A relational model of a parallel and non-deterministic λ-calculus

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Abstract. We recently introduced an extensional model of the pure λ-calculus living in a canonical cartesian closed category of sets and relations [6]. In the present paper, we study the non-deterministic features of this model. Unlike most traditional approaches, our way of interpreting non-determinism does not require any additional powerdomain construction: we show that our model provides a straightforward semantics of *non-determinism* (*may* convergence) by means of *unions* of interpretations as well as of *parallelism* (*must* convergence) by means of a binary, non-idempotent, operation available on the model, which is related to the *mix rule* of Linear Logic. More precisely, we introduce a λ-calculus extended with non-deterministic choice and parallel composition, and we define its operational semantics (based on the *may* and *must* intuitions underlying our two additional operations). We describe the interpretation of this calculus in our model and show that this interpretation is sensible with respect to our operational semantics: a term converges if, and only if, it has a non-empty interpretation.
Keywords: λ-calculus, relational model, non-determinism, parallel composition, denotational semantics.

1 Introduction

Pure and typed λ-terms are specifications of sequential and deterministic processes. Several extensions of the λ-calculus with parallel and/or non-deterministic constructs have been proposed in the literature, either to increase the expressive power of the language, in the typed [19, 17, 14] and untyped [4, 5] settings, or to study the interplay between higher order features and parallel/non-deterministic features [16, 8, 9].

When introducing non-determinism in a functional setting, it is crucial to specify what notion of convergence is chosen. Two widely used notions are:

- the *must* convergence: a non-deterministic choice converges if all its components do. This characterizes the *demonic* non-determinism.
- the *may* convergence: a non-deterministic choice converges if at least one of its components does. This characterizes the *angelic* non-determinism.

The usual denotational models of functional calculi do not accommodate *may* non-determinism: let TRUE and FALSE be two convergent terms[1], whose denotations in standard models are distinct.

What semantic value should take the non-deterministic term TRUE + FALSE, which *may* converges to TRUE and to FALSE? The value should be both TRUE and FALSE if we want the semantics to be invariant under reduction!

The typical way of interpreting "multi-valued" terms, like the one above, is to use models based on *powerdomains* [18], often defined as filter models with respect to suitable notions of intersection and union types [8, 9]. The semantics of TRUE + FALSE becomes some kind of join of both values, available in the powerdomain (similar techniques are also used for interpreting *must* non-determinism). In this framework, both kinds of non-determinism are modelled by some idempotent, commutative and associative operations.

In a recent paper [11], Faure and Miquel define a categorical counterpart of the syntactical notion of parallel execution: the *aggregation monad*. Powerdomains, sets with union and multisets with multi-union are all instances of aggregation monads (in categories of domains and of sets, respectively). In general, the notion of parallel composition modelled by an aggregation monad is neither idempotent, nor commutative, nor associative.

There are however models of the ordinary λ-calculus where aggregation, considered as parallel composition (that is, as *must* non-determinism), can be interpreted without introducing any additional structure, such as the above mentioned aggregation monads or powerdomain constructions.

This is the case in models of multiplicative exponential linear logic (MELL), where aggregation can be interpreted by the *mix rule*, if available. This rule allows to "put together" two proofs whatsoever [7]. More precisely, parallel composition is obtained by combining the mix rule with the contraction rule. Indeed, mix can be seen as a linear morphism $X \otimes Y \multimap X ⅋ Y$, so that there is a morphism $?A \otimes ?A \multimap ?A$, obtained by composing the mix morphism $?A \otimes ?A \multimap ?A ⅋ ?A$ with the contraction morphism $?A ⅋ ?A \multimap ?A$. This composite morphism defines a commutative algebra structure on $?A$, which is used to model the "parallel composition" of MELL proofs. Thus, to obtain a model of parallel λ-calculus, it is sufficient to solve the equation $\mathcal{D} \cong \mathcal{D} \Rightarrow \mathcal{D}$, with an object $\mathcal{D}$ of shape $?A$.

This is precisely what we did in [6], in a particularly simple model of linear logic: the model of sets and relations. Similar constructions are possible in other, richer models, such as the well known model of coherence spaces [12], or the model of hypercoherences [10]: the mix rule is available there, as well as in many other models. This shows that coherence (which prevents the above join of TRUE and FALSE) is not an obstacle to the interpretation of the *must* non-determinism in the pure λ-calculus[2]. Our model $\mathcal{D}$ of [6] satisfies the recursive equation $\mathcal{D} = ?(A)$ where $A = (\mathcal{D}^{\mathbb{N}})^{\perp}$, and therefore, $\mathcal{D}$ has the commutative

[1] They could be the actual boolean constants in a typed λ-calculus with constants, or the projections $\lambda xy.x$, $\lambda xy.y$ as pure λ-terms.

[2] In a typed language like PCF, this would be more problematic, since the object interpreting the type of booleans does not have the above mentioned structure.

algebra structure mentioned above. It is precisely this structure that we use for interpreting parallel composition, just as Danos and Krivine did in [7] for an extension of $\lambda\mu$-calculus with a parallel composition operation.

But the category of sets and relations has another feature, which allows for a direct interpretation of the *may* non-determinism as well: morphisms are arbitrary relations between sets (interpreting types), and hence morphisms are closed under arbitrary unions. Thanks to this union operation on morphisms, *may* non-determinism can be interpreted directly, without introducing any additional powerdomain construction or aggregation monad. Of course, this operation is not available in the coherence or hypercoherence space models. Note that, if we consider $M + N \to M$ as a reduction rule of our calculus, then our semantics is not invariant under reduction, since the process of performing non-deterministic choices entails a non recoverable loss of information. But the situation is fundamentally similar with the powerdomain-based interpretations.

To summarize, in our model $\mathcal{D}$, the semantic counterparts of *may* and *must* non-determinism are at hand: they are simply the set-theoretic union and the mix-based algebraic operation. In this framework, parallel composition is no longer idempotent. This is quite natural if we consider each component of a parallel composition as the specification of a process whose execution requires the consumption of some kind of resources.

Contents. We introduce an extension of λ-calculus with parallel composition and non-deterministic choice, called $\lambda_{+\|}$-*calculus*, and we define its operational semantics by associating with each term a generalized hnf (head normal form), which is a set of multisets of terms whose head subterms are variables[3]. Roughly speaking, the operational value of a term is the collection of all possible outcomes of its head reductions. When the head subterm is $M + N$ (*may* non-deterministic choice), the head reduction goes on by choosing either M or N, and when the head subterm is $M \| N$ (*must* parallelism), the head reduction forks.

We provide the denotational semantics of the $\lambda_{+\|}$-calculus in $\mathcal{D}$, considered as a λ-model, and endowed with two additional operations which turn it into a semiring. We prove the soundness with respect to β-reduction, and we show that the interpretations of the hnf's of a term M are included in the interpretation of M. Next, we generalize Krivine's realizability technique to our extended calculus, showing that our denotational model is *sensible*: the operational value of a term is non-empty (i.e., a term is solvable) if, and only if, its denotation is non-empty.

2 Preliminaries

To keep this article self-contained we summarize some definitions and results that will be used in the sequel. In particular, we present our semantic framework **MRel** and we recall the construction of a specific reflexive object $\mathcal{D}$ of **MRel**, that we have introduced in [6]. Our main reference for category theory is [1].

[3] This is reminiscent of the *capability semantics* of [8], but we consider different notions of convergence and of head normal form.

2.1 Multisets and sequences

Let S be a set. We denote by $\mathcal{P}(S)$ the collection of all subsets of S. A *multiset* m over S can be defined as an unordered list $m = [a_1, a_2, \ldots]$ with repetitions such that $a_i \in S$ for all i. A multiset m is called *finite* if it is a finite list, we denote by $[]$ the empty multiset. Given two multisets $m_1 = [a_1, a_2, \ldots]$ and $m_2 = [b_1, b_2, \ldots]$ the *multi-union* of m_1, m_2 is defined by $m_1 \uplus m_2 = [a_1, b_1, a_2, b_2, \ldots]$. We will write $\mathcal{M}_f(S)$ for the set of all finite multisets over S.

We denote by $\mathbb{N}$ the set of natural numbers. Given two $\mathbb{N}$-indexed sequences $\sigma = (\sigma_1, \sigma_2, \ldots), \tau = (\tau_1, \tau_2, \ldots)$ of multisets we define the *multi-union* of σ and τ componentwise as $\sigma \bar{\uplus} \tau = (\sigma_1 \uplus \tau_1, \sigma_2 \uplus \tau_2, \ldots)$. An $\mathbb{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all, but a finite number of indices i. If S is a set, then we denote by $\mathcal{M}_f(S)^{(\omega)}$ the set of all quasi-finite $\mathbb{N}$-indexed sequences of multisets over S. We write $\star$ for the $\mathbb{N}$-indexed sequence of empty multisets, i.e., $\star$ is the only inhabitant of $\mathcal{M}_f(\emptyset)^{(\omega)}$.

2.2 MRel: a cartesian closed category of sets and relations

We now present the category **MRel**, which is the Kleisli category of the functor $\mathcal{M}_f(-)$ over the $\star$-autonomous category **Rel** of sets and relations. We provide here a direct definition, since in the sequel we will not use explicitly the monoidal structure of **Rel**.

- The objects of **MRel** are all the sets.
- A morphism from S to T is a relation from $\mathcal{M}_f(S)$ to T, in other words, $\mathbf{MRel}(S,T) = \mathcal{P}(\mathcal{M}_f(S) \times T)$.
- The identity of S is the relation $Id_S = \{([a], a) \mid a \in S\} \in \mathbf{MRel}(S,S)$.
- The composition of $s \in \mathbf{MRel}(S,T)$ and $t \in \mathbf{MRel}(T,U)$ is defined by:

$$t \circ s = \{(m, c) \mid \begin{array}{l} \exists (m_1, b_1), \ldots, (m_k, b_k) \in s \text{ such that} \\ m = m_1 \uplus \ldots \uplus m_k \text{ and } ([b_1, \ldots, b_k], c) \in t\}. \end{array}$$

We now provide an overview of the proof of cartesian closedness.

Theorem 1. *The category* **MRel** *is cartesian closed.*

Proof. The terminal object $\mathbb{1}$ is the empty set $\emptyset$, and the unique element of $\mathbf{MRel}(S, \emptyset)$ is the empty relation.

Given two sets S_1 and S_2, their categorical product $S_1 \& S_2$ in **MRel** is their disjoint union:

$$S_1 \& S_2 = (\{1\} \times S_1) \cup (\{2\} \times S_2)$$

and the projections π_1, π_2 are given by:

$$\pi_i = \{([(i, a)], a) \mid a \in S_i\} \in \mathbf{MRel}(S_1 \& S_2, S_i), \text{ for } i = 1, 2.$$

Given $s \in \mathbf{MRel}(U, S_1)$ and $t \in \mathbf{MRel}(U, S_2)$, the corresponding morphism $\langle s, t \rangle \in \mathbf{MRel}(U, S_1 \& S_2)$ is given by:

$$\langle s, t \rangle = \{(m, (1, a)) \mid (m, a) \in s\} \cup \{(m, (2, b)) \mid (m, b) \in t\}.$$

We will consider the canonical bijection between $\mathcal{M}_f(S_1) \times \mathcal{M}_f(S_2)$ and $\mathcal{M}_f(S_1 \,\&\, S_2)$ as an equality, hence we will still denote by (m_1, m_2) the corresponding element of $\mathcal{M}_f(S_1 \& S_2)$.

Given two objects S and T the exponential object $S \Rightarrow T$ is $\mathcal{M}_f(S) \times T$ and the evaluation morphism is given by:

$$eval_{ST} = \{(([(m, b)], m), b) \mid m \in \mathcal{M}_f(S) \text{ and } b \in T\} \in \mathbf{MRel}((S \Rightarrow T) \& S, T)\,.$$

Given any set U and any morphism $s \in \mathbf{MRel}(U \,\&\, S, T)$, there is exactly one morphism $\Lambda(s) \in \mathbf{MRel}(U, S \Rightarrow T)$ such that:

$$eval_{ST} \circ \langle \Lambda(s), Id_S \rangle = s,$$

namely, $\Lambda(s) = \{(p, (m, b)) \mid ((p, m), b) \in s\}$. $\square$

The points of an object S, i.e., the elements of $\mathbf{MRel}(\mathbb{1}, S)$, are relations between $\mathcal{M}_f(\emptyset)$ and S. These are, up to isomorphism, the subsets of S.

2.3 An extensional reflexive object in MRel

A *reflexive object* of a cartesian closed category $\mathbf{C}$ (ccc, for short) is a triple $\mathcal{U} = (U, \mathcal{A}, \lambda)$ such that U is an object of $\mathbf{C}$, and $\lambda \in \mathbf{C}(U \Rightarrow U, U)$ and $\mathcal{A} \in \mathbf{C}(U, U \Rightarrow U)$ satisfy $\mathcal{A} \circ \lambda = Id_{U \Rightarrow U}$. $\mathcal{U}$ is called *extensional* if, moreover, $\lambda \circ \mathcal{A} = Id_U$; in this case we have that $U \cong U \Rightarrow U$.

We define a reflexive object $\mathcal{D}$ in $\mathbf{MRel}$, which is extensional by construction. We let $(D_n)_{n \in \mathbb{N}}$ be the increasing family of sets defined by:

- $D_0 = \emptyset$,
- $D_{n+1} = \mathcal{M}_f(D_n)^{(\omega)}$.

Finally, we set $D = \bigcup_{n \in \mathbb{N}} D_n$. So we have $D_0 = \emptyset$ and $D_1 = \{\star\} = \{([], [], \dots)\}$. The elements of D_2 are quasi-finite sequences of multisets over a singleton, i.e., quasi-finite sequences of natural numbers, and so on.

We say that $\sigma \in D$ *has rank* n if $n \in \mathbb{N}$ is minimum such that $\sigma \in D_n$.

In order to define an isomorphism in $\mathbf{MRel}$ between D and $D \Rightarrow D = \mathcal{M}_f(D) \times D$ just notice that every element $\sigma = (\sigma_1, \sigma_2, \dots) \in D$ stands for the pair $(\sigma_1, (\sigma_2, \dots))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_f(D)$, we write $m :: \sigma$ for the element $\tau = (\tau_1, \tau_2, \dots) \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_f(D) \times D$ and D, and hence an isomorphism in $\mathbf{MRel}$ as follows:

Proposition 1. *(Bucciarelli, et al. [6]) The triple $\mathcal{D} = (D, \mathcal{A}, \lambda)$ where:*
- $\lambda = \{([(m, \sigma)], m :: \sigma) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D \Rightarrow D, D)$,
- $\mathcal{A} = \{([m :: \sigma], (m, \sigma)) \mid m \in \mathcal{M}_f(D), \sigma \in D\} \in \mathbf{MRel}(D, D \Rightarrow D)$,

is an extensional reflexive object of $\mathbf{MRel}$.

3 A parallel and non-deterministic $\boldsymbol{\lambda}$-calculus

In this section we introduce the syntax and the operational semantics of a parallel and non-deterministic extension of λ-calculus that we call $\lambda_{+\|}$-*calculus*.

3.1 Syntax of $\lambda_{+\parallel}$-calculus

To begin with, we define the set $\Lambda_{+\parallel}$ of λ-terms enriched with two binary operators $+$ and $\parallel$, that is the set of terms generated by the following grammar (where x ranges over a countable set Var of variables):

$$M, N \ ::= \ x \mid \lambda x.M \mid MN \mid M + N \mid M \| N \ .$$

The elements of $\Lambda_{+\parallel}$ are called $\lambda_{+\parallel}$*-terms* and will be denoted by $M, N, P, \ldots$ Intuitively, $M + N$ denotes the *non-deterministic choice* between M and N, and $M\|N$ stands for their *parallel composition*.

As usual, we suppose that application associates to the left and λ-abstraction to the right. Moreover, to lighten the notation, we assume that application and λ-abstraction take precedence over $+$ and $\parallel$. The notions of *free* and *bound* variables of a term are defined in the obvious way.

A *substitution* is a finite set $s = \{(x_1, N_1), \ldots, (x_k, N_k)\}$ such that $x_i \neq x_j$ for all $1 \leq i < j \leq k$. Given a $\lambda_{+\parallel}$-term M and a substitution s as above, we denote by Ms the term obtained by substituting *simultaneously* the term N_j for all free occurrences of x_j (for $1 \leq j \leq k$) in M, subject to the usual proviso about renaming bound variables in M to avoid capture of free variables in the N_j's. If $s = \{(x, N)\}$ we will write $M[N/x]$ for Ms.

Note that, in general, $M\{(x_1, N_1), \ldots, (x_k, N_k)\} \neq M[N_1/x_1] \cdots [N_k/x_k]$. For instance, $x\{(x, y), (y, z)\} = y$, whereas $x[y/x][z/y] = z$. Actually, k-ary substitutions will be only used in Section 5 in the proof of the adequation lemma.

As a matter of notation, we will write $\vec{P}$ for a (possibly empty) finite sequence of $\lambda_{+\parallel}$-terms $P_1 \ldots P_k$ and $\ell(\vec{P})$ for the length of $\vec{P}$. It is easy to check that every $\lambda_{+\parallel}$-term M has the form $\lambda\vec{x}.N\vec{P}$ where N, which is called *the head subterm of* M, is either a variable, a non-deterministic choice, a parallel composition or a λ-abstraction. Notice that, in this last case, we must have $\ell(\vec{P}) > 0$.

3.2 Operational semantics

The set $\Lambda^h_{+\parallel} \subset \Lambda_{+\parallel}$ of *head normal forms*[4] (*hnf*'s, for short) is the set of $\lambda_{+\parallel}$-terms whose head subterm is a variable (called *head variable*).

The intuitive idea of the head reduction of $\lambda_{+\parallel}$-calculus underlying the notion of "value" (formalized below) is the following:

- when a term has the head subterm of the form $N_1 + N_2$, either of the alternatives may be chosen to pursue the head reduction, and the final value is the union of the values obtained by each choice. In particular, if one of the choices produces a non-empty value, then the global value is non-empty.
- when a term has the head subterm of the form $N_1\|N_2$, the head reduction forks, and the final value is obtained by "mixing" the values eventually obtained. In particular, if the value of one of the subprocesses is empty, then also the global value is.

[4] This terminology is coherent with the one usually adopted for λ-calculus (see [2, Def. 2.2.11]).

Instead of defining the operational semantics of $\lambda_{+\|}$-calculus via an explicit (head) rewriting system, we associate with each $M \in \Lambda_{+\|}$ the value eventually obtained by head reducing M. In particular, we use union (resp. multi-union) to get the value of $M_1 + M_2$ (resp. $M_1 \| M_2$) out of the values of M_1 and M_2.

Definition 1. *A* multiple hnf *is a finite multiset of hnf's of $\lambda_{+\|}$-calculus. A* value *is a set of multiple hnf's.*

To help the reader to get familiar with these notions, we first provide some simple examples of values (where[5] $\mathbf{I} \equiv \lambda x.x$, $\Delta \equiv \lambda x.xx$ and $\Omega \equiv \Delta\Delta$):

- the value of $\mathbf{I} + \Delta$ is $\{[\mathbf{I}], [\Delta]\}$. In other words, the term $\mathbf{I} + \Delta$ has two different multiple hnf's, which are singleton multisets;
- the value of $\mathbf{I} \| \Delta$ is $\{[\mathbf{I}, \Delta]\}$, then $\mathbf{I} \| \Delta$ has just one multiple hnf;
- the values of $\mathbf{I} + \Omega$ and $\mathbf{I} \| \Omega$ are $\{[\mathbf{I}]\}$ and $\emptyset$, respectively. This is a consequence of the fact that the value of Ω is the empty-set.

In general, the value $H(M)$ of a $\lambda_{+\|}$-term M can be characterized as the limit of an increasing sequence $(H_n(M))_{n \in \mathbb{N}}$ of "partial" values, which are defined by induction on $n \in \mathbb{N}$ and by cases on the form of the head subterm of M.

Definition 2. *Let $M \equiv \lambda \vec{x}.N\vec{P}$ be a $\lambda_{+\|}$-term.*

- $H_0(M) = \emptyset$;
- $$H_{n+1}(M) = \begin{cases} \{[M]\} & \text{if } N \equiv y, \\ H_n(\lambda \vec{x}.Q[P_1/y]P_2 \cdots P_{\ell(\vec{P})}) & \text{if } N \equiv \lambda y.Q, \\ H_n(\lambda \vec{x}.N_1\vec{P}) \cup H_n(\lambda \vec{x}.N_2\vec{P}) & \text{if } N \equiv N_1 + N_2, \\ \{m_1 \uplus m_2 \mid m_i \in H_n(\lambda \vec{x}.N_i\vec{P}) \text{ for } i = 1,2\} & \text{if } N \equiv N_1 \| N_2. \end{cases}$$

Notice that, for all $M \in \Lambda_{+\|}$ and $n \in \mathbb{N}$, the value $H_n(M) \subset \mathcal{M}_f(\Lambda^h_{+\|})$ is a finite set of multiple hnf's. Since the sequence $(H_n(M))_{n \in \mathbb{N}}$ is increasing, we can define the (final) value of M as its limit.

Definition 3. *The* value of a $\lambda_{+\|}$-term M *is defined by* $H(M) = \bigcup_{n \in \mathbb{N}} H_n(M)$.

Of course, $H(M)$ may be infinite as shown in the example below.

Example 1. Consider the $\lambda_{+\|}$-term $M \equiv \lambda n.\underline{0} + \mathbf{s}n$, where $\underline{0} \equiv \lambda xy.y$ is the 0-th Church numeral and $\mathbf{s} \equiv \lambda nxy.nx(xy)$ implements the successor function. Let now $C \equiv \mathbf{Y}M$ where $\mathbf{Y}$ is some fixpoint combinator. To have simpler calculations, we suppose that $\mathbf{Y}M$ reduces to $M(\mathbf{Y}M)$ in just one step of head β-reduction. Then, we get:

- $H_0(C) = \emptyset$,
- $H_1(C) = H_0(MC) = \emptyset$,
- $H_2(C) = H_1(MC) = H_0(\underline{0} + \mathbf{s}C) = \emptyset$,
- $H_3(C) = H_2(MC) = H_1(\underline{0} + \mathbf{s}C) = \{[\underline{0}]\} \cup H_0(\mathbf{s}C) = \{[\underline{0}]\}$.

Pursuing the calculation a little further, one gets $H_9(C) = \{[\underline{0}], [\underline{1}]\}$ and, eventually, $H(C) = \{[\underline{n}] \mid n \in \mathbb{N}\}$.

[5] The symbol $\equiv$ denotes syntactical equality.

3.3 Solvability

We now present the natural notion of solvability for the $\lambda_{+\|}$-calculus.

Definition 4. *A $\lambda_{+\|}$-term M is* solvable *if $H(M) \neq \emptyset$. The set of solvable terms will be denoted by $\mathcal{N}$.*

Among solvable terms, we single out the set $\mathcal{N}_0$ of hnf's starting with a variable, and the set $\mathcal{N}_1$ of solvable terms having a multiple hnf whose head variables are free.

Definition 5. *We set:*

- $\mathcal{N}_0 = \{x\vec{P} \mid x \in \mathrm{Var}$ *and* $\vec{P} \in \Lambda_{+\|}\}$, *and*
- $\mathcal{N}_1 = \{M \in \Lambda_{+\|} \mid \exists[\lambda\vec{x}_1.y_1\vec{P}_1, \ldots, \lambda\vec{x}_k.y_k\vec{P}_k] \in H(M) \wedge (\forall j = 1..k)\ y_j \notin \vec{x}_j\}$.

We end this section stating a technical proposition, which will be useful in Section 5. The proof is quite long and it is omitted.

Proposition 2. *Let $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, then we have that:*

(i) *if $Mx \in \mathcal{N}$ then $M \in \mathcal{N}$,*
(ii) *if $M\Omega \in \mathcal{N}_1$ then $M \in \mathcal{N}_1$,*
(iii) *if $M \in \mathcal{N}_1$ then $MN \in \mathcal{N}_1$ for all $N \in \Lambda_{+\|}$.*

Notice that in the case of the pure λ-calculus the analogous properties are trivial.

4 A relational model of $\boldsymbol{\lambda_{+\|}}$-calculus

Exploiting the existence of countable products in **MRel** we have shown in [6] that the reflexive object $\mathcal{D} = (D, \mathcal{A}, \lambda)$ built in Section 2.3 can be turned into a λ-*model* [2, Def. 5.2.1] (this was not clear before, since the category **MRel** does not have *enough points* [1, Def. 2.1.4]). The underlying set of the λ-model associated with D by our construction is the set of "finitary" morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$, where D^{Var} is the Var-indexed categorical product of countably many copies of D.

4.1 Finitary morphisms in MRel

The morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$ are sets of pairs whose first projection is a finite multiset of elements in D^{Var}, and whose second projection is an element of D. Since categorical products in **MRel** are disjoint unions, a typical such pair is of the form:

$$([(x_1, \sigma_1^1), \ldots, (x_1, \sigma_1^{n_1}), \ldots, (x_k, \sigma_k^1), \ldots, (x_k, \sigma_k^{n_k})], \sigma)$$

where $k, n_1, \ldots, n_k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathrm{Var}$ and $\sigma_1^1, \ldots, \sigma_k^{n_k}, \sigma \in D$.

Notation 1. *Given* $m \in \mathcal{M}_f(D^{\mathrm{Var}})$ *and* $x \in \mathrm{Var}$*, we set* $m_x = [\sigma \mid (x,\sigma) \in m] \in \mathcal{M}_f(D)$ *and* $m_{-x} = [(y,\sigma) \in m \mid y \neq x] \in \mathcal{M}_f(D^{\mathrm{Var}})$*.*

In general, given an object U of a ccc $\mathbf{C}$, we say that a morphism $f \in \mathbf{C}(U^{\mathrm{Var}}, U)$ is "finitary" if it can be decomposed as $f = f_I \circ \pi_I$ for some finite set I of variables (see [6, Sec. 3.1]). Working in **MRel** it is more convenient to take the following equivalent definition.

Definition 6. *A morphism* $r \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$ *is* finitary *if there exists a finite set* I *of variables such that for all* $(m,\sigma) \in r$ *and* $x \in \mathrm{Var}$ *we have that* $m_x \neq []$ *entails* $x \in I$*.*

We denote by $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ the set of all finitary morphisms.

4.2 The model

From [6, Thm. 1] we know that $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \bullet)$, where $\bullet$ is defined as usual by $r_1 \bullet r_2 = \mathit{eval} \circ \langle \mathcal{A} \circ r_1, r_2\rangle$, can be endowed with a structure of λ-model.

In order to interpret $\lambda_{+\|}$-terms as finitary morphisms of **MRel** we are going to define on $\mathbf{MRel}(D^{\mathrm{Var}}, D)$ two binary operations of *sum* and *aggregation* for modelling non-deterministic choice and parallel composition, respectively, and to prove that $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ is closed under these operations.

Definition 7. *Let* $r_1, r_2 \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$*, then:*

- *the* sum *of* r_1 *and* r_2 *is defined by* $r_1 \oplus r_2 = r_1 \cup r_2$*.*
- *the* aggregation *of* r_1 *and* r_2 *is defined by* $r_1 \odot r_2 = \{(m_1 \uplus m_2, \sigma_1 \bar{\uplus} \sigma_2) \mid (m_i, \sigma_i) \in r_i,$ *for* $i = 1, 2\}$*.*

Proposition 3. *The set* $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ *is closed under sum and aggregation.*

Proof. Straightforward. In both cases, the union of the finite sets of variables I_1 and I_2 given by the finiteness of the arguments of the operation, is a witness of the finiteness of the result. □

Composition is right-distributive over sum and aggregation.

Proposition 4. *Let* $r, s \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$ *and* $t \in \mathbf{MRel}(D^{\mathrm{Var}}, D^{\mathrm{Var}})$*, then:*

- $(r \oplus s) \circ t = (r \circ t) \oplus (s \circ t)$*,*
- $(r \odot s) \circ t = (r \circ t) \odot (s \circ t)$*.*

Proof. Straightforward. □

The units of the operations $\oplus$ and $\odot$ are $0 = \emptyset$ and $1 = \{([], \star)\}$, respectively; $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \oplus, 0)$ and $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \odot, 1)$ are commutative monoids. Moreover, 0 annihilates $\odot$ and aggregation distributes over sum. Summing up, the following proposition gives an overview of the algebraic properties of $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ equipped with application, sum and aggregation.

Proposition 5.
- $(\mathbf{MRel}_f(D^{\mathrm{Var}}, D), \oplus, \odot, 0, 1)$ *is a commutative semiring.*
- $\bullet$ *is right-distributive over* $\oplus$ *and* $\odot$*.*
- $\oplus$ *is idempotent (whereas* $\odot$ *is not).*

Proof. Straightforward.

4.3 The absolute interpretation

Before going through the formal definition of the interpretation of $\lambda_{+\|}$-terms, we present a short digression on the nature of such an interpretation.

In our framework, the $\lambda_{+\|}$-terms will be interpreted as morphisms in $\mathbf{MRel}_f(D^{\mathrm{Var}}, D)$, i.e., as subsets of $\mathcal{M}_f(D^{\mathrm{Var}}) \times D$. The occurrence of a particular pair $([(x_1, \sigma_1^1), \ldots, (x_1, \sigma_1^{n_1}), \ldots, (x_k, \sigma_k^1), \ldots, (x_k, \sigma_k^{n_k})], \sigma)$ in the interpretation of a term M may be read as "in an environment ρ such that $\rho(x_i) = [\sigma_i^1, \ldots, \sigma_i^{n_i}]$ (for all $i = 1, \ldots, k$) the interpretation $[\![M]\!]_\rho$ contains σ".

Hence, here there is no need of providing explicitly an environment to the interpretation function as classically done for λ-models [2, Def. 5.2.1(ii)] because the whole information is coded inside the elements of the λ-model itself.

On the other hand, the categorical interpretation of a term M is usually defined with respect to a finite list of variables, containing the free variables of M [2, Def. 5.5.3(vii)]. Intuitively, our interpretation is defined with respect to the list of *all* variables, encompassing then all categorical interpretations.

These considerations lead us to the definition of $[\![-]\!] : \Lambda_{+\|} \to \mathbf{MRel}_f(D^{\mathrm{Var}}, D)$ below, that we call the *absolute interpretation*[6] of $\lambda_{+\|}$-terms:

- $[\![x]\!] = \pi_x$, for $x \in \mathrm{Var}$,
- $[\![M_1 M_2]\!] = eval \circ \langle \mathcal{A} \circ [\![M_1]\!], [\![M_2]\!] \rangle$,
- $[\![\lambda x.M]\!] = \lambda \circ \Lambda([\![M]\!] \circ \eta_x)$,
- $[\![M_1 + M_2]\!] = [\![M_1]\!] \oplus [\![M_2]\!]$,
- $[\![M_1 \| M_2]\!] = [\![M_1]\!] \odot [\![M_2]\!]$,

where $\eta_x \in \mathbf{MRel}(D^{\mathrm{Var}} \& D, D^{\mathrm{Var}})$ is defined componentwise, for $y \in \mathrm{Var}$, by:

$$\pi_y \circ \eta_x = \begin{cases} \pi_2 & \text{if } x \equiv y, \\ \pi_y \circ \pi_1 & \text{if } x \not\equiv y. \end{cases}$$

In what follows, we will use the inductive characterization of the interpretation of (some) $\lambda_{+\|}$-terms provided by the proposition below:

Proposition 6. *(i)* $[\![x]\!] = \{([(x, \sigma)], \sigma) \mid \sigma \in D\}$,
(ii) $[\![MN]\!] = \{(m_0 \uplus m_1 \uplus \ldots \uplus m_k, \sigma) \mid \exists k \geq 0, (m_0, [\tau_1, \ldots, \tau_k] :: \sigma) \in [\![M]\!], (m_i, \tau_i) \in [\![N]\!] \ for\ 1 \leq i \leq k\}$,
(iii) $[\![\lambda x.M]\!] = \{(m_{-x}, m_x :: \sigma) \mid (m, \sigma) \in [\![M]\!]\}$.

Proof. Simple calculations based on the definitions of Section 2. □

We show now the soundness of the interpretation with respect to β-conversion, which relies on the following lemma.

Lemma 1. *If* $M, N \in \Lambda_{+\|}$ *and* $x \in \mathrm{Var}$*, then* $[\![M[N/x]]\!] = [\![M]\!] \circ \eta_x \circ \langle id, [\![N]\!] \rangle$.

[6] See [15, Sec. 2.3.2] for more details on the relations among the absolute, algebraic and categorical interpretations, and on how the former allows to recover the others.

Proof. By structural induction on M. The cases $M \equiv M_1 + M_2$ and $M \equiv M_1 \| M_2$ are settled by using Proposition 4. For the other cases, one can use Proposition 6 and the following characterization: $\eta_x \circ \langle id, [\![N]\!]\rangle = \{([(y,\sigma)],(y,\sigma)) \mid \sigma \in D, y \not\equiv x\} \cup \{(m,(x,\sigma)) \mid (m,\sigma) \in [\![N]\!]\} \in \mathbf{MRel}(D^{\mathrm{Var}}, D^{\mathrm{Var}})$. □

Lemma 2. *(Soundness) For all $M, N \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, we have $[\![(\lambda x.M)N]\!] = [\![M[N/x]]\!]$.*

Proof. $[\![(\lambda x.M)N]\!] = eval \circ \langle \mathcal{A} \circ \lambda \circ \Lambda([\![M]\!] \circ \eta_x), [\![N]\!]\rangle = eval \circ \langle \Lambda([\![M]\!] \circ \eta_x), [\![N]\!]\rangle = [\![M]\!] \circ \eta_x \circ \langle id, [\![N]\!]\rangle =$ by Lemma 1 $= [\![M[N/x]]\!]$. □

We aim to prove that our model is *sensible* w.r.t. the operational semantics: a $\lambda_{+\|}$-term M has a non-empty interpretation if, and only if, M is solvable.

We start showing that the interpretation of every solvable term is non-empty (for the converse we will adapt Krivine's realizability method [13], see Section 5). This is an immediate corollary of the following propositions stating that the interpretation of a $\lambda_{+\|}$-term includes the union of the interpretations of its multiple hnf's and that the interpretation of any hnf is non-empty.

Proposition 7. *For all $M \in \Lambda_{+\|}$, we have $(\bigoplus_{m \in H(M)} (\bigodot_{N \in m} [\![N]\!])) \subseteq [\![M]\!]$.*

Proof. It is enough to show that $(\bigoplus_{m \in H_n(M)} (\bigodot_{N \in m} [\![N]\!])) \subseteq [\![M]\!]$ holds for all $n \in \mathbb{N}$; we prove it by induction on n. The case $n = 0$ is trivial. The proof of the inductive step goes by case analysis on the head subterm M' of $M \equiv \lambda \vec{z}.M'\vec{P}$.

- The case $M' \equiv x$ is trivial, and the case $M' \equiv \lambda y.Q$ is settled by Lemma 2.
- If $M' \equiv Q_1 \| Q_2$, we start by observing that $[\![M]\!] = [\![\lambda \vec{z}.Q_1\vec{P}]\!] \odot [\![\lambda \vec{z}.Q_2\vec{P}]\!]$. This is an easy consequence of the right distributivity of $\bullet$ over $\odot$ (Proposition 5) and of the fact that, by Proposition 6(iii), we have $[\![\lambda \vec{x}.(R_1 \| R_2)]\!] = [\![\lambda \vec{x}.R_1]\!] \odot [\![\lambda \vec{x}.R_2]\!]$, for all $\vec{x} \in \mathrm{Var}$ and $R_1, R_2 \in \Lambda_{+\|}$. Then, we can conclude by the inductive hypothesis.
- The case $M' \equiv Q_1 + Q_2$ is similar, and simpler, once noted that $[\![M]\!] = [\![\lambda \vec{z}.Q_1\vec{P}]\!] \oplus [\![\lambda \vec{z}.Q_2\vec{P}]\!]$ (again, by Proposition 5 and Proposition 6(iii)). □

We now show that every hnf has a non-empty interpretation.

Proposition 8. *For all $x, \vec{y} \in \mathrm{Var}$ and $\vec{Q} \in \Lambda_{+\|}$ we have $[\![\lambda \vec{y}.x\vec{Q}]\!] \neq \emptyset$.*

Proof. By Proposition 6(iii), it is sufficient to prove that, for all $x \in \mathrm{Var}$ and $\vec{Q} \in \Lambda_{+\|}$, we have $[\![x\vec{Q}]\!] \neq \emptyset$. To conclude, it is easy to show by induction on k that $([(x,\star)],\star) \in [\![xQ_1 \dots Q_k]\!]$. □

Theorem 2. *For all $M \in \Lambda_{+\|}$, if $H(M) \neq \emptyset$ then $[\![M]\!] \neq \emptyset$.*

Proof. Let $[N_1, \dots, N_k] \in H(M)$. By Proposition 7, $\bigodot_{1 \le i \le k} [\![N_i]\!] \subseteq [\![M]\!]$, and by Proposition 8 $[\![N_i]\!] \neq \emptyset$ for $1 \le i \le k$. We conclude that $\emptyset \neq \bigodot_{1 \le i \le k} [\![N_i]\!] \subseteq [\![M]\!]$. □

5 Saturated sets and the realizability argument

In this section, we generalize Krivine's realizability technique [13] to $\lambda_{+\|}$-calculus and we use it for proving that $\lambda_{+\|}$-terms having a non-empty interpretation are all solvable. For notations and terminology, we mainly follow [3].

The saturation of a set S of terms expresses the fact that S is closed under weak head expansions. For the pure λ-calculus, this amounts to the well known condition of being closed under weak head β-expansion. For the extension of the λ-calculus we are dealing with, three cases of weak head expansions, corresponding to the possible shapes of the head term, must be considered.

Definition 8. *A set $S \subseteq \Lambda_{+\|}$ is* saturated *if the following conditions hold:*

- *if $M[N/x]\vec{P} \in S$ then $(\lambda x.M)N\vec{P} \in S$,*
- *if $(MQ\|NQ)\vec{P} \in S$ then $(M\|N)Q\vec{P} \in S$,*
- *if $M\vec{P} \in S$ and $N \in \Lambda_{+\|}$ then $(M+N)\vec{P} \in S$.*

We recall that the sets $\mathcal{N}_0, \mathcal{N}_1$ and $\mathcal{N}$ have been defined in Section 3.3. It is easy to check that $\mathcal{N}$ is saturated, whilst $\mathcal{N}_0$ is not. In the realizability argument, only saturated sets included within $\mathcal{N}_0$ and $\mathcal{N}$ will be considered.

Definition 9. *The set Sat_h of "small" saturated subsets of $\Lambda_{+\|}$ is defined by:*

$$Sat_h = \{S \subseteq \Lambda_{+\|} \mid \ S \text{ is saturated and } \mathcal{N}_0 \subseteq S \subseteq \mathcal{N}\}.$$

Given $A, B \subseteq \Lambda_{+\|}$, we define $A \to B = \{M \in \Lambda_{+\|} \mid (\forall N \in A)\ MN \in B\}$. The operator $\to$ is contravariant in its first argument and covariant in its second one, in other words, $A \to B \subseteq A' \to B'$ for all $A' \subseteq A$ and $B \subseteq B'$.

Lemma 3. $\mathcal{N}_0 \subseteq \Lambda_{+\|} \to \mathcal{N}_0 \subseteq \mathcal{N}_0 \to \mathcal{N} \subseteq \mathcal{N}$.

Proof. The first inclusion follows by definition, the second one is a consequence of the contravariance/covariance of the arrow. For the third one, it is enough to prove that, for all $M \in \Lambda_{+\|}$ and $x \in \mathrm{Var}$, $H(Mx) \neq \emptyset$ entails $H(M) \neq \emptyset$; this holds by Proposition 2(i). □

The set Sat_h enjoys the following closure properties.

Lemma 4. *The set Sat_h is closed under the arrow operator, finite unions, finite intersections, and under the map $\mathcal{F} : S \mapsto (\Lambda_{+\|} \to S)$.*

Proof. Given two sets $S_1, S_2 \in Sat_h$, it is straightforward to check that $S_1 \cap S_2$, $S_1 \cup S_2 \in Sat_h$ and that $S_1 \to S_2$ and $\Lambda_{+\|} \to S_2$ are saturated. The inclusions $\mathcal{N}_0 \subseteq S_1 \to S_2 \subseteq \mathcal{N}$ and $\mathcal{N}_0 \subseteq \Lambda_{+\|} \to S_2 \subseteq \mathcal{N}$ follow easily from Lemma 3 and contravariance/covariance of the arrow. □

We are going to define a function $(-)^\bullet : D \to Sat_h$, satisfying $(m :: \sigma)^\bullet = m^\bullet \to \sigma^\bullet$, where, for a multiset m of elements of D, $m^\bullet = \bigcap_{\alpha \in m} \alpha^\bullet$ and, in particular, $[]^\bullet = \Lambda_{+\|}$. Since $\star = [] :: \star$, the set $\star^\bullet$ must be a fixpoint of the function $\mathcal{F} : S \mapsto (\Lambda_{+\|} \to S)$. We now show that $\mathcal{N}_1$ is one of such fixpoints.

Proposition 9. *$\mathcal{N}_1 \in Sat_h$ and $\mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$.*

Proof. The saturation of $\mathcal{N}_1$ and the fact that $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}$ are both trivial. We now prove that $\mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$. Let $M \in \Lambda_{+\|} \to \mathcal{N}_1$. Since $M\Omega \in \mathcal{N}_1$, we get by Proposition 2(*ii*) that $M \in \mathcal{N}_1$. Conversely, let $M \in \mathcal{N}_1$ and $N \in \Lambda_{+\|}$. We conclude since, by Proposition 2(*iii*), we get $MN \in \mathcal{N}_1$. □

Observe that any element $\sigma \in D$ may be written in a unique way as $\sigma = \sigma_1 :: \cdots :: \sigma_n :: \star$, with $n \geq 0$ and $\sigma_n \neq []$ (and of course $\sigma_1, \ldots, \sigma_n$ have ranks strictly smaller than that of σ). This is called the *standard decomposition* of σ.

Definition 10. *Given $\sigma \in D$, we define $(\sigma)^\bullet \in Sat_h$ by induction on the rank k of σ. If $k = 0$, then $\sigma^\bullet = \star^\bullet = \mathcal{N}_1$. If $k > 0$ then $\sigma^\bullet = \sigma_1^\bullet \to \cdots \to \sigma_n^\bullet \to \mathcal{N}_1$, where $\sigma_1 :: \cdots :: \sigma_n :: \star$ is the standard decomposition of σ.*

Note that if $m \neq []$ or $\sigma \neq \star$, then the standard decomposition of $m :: \sigma$ is $m :: \sigma_1 :: \cdots :: \sigma_n :: \star$, where $\sigma_1 :: \cdots :: \sigma_n :: \star$ is the standard decomposition of σ. Hence, $(m :: \sigma)^\bullet = m^\bullet \to \sigma^\bullet$ holds in general, since $([] :: \star)^\bullet = \star^\bullet = \mathcal{N}_1 = \Lambda_{+\|} \to \mathcal{N}_1$.

We show now that the definition of $(-)^\bullet$ fits well with parallel composition.

Lemma 5. *Let $M, N \in \Lambda_{+\|}$, $\sigma = (\sigma_1, \sigma_2, \ldots), \tau = (\tau_1, \tau_2, \ldots) \in D$ and $\rho = \sigma \bar{\uplus} \tau$. If $M \in \sigma^\bullet$ and $N \in \tau^\bullet$, then $M\|N \in \rho^\bullet$.*

Proof. Let $\rho_n :: \cdots :: \rho_1 :: \star$ be the standard decomposition of ρ. We have to show that $M\|N \in \rho_n^\bullet \to \cdots \to \rho_1^\bullet \to \mathcal{N}_1$. We prove it by induction on n.

If $n = 0$, then $\sigma = \tau = \rho = \star$. Hence, we conclude since $\star^\bullet = \mathcal{N}_1$ and $\mathcal{N}_1$ is closed under parallel composition.

If $n > 0$, then we have to show that, for all $Q \in \rho_n^\bullet$, $(M\|N)Q \in (\rho')^\bullet$ where $\rho' = \rho_{n-1} :: \cdots :: \rho_1 :: \star$. Since $M \in \sigma_1^\bullet$ and $N \in \tau_1^\bullet$, we have that $MQ \in (\sigma')^\bullet$ and $NQ \in (\tau')^\bullet$, where $\sigma' = (\sigma_2, \sigma_3, \ldots)$ and $\tau' = (\tau_2, \tau_3, \ldots)^\bullet$. Moreover, $\rho' = \sigma' \bar{\uplus} \tau'$ and the standard decomposition of ρ' is strictly shorter than that of ρ. By the inductive hypothesis, we get $MQ\|NQ \in (\rho')^\bullet$. By saturation of $(\rho')^\bullet$, we conclude that $(M\|N)Q \in (\rho')^\bullet$, and hence $M\|N \in \rho^\bullet$. □

We are now able to prove the promised *adequation lemma*, which constitutes the key tool in the realizability argument.

Definition 11. *A substitution $s = \{(x_1, N_1), \ldots, (x_k, N_k)\}$ is* adequate *for a multiset $m \in \mathcal{M}_f(D^{\mathrm{Var}})$ if:*

- *$m_x \neq []$ implies $x \in \{x_1, \ldots, x_k\}$, for all $x \in \mathrm{Var}$,*
- *$N_i \in m_{x_i}^\bullet$ for all $1 \leq i \leq k$.*

Observe that, if a substitution is adequate for some multiset $m \in \mathcal{M}_f(D^{\mathrm{Var}})$, then it is adequate for all submultisets of m.

Lemma 6. (Adequation lemma) *Let $M \in \Lambda_{+\|}$, $(m, \sigma) \in [\![M]\!]$ and s be a substitution. If s is adequate for m, then $Ms \in \sigma^\bullet$.*

Proof. By structural induction on M.

- If $M \equiv x$, then $m = [(x,\sigma)]$ by Proposition 6(i). If s is adequate for m, then $(x, N) \in s$ for some $N \in [\sigma]^\bullet$. Hence, we have that $Ms = N \in [\sigma]^\bullet = \sigma^\bullet$.
- If $M \equiv PQ$, then by Proposition 6(ii), we have $m = m_0 \uplus m_1 \uplus \ldots \uplus m_k$ for some $k \geq 0$, and $\tau_1, \ldots, \tau_k \in D$ such that $(m_0, [\tau_1, \ldots, \tau_k] :: \sigma) \in [\![P]\!]$ and $(m_i, \tau_i) \in [\![Q]\!]\ for\ 1 \leq i \leq k$. Observe now that, if s is adequate for m then it is also adequate for $m_0, m_1, \ldots, m_k$, since they are all multisubsets of m. By the inductive hypothesis we have that:
  - $Ps \in ([\tau_1, \ldots, \tau_k] :: \sigma)^\bullet = [\tau_1, \ldots, \tau_k]^\bullet \to \sigma^\bullet$,
  - $Qs \in \tau_1^\bullet, \ldots, Qs \in \tau_k^\bullet$, which implies that $Qs \in [\tau_1, \ldots, \tau_k]^\bullet$.

  Hence, we can conclude that $(PQ)s \in \sigma^\bullet$.
- If $M \equiv \lambda x.P$, then by Proposition 6(iii), we have that $m = m'_{-x}$ and $\sigma = m'_x :: \sigma'$ for some $(m', \sigma') \in [\![P]\!]$. Let s be an adequate substitution for m'_{-x} and $Q \in (m'_x)^\bullet$. Since M is considered up to α-conversion, we can suppose without loss of generality that x does not occur in s. It is clear that $s' = s \cup \{(x, Q)\}$ is adequate for m' and hence, by the inductive hypothesis, we get $Ps' \in (\sigma')^\bullet$. Now we have that $Ps' = (Ps)[Q/x] \in (\sigma')^\bullet$ because x does not appear in s. Since $(\sigma')^\bullet$ is saturated and $(\lambda x.Ps) = (\lambda x.P)s$ we have that $(\lambda x.P)sQ \in (\sigma')^\bullet$. From the arbitrariness of $Q \in (m'_x)^\bullet$ we conclude that $(\lambda x.P)s \in (m'_x)^\bullet \to (\sigma')^\bullet = (m'_x :: \sigma')^\bullet$.
- If $M \equiv P + Q$, then (m, σ) belongs to, say, $[\![P]\!]$. Now, if s is adequate for m, then we get by the inductive hypothesis that $Ps \in \sigma^\bullet$ and we conclude, by saturation of $\sigma^\bullet$, that $(P+Q)s \in \sigma^\bullet$.
- If $M \equiv P \| Q$, then $m = m_1 \uplus m_2$ and $\sigma = \sigma_1 \bar{\uplus} \sigma_2$ with $(m_1, \sigma_1) \in [\![P]\!]$ and $(m_2, \sigma_2) \in [\![Q]\!]$. If s is adequate for m then it is also adequate for m_1, m_2 and, from the inductive hypothesis and Lemma 5, we conclude that $(P\|Q)s \in (\sigma_1 \bar{\uplus} \sigma_2)^\bullet$. □

Theorem 3. *For all $M \in \Lambda_{+\|}$, if $[\![M]\!] \neq \emptyset$ then $M \in \mathcal{N}$.*

Proof. Let $(m, \sigma) \in [\![M]\!]$. The substitution $s_{id} = \{(x, x) \mid m_x \neq []\}$ is adequate for m (note that $\mathrm{Var} \subset \mathcal{N}_0$), and $Ms_{id} = M$. Hence, by the adequation lemma, we conclude that $M \in \sigma^\bullet \subseteq \mathcal{N}$. □

By Theorem 2 and Theorem 3 we finally get our main result.

Theorem 4. *For all $M \in \Lambda_{+\|}$, $H(M) \neq \emptyset \Leftrightarrow [\![M]\!] \neq \emptyset$.*

6 Conclusions and Further works

We have defined a (relational) model $\mathcal{D}$ of a fairly standard parallel and non-deterministic extension of the pure λ-calculus, equipped with a notion of observation given by an operator H. In this framework, full abstraction spells out as follows: $(\forall M, N \in \Lambda_{+\|})[(\forall C[-])\ H(C[M]) \neq \emptyset \Rightarrow H(C[N]) \neq \emptyset]$ iff $[\![M]\!] \subseteq [\![N]\!]$. The "if" part of the previous statement (adequacy) is an easy consequence of Theorem 4. Nevertheless, the "only if" part fails. Indeed, given $\mathbf{I} \equiv \lambda x.x$, we have that $[\![\mathbf{I}]\!] = \{([], [\sigma] :: \sigma) \mid \sigma \in D)\}$ and $[\![\mathbf{I} \| \mathbf{I}]\!] = \{([], [\sigma, \sigma] :: (\sigma \uplus \sigma)) \mid \sigma \in D)\}$. Hence, $[\![\mathbf{I}]\!] \not\subseteq [\![\mathbf{I} \| \mathbf{I}]\!]$ whilst it is not difficult to check that $\mathbf{I}$ and $\mathbf{I} \| \mathbf{I}$ are not separable using contexts. As suggested by the counterexemple, the next step towards full abstraction should be to enrich the syntax of the language by some "resource sensitive" operator, to increase the discriminating power of contexts.

Finally, we already know from [15, Sec. 3.3] that the theory induced on the pure untyped λ-calculus by our model $\mathcal{D}$ is $\mathcal{H}^*$ (just as the theory induced by Scott's $\mathcal{D}_\infty$); it would be interesting to generalize such a result to the extended setting, as a step in the study and classification of $\lambda_{+\|}$-theories, and models.

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