\neg_{\mathsf{e}}\quad \begin{array}{c}\ulcorner B\urcorner^1,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi'''\\ \bot\end{array}}{\bot}\,\vee_{\mathsf{e}}^1}{C}\,\mathsf{raa}^2\\ \vdots\,\pi\end{array} = \Pi' \tag{17c}$$

where the last rule of the derivations π' and π'' is not an instance of the rule raa;

$$\Pi = \begin{array}{c}\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array}\quad \dfrac{\begin{array}{c}\ulcorner A\urcorner^3,\ulcorner\neg C\urcorner^1\\ \vdots\,\pi''\\ \bot\end{array}}{C}\,\mathsf{raa}^1\quad \dfrac{\begin{array}{c}\ulcorner B\urcorner^3,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi'''\\ \bot\end{array}}{C}\,\mathsf{raa}^2}{C}\,\vee_{\mathsf{e}}^3\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi'\\ A\vee B\end{array}\quad \begin{array}{c}\ulcorner A\urcorner^1,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi''\\ \bot\end{array}\quad \begin{array}{c}\ulcorner B\urcorner^1,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi'''\\ \bot\end{array}}{\bot}\,\vee_{\mathsf{e}}^1}{C}\,\mathsf{raa}^2\\ \vdots\,\pi\end{array} = \Pi' \tag{17d}$$

where the last rule of the derivation π' is not an instance of the rule raa;

$$\Pi = \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\ulcorner\neg(A\vee B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{array}}{A\vee B}\,\mathsf{raa}^1\quad \dfrac{\begin{array}{c}\ulcorner A\urcorner^3,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi''\\ \bot\end{array}}{C}\,\mathsf{raa}^2\quad \begin{array}{c}\ulcorner B\urcorner^3\\ \vdots\,\pi'''\\ C\end{array}}{C}\,\vee_{\mathsf{e}}^3\\ \vdots\,\pi\end{array} \quad\rightsquigarrow\quad \begin{array}{c}\dfrac{\dfrac{\ulcorner A\vee B\urcorner^2\quad \begin{array}{c}\ulcorner A\urcorner^1,\ulcorner\neg C\urcorner^3\\ \vdots\,\pi''\\ \bot\end{array}\quad \dfrac{\ulcorner\neg C\urcorner^3\quad \begin{array}{c}\ulcorner B\urcorner^1\\ \vdots\,\pi'''\\ C\end{array}}{\bot}\,\neg_{\mathsf{e}}}{\bot}\,\vee_{\mathsf{e}}^1}{\neg(A\vee B)}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \dfrac{\bot}{C}\,\mathsf{raa}^3\\ \vdots\,\pi\end{array} = \Pi' \tag{17e}$$

where the last rule of the derivation π''' is not an instance of the rule raa;

$$\Pi = \quad \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(A\lor B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A\lor B}\,\mathsf{raa}^1 \qquad \begin{matrix}\ulcorner A\urcorner^3\\ \vdots\,\pi''\\ C\end{matrix} \qquad \dfrac{\begin{matrix}\ulcorner B\urcorner^3,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi'''\\ \bot\end{matrix}}{C}\,\mathsf{raa}^2}{\begin{matrix}C\\ \vdots\,\pi\end{matrix}}\,\lor_{\mathsf{e}}^3 \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner A\lor B\urcorner^2 \qquad \dfrac{\ulcorner\neg C\urcorner^3 \qquad \begin{matrix}\ulcorner A\urcorner^1\\ \vdots\,\pi''\\ C\end{matrix}}{\bot}\,\neg_{\mathsf{e}} \qquad \begin{matrix}\ulcorner B\urcorner^1,\ulcorner\neg C\urcorner^3\\ \vdots\,\pi'''\\ \bot\end{matrix}}{\bot}\,\lor_{\mathsf{e}}^1}{\neg(A\lor B)}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \dfrac{\bot}{C}\,\mathsf{raa}^3\\ \vdots\,\pi\end{matrix} \quad = \Pi' \tag{17f}$$

where the last rule of the derivation π'' is not an instance of the rule raa;

$$\Pi = \quad \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(A\lor B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A\lor B}\,\mathsf{raa}^1 \qquad \dfrac{\begin{matrix}\ulcorner A\urcorner^4,\ulcorner\neg C\urcorner^2\\ \vdots\,\pi''\\ \bot\end{matrix}}{C}\,\mathsf{raa}^2 \qquad \dfrac{\begin{matrix}\ulcorner B\urcorner^4,\ulcorner\neg C\urcorner^3\\ \vdots\,\pi'''\\ \bot\end{matrix}}{C}\,\mathsf{raa}^3}{\begin{matrix}C\\ \vdots\,\pi\end{matrix}}\,\lor_{\mathsf{e}}^4 \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner A\lor B\urcorner^2 \qquad \begin{matrix}\ulcorner A\urcorner^1,\ulcorner\neg C\urcorner^3\\ \vdots\,\pi''\\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner B\urcorner^1,\ulcorner\neg C\urcorner^3\\ \vdots\,\pi'''\\ \bot\end{matrix}}{\bot}\,\lor_{\mathsf{e}}^1}{\neg(A\lor B)}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \dfrac{\bot}{C}\,\mathsf{raa}^3\\ \vdots\,\pi\end{matrix} \quad = \Pi' \tag{17g}$$

$\rightarrow$ introduction:

$$\Pi = \quad \begin{matrix}\dfrac{\dfrac{\begin{matrix}\ulcorner A\urcorner^2,\ulcorner\neg B\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{B}\,\mathsf{raa}^1}{A\to B}\,\to_{\mathsf{i}}^2\\ \vdots\,\pi\end{matrix} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner\neg(A\to B)\urcorner^3 \qquad \dfrac{\dfrac{\begin{matrix}\ulcorner A\urcorner^2, \quad \dfrac{\dfrac{\ulcorner\neg(A\to B)\urcorner^3 \qquad \dfrac{\ulcorner B\urcorner^1}{A\to B}\,\to_{\mathsf{i}}^0}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{B}\,\mathsf{efq}}{A\to B}\,\to_{\mathsf{i}}^2}{\bot}\,\neg_{\mathsf{e}}}{A\to B}\,\mathsf{raa}^3\\ \vdots\,\pi\end{matrix} \quad = \Pi' \tag{18}$$

→ elimination:

$$\Pi = \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(A\to B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A\to B}\,\mathsf{raa}^1 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{B}\,\to_{\mathsf{e}} \quad \begin{matrix}\\ \vdots\,\pi\end{matrix} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\dfrac{\ulcorner\neg B\urcorner^2 \qquad \dfrac{\ulcorner A\to B\urcorner^1 \qquad \begin{matrix}\vdots\,\pi''\\ A\end{matrix}}{B}\,\to_{\mathsf{e}}}{\bot}\,\neg_{\mathsf{e}}}{\neg(A\to B)}\,\neg_{\mathsf{i}}^1}{}\\ \vdots\,\pi'\\ \dfrac{\bot}{B}\,\mathsf{raa}^2\\ \vdots\,\pi\end{matrix} = \Pi' \tag{19a}$$

where the last rule of the derivation π'' is not an instance of the rule raa;

$$\Pi = \dfrac{\begin{matrix}\vdots\,\pi'\\ A\to B\end{matrix} \qquad \dfrac{\begin{matrix}\ulcorner\neg A\urcorner^1\\ \vdots\,\pi''\\ \bot\end{matrix}}{A}\,\mathsf{raa}^1}{B}\,\to_{\mathsf{e}} \quad \begin{matrix}\\ \vdots\,\pi\end{matrix} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner\neg B\urcorner^2 \qquad \dfrac{\begin{matrix}\vdots\,\pi'\\ A\to B\end{matrix} \qquad \ulcorner A\urcorner^1}{B}\,\to_{\mathsf{e}}}{\bot}\,\neg_{\mathsf{e}}}{\neg A}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi''\\ \dfrac{\bot}{B}\,\mathsf{raa}^2\\ \vdots\,\pi\end{matrix} = \Pi' \tag{19b}$$

where the last rule of the derivation π' is not an instance of the rule raa;

$$\Pi = \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(A\to B)\urcorner^1\\ \vdots\,\pi'\\ \bot\end{matrix}}{A\to B}\,\mathsf{raa}^1 \qquad \dfrac{\begin{matrix}\ulcorner\neg A\urcorner^2\\ \vdots\,\pi''\\ \bot\end{matrix}}{A}\,\mathsf{raa}^2}{B}\,\to_{\mathsf{e}} \quad \begin{matrix}\\ \vdots\,\pi\end{matrix} \quad\rightsquigarrow\quad \begin{matrix}\dfrac{\dfrac{\ulcorner\neg B\urcorner^3 \qquad \dfrac{\ulcorner A\to B\urcorner^2 \qquad \ulcorner A\urcorner^1}{B}\,\to_{\mathsf{e}}}{\bot}\,\neg_{\mathsf{e}}}{\neg A}\,\neg_{\mathsf{i}}^1\\ \vdots\,\pi''\\ \dfrac{\bot}{\neg(A\to B)}\,\neg_{\mathsf{i}}^2\\ \vdots\,\pi'\\ \dfrac{\bot}{B}\,\mathsf{raa}^3\\ \vdots\,\pi\end{matrix} = \Pi' \tag{19c}$$

∃ introduction:

$$\Pi = \begin{array}{c} \ulcorner\neg A\{t/x\}\urcorner^{1} \\ \vdots\ \pi' \\ \dfrac{\dfrac{\bot}{A\{t/x\}}\ \mathsf{raa}^{1}}{\exists x A}\ \exists_{\mathsf{i}} \\ \vdots\ \pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\ulcorner\neg\exists x A\urcorner^{2} \quad \dfrac{\ulcorner A\{t/x\}\urcorner^{1}}{\exists x A}\ \exists_{\mathsf{i}}}{\bot}\ \neg_{\mathsf{e}}}{\neg A\{t/x\}}\ \neg_{\mathsf{i}}^{1} \\ \vdots\ \pi' \\ \dfrac{\bot}{\exists x A}\ \mathsf{raa}^{2} \\ \vdots\ \pi \end{array} = \Pi' \tag{20}$$

∃ elimination:

$$\Pi = \begin{array}{c} \dfrac{\begin{array}{c}\ulcorner\neg\exists x A\urcorner^{1} \\ \vdots\ \pi' \\ \dfrac{\bot}{\exists x A}\ \mathsf{raa}^{1}\end{array} \quad \begin{array}{c}\ulcorner A\urcorner^{2} \\ \vdots\ \pi'' \\ C\end{array}}{C}\ \exists_{\mathsf{e}}^{2} \\ \vdots\ \pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\ulcorner\exists x A\urcorner^{2} \quad \dfrac{\ulcorner\neg C\urcorner^{3} \quad \begin{array}{c}\ulcorner A\urcorner^{1} \\ \vdots\ \pi'' \\ C\end{array}}{\bot}\ \neg_{\mathsf{e}}}{\bot}\ \exists_{\mathsf{e}}^{1}}{\neg\exists x A}\ \neg_{\mathsf{i}}^{2} \\ \vdots\ \pi' \\ \dfrac{\bot}{C}\ \mathsf{raa}^{3} \\ \vdots\ \pi \end{array} = \Pi' \tag{21a}$$

where the last rule of the derivation π'' is not an instance of the rule raa (notice that the variable x does not occur free in C and hence even in $\neg C$, thus the rule $\exists_{\mathsf{e}}$ is correctly instantiated in Π');

$$\Pi = \begin{array}{c} \dfrac{\begin{array}{c}\vdots\ \pi' \\ \exists x A\end{array} \quad \begin{array}{c}\ulcorner\neg C\urcorner^{1}, \ulcorner A\urcorner^{2} \\ \vdots\ \pi'' \\ \dfrac{\bot}{C}\ \mathsf{raa}^{1}\end{array}}{C}\ \exists_{\mathsf{e}}^{2} \\ \vdots\ \pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots\ \pi' \\ \exists x A\end{array} \quad \begin{array}{c}\ulcorner\neg C\urcorner^{2}, \ulcorner A\urcorner^{1} \\ \vdots\ \pi'' \\ \bot\end{array}}{\bot}\ \exists_{\mathsf{e}}^{1}}{C}\ \mathsf{raa}^{2} \\ \vdots\ \pi \end{array} = \Pi' \tag{21b}$$

where the last rule of the derivation π' is not an instance of the rule raa (notice that the variable x does not occur free in C and hence even in $\neg C$, thus the rule $\exists_{\mathsf{e}}$ is correctly instantiated in Π');

$$\Pi = \begin{array}{c} \dfrac{\begin{array}{c}\ulcorner\neg\exists x A\urcorner^{1} \\ \vdots\ \pi' \\ \dfrac{\bot}{\exists x A}\ \mathsf{raa}^{1}\end{array} \quad \begin{array}{c}\ulcorner\neg C\urcorner^{2}, \ulcorner A\urcorner^{3} \\ \vdots\ \pi'' \\ \dfrac{\bot}{C}\ \mathsf{raa}^{2}\end{array}}{C}\ \exists_{\mathsf{e}}^{3} \\ \vdots\ \pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\ulcorner\exists x A\urcorner^{2} \quad \begin{array}{c}\ulcorner\neg C\urcorner^{3}, \ulcorner A\urcorner^{1} \\ \vdots\ \pi'' \\ \bot\end{array}}{\bot}\ \exists_{\mathsf{e}}^{1}}{\neg\exists x A}\ \neg_{\mathsf{i}}^{2} \\ \vdots\ \pi' \\ \dfrac{\bot}{C}\ \mathsf{raa}^{3} \\ \vdots\ \pi \end{array} = \Pi' \tag{21c}$$

(notice that the variable x does not occur free in C and hence even in $\neg C$, thus the rule $\exists_{\mathsf{e}}$ is correctly instantiated in Π');

$\forall$ **elimination:**

$$\Pi = \begin{array}{c} \ulcorner\neg\forall x A\urcorner^1 \\ \vdots\,\pi' \\ \dfrac{\dfrac{\bot}{\forall x A}\,\mathsf{raa}^1}{A\{t/x\}}\,\forall_{\mathsf{e}} \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\ulcorner\neg A\{t/x\}\urcorner^2 \qquad \dfrac{\ulcorner\forall x A\urcorner^1}{A\{t/x\}}\,\forall_{\mathsf{e}}}{\bot}\,\neg_{\mathsf{e}}}{\neg\forall x A}\,\neg_{\mathsf{i}}^1 \\ \vdots\,\pi' \\ \dfrac{\bot}{A\{t/x\}}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} = \Pi' \tag{22}$$

efq **and** raa**:**

$$\Pi = \begin{array}{c} \ulcorner\neg\bot\urcorner^1 \\ \vdots\,\pi' \\ \dfrac{\dfrac{\bot}{\bot}\,\mathsf{raa}^1}{B}\,\mathsf{efq} \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\ulcorner\bot\urcorner^1}{\neg\bot}\,\neg_{\mathsf{i}}^1 \\ \vdots\,\pi' \\ \dfrac{\bot}{B}\,\mathsf{efq} \\ \vdots\,\pi \end{array} = \Pi' \tag{23a}$$

$$\Pi = \begin{array}{c} \ulcorner\neg B\urcorner^2, \ulcorner\neg\bot\urcorner^1 \\ \vdots\,\pi' \\ \dfrac{\dfrac{\bot}{\bot}\,\mathsf{raa}^1}{B}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \ulcorner\neg B\urcorner^2, \dfrac{\ulcorner\bot\urcorner^1}{\neg\bot}\,\neg_{\mathsf{i}}^1 \\ \vdots\,\pi' \\ \dfrac{\bot}{B}\,\mathsf{raa}^2 \\ \vdots\,\pi \end{array} = \Pi' \tag{23b}$$

Notation. For all derivations Π and Π' in NK, we write $\Pi \rightsquigarrow \Pi'$ if Π' is obtained from Π by applying one of the reduction steps listed above. The reflexive-transitive closure of $\rightsquigarrow$ is denoted by $\rightsquigarrow^*$.

Given $\Pi \rightsquigarrow \Pi'$, we say that each instance of raa in Π that is explicitly represented in the left-hand side of any reduction step listed above – with the exception of the reduction step (23b) – is *active*. Concerning the reduction step (23b) (i.e. the raa case), if r_1 and r_2 are the two instances of raa in Π that are explicitly represented in the left-hand side of the reduction step, and if r_1 (resp. r_2) is the instance whose conclusion is $\bot$ (resp. B), then only r_1 is *active*.

If $\Pi \rightsquigarrow \Pi'$ and $\mathsf{r}_1, \dots, \mathsf{r}_n$ are the active instances of raa in Π, we will write $\Pi \overset{\mathsf{r}_1,\dots,\mathsf{r}_n}{\rightsquigarrow} \Pi'$. According to the reduction rules listed above, $n \in \{1, 2, 3\}$.

It is easy to check that, for each reduction step listed above, there is at least one active instance of the rule raa in Π. These reduction steps might involve some non-local modifications over derivations. For example, when $\Pi \rightsquigarrow \Pi'$, a sub-derivation of Π might be erased or duplicated in Π', depending on the number of assumptions that are discharged by the active instances of raa in $\Pi \rightsquigarrow \Pi'$. Moreover, some sub-derivations of Π can be moved in Π' above some other sub-derivations of Π (this corresponds to an operation of proof composition).

When $\Pi \rightsquigarrow \Pi'$, there is no reduction step which introduces in Π' a new discharging instance of raa: any discharging instance of raa in Π' can thus be seen as a "residual" of an instance of raa in Π (possibly non-discharging or applied to another formula). However, it is not true that any instance of raa in Π has a residual in Π'; for example in the reduction steps (13c), (23a) or (23b) the active instance of raa in Π vanishes in Π'.

Remark 3. The case where $\Pi \rightsquigarrow \Pi'$ by applying the reduction step (18) (i.e. the $\to_\mathsf{i}$ case) is the only one introducing in Π' a new instance of efq: the instance of efq in Π' explicitly represented in the right-hand side of the reduction step (18) is not a residual of any instance of efq in Π. In a way, it is impossible to avoid adding an instance of efq in the $\to_\mathsf{i}$ case: this is deeply related to the fact that $(A \to \neg\neg B) \to \neg\neg(A \to B)$ is provable in NJ but not in NM. Indeed, if it were possible to define the following reduction step (where in the sub-derivations π and π' there is no instance of raa, and the formulas occurring in the non-discharged assumptions of Π' are a subset the formulas occurring the non-discharged assumptions of Π)

$$\Pi = \dfrac{\dfrac{\begin{array}{c}\ulcorner A \urcorner^2, \ulcorner \neg B \urcorner^1 \\ \vdots\, \pi \\ \bot\end{array}}{B}\,\mathsf{raa}^1}{A \to B}\,\to_\mathsf{i}^2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\ulcorner \neg(A \to B) \urcorner^2 \qquad \dfrac{\begin{array}{c}\ulcorner A \urcorner^1 \\ \vdots\, \pi' \\ B\end{array}}{A \to B}\,\to_\mathsf{i}^1}{\bot}\,\neg_\mathsf{e}}{A \to B}\,\mathsf{raa}^2 = \Pi'$$

then, by replacing the instances of raa in Π and Π' by instances of $\neg_\mathsf{i}$, the conclusions of Π and Π' would be $A \to \neg\neg B$ and $\neg\neg(A \to B)$, respectively: this would mean that the derivability of $A \to \neg\neg B$ in NM would imply the derivability of $\neg\neg(A \to B)$ in NM, which is impossible.

Remark 4. It is easy to check that if Π is a derivation in NK and $\Pi \rightsquigarrow \Pi'$, then Π' is a derivation in NK and Π is not a derivation in NM (Π contains at least one instance of raa, discharging or not discharging). In the reduction steps listed above, the $\forall_\mathsf{i}$ case (where in Π an instance of the rule $\forall_\mathsf{i}$ is immediately below the instance of raa under focus) is absent, otherwise Π' could not be a derivation in NK, as we have pointed out in §2.2. In other words, if $\Pi \overset{\mathsf{r}}{\rightsquigarrow} \Pi'$ then r is not an instance of raa in Π whose conclusion is the premise of an instance of $\forall_\mathsf{i}$.

Remark 5. The reduction steps (13c), (14c) and (19c) possess a certain degree of arbitrariness since they could also be defined so that, in Π', the sub-derivation π' would be put above π'', and not vice-versa.

Observe the similarities between the reduction steps in the $\wedge_\mathsf{i}$ and $\to_\mathsf{e}$ cases, or in the $\wedge_\mathsf{e}$, $\vee_\mathsf{i}$ and $\forall_\mathsf{e}$ cases, or in the $\vee_\mathsf{e}$ and $\exists_\mathsf{e}$ cases. On the contrary, the reduction steps for the $\neg_\mathsf{i}$ and $\to_\mathsf{i}$ cases (resp. $\neg_\mathsf{e}$ and $\to_\mathsf{e}$ cases) are rather different: the former – (12) (resp. (13)) – erases an instance of raa, whereas the latter – (18) (resp. (19)) – postpones an instance of raa after an instance of $\to_\mathsf{i}$ (resp. $\to_\mathsf{e}$), moreover in the $\to_\mathsf{i}$ case the reduction step introduces a new instance of efq, unlike the $\neg_\mathsf{i}$ case. This different behavior justifies our choice to consider $\neg$ as primitive and not to treat $\neg A$ as a shorthand for $A \to \bot$.

5 Postponement of raa

In this section we prove the first main result of this paper: the postponement of raa (Theorem 8, Corollary 9), i.e. the fact that it is possible to transform a derivation in NK in such a way that any possible instance of raa is pushed downward until it vanishes or it occurs only in the last rule, preserving the same conclusion and without adding any new non-discharged assumptions. More precisely, we show that, by repeated applications of the reduction steps of §4 following a suitable strategy:

- a derivation π in NK without the rule $\forall_\mathsf{i}$ reduces all its instances of raa to at most one instance of raa occurring as the last rule, the rest of the derivation being in NJ (Theorem 8.1);
- a derivation π in NK without the rules $\forall_\mathsf{i}$ and $\to_\mathsf{i}$ reduces all its instances of raa to at most one instance of raa occurring as the last rule, the rest of the derivation being in NM (Theorem 8.2).

This result will be then reformulated by focusing on the form of the conclusion and of the non-discharged assumptions of π, rather than on the kind of inference rules used in π (Corollary 9).

Two lemmas are used in the proof of Theorem 8. The first one (Lemma 6) says that when the reduction steps of §4 are applied, no new non-discharged assumptions are added in the reduced derivation, and the conclusion of the original derivation is preserved. Moreover, the reduction steps neither introduce nor erase any instance of the rules $\to_\mathsf{i}$ and $\forall_\mathsf{i}$. The second one (Lemma 7) says that the $\mathsf{size}_\mathsf{RAA}$ (resp. $\mathsf{size}^+_\mathsf{RAA}$) of a derivation strictly decreases when one applies a reduction rule whose active instance of raa is a RAA-maximal (resp. RAA^+-maximal) instance.

Lemma 6 (Preservations)**.** *Let Π and Π' be derivations in* NK *such that $\Pi \rightsquigarrow \Pi'$.*

1. *If $\Pi\colon \Gamma \vdash A$, then $\Pi'\colon \Gamma \vdash A$;*
2. *If Π has no instance of the rule $\to_\mathsf{i}$ (resp. $\forall_\mathsf{i}$), then Π' has no instance of the rule $\to_\mathsf{i}$ (resp. $\forall_\mathsf{i}$).*

Proof. By straightforward inspection of all the reduction steps listed in §4. ☐

The converse of Lemma 6.1 does not hold, as shown by the following counterexample (taking $\Gamma = \{\neg P, P\}$ and $A = Q \land R$, where P, Q and R are distinct proposition symbols):

$$\Pi = \frac{\dfrac{\dfrac{\neg P \qquad P}{\bot}\,\neg_\mathsf{e}}{Q}\,\mathsf{efq} \qquad R}{Q \land R}\,\land_\mathsf{i} \qquad \rightsquigarrow \qquad \frac{\dfrac{\neg P \qquad P}{\bot}\,\neg_\mathsf{e}}{Q \land R}\,\mathsf{efq} = \Pi' \tag{24}$$

Hence, in (24) we have $\Pi' : \Gamma \vdash A$ but we do not have $\Pi : \Gamma \vdash A$, since the set of non-discharged assumptions of Π also contains (an occurrence of) the formula R (see point (iii) about notations at p. 15). In other words, Lemma 6.1 says that if $\Pi \rightsquigarrow \Pi'$ then the formulas occurring among the non-discharged assumptions of Π' are a *subset* of the formulas occurring among the non-discharged assumptions of Π.

Derivability in NJ is not preserved by the reduction steps of §4: the fact that Π is a derivation in NJ and $\Pi \rightsquigarrow \Pi'$ does not always imply that Π' is a derivation in NJ, because an instance of the rule efq in Π can be transformed into a discharging instance of the rule raa in Π'. Consider, for example, the following situation:

$$\Pi = \frac{\neg P \lor Q \qquad \dfrac{\dfrac{\ulcorner\neg P\urcorner^1 \qquad P}{\bot}\,\neg_\mathsf{e}}{Q}\,\mathsf{efq} \qquad \ulcorner Q\urcorner^1}{Q}\,\lor^1_\mathsf{e} \qquad \rightsquigarrow \qquad \frac{\dfrac{\neg P \lor Q \qquad \dfrac{\ulcorner\neg P\urcorner^1 \qquad P}{\bot}\,\neg_\mathsf{e} \qquad \dfrac{\ulcorner\neg Q\urcorner^2 \qquad \ulcorner Q\urcorner^1}{\bot}\,\neg_\mathsf{e}}{\bot}\,\lor^1_\mathsf{e}}{Q}\,\mathsf{raa}^2 = \Pi'.$$

Lemma 7 (Size decreasing)**.** *Let Π and Π' be derivations in* NK *such that $\Pi \overset{\mathsf{r}_1,\dots,\mathsf{r}_n}{\rightsquigarrow} \Pi'$ where $n \in \mathbb{N}^+$.*

1. *If r_i is RAA_Π-maximal (resp. RAA^+_Π-maximal) for some $1 \le i \le n$, then r_j is RAA_Π-maximal (resp. RAA^+_Π-maximal) for all $1 \le j \le n$.*
2. *If Π contains no instance of the rule $\forall_\mathsf{i}$, and if r_j is RAA^+_Π-maximal for some $1 \le j \le n$, then $\mathsf{size}_{\mathsf{RAA}^+}(\Pi') < \mathsf{size}_{\mathsf{RAA}^+}(\Pi)$.*

3. *If Π contains no instance of the rule $\to_{\mathsf{i}}$ and no instance of the rule $\forall_{\mathsf{i}}$, and if r_j is RAA_Π-maximal for some $1 \leq j \leq n$, then $\mathsf{size}_{\mathsf{RAA}}(\Pi') < \mathsf{size}_{\mathsf{RAA}}(\Pi)$.*

Proof. By straightforward inspection of all the reduction steps listed in §4. The hypothesis of maximality for the active instances of raa is crucial: it ensures that when a sub-derivation is moved above another sub-derivation according to the reduction step, no instance of raa is moved away from the conclusion of the derivation. □

Note that, in the proof of Lemma 7, the complexity of the formulas occurring in the conclusions of the instances of raa play no role (see Definition 1): the fact that the sizes $\mathsf{size}_{\mathsf{RAA}}$ and $\mathsf{size}_{\mathsf{RAA}^+}$ decrease applying the reduction steps listed in §4 is purely "geometrical", due to the decrease of the distance of a maximal raa from the conclusion of the derivation.

Lemma 7.3 becomes false if Π contains an instance of the rule $\to_{\mathsf{i}}$, as shown by the following counterexample (see also Remark 3):

$$\Pi = \dfrac{\dfrac{\dfrac{\ulcorner P \urcorner^2 \qquad \ulcorner \neg P \urcorner^1}{\bot}\,\neg_{\mathsf{e}}}{P}\,\mathsf{raa}^1}{P \to P}\,\to_{\mathsf{i}}^2 \quad \rightsquigarrow \quad \dfrac{\ulcorner \neg(P \to P) \urcorner^3 \qquad \dfrac{\dfrac{\dfrac{\ulcorner P \urcorner^2 \qquad \dfrac{\dfrac{\ulcorner \neg(P \to P) \urcorner^3 \qquad \dfrac{\ulcorner P \urcorner^1}{P \to P}\,\to_{\mathsf{i}}^0}{\bot}\,\neg_{\mathsf{e}}}{\neg P}\,\neg_{\mathsf{i}}^1}{\bot}\,\neg_{\mathsf{e}}}{P}\,\mathsf{efq}}{P \to P}\,\to_{\mathsf{i}}^2}{\dfrac{\bot}{P \to P}\,\mathsf{raa}^3}\,\neg_{\mathsf{e}} \quad = \Pi'$$

where $\mathsf{size}_{\mathsf{RAA}}(\Pi) = 1 < 3 = \mathsf{size}_{\mathsf{RAA}}(\Pi')$, since the instance of efq in Π' belongs to $\mathsf{RAA}_{\Pi'}$.

Theorem 8 (Postponement of raa, version 1)**.** *Let $\Pi \colon \Gamma \vdash A$ be a derivation in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$.*

1. *There exists some j-standard derivation $\Pi' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$ such that $\Pi \rightsquigarrow^* \Pi'$.*
2. *If Π contains no instance of the rule $\to_{\mathsf{i}}$, then there exists some m-standard derivation $\Pi' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ such that $\Pi \rightsquigarrow^* \Pi'$.*

Proof.

1. By induction on $\mathsf{size}_{\mathsf{RAA}^+}(\Pi) \in \mathbb{N}$.

   If $\mathsf{size}_{\mathsf{RAA}^+}(\Pi) = 0$, then just take $\Pi' = \Pi$, according to Remark 2.1.

   Otherwise, $\mathsf{size}_{\mathsf{RAA}^+}(\Pi) > 0$ and there exists $\mathsf{r} \in \mathsf{RAA}^+_\Pi$ which is RAA^+_Π-maximal and which is not the last rule of Π. Since there is no instance of the rule $\forall_{\mathsf{i}}$ in Π, there necessarily exists a Π' such that $\Pi \overset{\mathsf{r}_1,\dots,\mathsf{r}_n}{\rightsquigarrow} \Pi'$ where $n \in \mathbb{N}^+$ and $\mathsf{r} = \mathsf{r}_j$ for some $1 \leq j \leq n$. According to Lemma 7.1, all $\mathsf{r}_1, \dots, \mathsf{r}_n$ are RAA^+_Π-maximal. By Lemma 6.1, $\Pi' \colon \Gamma \vdash A$, and, by Lemma 6.2, Π' has no instance of the rule $\forall_{\mathsf{i}}$. According to Lemma 7.2, $\mathsf{size}_{\mathsf{RAA}^+}(\Pi') < \mathsf{size}_{\mathsf{RAA}^+}(\Pi)$. Hence, by the induction hypothesis, there is a j-standard derivation $\Pi'' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$ such that $\Pi \rightsquigarrow \Pi' \rightsquigarrow^* \Pi''$.

2. By induction on $\mathsf{size}_{\mathsf{RAA}}(\Pi) \in \mathbb{N}$.

   If $\mathsf{size}_{\mathsf{RAA}}(\Pi) = 0$, then just take $\Pi' = \Pi$, according to Remark 2.2.

   Otherwise, $\mathsf{size}_{\mathsf{RAA}}(\Pi) > 0$ and there exists $\mathsf{r} \in \mathsf{RAA}_\Pi$ which is RAA_Π-maximal and which is not the last rule of Π. Since there is no instance of the rule $\forall_{\mathsf{i}}$ in Π, there necessarily exists a Π' such that $\Pi \overset{\mathsf{r}_1,\dots,\mathsf{r}_n}{\rightsquigarrow} \Pi'$, where $n \in \mathbb{N}^+$ and $\mathsf{r} = \mathsf{r}_j$ for some $1 \leq j \leq n$. According to Lemma 7.1, all $\mathsf{r}_1, \dots, \mathsf{r}_n$ are RAA_Π-maximal. By Lemma 6.1, $\Pi' \colon \Gamma \vdash A$, and, by Lemma 6.2, Π' has no instance of the rules $\forall_{\mathsf{i}}$ and $\to_{\mathsf{i}}$. According to Lemma 7.3, $\mathsf{size}_{\mathsf{RAA}}(\Pi') < \mathsf{size}_{\mathsf{RAA}}(\Pi)$. Hence, by the induction hypothesis, there is a m-standard derivation $\Pi'' \colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ such that $\Pi \rightsquigarrow \Pi' \rightsquigarrow^* \Pi''$. □

Theorem 8 can be see as a *weak* standardization theorem: for every derivation Π in NK (fulfilling suitable conditions), we have shown that there exists a particular strategy for the application of the reduction steps of §4 (we fire only maximal instances of raa) transforming Π into a "standard" derivation in NK, where "standard" is here understood in the sense of Definition 1. We conjecture that Theorem 8 can be strengthened in a *strong* standardization theorem: whatever strategy in the application of the reduction steps of §4 terminates into a "standard" derivation in NK. To prove that, one should refine the notion of size of a derivation and proceed by a more complex induction.

Thanks to the normalization theorem and the suitable subformula property for NK proved by Stålmarck in [27, pp. 130, 135],[20] we can reformulate Theorem 8.1 (resp. Theorem 8.2) with a more satisfactory hypothesis: instead of supposing that the derivation $\Pi\colon \Gamma \vdash A$ in NK is without any instance of the rule $\forall_{\mathsf{i}}$ (resp. rules $\forall_{\mathsf{i}}$ and $\to_{\mathsf{i}}$), it is sufficient to suppose that A and the formulas in Γ do not contain any occurrence of $\forall$ (resp. $\forall$ and $\to$). Note that, since in Stålmark's normalization strategy for classical logic raa is pushed downward only with respect to elimination rules, a proof in classical normal form in his sense can still contain instances of raa that can reduced via the reduction steps proposed in §4. In this sense the following corollary of Theorem 8 is not completely trivial.

Corollary 9 (Postponement of raa, version 2)**.** *Let $\Gamma \vdash A$ be derivable in* NK.

1. *If A and the formulas in Γ do not contain any occurrence of $\forall$, then there exists a* j*-standard derivation $\Pi'\colon \Gamma \vdash A$ in* NK.

2. *If A and the formulas in Γ do not contain any occurrence of $\forall$ nor $\to$, then there exists a* m*-standard derivation $\Pi'\colon \Gamma \vdash A$ in* NK.

Proof. First, we recall some facts that will be used to prove Corollaries 9.1-2. As $\Gamma \vdash A$ is derivable in NK, there exists a normal (in the sense of [27, p. 130]) derivation $\Pi\colon \Gamma \vdash A$ in NK and hence, in conformity to the aforementioned *subformula principle* for NK (see [27, p. 130]), each formula occurrence B in Π satisfies one of the clauses (i)-(iii) below:

(i) B is an occurrence of a subformula of A or of some formula in Γ;

(ii) B is an assumption discharged by some instance of the rule raa, B has the form $\neg C$, and C is a subformula of A or of some formula in Γ;

(iii) B has the form $\bot$ and stands immediately below an assumption which satisfies (ii) above.

We can now prove Corollaries 9.1-2.

1. According to the subformula principle, there are no instances of the rule $\forall_{\mathsf{i}}$ in Π. By Theorem 8.1, there exists a j-standard derivation $\Pi'\colon \Gamma \vdash A$ in NK.

2. According to the subformula principle, there is no instance of the rule $\to_{\mathsf{i}}$ and $\forall_{\mathsf{i}}$ in Π. By Theorem 8.2, there is a m-standard derivation $\Pi'\colon \Gamma \vdash A$ in NK. □

Thus, according to Corollary 9.1 (resp. Corollary 9.2), when we look for a derivation of A from Γ, where A and all formulas in Γ do not contain any occurrence of $\forall$ (resp. $\forall$ nor $\to$), we can consider the use of a discharging (resp. either discharging or non-discharging) instance of raa only at the end of the derivation, if this is use is required.

In what follows we give two examples of the postponement of raa, according Theorems 8.1 and 8.2.

[20] See also [21, p. 208].

Example 10. Given two proposition symbols P and Q, let $\Pi\colon\ \vdash (\neg\neg P \to P) \wedge (\neg\neg Q \to Q)$ be the following derivation in NK (note that there are no instances of $\forall_{\mathsf{i}}$ in Π):

$$\Pi \;=\; \dfrac{\dfrac{\dfrac{\dfrac{\ulcorner\neg\neg P\urcorner^{2} \qquad \ulcorner\neg P\urcorner^{1}}{\bot}\,\neg_{\mathsf{e}}}{P}\,\mathsf{raa}^{1}}{\neg\neg P \to P}\,\to_{\mathsf{i}}^{2} \qquad \dfrac{\dfrac{\dfrac{\ulcorner\neg\neg Q\urcorner^{4} \qquad \ulcorner\neg Q\urcorner^{3}}{\bot}\,\neg_{\mathsf{e}}}{Q}\,\mathsf{raa}^{3}}{\neg\neg Q \to Q}\,\to_{\mathsf{i}}^{4}}{(\neg\neg P \to P) \wedge (\neg\neg Q \to Q)}\,\wedge_{\mathsf{i}}$$

For every proposition symbol Z, let $\pi(Z)\colon \neg(\neg\neg Z \to Z) \vdash \bot$ be the following derivation in NJ:

$$\pi(Z) \;=\; \dfrac{\neg(\neg\neg Z \to Z) \qquad \dfrac{\dfrac{\dfrac{\ulcorner\neg\neg Z\urcorner^{2} \qquad \dfrac{\dfrac{\neg(\neg\neg Z \to Z) \qquad \dfrac{\ulcorner Z\urcorner^{1}}{\neg\neg Z \to Z}\,\to_{\mathsf{i}}^{0}}{\bot}\,\neg_{\mathsf{e}}}{\neg Z}\,\neg_{\mathsf{i}}^{1}}{\bot}\,\neg_{\mathsf{e}}}{Z}\,\mathsf{efq}}{\neg\neg Z \to Z}\,\to_{\mathsf{i}}^{2}}{\bot}\,\neg_{\mathsf{e}}$$

By applying the procedure defined in the proof of Theorem 8.1, we get:

$$\Pi \;\rightsquigarrow\; \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(\neg\neg P \to P)\urcorner^{1}\\ \vdots\ \pi(P)\\ \bot\end{matrix}}{\neg\neg P \to P}\,\mathsf{raa}^{1} \qquad \dfrac{\dfrac{\dfrac{\ulcorner\neg\neg Q\urcorner^{3} \qquad \ulcorner\neg Q\urcorner^{2}}{\bot}\,\neg_{\mathsf{e}}}{Q}\,\mathsf{raa}^{2}}{\neg\neg Q \to Q}\,\to_{\mathsf{i}}^{3}}{(\neg\neg P \to P) \wedge (\neg\neg Q \to Q)}\,\wedge_{\mathsf{i}} \;\rightsquigarrow\; \dfrac{\dfrac{\begin{matrix}\ulcorner\neg(\neg\neg P \to P)\urcorner^{1}\\ \vdots\ \pi(P)\\ \bot\end{matrix}}{\neg\neg P \to P}\,\mathsf{raa}^{1} \qquad \dfrac{\begin{matrix}\ulcorner\neg(\neg\neg Q \to Q)\urcorner^{2}\\ \vdots\ \pi(Q)\\ \bot\end{matrix}}{\neg\neg Q \to Q}\,\mathsf{raa}^{2}}{(\neg\neg P \to P) \wedge (\neg\neg Q \to Q)}\,\wedge_{\mathsf{i}}$$

$$\rightsquigarrow\; \dfrac{\begin{matrix}\dfrac{\begin{matrix}\dfrac{\begin{matrix}\dfrac{\ulcorner\neg((\neg\neg P \to P) \wedge (\neg\neg Q \to Q))\urcorner^{3} \qquad \dfrac{\ulcorner\neg\neg P \to P\urcorner^{2} \qquad \ulcorner\neg\neg Q \to Q\urcorner^{1}}{(\neg\neg P \to P) \wedge (\neg\neg Q \to Q)}\,\wedge_{\mathsf{i}}}{\bot}\,\neg_{\mathsf{e}}\end{matrix}}{\neg(\neg\neg Q \to Q)}\,\neg_{\mathsf{i}}^{1}\\ \vdots\ \pi(Q)\\ \bot\end{matrix}}{\neg(\neg\neg P \to P)}\,\neg_{\mathsf{i}}^{2}\\ \vdots\ \pi(P)\\ \bot\end{matrix}}{(\neg\neg P \to P) \wedge (\neg\neg Q \to Q)}\,\mathsf{raa}^{3}\;.$$

Example 11. Given two proposition symbols P and Q, let $\Pi\colon P \vdash P \vee Q$ be the following derivation in NK (note that there is no instance of $\to_{\mathsf{i}}$ and $\forall_{\mathsf{i}}$ in Π):

$$\Pi \;=\; \dfrac{\dfrac{\dfrac{\ulcorner\neg P\urcorner^{1} \qquad \dfrac{\dfrac{\dfrac{\ulcorner\neg P\urcorner^{1} \qquad P}{\bot}\,\neg_{\mathsf{e}}}{P \wedge Q}\,\mathsf{efq}}{P}\,\wedge_{\mathsf{e}_1}}{\bot}\,\neg_{\mathsf{e}}}{P}\,\mathsf{raa}^{1}}{P \vee Q}\,\vee_{\mathsf{i}_1}$$

By applying the procedure defined in the proof of Theorem 8.2, we get:

$$\Pi \leadsto \dfrac{\dfrac{\ulcorner\neg P\urcorner^1 \qquad \dfrac{\dfrac{\ulcorner\neg P\urcorner^1 \quad P}{\bot}\,\neg_\mathsf{e}}{P}\,\mathsf{efq}}{\dfrac{\bot}{P}\,\mathsf{raa}^1}\,\neg_\mathsf{e}}{P \vee Q}\,\vee_{\mathsf{i}_1} \leadsto \dfrac{\dfrac{\dfrac{\ulcorner\neg P\urcorner^1 \quad P}{\bot}\,\neg_\mathsf{e}}{P}\,\mathsf{raa}^1}{P \vee Q}\,\vee_{\mathsf{i}_1} \leadsto \dfrac{\dfrac{\dfrac{\dfrac{\ulcorner\neg(P \vee Q)\urcorner^2 \qquad \dfrac{\ulcorner P\urcorner^1}{P \vee Q}\,\vee_{\mathsf{i}_1}}{\bot}\,\neg_\mathsf{e}}{\neg P}\,\neg_\mathsf{i}^1 \qquad P}{\bot}\,\neg_\mathsf{e}}{P \vee Q}\,\mathsf{raa}^2 .$$

6 Generalized Glivenko's theorem

As already mentioned in §1-2, an immediate consequence of the postponement of raa is the weak normalization of $\mathsf{NK} \smallsetminus \{\forall_\mathsf{i}\}$. Indeed, a normalization strategy is the following: by Theorem 8.1, any derivation $\pi\colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_\mathsf{i}\}$ reduces to a j-standard derivation $\pi'\colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_\mathsf{i}\}$ (which is a derivation in NJ, possibly except for its last rule which could be a discharging instance of raa), then one can apply Prawitz's original weak normalization theorem for NJ [22, p. 50] to π' (or to π' without its last rule), so as to obtain a normal derivation $\pi''\colon \Gamma \vdash A$ in $\mathsf{NK} \smallsetminus \{\forall_\mathsf{i}\}$.

Another consequence of the postponement of raa (Theorem 8, Corollary 9) is a strengthened form of Glivenko's theorem embedding full first-order classical logic not only into the fragment $\{\bot, \top, \neg, \wedge, \vee, \to, \exists\}$ of intuitionistic logic (Theorem 16), but also into the fragment $\{\bot, \top, \neg, \wedge, \vee, \exists\}$ of minimal logic (Theorem 15). The idea is that, given a derivation Π in NK whose last rule is an instance of raa, the rest of Π is a sub-derivation in NJ or NM; the instance of raa can thus be replaced by an instance of $\neg_\mathsf{i}$.

We define a translation $(\cdot)^\mathsf{m}$ (resp. $(\cdot)^\mathsf{j}$) on formulas that just redefines the implication and the universal quantifier (resp. only the universal quantifier) in a classical way, using the negation, the disjunction and the existential quantifier (resp. the negation and the existential quantifier). All other connectives and the existential quantifier are not modified by $(\cdot)^\mathsf{m}$ (resp. $(\cdot)^\mathsf{j}$).

Definition 12 (Minimal and intuitionistic translations)**.** *The* minimal translation *is a function* $(\cdot)^\mathsf{m}$ *associating with every formula* A *a formula* A^m *defined by induction on* A *as follows:*

$$\begin{array}{ccc} (P(t_1,\dots,t_n))^\mathsf{m} = P(t_1,\dots,t_n) & \top^\mathsf{m} = \top & \bot^\mathsf{m} = \bot \\ (A \wedge B)^\mathsf{m} = A^\mathsf{m} \wedge B^\mathsf{m} & (A \vee B)^\mathsf{m} = A^\mathsf{m} \vee B^\mathsf{m} & (\neg A)^\mathsf{m} = \neg A^\mathsf{m} \\ (A \to B)^\mathsf{m} = \neg A^\mathsf{m} \vee B^\mathsf{m} & (\forall x A)^\mathsf{m} = \neg\exists x\, \neg A^\mathsf{m} & (\exists x A)^\mathsf{m} = \exists x A^\mathsf{m} \end{array}$$

The intuitionistic translation *is a function* $(\cdot)^\mathsf{j}$ *associating with every formula* A *a formula* A^j *defined by induction on* A *as follows:*

$$\begin{array}{ccc} (P(t_1,\dots,t_n))^\mathsf{j} = P(t_1,\dots,t_n) & \top^\mathsf{j} = \top & \bot^\mathsf{j} = \bot \\ (A \wedge B)^\mathsf{j} = A^\mathsf{j} \wedge B^\mathsf{j} & (A \vee B)^\mathsf{j} = A^\mathsf{j} \vee B^\mathsf{j} & (\neg A)^\mathsf{j} = \neg A^\mathsf{j} \\ (A \to B)^\mathsf{j} = A^\mathsf{j} \to B^\mathsf{j} & (\forall x A)^\mathsf{j} = \neg\exists x\, \neg A^\mathsf{j} & (\exists x A)^\mathsf{j} = \exists x A^\mathsf{j} \end{array}$$

Given a set of formulas Γ*, we set* $\Gamma^\mathsf{m} = \{A^\mathsf{m} \mid A \in \Gamma\}$ *and* $\Gamma^\mathsf{j} = \{A^\mathsf{j} \mid A \in \Gamma\}$*.*

The difference between $(\cdot)^\mathsf{m}$ and $(\cdot)^\mathsf{j}$ is only in the translation of $A \to B$. Our minimal and intuitionistic translations are deeply related to Kuroda's negative translation. More precisely, if $(\cdot)^{\mathsf{m}'}$ and $(\cdot)^{\mathsf{j}'}$ are the translations defined as in Definition 12, except for

$$(\forall x A)^{\mathsf{m}'} = \forall x\, \neg\neg A^{\mathsf{m}'} \qquad\qquad (\forall x A)^{\mathsf{j}'} = \forall x\, \neg\neg A^{\mathsf{j}'},$$

then the negative translation $A \mapsto \neg\neg A^{\mathsf{j}'}$ is the one defined by Kuroda in [11], while the negative translation $A \mapsto \neg\neg A^{\mathsf{m}'}$ is a variant of Kuroda's one introduced in [6, p. 231].

Using the terminology of [6], $(\cdot)^{\mathsf{m}}$ and $(\cdot)^{\mathsf{j}}$ are modular translations in the sense that the translation of a formula is based on the translation of its immediate subformulas. The names "minimal" and "intuitionistic" associated with $(\cdot)^{\mathsf{m}}$ and $(\cdot)^{\mathsf{j}}$, respectively, are due to the derivability relation they preserve: this will be clarified in Theorems 15.2-16.2 and Propositions 18.1-2 below, which imply that

$$\vdash_{\mathsf{NK}} A \quad \text{iff} \quad \vdash_{\mathsf{NJ}} \neg\neg A^{\mathsf{j}} \quad \text{iff} \quad \vdash_{\mathsf{NM}} \neg\neg A^{\mathsf{m}}.$$

A consequence of this fact, together with Remark 13.1 below, is that the translation $A \mapsto \neg\neg A^{\mathsf{m}}$ (resp. $A \mapsto \neg\neg A^{\mathsf{j}}$) is modular *negative* according to [6]. Besides, the modular negative translations $A \mapsto \neg\neg A^{\mathsf{m}}$ and $A \mapsto \neg\neg A^{\mathsf{m}'}$ (resp. $A \mapsto \neg\neg A^{\mathsf{j}}$ and $A \mapsto \neg\neg A^{\mathsf{j}'}$) are the same according to [6, Definition 2], in the sense that they are interderivable with respect to minimal logic.[21] However, quite interestingly, they have a very different behavior with respect to the postponement of raa: only the negative translation $A \mapsto \neg\neg A^{\mathsf{m}}$ (resp. $A \mapsto \neg\neg A^{\mathsf{j}}$) allows one to use Theorem 8.2 (resp. Theorem 8.1), since in $A^{\mathsf{m}'}$ (resp. $A^{\mathsf{j}'}$) the universal quantifier might occur, with the disturbing effect pointed out in §2.2 and Remark 4. A variant of our minimal translation is discussed in Appendix A.3.

Remark 13. For every formula A, by induction on A we can prove that:

1. $\vdash_{\mathsf{NK}} A$ if and only if $\vdash_{\mathsf{NK}} A^{\mathsf{m}}$ if and only if $\vdash_{\mathsf{NK}} A^{\mathsf{j}}$;
2. A^{m} contains no occurrences of $\to$ and $\forall$; A^{j} contains no occurrences of $\forall$;
3. the free variables in A^{m} and A^{j} are the same as in A, and $(A\{t/x\})^{\mathsf{m}} = A^{\mathsf{m}}\{t/x\}$ and $(A\{t/x\})^{\mathsf{j}} = A^{\mathsf{j}}\{t/x\}$ for any term t;
4. $A^{\mathsf{m}} = A$ if A contains no occurrences of $\to$ and $\forall$; $A^{\mathsf{j}} = A$ if A contains no occurrences of $\forall$.

Actually, one direction of the equivalences in Remark 13.1 can be reformulated in a more informative and constructive way from a proof-theoretic viewpoint, thanks to (the proof of) the following lemma.

Lemma 14 (Preservation of derivability in NK w.r.t. translations)**.** *For every derivation $\Pi\colon \Gamma \vdash A$ in NK there exist a derivation $\Pi'\colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \setminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ and a derivation $\Pi''\colon \Gamma^{\mathsf{j}} \vdash A^{\mathsf{j}}$ in $\mathsf{NK} \setminus \{\forall_{\mathsf{i}}\}$.*

Proof. By induction on the derivation Π in NK. Let us consider its last rule r. Due to Definition 12, to prove the existence of a derivation $\Pi'\colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \setminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ the only interesting cases are when r is an instance of $\to_{\mathsf{i}}$ or $\to_{\mathsf{e}}$ or $\forall_{\mathsf{i}}$ or $\forall_{\mathsf{e}}$.

If r is an instance of $\to_{\mathsf{i}}$, then $\Pi\colon \Gamma \vdash A \to B$ and there is a derivation $\pi\colon \Gamma, A \vdash B$ in NK such that

$$\Pi = \dfrac{\begin{matrix}\ulcorner A \urcorner^{1} \\ \vdots\,\pi \\ B\end{matrix}}{A \to B}\,{\to_{\mathsf{i}}^{1}} \; .$$

By induction hypothesis, there is a derivation $\pi'\colon \Gamma^{\mathsf{m}}, A^{\mathsf{m}} \vdash B^{\mathsf{m}}$ in $\mathsf{NK} \setminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$. Let

$$\Pi' = \dfrac{\dfrac{\dfrac{\ulcorner \neg(\neg A^{\mathsf{m}} \vee A^{\mathsf{m}}) \urcorner^{2} \qquad \dfrac{\dfrac{\dfrac{\ulcorner \neg(\neg A^{\mathsf{m}} \vee A^{\mathsf{m}}) \urcorner^{2} \qquad \dfrac{\ulcorner A^{\mathsf{m}} \urcorner^{1}}{\neg A^{\mathsf{m}} \vee A^{\mathsf{m}}}\,\vee_{\mathsf{i}_1}}{\bot}\,\neg_{\mathsf{e}}}{\neg A^{\mathsf{m}}}\,\neg_{\mathsf{i}}^{1}}{\neg A^{\mathsf{m}} \vee A^{\mathsf{m}}}\,\vee_{\mathsf{i}_2}}{\bot}\,\neg_{\mathsf{e}}}{\neg A^{\mathsf{m}} \vee A^{\mathsf{m}}}\,\mathsf{raa}^{2} \qquad \dfrac{\ulcorner \neg A^{\mathsf{m}} \urcorner^{3}}{\neg A^{\mathsf{m}} \vee B^{\mathsf{m}}}\,\vee_{\mathsf{i}_1} \qquad \dfrac{\begin{matrix}\ulcorner A^{\mathsf{m}} \urcorner^{3} \\ \vdots\,\pi' \\ B^{\mathsf{m}}\end{matrix}}{\neg A^{\mathsf{m}} \vee B^{\mathsf{m}}}\,\vee_{\mathsf{i}_2}}{\neg A^{\mathsf{m}} \vee B^{\mathsf{m}}}\,\vee_{\mathsf{e}}^{3} \; .$$

[21] Note, in particular, that $\neg\exists x \neg A$ is provably equivalent to $\forall x \neg\neg A$ in minimal logic.

Thus, $\Pi' \colon \Gamma^{\mathsf{m}} \vdash \neg A^{\mathsf{m}} \vee B^{\mathsf{m}}$ is a derivation in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$, where $\neg A^{\mathsf{m}} \vee B^{\mathsf{m}} = (A \to B)^{\mathsf{m}}$.

If r is an instance of $\to_{\mathsf{e}}$, then $\Pi \colon \Gamma \vdash B$ and there are derivations $\pi_1 \colon \Gamma_1 \vdash A \to B$ and $\pi_2 \colon \Gamma_2 \vdash A$ in NK such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and

$$\Pi = \dfrac{\begin{array}{c}\vdots\ \pi_1\\ A \to B\end{array} \qquad \begin{array}{c}\vdots\ \pi_2\\ A\end{array}}{B}\ {\to_{\mathsf{e}}}\ .$$

By induction hypothesis, there are derivations $\pi_1' \colon \Gamma_1^{\mathsf{m}} \vdash \neg A^{\mathsf{m}} \vee B^{\mathsf{m}}$ and $\pi_2' \colon \Gamma_2^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$. Let

$$\Pi' = \dfrac{\begin{array}{c}\vdots\ \pi_1'\\ \neg A^{\mathsf{m}} \vee B^{\mathsf{m}}\end{array} \qquad \dfrac{\dfrac{\ulcorner \neg A^{\mathsf{m}} \urcorner^1 \qquad \begin{array}{c}\vdots\ \pi_2'\\ A^{\mathsf{m}}\end{array}}{\bot}\ \neg_{\mathsf{e}}}{B^{\mathsf{m}}}\ \mathsf{efq} \qquad \ulcorner B^{\mathsf{m}} \urcorner^1}{B^{\mathsf{m}}}\ \vee_{\mathsf{e}}^1\ .$$

Thus, $\Pi' \colon \Gamma^{\mathsf{m}} \vdash B^{\mathsf{m}}$ is a derivation in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$.

If r is an instance of $\forall_{\mathsf{i}}$, then $\Pi \colon \Gamma \vdash \forall x A$ and there is a derivation $\pi \colon \Gamma \vdash A$ in NK such that the variable x is not free in any formula of Γ and

$$\Pi = \dfrac{\begin{array}{c}\vdots\ \pi\\ A\end{array}}{\forall x A}\ \forall_{\mathsf{i}}\ .$$

By induction hypothesis, there is a derivation $\pi' \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$. By Remark 13.3, the variable x is not free in any formula of Γ^{m}. Let

$$\Pi' = \dfrac{\dfrac{\ulcorner \exists x \neg A^{\mathsf{m}} \urcorner^2 \qquad \dfrac{\ulcorner \neg A^{\mathsf{m}} \urcorner^1 \qquad \begin{array}{c}\vdots\ \pi'\\ A^{\mathsf{m}}\end{array}}{\bot}\ \neg_{\mathsf{e}}}{\bot}\ \exists_{\mathsf{e}}^1}{\neg \exists x \neg A^{\mathsf{m}}}\ \neg_{\mathsf{i}}^2\ .$$

Thus, $\Pi' \colon \Gamma^{\mathsf{m}} \vdash \neg \exists x \neg A^{\mathsf{m}}$ is a derivation in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$, where $(\forall x A)^{\mathsf{m}} = \neg \exists x \neg A^{\mathsf{m}}$.

If r is an instance of $\forall_{\mathsf{e}}$, then $\Pi \colon \Gamma \vdash A\{t/x\}$ and there is a derivation $\pi \colon \Gamma \vdash \forall x A$ in NK such that

$$\Pi = \dfrac{\begin{array}{c}\vdots\ \pi\\ \forall x A\end{array}}{A\{t/x\}}\ \forall_{\mathsf{e}}\ .$$

By induction hypothesis, there is a derivation $\pi' \colon \Gamma^{\mathsf{m}} \vdash \neg \exists x \neg A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$. Let

$$\Pi' = \dfrac{\dfrac{\begin{array}{c}\vdots\ \pi'\\ \neg \exists x \neg A^{\mathsf{m}}\end{array} \qquad \dfrac{\ulcorner \neg A^{\mathsf{m}}\{t/x\} \urcorner^1}{\exists x \neg A^{\mathsf{m}}}\ \exists_{\mathsf{i}}}{\bot}\ \neg_{\mathsf{e}}}{A^{\mathsf{m}}\{t/x\}}\ \mathsf{raa}^1\ .$$

Thus, $\Pi' \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}\{t/x\}$ is a derivation in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$, where $(A\{t/x\})^{\mathsf{m}} = A^{\mathsf{m}}\{t/x\}$ by Remark 13.3.

The proof of the existence of a derivation $\Pi'' \colon \Gamma^{\mathsf{j}} \vdash A^{\mathsf{j}}$ in $\mathsf{NK} \smallsetminus \{\forall_{\mathsf{i}}\}$ is analogous to the proof of the existence of a derivation $\Pi' \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$, but the only interesting cases are when r is an instance of $\forall_{\mathsf{i}}$ or $\forall_{\mathsf{e}}$. $\square$

In general, derivability in NM is not preserved via the translation $(\cdot)^{\mathsf{m}}$: for instance, $\vdash_{\mathsf{NM}} P \to P$ but $\nvdash_{\mathsf{NM}} \neg P \vee P$, where $(P \to P)^{\mathsf{m}} = \neg P \vee P$. Also, a formula A in general is not derivably equivalent to A^{j} in NJ (resp. A^{m} in NM), since $\forall x A$ is not equivalent to $\neg\exists x \neg A$ in minimal (resp. intuitionistic) logic.

Theorem 15 (Generalized Glivenko's theorem, minimal version)**.**

1. *If $\Gamma \vdash_{\mathsf{NK}} A$, then $\Gamma^{\mathsf{m}} \vdash_{\mathsf{D}} \neg\neg A^{\mathsf{m}}$ and $\Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash_{\mathsf{D}} \bot$, where $\mathsf{D} = \mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \to_{\mathsf{e}}, \forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$.*
2. *If $\to$ and $\forall$ occur neither in A nor in any formula of Γ, then the following are equivalent:*
   (a) $\Gamma \vdash_{\mathsf{NK}} A$, *(b) $\Gamma \vdash_{\mathsf{NM}} \neg\neg A$,* *(c) $\Gamma, \neg A \vdash_{\mathsf{NM}} \bot$.*

   If moreover $A = \neg B$, then: $\Gamma \vdash_{\mathsf{NK}} \neg B$ if and only if $\Gamma \vdash_{\mathsf{NM}} \neg B$.

Proof.

1. Since $\Gamma \vdash A$ is derivable in NK, according to Lemma 14, there exists a derivation $\Pi \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$, and by Theorem 8.2, there exists a derivation $\Pi' \colon \Gamma^{\mathsf{m}} \vdash A^{\mathsf{m}}$ in $\mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ with at most one instance of the rule raa; this instance, if any, is the last rule of Π', the rest of Π' being a derivation in NM. Only two cases are possible:

   - either the last rule of Π' is not an instance of raa, and thus Π' is a derivation in NM, so that $\Pi'' \colon \Gamma^{\mathsf{m}} \vdash \neg\neg A^{\mathsf{m}}$ and $\Pi''' \colon \Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash \bot$ are derivations in NM, where:

   $$\Pi'' = \cfrac{\cfrac{\overset{\vdots\,\Pi'}{A^{\mathsf{m}}} \qquad \ulcorner\neg A^{\mathsf{m}}\urcorner^{1}}{\bot}\;\neg_{\mathsf{e}}}{\neg\neg A^{\mathsf{m}}}\;\neg_{\mathsf{i}}^{1} \qquad \text{and} \qquad \Pi''' = \cfrac{\overset{\vdots\,\Pi'}{A^{\mathsf{m}}} \qquad \neg A^{\mathsf{m}}}{\bot}\;\neg_{\mathsf{e}} \;;$$

   - or the last rule of Π' is an instance of raa, i.e.

   $$\Pi' = \cfrac{\begin{matrix}\ulcorner\neg A^{\mathsf{m}}\urcorner^{1}\\ \vdots\,\pi\\ \bot\end{matrix}}{A^{\mathsf{m}}}\;\mathsf{raa}^{1}$$

   where $\pi \colon \Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash \bot$ is a derivation in NM. So, $\Pi'' \colon \Gamma^{\mathsf{m}} \vdash \neg\neg A^{\mathsf{m}}$ is a derivation in NM where Π'' is obtained from Π' by replacing the instance of raa with an instance of $\neg_{\mathsf{i}}$ discharging the same assumptions, i.e.

   $$\Pi'' = \cfrac{\begin{matrix}\ulcorner\neg A^{\mathsf{m}}\urcorner^{1}\\ \vdots\,\pi\\ \bot\end{matrix}}{\neg\neg A^{\mathsf{m}}}\;\neg_{\mathsf{i}}^{1} \quad .$$

   We have thus proved that $\Gamma^{\mathsf{m}} \vdash \neg\neg A^{\mathsf{m}}$ and $\Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash \bot$ are derivable in NM. According to Remark 13.2, A^{m} and any formula in Γ^{m} do not contain any occurrence of $\to$ and $\forall$; hence, according to the normalization theorem and the subformula principle for NM [22, p. 53], $\Gamma^{\mathsf{m}} \vdash \neg\neg A^{\mathsf{m}}$ and $\Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash \bot$ are derivable in $\mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \to_{\mathsf{e}}, \forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$.

2. (a) **implies** (b)**:** By Theorem 15.1, since $\Gamma \vdash A$ is derivable in NK, there is a derivation $\Pi \colon \Gamma^{\mathsf{m}} \vdash \neg\neg A^{\mathsf{m}}$ in NM. According to Remark 13.4, $\Gamma^{\mathsf{m}} = \Gamma$ and $A^{\mathsf{m}} = A$. So, $\Pi \colon \Gamma \vdash \neg\neg A$ (in NM).

(b) **implies** (c)**:** If $\Pi\colon \Gamma \vdash \neg\neg A$ is a derivation in NM, then $\Pi'\colon \Gamma, \neg A \vdash \bot$ is a derivation in NM, where

$$\Pi' \;=\; \dfrac{\begin{array}{c}\vdots\,\Pi\\ \neg\neg A\end{array} \qquad \neg A}{\bot}\,\neg_{\mathsf{e}}\;.$$

(c) **implies** (a)**:** Since $\mathsf{NM} \subseteq \mathsf{NK}$, if $\Pi\colon \Gamma, \neg A \vdash \bot$ is a derivation in NM, then Π is a derivation in NK. Therefore, $\Pi'\colon \Gamma \vdash A$ is a derivation in NK, where

$$\Pi' \;=\; \dfrac{\begin{array}{c}\ulcorner\neg A\urcorner^{1}\\ \vdots\,\Pi\\ \bot\end{array}}{A}\,\mathsf{raa}^{1}\;.$$

This proves the equivalences: (a) iff (b) iff (c).

Suppose now that moreover $A = \neg B$.

if: Every derivation $\pi\colon \Gamma \vdash_{\mathsf{NM}} \neg B$ is also a derivation in NK because $\mathsf{NM} \subseteq \mathsf{NK}$.

only if: Since $\Gamma \vdash_{\mathsf{NK}} \neg B$, there exists $\pi\colon \Gamma \vdash_{\mathsf{NM}} \neg\neg\neg B$ according to the implication (a)$\Rightarrow$(b) we have just proved. Therefore, $\pi'\colon \Gamma \vdash_{\mathsf{NM}} \neg B$ where

$$\pi' \;=\; \dfrac{\begin{array}{c}\vdots\,\pi\\ \neg\neg\neg B\end{array} \qquad \dfrac{\dfrac{\dfrac{\ulcorner\neg\neg\neg B\urcorner^{3} \qquad \dfrac{\dfrac{\ulcorner\neg B\urcorner^{1} \qquad \ulcorner B\urcorner^{2}}{\bot}\,\neg_{\mathsf{e}}}{\neg\neg B}\,\neg_{\mathsf{i}}^{1}}{\bot}\,\neg_{\mathsf{e}}}{\neg B}\,\neg_{\mathsf{i}}^{2}}{\neg\neg\neg B \to \neg B}\,\to_{\mathsf{i}}^{3}}{\neg B}\,\to_{\mathsf{e}}\;.$$

□

Theorem 16 (Generalized Glivenko's theorem, intuitionistic version)**.**

1. *If $\Gamma \vdash_{\mathsf{NK}} A$, then $\Gamma^{\mathsf{j}} \vdash_{\mathsf{D}} \neg\neg A^{\mathsf{j}}$ and $\Gamma^{\mathsf{j}}, \neg A^{\mathsf{j}} \vdash_{\mathsf{D}} \bot$ where $\mathsf{D} = \mathsf{NJ} \smallsetminus \{\forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$.*
2. *If $\forall$ occurs neither in A nor in any formula of Γ, then the following are equivalent:*

   (a) $\Gamma \vdash_{\mathsf{NK}} A$, *(b) $\Gamma \vdash_{\mathsf{NJ}} \neg\neg A$,* *(c) $\Gamma, \neg A \vdash_{\mathsf{NJ}} \bot$.*

   If moreover $A = \neg B$, then: $\Gamma \vdash_{\mathsf{NK}} \neg B$ if and only if $\Gamma \vdash_{\mathsf{NJ}} \neg B$.

Proof. The proof of both points of Theorem 16 is analogous to the proof of both points of Theorem 15 below, replacing $(\cdot)^{\mathsf{m}}$ with $(\cdot)^{\mathsf{j}}$, NM with NJ, $\mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ with $\mathsf{NJ} \smallsetminus \{\forall_{\mathsf{i}}\}$, $\mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \to_{\mathsf{e}}, \forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$ with $\mathsf{NJ} \smallsetminus \{\forall_{\mathsf{i}}, \forall_{\mathsf{e}}\}$. To prove Theorem 16.1 we use Theorem 8.1 instead of Theorem 8.2, and the normalization theorem and the subformula principle for NJ instead of the normalization theorem and the subformula principle for NM (see [22, p. 53]). To prove Theorem 16.2 we use Theorem 16.1 instead of Theorem 15.1. □

An immediate consequence of Theorem 16.2 (resp. Theorem 15.2) is the next corollary: in the fragment $\{\bot, \top, \neg, \vee, \wedge, \to, \exists\}$ (resp. $\{\bot, \top, \neg, \vee, \wedge, \exists\}$) of first-order logic, the consistency of a set of formulas in classical logic is equivalent to its consistency in intuitionistic (resp. minimal) logic.

Corollary 17 (Relative consistency of a theory)**.** *Let a theory be a set of formulas Γ.*

1. *If $\forall$ does not occur in any formula of Γ, then: $\Gamma \vdash_{\mathsf{NK}} \bot$ if and only if $\Gamma \vdash_{\mathsf{NJ}} \bot$.*

2. *If $\to$ and $\forall$ do not occur in any formula of Γ, then: $\Gamma \vdash_{\mathsf{NK}} \bot$ if and only if $\Gamma \vdash_{\mathsf{NM}} \bot$.*

Proof. The proof of Corollary 17.1 is analogous to the proof of Corollary 17.2: it is sufficient to replace NM with NJ, and use Theorem 16.2 instead of Theorem 15.2. We prove now Corollary 17.2.

only if: Since $\Gamma \vdash \bot$ is a derivable in NK, then by Theorem 15.2 there is a derivation $\pi \colon \Gamma, \neg\bot \vdash \bot$ in NM. So, $\Pi \colon \Gamma \vdash \bot$ is a derivation in NM where

$$\Pi \;=\; \begin{array}{c} \dfrac{\ulcorner \bot \urcorner^{1}}{\neg\bot}\,\neg_{\mathsf{i}}^{1} \\ \vdots\,\pi \\ \bot \end{array} .$$

if: Since $\mathsf{NM} \subseteq \mathsf{NK}$, every derivation $\pi \colon \Gamma \vdash \bot$ in NM is also a derivation in NK. □

The fact that Theorem 15.2 and Corollary 17.2 (resp. Theorem 16.2 and Corollary 17.1) are restricted to the fragment $\{\bot, \top, \neg, \wedge, \vee, \exists\}$ (resp. $\{\bot, \top, \neg, \wedge, \vee, \to, \exists\}$) of first-order language of classical logic is not a limit because this fragment is equally expressive as the full first-order classical logic (with respect to the derivability relation).

For the sake of completeness, also (a slightly strengthened version of) the converses of Theorems 15.1 and 16.1 hold (the proof is straightforward and left to the reader):

Proposition 18. *Let A be a formula and Γ be a set of formulas.*

1. *If $\Gamma^{\mathsf{m}} \vdash_{\mathsf{NM}} \neg\neg A^{\mathsf{m}}$ (or equivalently $\Gamma^{\mathsf{m}}, \neg A^{\mathsf{m}} \vdash_{\mathsf{NM}} \bot$), then $\Gamma \vdash_{\mathsf{NK}} A$.*

2. *If $\Gamma^{\mathsf{j}} \vdash_{\mathsf{NJ}} \neg\neg A^{\mathsf{j}}$ (or equivalently $\Gamma^{\mathsf{j}}, \neg A^{\mathsf{j}} \vdash_{\mathsf{NJ}} \bot$), then $\Gamma \vdash_{\mathsf{NK}} A$.*

7 Conclusion

The literature concerning the connexions between classical and constructive logics is extremely rich and prolific. Our aim in this paper was to give a sort of unifying view of some of these results, by adopting a proof-theoretic perspective, and, in particular, by focusing on a very specific technique: that of postponing the application of the rule of *reduction ad absurdum* in the proofs of classical logic.

After having sketched the evolution of this technique starting from the seminal work of Prawitz in his monograph on natural deduction [22], we have focused our attention on a particular strategy of postponement: the one adopted by Seldin [25]. The interest of this strategy is that it can be characterized in a sort of geometrical way. In this sense, we proposed a modified version of it by reasoning only on the distance from the conclusion of the instances of raa present in a certain derivation, and we left aside any consideration on the syntactic structure of the formulas introduced by the instances of raa. This non-sensitivity to syntactic considerations makes the technique extensible to logic systems going beyond first-order classical logic, like modal classical logic and, especially, second-order classical logic.[22] We also conjecture that, even if our postponing strategy is a weak one, it is possible to transform it into a strong one, in the sense that the order of application of our reduction steps is not essential: any order of application should allow one to push raa downward with respect to all the other rules.

The other aspect on which we focused our attention is the possibility of extracting some constructive content from the postponing strategy that we presented. In particular, we have been able to obtain Glivenko's theorem in a uniform form, that is, working both for intuitionistic and minimal logic. As

[22] Clearly, in second order classical logic, the postponement of raa does not implies also the normalization theorem.

for the postponement of raa, it should not be difficult to extend it to systems going beyond first-order logic, like modal logic and second-order logic.

Finally, since the proof of Glivenko's theorem rests on the use of a negative translation, and since negative translation are closely related to the *continuation-passing style* (CPS) transformations in functional programming, it would be interesting to investigate which is the proper computational interpretation (in terms of λ-calculus) that can be assigned to the negative translation induced by our postponing strategy. In particular, since our translation of classical logic into intuitionistic logic is just a variant of the Kuroda translation, it is reasonable to expect that our translation simulates a call-by-value evaluation strategy in a call-by-name interpreter (see [6, p. 255], [12, p. 158 ff.]). And since we can define also a translation of classical logic into minimal logic, it would be interesting to understand whether this second translation generates a different CPS transformation or not.

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A Appendix

A.1 Hidden negative translations

In Stålmarck's reduction for $\lor$ classical detours – see (9) – the application of raa used for introducing the formula $A \lor B$ disappears, and it is not replaced by any application of raa on (one of) the sub-formulas A and B, as it happens, instead, in the case of conjunction. It is thus not possible, in the reduced derivation of (9), to operate the substitution of raa with a $\neg_{\mathsf{i}}$ in order to obtain $\neg A$ or $\neg B$. It seems then that the procedure used for obtaining a negative translation from the normalization of classical logic cannot be applied in the case of disjunction. However, in this reduced derivation, the $\lor_{\mathsf{e}}$ is used in a restricted way, namely with $C = \bot$, i.e.

$$\dfrac{A \lor B \qquad \begin{matrix}\ulcorner A \urcorner^1 \\ \vdots\,\pi' \\ \bot\end{matrix} \qquad \begin{matrix}\ulcorner B \urcorner^1 \\ \vdots\,\pi'' \\ \bot\end{matrix}}{\bot}\lor_{\mathsf{e}}^1$$

Now, from the two sub-derivations π' and π'' it is possible to obtain the formula $\neg A \land \neg B$:

$$\dfrac{\dfrac{\begin{matrix}\ulcorner A \urcorner^1 \\ \vdots\,\pi' \\ \bot\end{matrix}}{\neg A}\neg_{\mathsf{i}}^1 \qquad \dfrac{\begin{matrix}\ulcorner B \urcorner^2 \\ \vdots\,\pi'' \\ \bot\end{matrix}}{\neg B}\neg_{\mathsf{i}}^2}{\neg A \land \neg B}\land_{\mathsf{i}}$$

And since $A \lor B$ and $\neg A \land \neg B$ are contradictory in minimal logic, one can infer $\neg(\neg A \land \neg B)$. This means that, in the process of normalization, $A \lor B$ behaves like $\neg(\neg A \land \neg B)$. There is thus no substantial difference with Prawitz's treatment of disjunction, which is from the beginning defined via negation and conjunction (see p. 6, *supra*).

It is worth noting that this is not a peculiarity of the $\lor_{\mathsf{e}}$ rule. It can be claimed that all general elimination rules hide a sort of negative translation. According to Schroeder-Heister and Olkhovikov [24], it is possible to flatten the general elimination rules used by von Plato and Siders (see p. 9, *supra*) by translating them intro formulas of second-order propositional logic. In particular, the $\land_{\mathsf{e}}$ becomes the formula $\forall X(((A \land B) \to X) \to X)$, the $\lor_{\mathsf{e}}$ becomes $\forall X(((A \to X) \land (B \to X)) \to X)$, and the $\to_{\mathsf{e}}$ becomes $\forall X((A \land (B \to X)) \to X)$. Then, by instantiating X with $\bot$, the $\land_{\mathsf{e}}$ gets associated with $\neg\neg(A \land B)$, the $\lor_{\mathsf{e}}$ with $\neg(\neg A \land \neg B)$, and the $\to_{\mathsf{e}}$ with $\neg(A \land \neg B)$. Disjunction and implication are then treated like in the Gödel-Gentzen's translation.

A.2 Tennant's proof of the Glivenko's theorem

In [29, pp. 266–274] Tennant gives a proof of the Glivenko's theorem for minimal and intuitionistic logic by using a double-negation translation which allows him to transform classical rules into derivable rules of minimal and intuitionistic logic.

What he proves is the following theorem:

Theorem 19. *Let Γ a set of formulas, and $(\cdot)^*$ a function that, when applied to Γ, adds two negations ($\neg\neg$) in front of any formula of Γ.*

1. *Every derivation $\Pi : \Gamma \vdash A$ in $\mathsf{D_c}$ can be converted into a derivation $\Pi' : (\Gamma)^* \vdash (A)^*$ in $\mathsf{D_m}$, where $\mathsf{D_c} = \mathsf{NK} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$ and $\mathsf{D_m} = \mathsf{NM} \smallsetminus \{\to_{\mathsf{i}}, \forall_{\mathsf{i}}\}$.*

2. *Every derivation $\Pi : \Gamma \vdash A$ in $\mathsf{D_c}$ can be converted into a derivation $\Pi' : (\Gamma)^* \vdash (A)^*$ in $\mathsf{D_i}$, where $\mathsf{D_c} = \mathsf{NK} \smallsetminus \{\forall_\mathsf{i}\}$ and $\mathsf{D_m} = \mathsf{NJ} \smallsetminus \{\forall_\mathsf{i}\}$.*

In order to better explain how Tennant's idea works, it is easier to consider a natural deduction setting presented in a sequent calculus style. Consider then a rule of $\mathsf{D_c}$ of the form

$$\dfrac{\Gamma_1 \vdash A_1 \quad \dots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A}\ \mathsf{s}$$

What is first shown by Tenannt is that the rule

$$\dfrac{(\Gamma_1)^* \vdash (A_1)^* \quad \dots \quad (\Gamma_n)^* \vdash (A_n)^*}{(\Gamma)^* \vdash (A)^*}\ \mathsf{s}^* \tag{25}$$

is derivable either in $\mathsf{D_m}$ or in $\mathsf{D_i}$.

Consider, for example, the rule

$$\dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge_\mathsf{i}$$

In order to show that the rule

$$\dfrac{(\Gamma)^* \vdash (A)^* \quad (\Delta)^* \vdash (B)^*}{(\Gamma)^*, (\Delta)^* \vdash (A \wedge B)^*}\ \wedge_\mathsf{i}^*$$

is derivable in $\mathsf{D_m}$ or $\mathsf{D_i}$, it is sufficient to consider the following derivation

$$\dfrac{(\Gamma)^* \vdash \neg\neg A \qquad \dfrac{\dfrac{(\Delta)^* \vdash \neg\neg B \qquad \dfrac{\dfrac{\dfrac{}{\neg(A \wedge B) \vdash \neg(A \wedge B)}\ \mathrm{Ax} \qquad \dfrac{\dfrac{}{A \vdash A}\ \mathrm{Ax} \quad \dfrac{}{B \vdash B}\ \mathrm{Ax}}{A, B \vdash A \wedge B}\ \wedge_\mathsf{i}}{\neg(A \wedge B), A, B \vdash \bot}\ \neg_\mathsf{e}}{\neg(A \wedge B), A \vdash \neg B}\ \neg_\mathsf{i}}{\neg(A \wedge B), A, (\Delta)^* \vdash \bot}\ \neg_\mathsf{e}}{\neg(A \wedge B), (\Delta)^* \vdash \neg A}\ \neg_\mathsf{i}}{\dfrac{\neg(A \wedge B), (\Gamma)^*, (\Delta)^* \vdash \bot}{(\Gamma)^*, (\Delta)^* \vdash \neg\neg(A \wedge B)}\ \neg_\mathsf{i}}\ \neg_\mathsf{e}$$

The situation is a little bit more complicated when one has to deal with rules involving a discharge of hypothesis. The problem is that if A is one of the hypothesis to be discharged, then it is not sufficient to translate it into $(A)^*$, i.e. $\neg\neg A$. Otherwise, trying to discharge $\neg\neg A$ will perturb the structure of the derivation, and one will be unable to get the desired conclusion. It becomes then necessary to operate a composition with a proof of $A \vdash \neg\neg A$ in order to operate the right discharge on A.

Consider, for example, the case of the implication introduction rule:

$$\dfrac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ \to_\mathsf{i}$$

In order to show that the rule

$$\dfrac{(A)^*, (\Gamma)^* \vdash (B)^*}{(\Gamma)^* \vdash (A \to B)^*}\ \to_\mathsf{i}^*$$

is derivable, one has to consider the following derivation

$$
\dfrac{\dfrac{}{\neg(A \to B) \vdash \neg(A \to B)}\,\mathrm{Ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\,\mathrm{Ax} \quad \dfrac{}{A \vdash A}\,\mathrm{Ax}}{\neg A, A \vdash \bot}\,\neg_{\mathsf{e}}}{A \vdash \neg\neg A}\,\neg_{\mathsf{i}} \quad \neg\neg A, (\Gamma)^* \vdash \neg\neg B}{A, (\Gamma)^* \vdash \neg\neg B}\,\mathsf{Comp} \quad \dfrac{\dfrac{\dfrac{}{\neg(A \to B) \vdash \neg(A \to B)}\,\mathrm{Ax} \quad \dfrac{\dfrac{}{B \vdash B}\,\mathrm{Ax}}{B \vdash A \to B}\,\to_{\mathsf{i}}}{B, \neg(A \to B) \vdash \bot}\,\neg_{\mathsf{e}}}{\neg(A \to B) \vdash \neg B}\,\neg_{\mathsf{i}}}{A, \neg(A \to B), (\Gamma)^* \vdash \bot}\,\neg_{\mathsf{e}}}{A, \neg(A \to B), (\Gamma)^* \vdash B}\,\mathsf{efq}}{\neg(A \to B), (\Gamma)^* \vdash A \to B}\,\to_{\mathsf{i}}}{\dfrac{\neg(A \to B), (\Gamma)^* \vdash \bot}{(\Gamma)^* \vdash \neg\neg(A \to B)}\,\neg_{\mathsf{i}}}\,\neg_{\mathsf{e}}
$$

where Comp is a rule for explicit composition (of derivations), like the one used in [18].

Note that the step of efq cannot be avoided. It is for this reason that the $\to_{\mathsf{i}}$ rule has to be dropped in order to obtain the Glivenko's theorem for minimal logic (cf. Remark 3).

The rule of $\neg_{\mathsf{i}}$ is treated similarly to $\to_{\mathsf{i}}$. However, Tennant works modulo the minimal equivalences $\neg\neg\neg A \leftrightarrow \neg A$ and $\neg\neg\bot \leftrightarrow \bot$; he can thus simplify the form taken by the conclusion of the translated rule, obtaining

$$
\dfrac{\neg\neg A, (\Gamma)^* \vdash \bot}{(\Gamma)^* \vdash \neg A}\,\neg_i^*
$$

Moreover, the translation of the raa rule can be simply obtained as a special case of the previous rule, namely by taking $A = \neg B$, i.e.

$$
\dfrac{\neg\neg\neg B, (\Gamma)^* \vdash \bot}{(\Gamma)^* \vdash \neg\neg B}\,\mathsf{raa}^*
$$

Once the rules of classical logic have been translated into derivable rules of minimal or of intuitionistic logic of the form we just described, it is easy to show that each derivation $\Pi : \Gamma \vdash A$ in classical logic can be translated into a derivation $\Pi' : (\Gamma)^* \vdash (A)^*$ in minimal or intuitionistic logic, by translating step-by-step each rule of Π. More precisely, the proof of Theorem 19 is given by induction on the length of the proof $\Pi : \Gamma \vdash A$ ([29, p. 273]). The base step is that of the axiom rule, which is trivial. The inductive step depends on the last rule s applied in Π. The inductive hypothesis guarantees, for each sub-derivation Π_i of Π, a derivation Π_i' of the desired form. Then, in order to obtain Π', it is sufficient to apply the translation of the rule s to these derivations Π_i'.

In some particular situations, Π could already contain a sub-derivation Π_i of the same conclusion $\neg\neg A$ (i.e. $(A)^*$) of the desired derivation Π', but from a set of assumptions which is only a subset of $(\Gamma)^*$. The two derivations Π_i' and Π' are thus the same, modulo some applications of the weakening rule. One could then simply take the sub-derivation Π_i', instead of reconstructing Π' step-by-step from Π. It is for this reason that Tennant formulates the Theorem 19 by considering a derivation Π' of $(\Gamma')^* \vdash (A)^*$, where $\Gamma' \subseteq \Gamma$.[23]

Note that if in the case of minimal logic we restrict to the fragment $\{\neg, \wedge, \vee, \bot, \exists\}$, and in the case of intuitionistic logic to the fragment $\{\neg, \wedge, \vee, \to, \bot, \exists\}$, then the translation function $(\cdot)^*$ is nothing but the Kuroda translation (see §6). More precisely, in the case of minimal logic we have that $(A)^* = \neg\neg(A)^{\mathsf{m}}$, while in the case of intuitionistic logic $(A)^* = \neg\neg(A)^{\mathsf{j}}$. In particular, when the translations $(\cdot)^{\mathsf{m}}$ and $(\cdot)^{\mathsf{j}}$ are applied to the two aforementioned fragments, they behave like the identity function, keeping the formulas on which they are applied invariant (see Definition 12). Thus, in order

[23]Note also that Tennant proves the Glivenko's theorem for the relevant versions of minimal and intuitionistic logic. However, we will not discuss this point here, since our article simply focuses on standard logical systems, and not on relevant ones.

to obtain a result analogue to Theorems 15.2 and 16.2., it would be sufficient to apply Theorem 19 to a certain derivation $\Pi : \Gamma \vdash A$, obtain then a derivation $\Pi' : (\Gamma)^* \vdash (A)^*$, and finally compose it with the derivation

$$\dfrac{\dfrac{\overline{\neg C \vdash \neg C}\ \text{Ax} \qquad \overline{C \vdash C}\ \text{Ax}}{\neg C, C \vdash \bot}\ \neg_{\mathsf{e}}}{C \vdash \neg\neg C}\ \neg_{\mathsf{i}} \tag{26}$$

for every formula $\neg\neg C$ present in $(\Gamma)^*$. In this way a new derivation $\Pi'' : \Gamma \vdash (A)^*$ is obtained (see [29, p. 274]).[24]

Tennant remarks that if one does not want to drop the universal quantifier – both in the case of minimal and intuitionistic logic – then he has to proceed in the following way. As it concerns the $\forall_{\mathsf{e}}$ rule, one can simply translate it according to the the schema (25). While, as it concerns the $\forall_{\mathsf{i}}$ rule, one has to proceed inductively (see [29, p. 247]). Consider a derivation $\Pi : \Gamma \vdash \forall x A$ in NK, ending with $\forall_{\mathsf{i}}$. By the induction hypothesis, we can apply Theorem 19 to the sub-derivation $\Pi_1 : \Gamma \vdash A$ and obtain the derivation $\Pi_1' : (\Gamma)^* \vdash (A)^*$. Apply then the $\forall_{\mathsf{i}}$ rule, and finally add a double negation using (26). In other words, the derivation

$$\Pi \;=\; \dfrac{\begin{matrix} \vdots\, \Pi_1 \\ \Gamma \vdash A \end{matrix}}{\Gamma \vdash \forall x A}\ \forall_{\mathsf{i}}$$

of NK is transformed into the derivation

$$\Pi' \;=\; \dfrac{\dfrac{\overline{\neg\forall x(A)^* \vdash \neg\forall x(A)^*} \qquad \dfrac{\begin{matrix} \vdots\, \Pi_1' \\ (\Gamma)^* \vdash (A)^* \end{matrix}}{(\Gamma)^* \vdash \forall x(A)^*}\ \forall_{\mathsf{i}}}{(\Gamma)^*, \neg\forall x(A)^* \vdash \bot}\ \neg_{\mathsf{e}}}{(\Gamma)^* \vdash \neg\neg\forall x(A)^*}\ \neg_{\mathsf{i}}$$

of either $\mathsf{NM} \smallsetminus \{\rightarrow_{\mathsf{i}}, \rightarrow_{\mathsf{e}}\}$ or NJ.

This seems to suggest that universal formulas can be translated in the following way: $(\forall x A)^* = \neg\neg\forall x(A)^*$, which corresponds to the original Kuroda translation, since $\neg\neg\forall x(A)^*$ corresponds either to $\neg\neg\forall x(A)^{\mathsf{m}'}$ or to $\neg\neg\forall x(A)^{\mathsf{j}'}$ (for the definition of $(\cdot)^{\mathsf{m}'}$ and $(\cdot)^{\mathsf{j}'}$ see p. 30).

However, this is not the case. The reason is that, by following Tennant's idea, the way in which universal formulas are translated is not a uniform one, but it depends on their position inside a derivation. In particular, it depends if they are in the position of hypothesis or if they are the conclusion of a $\forall_{\mathsf{i}}$. In the first case, only a double negation is put in front of them, while in the second case a double negation is also put in front of the quantified formula. This means that when the universal quantifier is added to the language, the translation set up by Tennant is no more functional at the level of formulas.

This same situation would occur if one tried to adapt Tennant's treatment of the universal quantifier in order to deal with implication in minimal logic: it would be possible to prevent from giving up implication, but the price to pay would be to have a translation from NK to NM which is not functional at the level of formulas.

[24] Note that this is very similar to the procedure we already discussed in order to deal with the translation of rules that discharge hypothesis (see p. 39, *supra*).

A.3 An alternative reduction strategy

As we already noticed (see Remark 3), the reduction steps that we defined in §4 make an essential use of intuitionistic logic only in the $\to_{\mathsf{i}}$ case – where the efq rule is explicitly introduced for defining the reduction step (18) at p. 21 – while in all the other cases the appeal to minimal logic is already sufficient for defining the sub-derivations preceding the application of the raa rule. It is for this reason that in order to prove the Glivenko's theorem for minimal logic, we had to give up implication, and define a translation $(\cdot)^{\mathsf{m}}$ such that $(A \to B)^{\mathsf{m}} = \neg A^{\mathsf{m}} \vee B^{\mathsf{m}}$.

We show here that, in fact, implication could be preserved, but the price to pay is to impose important restrictions on the form of the consequent of an implication, so that a certain uniformity and generality of the translation is lost.

Consider the following reduction:

$$\Pi = \begin{array}{c} \ulcorner A \urcorner^2, \ulcorner \neg B \urcorner^1 \\ \vdots\ \pi' \\ \dfrac{\dfrac{\bot}{B}\ \mathsf{raa}^1}{A \to B}\ {\to_{\mathsf{i}}^2} \\ \vdots\ \pi \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \ulcorner A \urcorner^2, \quad \dfrac{\dfrac{\ulcorner \neg (A \to B) \urcorner^3 \qquad \dfrac{\ulcorner B \urcorner^1}{A \to B}\ {\to_{\mathsf{i}}^0}}{\bot}\ \neg_{\mathsf{e}}}{\neg B}\ \neg_{\mathsf{i}}^1 \\ \vdots\ \pi' \\ \dfrac{\dfrac{\ulcorner \neg (A \to B) \urcorner^3 \qquad \dfrac{\dfrac{\dfrac{\bot}{\neg\neg B}\ \neg_{\mathsf{i}}^0}{B}\ (28)}{A \to B}\ {\to_{\mathsf{i}}^2}}{\bot}\ \neg_{\mathsf{e}}}{A \to B}\ \mathsf{raa}^3 \\ \vdots\ \pi \end{array} = \Pi' \tag{27}$$

The rule (28) is a derivable rule, obtained by appealing to the following theorem of minimal logic:

$$\vdash_{\mathsf{NM}} B \leftrightarrow \neg\neg B \tag{28}$$

where B is a *negative formula*, i.e. atomic formulas occur only negated in B, and B does not contain $\vee$ nor $\exists$ (see [30, p. 48]).

According to this new reduction, before applying the raa rule, only inferential steps coming from minimal logic are used. However, such reduction can be applied only when the consequent of the implication is a negative formula. The consequence is that, even if we can extract a translation from classical to minimal logic from this new reduction, this translation will not be uniform and general as the one given in Definition 12.

More precisely, when we restrict to the propositional case, the translation induced by replacing the reduction rule (18) at p. 21 with this new one corresponds to a translation $(\cdot)^{\mathsf{m}^*}$ which behaves like $(\cdot)^{\mathsf{m}}$, except for the implication, which is defined as follows:

$$(A \to B)^{\mathsf{m}^*} = A^{\mathsf{m}^*} \to \neg\neg B^{\mathsf{m}^*} \qquad \text{(where } B \text{ is a negative formula)}$$

The same considerations can be made in the case of the $\forall_{\mathsf{i}}$ rule. In particular, we could use the following reduction:

$$\Pi = \begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\, \pi' \\ \dfrac{\dfrac{\bot}{A}\,\mathsf{raa}^1}{\forall x A}\,\forall_\mathsf{i} \\ \vdots\, \pi \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \ulcorner \neg A \urcorner^1 \\ \vdots\, \pi' \\ \dfrac{\bot}{\neg\neg A}\,\neg_\mathsf{i}^1 \\ \text{- - - -}\ (28) \\ \dfrac{A}{\forall x A}\,\forall_\mathsf{i} \\ \vdots\, \pi \end{array} = \Pi' \tag{29}$$

and then define $(\cdot)^{\mathsf{m}^*}$ for the universal quantifier as follows:

$$(\forall x A)^{\mathsf{m}^*} = \forall x (A)^{\mathsf{m}^*} \qquad\qquad \text{(where } A \text{ is a negative formula)}$$

This new translation $(\cdot)^{\mathsf{m}^*}$ is the same as the one defined in [6, p. 249],[25] except for the restriction about negative formulas. The need of this restriction seems to be explained by the fact the our translation is *directly* defined from classical to minimal logic (at the level of proofs' reduction), while in [6, §6] it is obtained by making an intermediary step through intuitionistic logic (at the level of formulas): first, classical logic is embedded into intuitionistic logic via the Kolmogorov translation; secondly, the translated formulas obtained in this way are embedded into minimal logic via a set of simplification rules definable in minimal logic itself and reducing the number of negations present in a formula. In particular, passing through the Kolmogorov translation allows one to put negations in front of atomic formulas and to eliminate the occurrences of $\vee$ or $\exists$ by using equivalences provable in minimal logic, like $\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow \neg A \wedge \neg B$ or $\neg\exists x \neg\neg A \leftrightarrow \forall x \neg A$. In this way, the condition on (28) is always respected, and thus no restrictions have to be explicitly imposed.

[25] In fact, this translation was already present in [12, p. 159], but only for the intuitionistic case.