The Visser topology of lambda calculus

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Abstract

A longstanding open problem in lambda calculus is whether there exists a non-syntactical model of the untyped lambda calculus whose theory is exactly the least λ-theory $\lambda\beta$. In this paper we make use of the Visser topology for investigating the related question of whether the equational theory of a model can be recursively enumerable (r.e. for brevity). We introduce the notion of an effective model *of lambda calculus and prove the following results: (i) The equational theory of an effective model cannot be $\lambda\beta$, $\lambda\beta\eta$; (ii) The order theory of an effective model cannot be r.e.; (iii) No effective model living in the stable or strongly stable semantics has an r.e. equational theory. Concerning Scott's semantics, we investigate the class of graph models and prove the following, where "graph theory" is a shortcut for "theory of a graph model": (iv) There exists a minimum order graph theory (for equational graph theories this was proved in [9, 10]). (v) The minimum equational/order graph theory is the theory of an effective graph model. (vi) No order graph theory can be r.e. (vii) Every equational/order graph theory is the theory of a graph model having a countable web. This last result proves that the class of graph models enjoys a kind of (downwards) Löwenheim-Skolem theorem, and it answers positively Question 3 in [4, Section 6.3] for the class of graph models.*

1. Introduction

Lambda theories are equational extensions of the untyped lambda calculus closed under derivation. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to a possible operational (observational) semantics of lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function. Although researchers have mainly focused their interest on a limited number of them, the class of λ-theories constitutes a very rich and complex structure (see e.g. [2, 4, 5]). Syntactical techniques are usually difficult to use in the study of λ-theories. Therefore, semantic methods have been extensively investigated.

Topology is at the center of the known approaches to give models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, was successfully used to show that all the unsolvable λ-terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 2, 4, 12, 23]. Scott continuous semantics [26] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [7]) and the strongly stable semantics (Bucciarelli-Ehrhard [8]) are refinements of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. All these semantics are structurally and equationally rich [19, 20] in the sense that, in each of them, it is possible to build up $2^{\aleph_0}$ models inducing pairwise distinct λ-theories. Nevertheless, the above denotational semantics are equationally *incomplete*: they do not even match all possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [16] for the continuous semantics and by Bastonero-Gouy [3] for the stable semantics. In [24, 25] Salibra has shown in a uniform way that all semantics (including the strongly stable semantics), which involve monotonicity with respect to some partial order and have a bottom element, fail to induce a continuum of λ-theories. Manzonetto and Salibra [22] have recently shown an algebraic incompleteness theorem for lambda calculus: the semantics of lambda calculus given in terms of models, which are directly indecomposable as combinatory algebras (i.e., they cannot be decomposed as the Cartesian product of two other non-trivial combinatory algebras), is incomplete, although it strictly includes the continuous semantics, its stable and strongly stable refinements and the term models of all semi-sensible λ-theories. The proof of incompleteness is based on a generalization of the Stone representation theorem for Boolean algebras to combinatory algebras.

Berline has raised in [4] the natural question of whether, given a class of models of lambda calculus, there is a minimum λ-theory represented by it. This question relates to the longstanding open problem proposed by Barendregt about the existence of a continuous model or, more generally, of a non-syntactical model of $\lambda\beta$ ($\lambda\beta\eta$). Di Gianantonio, Honsell and Plotkin [13] have shown that Scott continuous semantics admits a minimum theory, at least if we restrict to extensional models. Another result of [13], in the same spirit, is the construction of an extensional model whose theory is $\lambda\beta\eta$, a fortiori minimal, in a weakly-continuous semantics. However, the construction of this model starts from the term model of $\lambda\beta\eta$, and hence it cannot be seen as having a purely non syntactical presentation. More recently, Bucciarelli and Salibra [9, 10] have shown that the graph semantics (that is, the semantics of lambda calculus given in terms of graph models) admits a minimum λ-theory different from $\lambda\beta$. Graph models, isolated in the seventies by Plotkin, Scott and Engeler (see e.g. [2]) within the continuous semantics, have been proved useful for giving proofs of consistency of extensions of lambda calculus and for studying operational features of lambda calculus (see [4]).

Topology is an important instrument not only to obtain models of lambda calculus, but also to investigate syntactical properties of λ-terms. The Scott topology is the most frequently used in this field. The Visser topology has been defined by Visser [29] in the eighties in an orthogonal way with respect to the Scott topology: the former is strictly connected with classical recursion theory whilst the latter is related to domain theory. In this paper we make use of the Visser topology for investigating the question of whether the equational theory of a model can be recursively enumerable (r.e. for brevity). As far as we know, this problem was first raised in [5], where it is conjectured that no graph model can have an r.e. theory. But we expect that this could indeed be true for all models living in the continuous semantics, and its refinements, and in the present paper we extend this conjecture.

Conjecture 1 *The equational theory of every model living in Scott continuous semantics, or in one of its refinements, is not r.e.*

It should be noted that, since sensible λ-theories cannot be r.e., our conjecture is only open for non sensible models.

We find natural to concentrate our investigation on models with built-in effectivity properties, i.e. models based on effective domains (see Section 2.2). It seems indeed reasonable to think that, if effective models do not even succeed to have an r.e. theory, then the other ones have no chance to succeed.

In this paper, starting from the known notion of an effective domain, we introduce a general notion of an *effective model* of lambda calculus and we study the main properties of these models. Effective models are omni-present in the continuous, stable and strongly stable semantics. In particular, all the models which have been introduced individually in the literature can easily be proved effective. As far as we know, only Giannini and Longo [14] have introduced a notion of an effective model; but their definition is *ad hoc* for two particular models (Scott's P_ω and Plotkin's T_ω) and their results depend on the fact that these models have a very special (and well known) common theory.

In this paper we show the following results:

(i) The equational theory of an effective model cannot be $\lambda\beta$, $\lambda\beta\eta$.

(ii) No effective model can have an r.e. order theory.

(iii) No effective model living in the stable or strongly stable semantics has an r.e. equational theory.

Concerning Scott continuous semantics we continue our investigation of the class of graph models (see [6, 9, 10, 11]) and show that:

(iv) No effective graph model, freely generated by a "partial model" which is finite modulo its group of automorphisms, has an r.e. equational theory.

(v) There exists a minimum order graph theory (where "graph theory" means "theory of a graph model").

(vi) No order graph theory can be r.e.

(vii) For any β-normal form M, there exists a non-empty β-closed co-r.e. set $\mathcal{U}$ of unsolvables whose interpretations are, in all graph models, under the interpretation of M.

The last two results are a consequence of one of the main results of the paper: the minimum equational/order graph theory is the theory of an effective graph model.

Another main result of the paper is a kind of (downwards) Löwenheim-Skolem theorem for the class of graph models: every equational/order graph theory is the theory of a graph model having a countable web (this result positively answers Question 3 in [4, Section 6.3] for the class of graph models). As a consequence, every graph theory (we know from Kerth [19] that there exists a continuum of them) is the theory of a graph model whose web is the set of natural numbers.

2. Preliminaries

To keep this article self-contained, we summarize some definitions and results concerning lambda calculus and topology that we need in the subsequent part of the paper. With regard to the lambda calculus we follow the notation and terminology of [2]. The main reference for topology is [17].

We denote by $\mathbb{N}$ the set of natural numbers. The complement of a recursively enumerable set (r.e. set for short) is called a *co-r.e.* set. If both A and its complement are r.e., A is called *decidable*. We will denote by $\mathcal{RE}$ the collection of all r.e. subsets of $\mathbb{N}$.

A numeration of a set A is a map from $\mathbb{N}$ onto A. $\mathcal{W} : \mathbb{N} \to \mathcal{RE}$ denotes the usual numeration of r.e. sets (i.e., $\mathcal{W}_n$ is the domain of the n-th computable function ϕ_n).

A family $\mathcal{B}$ of open sets of a topological space (X, τ_X) is a *base* for the topology τ_X if every open set is a union of elements of $\mathcal{B}$.

Let X, Y be topological spaces and $\mathcal{B}, \mathcal{B}'$ be bases respectively for τ_X and τ_Y. A function $f : X \to Y$ is called *strongly continuous* w.r.t. $\mathcal{B}, \mathcal{B}'$ if $f^{-1}(U) \in \mathcal{B}$ for all $U \in \mathcal{B}'$.

2.1. Lambda calculus and lambda models

Λ and Λ^o are, respectively, the set of λ-terms and of closed λ-terms. Concerning specific λ-terms we set: $\mathbf{i} \equiv \lambda x.x$; $\mathrm{T} \equiv \lambda xy.x$; $\mathrm{F} \equiv \lambda xy.y$; $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

We denote $\alpha\beta$-conversion by $\lambda\beta$. A *λ-theory* $\mathcal{T}$ is a congruence on Λ (with respect to the operators of abstraction and application) which contains $\lambda\beta$. We write $M =_\mathcal{T} N$ for $(M, N) \in \mathcal{T}$. If $\mathcal{T}$ is a λ-theory, then $[M]_\mathcal{T}$ denotes the set $\{N : N =_\mathcal{T} M\}$. A λ-theory $\mathcal{T}$ is: *consistent* if $\mathcal{T} \neq \Lambda \times \Lambda$; *extensional* if it contains the equation $\mathbf{i} = \lambda xy.xy$; *recursively enumerable* if the set of Gödel numbers of all pairs of $\mathcal{T}$-equivalent λ-terms is r.e. Finally, $\lambda\beta\eta$ is the least extensional λ-theory.

The λ-theory $\mathcal{H}$, generated by equating all the unsolvable λ-terms, is consistent by [2, Theorem 16.1.3]. A λ-theory $\mathcal{T}$ is *sensible* if $\mathcal{H} \subseteq \mathcal{T}$, while it is *semi-sensible* if it contains no equations of the form $U = S$ where S is solvable and U unsolvable. Sensible theories are semi-sensible and are never r.e. (see [2]).

It is well known [2, Chapter 5] that a model of lambda calculus (*λ-model*, for short) can be defined as a reflexive object in a ccc (Cartesian closed category) $\mathbf{C}$, that is to say a triple $(D, \mathcal{F}, \lambda)$ such that D is an object of $\mathbf{C}$ and $\mathcal{F} : D \to [D \to D]$, $\lambda : [D \to D] \to D$ are morphisms such that $\mathcal{F} \circ \lambda = id_{[D \to D]}$. In the following we will mainly be interested in Scott's ccc of cpos and Scott continuous functions (*continuous semantics*), but we will also draw conclusions for Berry's ccc of DI–domains and stable functions (*stable semantics*), and for Ehrhard's ccc of DI-domains with coherence and strongly stable functions between them (*strongly stable semantics*).

We recall that DI-domains are special Scott domains, and that Scott domains are special cpos (see, e.g., [28]).

Let D be a cpo. The partial order of D will be denoted by $\sqsubseteq_D$. We let Env_D be the set of environments ρ mapping the set Var of variables of lambda calculus into D. For every $x \in Var$ and $d \in D$ we denote by $\rho[x := d]$ the environment ρ' which coincides with ρ, except on x, where ρ' takes the value d. A reflexive cpo D generates a λ-model $\mathcal{D} = (D, \mathcal{F}, \lambda)$ of lambda calculus with the interpretation of a λ-term defined as follows:

$$x^\mathcal{D}_\rho = \rho(x);\ (MN)^\mathcal{D}_\rho = \mathcal{F}(M^\mathcal{D}_\rho)(N^\mathcal{D}_\rho);\ (\lambda x.M)^\mathcal{D}_\rho = \lambda(f),$$

where f is defined by $f(d) = M^\mathcal{D}_{\rho[x:=d]}$ for all $d \in D$.
In the following $\mathcal{F}(d)(e)$ will also be written $d \cdot e$ or de.

Each λ-model $\mathcal{D}$ induces a λ-theory, denoted here by Eq$(\mathcal{D})$, and called *the equational theory of* $\mathcal{D}$. Thus, $M = N \in$ Eq$(\mathcal{D})$ if, and only if, M and N have the same interpretation in $\mathcal{D}$.

A reflexive cpo $\mathcal{D}$ induces also an *order theory* Ord$(\mathcal{D}) = \{M \sqsubseteq N : M^\mathcal{D}_\rho \sqsubseteq_D N^\mathcal{D}_\rho$ for all environments $\rho\}$.

2.2. Effective domains

We give here the definition of an *effective domain*, also called in the literature "effectively given domain" (see, e.g., [28, Chapter 10]).

A triple $\mathcal{D} = (D, \sqsubseteq_D, d)$ is called an *effective domain* if $(D, \sqsubseteq_D)$ is a Scott domain and d is a numeration of the set $K(\mathcal{D})$ of its compact elements such that the relations "d_m and d_n have an upper bound" and "$d_n = d_m \sqcup d_k$" are both decidable.

We recall that an element v of an effective domain $\mathcal{D}$ is said *r.e.* (*decidable*) if the set $\{n : d_n \sqsubseteq_D v\}$ is r.e. (decidable); we will write $\mathcal{D}^{r.e.}$ ($\mathcal{D}^{dec}$) for the set of r.e. (decidable) elements of $\mathcal{D}$. The set $K(\mathcal{D})$ of compact elements is included within $\mathcal{D}^{dec}$.

Using standard techniques of recursion theory it is possible to get in a uniform way a numeration $\xi : \mathbb{N} \to \mathcal{D}^{r.e.}$ which is *adequate* in the sense that the relation $d_k \sqsubseteq_D \xi_n$ is r.e. in (k, n) and the inclusion mapping $\iota : K(\mathcal{D}) \to \mathcal{D}^{r.e.}$ is computable w.r.t. d, ξ.

If $\mathcal{D}$ and $\mathcal{D}'$ are effective domains, then the Cartesian product $D \times D'$ as well as the set $[D \to D']$ of Scott continuous functions from $(D, \sqsubseteq_D)$ into $(D', \sqsubseteq_{D'})$, ordered pointwise, can be endowed canonically with a structure of effective domain. These effective domains will be denoted, respectively, by $\mathcal{D} \times \mathcal{D}'$ and $[\mathcal{D} \to \mathcal{D}']$. Then the full subcategory $\mathbf{ED}$ of the category of Scott-domains with effective domains as objects and continuous functions as morphisms is a ccc.

We recall that a continuous function $f : D \to D'$ is an r.e. element in the effective domain of Scott continuous functions (i.e., $f \in [\mathcal{D} \to \mathcal{D}']^{r.e.}$) if, and only if, its restriction $f\!\restriction : \mathcal{D}^{r.e.} \to \mathcal{D}'^{r.e.}$ is computable w.r.t. ξ, ξ', i.e., there is a computable map $g : \mathbb{N} \to \mathbb{N}$ such that $f(\xi_n) = \xi'_{g(n)}$. In such a case we say that g *tracks* f.

The key example of an effective domain is $(\mathcal{P}(\mathbb{N}), \subseteq, d)$ where d is some standard effective numeration of the finite subsets of $\mathbb{N}$. Here r.e. (decidable) elements are the r.e. (decidable) subsets of $\mathbb{N}$ and the adequate numeration of r.e. elements is the usual map $n \mapsto \mathcal{W}_n$ (where $\mathcal{W}_n$ is the domain of the n-th computable function).

A subset V of $\mathcal{D}^{r.e.}$ is called *completely r.e.* if $\{n : \xi_n \in V\}$ is r.e. In a similar way we can define *completely co-r.e.* and *completely decidable* sets, but a completely decidable set V is always trivial, i.e., $V = \emptyset$ or $V = \mathcal{D}^{r.e.}$.

The generalized Myhill-Shepherdson theorem [28, Theorem 10.5.2] states that, for every completely r.e. set $V \subseteq \mathcal{D}^{r.e.}$, there exists an r.e. set $A \subseteq \mathbb{N}$ such that:

$$V = \{v \in \mathcal{D}^{r.e.} : (\exists n \in A)\ \xi_n \sqsubseteq_D v \text{ and } \xi_n \in K(\mathcal{D})\}.$$

In particular, completely r.e. (co-r.e.) sets are upper (lower) subsets of $\mathcal{D}^{r.e.}$ with respect to $\sqsubseteq_D$.

2.3. Graph models

The class of graph models belongs to Scott continuous semantics (see [5] for a complete survey on this class of models). Historically, the first graph model was Scott's P_ω, which is also known in the literature as "the graph model". "Graph" referred to the fact that the continuous functions were encoded in the model via (a sufficient fragment of) their graph.

As a matter of notation, for every set G, G^* is the set of all finite subsets of G, while $\mathcal{P}(G)$ is the powerset of G.

Definition 2.1 *A* graph model $\mathcal{G}$ *is a pair* $(G, c_\mathcal{G})$*, where* G *is an infinite set, called the* web *of* $\mathcal{G}$*, and* $c_\mathcal{G} : G^* \times G \to G$ *is an injective total function.*

The function $c_\mathcal{G}$ is used to encode a fragment of the graph of a Scott continuous function $f : \mathcal{P}(G) \to \mathcal{P}(G)$ as a subset $\lambda(f)$ of G:

$$\lambda(f) = \{c_\mathcal{G}(a, \alpha) : \alpha \in f(a) \text{ and } a \in G^*\}. \qquad (1)$$

Any graph model $\mathcal{G}$ allows us to define a λ-model through the reflexive cpo $(\mathcal{P}(G), \subseteq)$ determined by two Scott continuous functions $\lambda : [\mathcal{P}(G) \to \mathcal{P}(G)] \to \mathcal{P}(G)$ and $\mathcal{F} : \mathcal{P}(G) \to [\mathcal{P}(G) \to \mathcal{P}(G)]$. The function λ is defined in (1), while $\mathcal{F}$ is defined as follows, for all $X, Y \subseteq G$:

$$\mathcal{F}(X)(Y) = \{\alpha \in G : (\exists a \subseteq Y)\ c_\mathcal{G}(a, \alpha) \in X\}.$$

For more details we refer the reader to Berline [4] and Barendregt [2].

The interpretation of a λ-term M into a λ-model has been defined in Section 2.1. However, in this context we can make explicit the interpretation $M^\mathcal{G}_\rho$ of a λ-term M as follows:

- $x^\mathcal{G}_\rho = \rho(x)$
- $(MN)^\mathcal{G}_\rho = \{\alpha \in G : (\exists a \subseteq N^\mathcal{G}_\rho)\ c_\mathcal{G}(a, \alpha) \in M^\mathcal{G}_\rho\}$
- $(\lambda x.M)^\mathcal{G}_\rho = \{c_\mathcal{G}(a, \alpha) : \alpha \in M^\mathcal{G}_{\rho[x:=a]}\}$

We turn now to the interpretation of Ω in graph models (the details of the proof are, for example, worked out in [6, Lemma 4]).

Lemma 2.2 $\alpha \in \Omega^\mathcal{G}$ *if, and only if, there exists* $a \subseteq (\lambda x.xx)^\mathcal{G}$ *such that* $c_\mathcal{G}(a, \alpha) \in a$.

In the following we use the terminology "*graph theory*" as a shortcut for "theory of a graph model".

It is well known that the equational graph theories are never extensional and that there exists a continuum of them (see [19]). In [9, 10] the existence of a minimum equational graph theory was proved and it was also shown that this minimum theory is different from $\lambda\beta$.

The completion method for building graph models from "partial pairs" was initiated by Longo in [21] and developed on a wide scale by Kerth in [19, 20]. This method is useful to build models satisfying prescribed constraints, such as domain equations and inequations, and it is particularly convenient for dealing with the equational theories of graph models.

Definition 2.3 *A* partial pair $\mathcal{A}$ *is given by a set* A *and by a partial, injective function* $c_\mathcal{A} : A^* \times A \to A$.

A partial pair is *finite* if A is a finite set, and it is a graph model if, and only if, $c_\mathcal{A}$ is total.

The interpretation of a λ-term in a partial pair $\mathcal{A}$ is defined in the obvious way. For example, we have:

- $(MN)^\mathcal{A}_\rho = \{\alpha \in A : (\exists a \subseteq N^\mathcal{A}_\rho)\ [(a, \alpha) \in dom(c_\mathcal{A}) \wedge c_\mathcal{A}(a, \alpha) \in M^\mathcal{A}_\rho]\}$.

Definition 2.4 *Let* $\mathcal{A}$ *be a partial pair. The* completion *of* $\mathcal{A}$ *is the graph model* $\mathcal{E}_\mathcal{A}$ *defined as follows:*

- $E_\mathcal{A} = \bigcup_{n \in \mathbb{N}} (E_\mathcal{A})_n$*, where* $(E_\mathcal{A})_0 = A$ *and* $(E_\mathcal{A})_{n+1} = (E_\mathcal{A})_n \cup (((E_\mathcal{A})^*_n \times (E_\mathcal{A})_n) - dom(c_\mathcal{A}))$.
- *Given* $a \in E^*_\mathcal{A}$, $\alpha \in E_\mathcal{A}$,

$$c_{\mathcal{E}_\mathcal{A}}(a, \alpha) = \begin{cases} c_\mathcal{A}(a, \alpha) & \text{if } c_\mathcal{A}(a, \alpha) \text{ is defined} \\ (a, \alpha) & \text{otherwise} \end{cases}$$

A notion of *rank* can be naturally defined on the completion $\mathcal{E}_\mathcal{A}$ of a partial pair $\mathcal{A}$. The elements of A are the elements of rank 0, while an element $\alpha \in E_\mathcal{A} - A$ has rank n if $\alpha \in (E_\mathcal{A})_n$ and $\alpha \notin (E_\mathcal{A})_{n-1}$.

Graph models, such as Scott's P_ω [2], Park's $\mathcal{P}$ [4] and Engeler's $\mathcal{E}$ [4], can be viewed as the completions of suitable partial pairs. In fact, P_ω, $\mathcal{P}$ and $\mathcal{E}$ are respectively isomorphic to the completions of $\mathcal{A} = (\{0\}, c_A)$ (with $c_\mathcal{A}(\emptyset, 0) = 0$), $\mathcal{B} = (\{0\}, c_\mathcal{B})$ (with $c_\mathcal{B}(\{0\}, 0) = 0$) and $\mathcal{C} = (\{0\}, c_\mathcal{C})$ (with $c_\mathcal{C}$ the empty function).

Let $\mathcal{A}$ and $\mathcal{B}$ be two partial pairs. A *morphism* from $\mathcal{A}$ into $\mathcal{B}$ is a map $f : A \to B$ such that $(a, \alpha) \in dom(c_\mathcal{A})$ if, and only if, $(fa, f\alpha) \in dom(c_\mathcal{B})$ and, in such a case $f(c_\mathcal{A}(a, \alpha)) = c_\mathcal{B}(fa, f\alpha)$. Isomorphisms and automorphisms can be defined in the obvious way. $Aut(\mathcal{A})$ denotes the group of automorphisms of the partial pair $\mathcal{A}$.

Lemma 2.5 *Let $\mathcal{G}, \mathcal{G}'$ be graph models and $f : \mathcal{G} \to \mathcal{G}'$ be a morphism. If $M \in \Lambda$ and $\alpha \in M^{\mathcal{G}}_\rho$, then $f\alpha \in M^{\mathcal{G}'}_{f\circ\rho}$.*

3. The Visser topology

In this section we recall the definition of the Visser topology over the set of λ-terms and we show some of its basic properties.

Definition 3.1 ([29]) *The Visser topology on Λ (or Λ^o) is the topology whose basic open sets are the co-r.e. subsets of Λ (Λ^o) closed under β-conversion.*

In the remaining part of this paper we always assume that the set Λ (or Λ^o) of λ-terms is endowed with the Visser topology.

Proposition 3.2 ([29, Theorem 2.5]) *The Visser topology is hyperconnected (i.e., the intersection of two arbitrary non-empty open sets is non-empty).*

The set of unsolvables is a basic Visser open set. It follows from hyperconnectedness that every Visser open set contains unsolvable λ-terms and that a non trivial Visser open set can never be r.e.

Proposition 3.3 *Every unary λ-definable map on Λ is Visser strongly continuous.*

Proof. The inverse image of an r.e. set via a computable function is an r.e. set. ■

Proposition 3.4 *The application function $\cdot : \Lambda \times \Lambda \to \Lambda$ is Visser strongly continuous in each coordinate, but it is not continuous when $\Lambda \times \Lambda$ is equipped with the product topology of the Visser topology on Λ.*

Proof. The continuity in each coordinate follows from the fact that the application operator is computable. We now show that the application function is not continuous. Let ψ be defined on $\Lambda \times \Lambda$ by $\psi(M, N) = \Omega MN$, and let $\mathcal{T}$ be the r.e. λ-theory generated by the equation $\Omega xx = \Omega$, where x is a variable. Since $\mathcal{T} \subseteq \mathcal{H}$ (where $\mathcal{H}$ is the least sensible λ-theory) obviously $\mathcal{T}$ is consistent. It was shown in Salibra [25] that $\Omega MN =_\mathcal{T} \Omega$ if, and only if, $M =_\mathcal{T} N$. This implies that the non-empty set $V = \{(M, N) : \Omega MN \neq_\mathcal{T} \Omega\}$ does not meet the diagonal. Since $\mathcal{T}$ is r.e., $[\Omega]_\mathcal{T}$ is r.e. and hence its complement O is Visser open (and non empty). If the application were continuous, then ψ would be continuous, and $V = \psi^{-1}(O)$ would be open in the product topology. Then there would exist two Visser open sets W and W' such that $W \times W' \subseteq V$. Since the Visser topology is hyperconnected, $W \cap W' \neq \emptyset$. Hence V would meet the diagonal, which lead us to a contradiction. ■

We conclude this section by showing a new topological proof, based on Visser topology, of the genericity lemma of lambda calculus. We recall that a classic proof of the genericity lemma, due to Barendregt (see [2, Theorem 14.3.24]), is obtained by using the tree topology on Λ which is induced by the Scott topology on the set of Böhm trees. The most difficult part of Barendregt's proof is to show that the contexts (i.e., λ-terms with occurrences of a hole $[-]$) are continuous maps w.r.t. the tree topology.

Lemma 3.5 *(Genericity Lemma) Let $U, N \in \Lambda$, where U is unsolvable and N is β-normal. Then for all contexts $C[-]$*

$$C[U] =_{\lambda\beta} N \Rightarrow \forall M \in \Lambda \quad C[M] =_{\lambda\beta} N.$$

Proof. The set $\{M : M \neq_{\lambda\beta} N\}$ is Visser open since it is co-r.e. and β-closed. As the map defined by $M \mapsto C[M]$ is computable and hence Visser strongly continuous, the set $O = \{M : C[M] \neq_{\lambda\beta} N\}$ is a Visser open set, *not* containing the unsolvable U. Since N is normal $[N]_\mathcal{H} = [N]_{\lambda\beta}$ [2, Theorem 16.1.9], where $\mathcal{H}$ is the least sensible λ-theory. Hence the Visser open set O is a union of $\mathcal{H}$-equivalence classes, not containing unsolvables; since the set of unsolvable is Visser open, then hyperconnectedness implies that $O = \emptyset$, which proves the genericity lemma. ■

4. Effective lambda models

In this section we introduce the notion of an effective λ-model and we study the main properties of these models. We show that the order theory of an effective λ-model is not r.e. and that its equational theory is different from $\lambda\beta$, $\lambda\beta\eta$. Effective λ-models are omni-present in the continuous, stable and strongly stable semantics (see Section 4.1). In particular, all the λ-models which have been introduced individually in the literature, to begin with Scott's $\mathcal{D}_\infty$, can be easily proved effective.

We introduce the Visser topology on the set $\mathcal{D}^{r.e.}$ consisting of the r.e. elements of the effective domain $\mathcal{D}$.

Definition 4.1 *Let $\mathcal{D}$ be an effective domain. The* Visser topology *on $\mathcal{D}^{r.e.}$ is the topology whose basic open sets are the completely co-r.e. subsets of $\mathcal{D}^{r.e.}$.*

Proposition 4.2 *1. The Visser topology on $\mathcal{D}^{r.e.}$ is hyperconnected.*

2. If $e \in \mathcal{D}^{dec}$, then $\{c \in \mathcal{D}^{r.e.} : c \sqsubseteq_D e\}$ is a basic Visser open set.

Proof. (1) Any non-empty Visser open set contains $\bot_D$, since the completely co-r.e. sets are lower sets with respect to $\sqsubseteq_D$.

(2) Straightforward. ∎

In the following we assume that the set $\mathcal{D}^{r.e.}$ of r.e. elements of an effective domain $\mathcal{D}$ is always equipped with the Visser topology.

The following natural definition is enough to force the interpretation function of λ-terms to be strongly Visser continuous from Λ^o into $\mathcal{D}^{r.e.}$. However, other results of this paper will need a more powerful notion. That is the reason why we only speak of "weak effectivity" here.

Definition 4.3 *A λ-model is called* weakly effective *if it is a reflexive object $(\mathcal{D}, \mathcal{F}, \lambda)$ in the category* **ED** *and, $\mathcal{F} \in [\mathcal{D} \to [\mathcal{D} \to \mathcal{D}]]$ and $\lambda \in [[\mathcal{D} \to \mathcal{D}] \to \mathcal{D}]$ are r.e. elements.*

In the following, for brevity, a weakly effective λ-model $(\mathcal{D}, \mathcal{F}, \lambda)$ will be denoted by $\mathcal{D}$.

We fix bijective effective numerations $\nu_\Lambda : \mathbb{N} \to \Lambda$ of the set of λ-terms and $\nu_{var} : \mathbb{N} \to Var$ of the set of variables of lambda calculus. In particular this gives to the set Env_D of all environments a structure of effective domain. $\Lambda_\perp = \Lambda \cup \{\perp\}$ is the usual flat domain of λ-terms. The element $\perp$ is always interpreted as $\perp_D$ in a cpo $(D, \sqsubseteq_D)$.

Proposition 4.4 *Let $\mathcal{D}$ be a weakly effective λ-model. Then the function f mapping $(\rho, M) \mapsto M^{\mathcal{D}}_\rho$ is an element of $[Env_D \times \Lambda_\perp \to \mathcal{D}]^{r.e.}$.*

Proof. (Sketch) The proof is by structural induction on M. It is possible to show the existence of a partial computable map ϕ tracking f. The only difficult case is $M \equiv \lambda x.N$. Since λ is r.e. it is sufficient to show that the function $g : e \mapsto N^{\mathcal{D}}_{\rho[x:=e]}$ is also r.e. Once shown that $h : (\rho, x, e) \mapsto \rho[x := e]$ is r.e., from the induction hypothesis it follows that the function $g'(\rho, x, e) = f(h(\rho, x, e), N)$ is r.e. Then by appying the s-m-n theorem of classic recursion theory to the computable function tracking g' we obtain a computable function tracking g, which is then r.e. ∎

Corollary 4.5 *For every environment $\rho \in (Env_D)^{r.e.}$, the interpretation function $M \mapsto M^{\mathcal{D}}_\rho$ of λ-terms is strongly Visser continuous from Λ into $\mathcal{D}^{r.e.}$.*

Proof. The interpretation map is computable and the inverse image of an r.e. set via a computable map is r.e. ∎

From Corollary 4.5 it follows that the restriction of the interpretation function to closed λ-terms is also strongly Visser continuous from Λ^o into $\mathcal{D}^{r.e.}$.

Notation 4.6 *We define for any $e \in D$ and $M \in \Lambda^o$:*
(i) $e^- \equiv \{P \in \Lambda^o : P^{\mathcal{D}} \sqsubseteq_D e\}$;
(ii) $M^- \equiv \{P \in \Lambda^o : P^{\mathcal{D}} \sqsubseteq_D M^{\mathcal{D}}\}$.

Corollary 4.7 *If $e \in \mathcal{D}^{dec}$, then e^- is a basic Visser open set.*

Definition 4.8 *A weakly effective λ-model $\mathcal{D}$ is called* effective *if satisfies the following two further conditions:*

(i) If $d \in K(\mathcal{D})$ and $e_i \in \mathcal{D}^{dec}$, then $de_1 \dots e_n \in \mathcal{D}^{dec}$.

(ii) If $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$ and $f(e) \in \mathcal{D}^{dec}$ for all compact elements e, then $\lambda(f) \in \mathcal{D}^{dec}$.

An environment ρ is *compact* in the effective domain Env_D (i.e., $\rho \in K(Env_D)$) if $\rho(x) \in K(\mathcal{D})$ for all variables x and $\{x : \rho(x) = \perp_D\}$ is cofinite.

Notation 4.9 *We define:*

$$\Lambda^{dec}_{\mathcal{D}} \equiv \{M \in \Lambda : M^{\mathcal{D}}_\rho \in \mathcal{D}^{dec} \text{ for all } \rho \in K(Env_D)\}.$$

Theorem 4.10 *Suppose $\mathcal{D}$ is an effective λ-model. Then the set $\Lambda^{dec}_{\mathcal{D}}$ is closed under the following rules:*

1. *$x \in \Lambda^{dec}_{\mathcal{D}}$ for every variable x.*
2. *$M_1, \dots, M_k \in \Lambda^{dec}_{\mathcal{D}} \Rightarrow yM_1 \dots M_k \in \Lambda^{dec}_{\mathcal{D}}$.*
3. *$M \in \Lambda^{dec}_{\mathcal{D}} \Rightarrow \lambda x.M \in \Lambda^{dec}_{\mathcal{D}}$.*

In particular, $\Lambda^{dec}_{\mathcal{D}}$ contains all the β-normal forms.

Proof. Let $\rho \in K(Env_D)$. We have three cases.
(1) $x^{\mathcal{D}}_\rho = \rho(x)$ is compact, hence it is decidable.
(2) By definition $(yM_1 \dots M_k)^{\mathcal{D}}_\rho = \rho(y)(M_1)^{\mathcal{D}}_\rho \dots (M_k)^{\mathcal{D}}_\rho$. Hence the result follows from Definition 4.8(i), $\rho(y) \in K(\mathcal{D})$ and $(M_i)^{\mathcal{D}}_\rho \in \mathcal{D}^{dec}$.
(3) By definition we have that $(\lambda x.M)^{\mathcal{D}}_\rho = \lambda(f)$, where $f(e) = M^{\mathcal{D}}_{\rho[x:=e]}$ for all $e \in D$. Note that $\rho[x := e]$ is also compact for all $e \in K(D)$. Hence the conclusion follows from $M^{\mathcal{D}}_{\rho[x:=e]} \in D^{dec}$ ($e \in K(\mathcal{D})$), Definition 4.8(ii) and $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$. ∎

Recall that Eq($\mathcal{D}$) and Ord($\mathcal{D}$) are respectively the equational theory and the order theory of $\mathcal{D}$.

Theorem 4.11 *Let $\mathcal{D}$ be an effective λ-model, and let $M_1, \dots M_k \in \Lambda^{dec}_{\mathcal{D}}$ ($k \geq 1$) be closed terms. Then we have:*

(i) The set $M_1^- \cap \dots \cap M_k^-$ is a non-empty β-closed co-r.e. set of terms.

(ii) If $e \in \mathcal{D}^{dec}$ and e^- is non-empty and finite modulo Eq($\mathcal{D}$), then Eq($\mathcal{D}$) is not r.e. (in particular, if $\perp^-_{\mathcal{D}} \neq \emptyset$ then Eq($\mathcal{D}$) is not r.e.).

(iii) Ord($\mathcal{D}$) is not r.e.

(iv) Eq($\mathcal{D}$) $\neq \lambda\beta, \lambda\beta\eta$.

Proof. (i) By Theorem 4.10, Corollary 4.7, and the hyperconnectedness of the Visser topology.

(ii) By Corollary 4.7.

(iii) Let $M \in \Lambda^{dec}_{\mathcal{D}}$ be a closed term. If Ord($\mathcal{D}$) were r.e., then we could enumerate the set M^-. However, by (i) this set is non-empty and co-r.e. The conclusion follows because there is no decidable set of λ-terms closed under β-conversion.

(iv) Because of (iii), if Eq($\mathcal{D}$) is r.e. then Ord($\mathcal{D}$) strictly-contains Eq($\mathcal{D}$). Hence the conclusion follows from Selinger's result stating that in any partially ordered λ-model, whose theory is $\lambda\beta$, the interpretations of distinct closed terms are incomparable [27, Corollary 4]. Similarly for $\lambda\beta\eta$. ■

4.1. Models with non-r.e. theories

In this section we give a sufficient condition for a wide class of graph models to be effective and show that no effective graph model generated freely by a partial pair, which is finite modulo its group of automorphisms, can have an r.e. equational theory. Finally, we show that no effective λ-model living in the stable or strongly stable semantics can have an r.e. equational theory.

In Section 5 we will show that every equational/order graph theory is the theory of a graph model $\mathcal{G}$ whose web is the set $\mathbb{N}$ of natural numbers. In the next theorem we characterize the effectivity of these models.

Theorem 4.12 *Let $\mathcal{G}$ be a graph model such that, after encoding, $G = \mathbb{N}$ and $c_\mathcal{G}$ is a computable map. Then $\mathcal{G}$ is weakly effective. Moreover, $\mathcal{G}$ is effective in the hypothesis that $c_\mathcal{G}$ has a decidable range.*

Proof. It is easy to check, using the definitions given in Section 2.3, that $\mathcal{F}, \lambda$ are r.e. in their respective domains and that condition (i) of Definition 4.8 is satisfied. Then $\mathcal{G}$ is weakly effective. Moreover, Definition 4.8(ii) holds under the hypothesis that the range of $c_\mathcal{G}$ is decidable. ■

Completions of partial pairs have been extensively studied in literature. They are useful for solving equational and inequational constraints (see [4, 5, 18, 10, 11]). In [11] Bucciarelli and Salibra have recently proved that the theory of the completion of a partial pair, which is not a graph model, is semi-sensible. The following theorem shows, in particular, that the theory of the completion of a finite partial pair is not r.e.

Theorem 4.13 *Let $\mathcal{A}$ be a partial pair such that A is finite or equal to $\mathbb{N}$ after encoding, and $c_\mathcal{A}$ is a computable map with a decidable domain. Then we have:*

(i) The completion $\mathcal{E}_\mathcal{A}$ of $\mathcal{A}$ is weakly effective;

(ii) If the range of $c_\mathcal{A}$ is decidable, then $\mathcal{E}_\mathcal{A}$ is effective;

(iii) If $\mathcal{A}$ is finite modulo its group of automorphisms (in particular, if the set A is finite), then Eq$(\mathcal{E}_\mathcal{A})$ is not r.e.

Proof. Since A is finite or equal to $\mathbb{N}$ we have that $E_\mathcal{A}$ is also decidable (see Definition 2.4). Moreover, the map $c_{\mathcal{E}_\mathcal{A}} : E^*_\mathcal{A} \times E_\mathcal{A} \to E_\mathcal{A}$ is computable, because it is an extension of a computable function $c_\mathcal{A}$ with decidable domain, and it is the identity on the decidable set $(E^*_\mathcal{A} \times E_\mathcal{A}) - dom(c_\mathcal{A})$. Then (i)-(ii) follow from Theorem 4.12.

Clearly A is a decidable subset of $E_\mathcal{A}$; then by Corollary 4.7 A^- is a co-r.e. set of λ-terms. We now show that this set is non-empty. This follows from the following claim.

Claim 4.14 $\Omega^{\mathcal{E}_\mathcal{A}} \subseteq A$.

By Lemma 2.2 we have that $\alpha \in \Omega^{\mathcal{E}_\mathcal{A}}$ implies that $c_{\mathcal{E}_\mathcal{A}}(a, \alpha) \in a$ for some $a \in E^*_\mathcal{A}$. Immediate considerations on the rank show that this is only possible if $(a, \alpha) \in dom(c_\mathcal{A})$, which forces $\alpha \in A$.

The orbit of $\alpha \in A$ modulo $Aut(\mathcal{A})$ is defined by $O(\alpha) = \{\theta(\alpha) : \theta \in Aut(\mathcal{A})\}$.

Claim 4.15 *If the set of orbits of $\mathcal{A}$ has cardinality k for some $k \in \mathbb{N}$, then the cardinality of A^- modulo Eq$(\mathcal{E}_\mathcal{A})$ is less than or equal to 2^k.*

Assume $p \in M^{\mathcal{E}_\mathcal{A}} \subseteq A$. Then by Lemma 2.5 the orbit of p modulo $Aut(\mathcal{A})$ is included within $M^{\mathcal{E}_\mathcal{A}}$. By hypothesis the number of the orbits is k; hence, the number of all possible values for $M^{\mathcal{E}_\mathcal{A}}$ cannot overcome 2^k.

In conclusion, A^- is non-empty, co-r.e. and modulo Eq$(\mathcal{E}_\mathcal{A})$ is finite. Then (iii) follows from Theorem 4.11. ■

Example 4.16 *Consider the mixed-Scott-Park graph model generated by the partial pair $\mathcal{A}$ defined as follows: take a non trivial partition Q, R of A and let $c_\mathcal{A}(\emptyset, p) = p$ for all $p \in Q$ and $c_\mathcal{A}(\{p\}, p) = p$ for all $p \in R$. Then only the permutations of A which leave R and S invariant will be automorphisms of $\mathcal{A}$, and there will be two orbits.*

All the material developed in Section 4 could be adapted to the stable semantics (Berry's ccc of DI–domains and stable functions) and strongly stable semantics (Ehrhard's ccc of DI-domains with coherence and strongly stable functions). We recall that the notion of an effectively given DI-domain has been introduced by Gruchalski in [15], where it is shown that the category having effective DI-domains as objects and stable functions as morphisms is a ccc. There are also many effective models in the stable and strongly stable semantics. Indeed, the stable semantics contains a class which is analogous to the class of graph models (see [4]), namely Girard's class of *reflexive coherent spaces* called *G-models* in [4]. The results shown in Theorem 4.12 and in Theorem 4.13 for graph models could also be adapted for G-models, even if it is more delicate to complete partial pairs in this case (the completion process has been described in Kerth [18, 20]). It could also be developed for Ehrhard's class of strongly stable H-models (see [4]) even though working in the strongly stable semantics certainly adds technical difficulties.

Theorem 4.17 *Let $\mathcal{D}$ be an effective λ-model in the stable or strongly stable semantics. Then Eq$(\mathcal{D})$ is not r.e.*

Proof. Since $\perp_D \in \mathcal{D}^{dec}$ and the interpretation function is Visser strongly continuous, then $\perp_D^- = \{M \in \Lambda^o : M^\mathcal{D} = \perp_D\}$ is co-r.e. If we show that this set is non-empty, then Eq$(\mathcal{D})$ cannot be r.e. Since $\mathcal{D}$ is effective, then by Theorem 4.11(i) $\mathrm{F}^- \cap \mathrm{T}^-$ is a non-empty and co-r.e. set of λ-terms. Let $N \in \mathrm{F}^- \cap \mathrm{T}^-$ and let $f, g, h : \mathcal{D} \to \mathcal{D}$ be three (strongly) stable functions such that $f(x) = \mathrm{T}^{\mathcal{D}} \cdot x$, $g(x) = \mathrm{F}^{\mathcal{D}} \cdot x$ and $h(x) = N^{\mathcal{D}} \cdot x$ for all $x \in D$. By monotonicity we have $h \leq_s f, g$ in the stable ordering. Now, g is the constant function taking value $\mathbf{i}^{\mathcal{D}}$, and $f(\bot_D) = \mathrm{T}^{\mathcal{D}} \cdot \bot_D$. The first assertion forces h to be a constant function, because in the stable ordering all functions under a constant map are also constant, while the second assertion together with the fact that h is pointwise smaller than f forces the constant function h to satisfy $h(x) = \mathrm{T}^{\mathcal{D}} \cdot \bot_D$ for all x. Then an easy computation provides that $(NPP)^{\mathcal{D}} = \bot_D$ for every closed term P. In conclusion, we have that $\{M \in \Lambda^o : M^{\mathcal{D}} = \bot_D\} \neq \emptyset$ and the theory of $\mathcal{D}$ is not r.e. ∎

5. The Löwenheim-Skolem theorem

In this section we show a kind of Löwenheim-Skolem theorem for graph models: every equational/order graph theory is the theory of a graph model having a countable web (this result positively answers Question 3 in [4, Section 6.3] for the class of graph models). The Löwenheim-Skolem theorem for graph models, and, more generally, for any class of webbed models (in the sense of [4]) does not follow from the analogous theorem for first-order logic, because these classes of λ-models are not closed under subalgebras.

Let $\mathcal{A}, \mathcal{B}$ be partial pairs. We say that $\mathcal{A}$ is a *subpair* of $\mathcal{B}$, and we write $\mathcal{A} \leq \mathcal{B}$, if $A \subseteq B$ and $c_{\mathcal{B}}(a, \alpha) = c_{\mathcal{A}}(a, \alpha)$ for all $(a, \alpha) \in dom(c_{\mathcal{A}})$.

As a matter of notation, if ρ, σ are environments and C is a set, we let $\sigma = \rho \cap C$ mean $\sigma(x) = \rho(x) \cap C$ for every variable x, and $\rho \subseteq \sigma$ mean $\rho(x) \subseteq \sigma(x)$ for every variable x.

The proof of the following lemma is straightforward. Recall that the definition of interpretation in a partial pair is defined in Section 2.3.

Lemma 5.1 *Suppose $\mathcal{A} \leq \mathcal{B}$, then $M^{\mathcal{A}}_{\rho} \subseteq M^{\mathcal{B}}_{\sigma}$ for all environments $\rho : Var \to \mathcal{P}(A)$ and $\sigma : Var \to \mathcal{P}(B)$ such that $\rho \subseteq \sigma$.*

Lemma 5.2 *Let M be a λ-term, $\mathcal{G}$ be a graph model and $\alpha \in M^{\mathcal{G}}_{\rho}$ for some environment ρ. Then there exists a finite subpair $\mathcal{A}$ of $\mathcal{G}$ such that $\alpha \in M^{\mathcal{A}}_{\rho \cap A}$.*

Proof. The proof is by induction on M.

If $M \equiv x$, then $\alpha \in \rho(x)$, so that we define $A = \{\alpha\}$ and $dom(c_{\mathcal{A}}) = \emptyset$.

If $M \equiv \lambda x.P$, then $\alpha \equiv c_{\mathcal{G}}(b, \beta)$ for some b and β such that $\beta \in P^{\mathcal{G}}_{\rho[x:=b]}$. By induction hypothesis there exists a finite subpair $\mathcal{B}$ of $\mathcal{G}$ such that $\beta \in P^{\mathcal{B}}_{\rho[x:=b] \cap B}$. We define another finite subpair $\mathcal{A}$ of $\mathcal{G}$ as follows:

1. $A = B \cup b \cup \{\beta, \alpha\}$;
2. $dom(c_{\mathcal{A}}) = dom(c_{\mathcal{B}}) \cup \{(b, \beta)\}$;

Then we have that $\mathcal{B} \leq \mathcal{A}$ and $\rho[x := b] \cap B \subseteq \rho[x := b] \cap A$. From $\beta \in P^{\mathcal{B}}_{\rho[x:=b] \cap B}$ and from Lemma 5.1 it follows that $\beta \in P^{\mathcal{A}}_{\rho[x:=b] \cap A} = P^{\mathcal{A}}_{(\rho \cap A)[x:=b]}$. Then we have that $\alpha \equiv c_{\mathcal{A}}(b, \beta) \in (\lambda x.P)^{\mathcal{A}}_{\rho \cap A}$.

If $M \equiv PQ$, then there is $a = \{\alpha_1, \ldots, \alpha_n\}$ such that $c_{\mathcal{G}}(a, \alpha) \in P^{\mathcal{G}}_{\rho}$ and $a \subseteq Q^{\mathcal{G}}_{\rho}$. By induction hypothesis there exist finite subpairs $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{G}$ such that $c_{\mathcal{G}}(a, \alpha) \in P^{\mathcal{A}_0}_{\rho \cap A_0}$ and $\alpha_k \in Q^{\mathcal{A}_k}_{\rho \cap A_k}$ for $k = 1, \ldots, n$. We define another finite subpair $\mathcal{A}$ of $\mathcal{G}$ as follows: $A = \cup_{0 \leq k \leq n} A_k \cup a \cup \{\alpha\}$ and $dom(c_{\mathcal{A}}) = (\cup_{0 \leq k \leq n} dom(c_{\mathcal{A}_k})) \cup \{(a, \alpha)\}$. From Lemma 5.1 it follows the conclusion. ∎

Proposition 5.3 *Let $\mathcal{G}$ be a graph model, and suppose $\alpha \in M^{\mathcal{G}} - N^{\mathcal{G}}$ for some $M, N \in \Lambda^o$. Then there exists a finite $\mathcal{A} \leq \mathcal{G}$ such that: for all pairs $\mathcal{C} \geq \mathcal{A}$, if there is a morphism $f : \mathcal{C} \to \mathcal{G}$ such that $f(\alpha) = \alpha$, then $\alpha \in M^{\mathcal{C}} - N^{\mathcal{C}}$.*

Proof. By Lemma 5.2 there is a finite pair $\mathcal{A}$ such that $\alpha \in M^{\mathcal{A}}$. By Lemma 5.1 we have $\alpha \in M^{\mathcal{C}}$. Now, if $\alpha \in N^{\mathcal{C}}$ then, by Lemma 2.5 $\alpha = f(\alpha) \in N^{\mathcal{G}}$, which is a contradiction. ∎

Corollary 5.4 *Let $\mathcal{G}$ be a graph model, and suppose $\alpha \in M^{\mathcal{G}} - N^{\mathcal{G}}$ for some $M, N \in \Lambda^o$. Then there exists a finite $\mathcal{A} \leq \mathcal{G}$ such that: for all pairs $\mathcal{B}$ satisfying $\mathcal{A} \leq \mathcal{B} \leq \mathcal{G}$ we have $\alpha \in M^{\mathcal{B}} - N^{\mathcal{B}}$.*

Let $\mathcal{G}$ be a graph model. A graph model $\mathcal{P}$ is called a *sub graph model* of $\mathcal{G}$ if $\mathcal{P} \leq \mathcal{G}$. It is easy to check that the class of sub graph models of $\mathcal{G}$ is closed under (finite and infinite) intersection. If $\mathcal{A} \leq \mathcal{G}$ is a partial pair, then the *sub graph model generated by* $\mathcal{A}$ is defined as the intersection of all graph models $\mathcal{P}$ such that $\mathcal{A} \leq \mathcal{P} \leq \mathcal{G}$.

Theorem 5.5 (Löwenheim-Skolem Theorem for graph models) *For all graph models $\mathcal{G}$ there exists a graph model $\mathcal{P}$ with a countable web and such that Ord($\mathcal{P}$) = Ord($\mathcal{G}$), and hence Eq($\mathcal{P}$) = Eq($\mathcal{G}$).*

Proof. We will define an increasing sequence of countable subpairs $\mathcal{A}_n$ of $\mathcal{G}$, and take for $\mathcal{P}$ the sub graph model of $\mathcal{G}$ generated by $\mathcal{A} \equiv \cup \mathcal{A}_n$.

First we define $\mathcal{A}_0$. Let I be the countable set of inequations between closed λ-terms which fail in $\mathcal{G}$. Let $e \in I$. By Corollary 5.4 there exists a finite partial pair $\mathcal{A}_e \leq \mathcal{G}$ such that e fails in every partial pair $\mathcal{B}$ satisfying $\mathcal{A}_e \leq \mathcal{B} \leq \mathcal{G}$. Then we define $\mathcal{A}_0 = \cup_{e \in I} \mathcal{A}_e \leq \mathcal{G}$. Assume now that $\mathcal{A}_n$ has been defined. We define $\mathcal{A}_{n+1}$ as follows. For each inequation $e \equiv M \sqsubseteq N$ which holds in $\mathcal{G}$ and fails in the sub graph model $\mathcal{P}_n \leq \mathcal{G}$ generated by $\mathcal{A}_n$, we consider the set $L_e = \{\alpha \in P_n : \alpha \in M^{\mathcal{P}_n} - N^{\mathcal{P}_n}\}$. Let $\alpha \in L_e$. Since $\mathcal{P}_n \leq \mathcal{G}$ and $\alpha \in M^{\mathcal{P}_n}$, then by Lemma 5.1 we have that $\alpha \in M^{\mathcal{G}}$. By $\mathcal{G} \models M \sqsubseteq N$ we also obtain $\alpha \in N^{\mathcal{G}}$. By Lemma 5.2 there exists a partial pair $\mathcal{F}_{\alpha,e} \leq \mathcal{G}$ such that

$\alpha \in N^{\mathcal{F}_{\alpha,e}}$. We define $\mathcal{A}_{n+1}$ as the union of the partial pair $\mathcal{A}_n$ and the partial pairs $\mathcal{F}_{\alpha,e}$ for every $\alpha \in L_e$.

Finally take for $\mathcal{P}$ the sub graph model of $\mathcal{G}$ generated by $\mathcal{A} \equiv \cup \mathcal{A}_n$. By construction we have, for every inequation e which fails in $\mathcal{G}$: $\mathcal{A}_e \leq \mathcal{P}_n \leq \mathcal{P} \leq \mathcal{G}$. Now, Ord($\mathcal{P}$) $\subseteq$ Ord($\mathcal{G}$) follows from Corollary 5.4 and from the choice of $\mathcal{A}_e$.

Let now $M \sqsubseteq N$ be an inequation which fails in $\mathcal{P}$ but not in $\mathcal{G}$. Then there is an $\alpha \in M^{\mathcal{P}} - N^{\mathcal{P}}$. By Corollary 5.4 there is a finite partial pair $\mathcal{B} \leq \mathcal{P}$ satisfying the following condition: for every partial pair $\mathcal{C}$ such that $\mathcal{B} \leq \mathcal{C} \leq \mathcal{P}$, we have $\alpha \in M^{\mathcal{C}} - N^{\mathcal{C}}$. Since B is finite, we have that $\mathcal{B} \leq \mathcal{P}_n$ for some n. This implies that $\alpha \in M^{\mathcal{P}_n} - N^{\mathcal{P}_n}$. By construction of $\mathcal{P}_{n+1}$ we have that $\alpha \in N^{\mathcal{P}_{n+1}}$; this implies $\alpha \in N^{\mathcal{P}}$. Contradiction. ■

6. The minimum order graph theory

In this section we show the other main theorem of the paper: the minimum order graph theory exists and it is the theory of an effective graph model. This result has the following interesting consequences: (i) no order graph theories can be r.e.; (ii) for any β-normal form M, there exists a non-empty β-closed co-r.e. set $\mathcal{U}$ of unsolvables such that, in all graph models, the interpretations of the elements of $\mathcal{U}$ are below that of M.

Lemma 6.1 *Suppose $\mathcal{A} \leq \mathcal{G}$ and let $f : E_{\mathcal{A}} \to G$ be defined by induction over the rank of $x \in E_{\mathcal{A}}$ as follows:*

$$f(x) = \begin{cases} x & \text{if } x \in A \\ c_{\mathcal{G}}(fa, f\alpha) & \text{if } x \notin A \text{ and } x \equiv (a, \alpha). \end{cases}$$

Then f is a morphism from $\mathcal{E}_{\mathcal{A}}$ into $\mathcal{G}$.

Lemma 6.2 *Suppose $\alpha \in M^{\mathcal{G}} - N^{\mathcal{G}}$ for some $M, N \in \Lambda^o$. Then there exists a finite $\mathcal{A} \leq \mathcal{G}$ such that: for all pairs $\mathcal{B}$ satisfying $\mathcal{A} \leq \mathcal{B} \leq \mathcal{G}$, we have $\alpha \in M^{\mathcal{E}_{\mathcal{B}}} - N^{\mathcal{E}_{\mathcal{B}}}$.*

Proof. By Proposition 5.3 and Lemma 6.1. ■

Theorem 6.3 *There exists an effective graph model whose order/equational theory is the minimum order/equational graph theory.*

Proof. It is not difficult to define an effective bijective numeration $\mathcal{N}$ of all finite partial pairs whose web is a subset of $\mathbb{N}$. We denote by $\mathcal{N}_k$ the k-th finite partial pair with $N_k \subseteq \mathbb{N}$. We now make the webs N_k ($k \in \mathbb{N}$) disjoint. Let p_k be the k-th prime natural number. Then we define another finite partial pair $\mathcal{P}_k$ as follows: $P_k = \{p_k^{x+1} : x \in N_k\}$ and $c_{\mathcal{P}_k}(\{p_k^{\alpha_1+1}, \dots, p_k^{\alpha_n+1}\}, p_k^{\alpha+1}) = p_k^{c_{\mathcal{N}_k}(\{\alpha_1,\dots,\alpha_n\},\alpha)+1}$ for all $(\{\alpha_1, \dots, \alpha_n\}, \alpha) \in dom(c_{\mathcal{N}_k})$. In this way we get an effective bijective numeration of all finite partial pairs $\mathcal{P}_k$. Finally, we take $\mathcal{P} \equiv \cup_{k\in\mathbb{N}} \mathcal{P}_k$. It is an easy matter to prove that P is a decidable subset of $\mathbb{N}$ and that, after encoding, $c_{\mathcal{P}} = \cup_{k\in\mathbb{N}} c_{\mathcal{P}_k}$ is a computable map with a decidable domain and range. Then by Theorem 4.13(ii) $\mathcal{E}_{\mathcal{P}}$ is an effective graph model. Notice that $\mathcal{E}_{\mathcal{P}}$ is also isomorphic to the completion of the union $\cup_{k\in\mathbb{N}} \mathcal{E}_{\mathcal{P}_k}$, where $\mathcal{E}_{\mathcal{P}_k}$ is the completion of the partial pair $\mathcal{P}_k$.

We now prove that the order theory of $\mathcal{E}_{\mathcal{P}}$ is the minimum one. Let $e \equiv M \sqsubseteq N$ be an inequation which fails in some graph model $\mathcal{G}$. By Lemma 6.2 e fails in the completion of a finite partial pair $\mathcal{A}$. Without loss of generality, we may assume that the web of $\mathcal{A}$ is a subset of $\mathbb{N}$, and then that $\mathcal{A}$ is one of the partial pairs $\mathcal{P}_k$. For such a $\mathcal{P}_k$, e fails in $\mathcal{E}_{\mathcal{P}_k}$. Now, it was shown by Bucciarelli and Salibra in [9, Proposition 2] that, if a graph model $\mathcal{G}$ is the completion of the disjoint union of a family of graph models $\mathcal{G}_i$, then $Q^{\mathcal{G}_i} = Q^{\mathcal{G}} \cap G_i$ for any closed λ-term Q. Then we can conclude the proof as follows: if the inequation e holds in $\mathcal{E}_{\mathcal{P}}$, then by [9, Proposition 2] we get a contradiction: $M^{\mathcal{E}_{\mathcal{P}_k}} = M^{\mathcal{E}_{\mathcal{P}}} \cap E_{P_k} \subseteq N^{\mathcal{E}_{\mathcal{P}}} \cap E_{P_k} = N^{\mathcal{E}_{\mathcal{P}_k}}$, where $\mathcal{E}_{\mathcal{P}_k}$ is the completion of the partial pair $\mathcal{P}_k$. ■

Theorem 6.4 *Let $\mathcal{T}_{min}$ and $\mathcal{O}_{min}$ be, respectively, the minimum equational graph theory and the minimum order graph theory. Then the following conditions hold:*

(i) $\mathcal{O}_{min}$ is not r.e.

(ii) $\mathcal{T}_{min}$ is an intersection of a countable set of non-r.e. equational graph theories.

(iii) For all closed β-normal forms M and N, there exists a non-empty β-closed co-r.e. set $\mathcal{U}$ of closed unsolvable terms such that $(\forall U \in \mathcal{U})\ \mathcal{O}_{min} \vdash U \sqsubseteq M \wedge \mathcal{O}_{min} \vdash U \sqsubseteq N$.

Proof. (i) follows from Theorem 6.3 and from Theorem 4.11(iii), because $\mathcal{O}_{min}$ is the theory of an effective λ-model.

(ii) By the proof of Theorem 6.3 we have that $\mathcal{T}_{min}$ is an intersection of a countable set of graph theories, which are theories of completions of finite partial pairs. By Theorem 4.13(iii) these theories are not r.e.

(iii) Recall from Theorem 6.3 that $\mathcal{O}_{min}$ is the order theory of an effective completion $\mathcal{E}_{\mathcal{P}}$, where $\mathcal{P} \equiv \cup \mathcal{P}_k$, P_k ($k \in \mathbb{N}$) is a finite set, and $P \equiv \cup P_k$ is a decidable subset of $\mathbb{N}$. Moreover, if the k-th prime number divides $\alpha \in P$ then $\alpha \in P_k$.

We will now show that $\mathcal{U} \equiv \Omega^- \cap M^- \cap N^-$ is a non-empty β-closed co-r.e. set of unsolvables, where X^- is defined with respect to $\mathcal{E}_{\mathcal{P}}$. Since $\mathcal{E}_{\mathcal{P}}$ is effective, if we prove that Ω^- only consists of unsolvable terms, and that $\Omega \in \Lambda^{dec}_{\mathcal{D}}$, then we are done (by Theorems 4.11(i) and 4.10). This is the object of the proofs of the two claims below.

Claim 6.5 $\Omega^{\mathcal{E}_{\mathcal{P}}}$ *is a decidable subset of P.*

By Lemma 2.2 we have that, if $\alpha \in \Omega^{\mathcal{E}_{\mathcal{P}}}$, then there exists a such that $c_{\mathcal{E}_{\mathcal{P}}}(a, \alpha) \in a$. Notice that this condition implies that $c_{\mathcal{E}_{\mathcal{P}}}(a, \alpha)$ has rank 0 (i.e., $c_{\mathcal{E}_{\mathcal{P}}}(a, \alpha) \in P$), so that

α has also rank 0 (i.e., $\alpha \in P$). To check whether an element $\alpha \in P$ belongs to $\Omega^{\mathcal{E}_\mathcal{P}}$ we first find the prime number p_k such that p_k divides α, so that $\alpha \in P_k$. Then we check whether there is a finite set $a \subseteq P_k$ such that $c_{\mathcal{E}_\mathcal{P}}(a, \alpha) \in a$ and we have to check a finite number of elements because P_k is finite.

Claim 6.6 *If $M^{\mathcal{E}_\mathcal{P}} \subseteq \Omega^{\mathcal{E}_\mathcal{P}}$, then M is unsolvable.*

Solvable terms have an interpretation which contains elements of any rank, while $\Omega^{\mathcal{E}_\mathcal{P}}$ contains only elements of rank 0. ■

Corollary 6.7 *For all graph models $\mathcal{G}$, Ord($\mathcal{G}$) is not r.e.*

Proof. If $\mathrm{Ord}(\mathcal{G})$ is r.e. and M is a closed β-normal form, then $M^- = \{N \in \Lambda^o : N^\mathcal{G} \subseteq M^\mathcal{G}\}$ is a basic Visser closed set (i.e., it is an r.e. β-closed set of λ-terms), which contains the basic Visser open set $\{N \in \Lambda^o : \mathcal{O}_{min} \vdash N \sqsubseteq M\}$. By hyperconnectedness $M^- = \Lambda^o$. By the arbitrariness of M, it follows that $\mathrm{T}^- = \mathrm{F}^-$. Since $\mathrm{F} \in \mathrm{T}^-$ and conversely we get $\mathrm{F} = \mathrm{T}$ in $\mathcal{G}$, contradiction. ■

Corollary 6.8 *Let $\mathfrak{G}$ be the class of all graph models. For all closed β-normal forms M and N, there exists a non-empty β-closed co-r.e. set $\mathcal{U}$ of closed unsolvable terms such that $(\forall \mathcal{G} \in \mathfrak{G})(\forall U \in \mathcal{U})\ U^\mathcal{G} \subseteq M^\mathcal{G} \cap N^\mathcal{G}$.*

The authors do not know any example of unsolvable satisfying the above condition.

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