Boolean algebras for lambda calculus*

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Abstract

In this paper we show that the Stone representation theorem for Boolean algebras can be generalized to combinatory algebras. In every combinatory algebra there is a Boolean algebra of central elements *(playing the role of idempotent elements in rings), whose operations are defined by suitable combinators. Central elements are used to represent any combinatory algebra as a Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other nontrivial algebras). Central elements are also used to provide applications of the representation theorem to lambda calculus. We show that the* indecomposable semantics *(i.e., the semantics of lambda calculus given in terms of models of lambda calculus, which are directly indecomposable as combinatory algebras) includes the continuous, stable and strongly stable semantics, and the term models of all semisensible lambda theories. In one of the main results of the paper we show that the indecomposable semantics is equationally incomplete, and this incompleteness is as wide as possible: for every recursively enumerable lambda theory T, there is a continuum of lambda theories including T which are omitted by the indecomposable semantics.*

1. Introduction

The lambda calculus is not a true equational theory since the variable-binding properties of lambda abstraction prevent variables in lambda calculus from operating as real algebraic variables. Consequently the general methods that have been developed in universal algebra, for defining the semantics of an arbitrary algebraic theory for instance, are not directly applicable. There have been several attempts to reformulate the lambda calculus as a purely algebraic theory. The earliest, and best known, algebraic models are the combinatory algebras of Curry [14] and Schönfinkel [27]. Combinatory algebras have a simple purely equational characterization and were used to provide an intrinsic first-order, but not equational, characterization of the models of lambda calculus, as a special class of combinatory algebras called λ-models [3, Def. 5.2.7].

Topology is at the center of the known approaches to giving models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of algebraic lattices, was successfully used to show that all unsolvable λ-terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 3, 5, 6, 23]. Scott's continuous semantics [28] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [8]) and the strongly stable semantics (Bucciarelli-Ehrhard [10]) are a strengthening of the continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. All these semantics are structurally and equationally rich [7, 17, 18] in the sense that it is possible to build up $2^{\aleph_0}$ λ-models in each of them inducing, through the kernel congruence relation of the interpretation function, pairwise distinct lambda theories. Nevertheless, the above denotational semantics are equationally *incomplete*: they do not match all possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [16] for the continuous semantics and by Bastonero-Gouy [4, 15] for the stable semantics. In [24, 25] Salibra has shown in a uniform way that all semantics (including the strongly stable semantics), which involve monotonicity with respect to some partial order and have a bottom element, fail to induce a continuum of lambda theories.

* Work partially supported by the Equipe PPS of the University Paris 7-Denis Diderot, and by MIUR Cofin'04 FOLLIA Project.

Salibra [20, 25, 26] has recently launched a research program for exploring lambda calculus and combinatory logic using techniques of universal algebra. In [20] Lusin and Salibra have shown that a lattice identity is satisfied by all congruence lattices of combinatory algebras iff it is trivial (i.e, true in all lattices). As a consequence, it is not possible to apply to combinatory algebras the nice results developed in universal algebra (see [12, 21]) in the last thirty years, which essentially connect lattice identities satisfied by all congruence lattices of algebras in a variety, and Mal'cev conditions (that characterize properties in varieties by the existence of suitable terms involved in certain identities). Thus there is a common belief that lambda calculus and combinatory logic are algebraically pathological.

On the contrary, in this paper we show that combinatory algebras satisfy interesting algebraic properties. One of the milestones of modern algebra is the Stone representation theorem for Boolean algebras, which was generalized by Peirce to commutative rings with unit and next by Comer to the class of algebras with Boolean factor congruences (see [13, 19, 22]). By applying a theorem by Vaggione [30], we show that Comer's generalization of Stone representation theorem holds also for combinatory algebras: any combinatory algebra is isomorphic to a weak Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other nontrivial algebras). Another way to express the representation theorem is in terms of sheaves: any combinatory algebra is isomorphic to the algebra of global sections of a sheaf of indecomposable combinatory algebras over a Boolean space.

The proof of the representation theorem for combinatory algebras is based on the fact that every combinatory algebra has *central elements*, i.e., elements which define a direct decomposition of the algebra as the Cartesian product of two other combinatory algebras, just like idempotent elements in rings or complemented elements in bounded distributive lattices. We show that central elements constitute a Boolean algebra, whose Boolean operations can be defined by suitable combinators. This result highlights a connection between propositional classic logic and combinatory logic. What is the real meaning of this flavour of classic logic within combinatory logic remains to be investigated in the future. What we would like to emphasize here is that central elements have been shown fundamental in the application of the representation theorem to lambda calculus, as it will be explained in the next paragraph.

The representation theorem can be roughly summarized as follows: the directly indecomposable combinatory algebras are the 'building blocks' in the variety of combinatory algebras. On the other hand, the result of incompleteness [25], stating that any semantics of lambda calculus given in terms of partial orderings with a bottom element is incomplete, removes the belief that partial orderings are intrinsic to λ-models. It would be interesting to find new Cartesian closed categories, where the partial orderings play no role and the reflexive objects are directly indecomposable as combinatory algebras. In this paper we investigate the class of all models of lambda calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short). We show that the indecomposable semantics includes: (i) the continuous semantics; (ii) the stable and strongly stable semantics restricted to models whose underlying domain is algebraic; (iii) the term models of all semisensible lambda theories (theories which do not equate solvable and unsolvable terms). In one of the main results of the paper we show that the indecomposable semantics is incomplete, and this incompleteness is as wide as possible: for every recursively enumerable lambda theory T, there is a continuum of lambda theories including T which are omitted by the indecomposable semantics.

It is unknown, in general, whether the set of lambda theories, which are representable in a semantics of lambda calculus, is a lattice with respect to the inclusion ordering. In the last result of the paper we show that the set of lambda theories representable in the continuous (stable) semantics is not closed under finite intersection, so that it cannot constitute a sublattice of the lattice of all lambda theories.

The paper is organized as follows. In Section 2 we review the basic definitions of lambda calculus, combinatory logic and universal algebra. In particular, we recall the formal definitions of a model of lambda calculus and of a Boolean product. The Stone representation theorem for combinatory algebras is presented in Section 3, where it is also shown that the central elements of a combinatory algebra constitute a Boolean algebra, whose operations are defined by suitable combinators. Section 4 is devoted to the algebraic incompleteness of lambda calculus.

2. Notation and basic definitions

We will generally use the notation of Barendregt's classic work [3] for lambda calculus and combinatory logic and the notation of Burris and Sankappanavar [12] for universal algebra.

Since this paper spans several fields (logic, algebra, and computation), which may each have their own vocabularies, it may be useful to recall some basic terminology.

2.1. The untyped lambda calculus

Λ and Λ^o are, respectively, the set of λ-terms and of closed λ-terms. Concerning specific λ-terms we set:

$$\mathbf{I} \equiv \lambda x.x; \boldsymbol{T} \equiv \lambda xy.x; \boldsymbol{F} \equiv \lambda xy.y; \Omega \equiv (\lambda x.xx)(\lambda x.xx).$$

A more traditional notation for $\boldsymbol{T}$ is $\mathbf{K}$ (when not viewed as a boolean).

We will denote $\alpha\beta$-conversion by $\lambda\beta$. A *lambda theory* is a congruence on Λ (with respect to the operators of abstraction and application) which contains $\lambda\beta$; it can also be seen as a (specific) set of equations between λ-terms. The set of all lambda theories is naturally equipped with a structure of complete lattice, hereafter denoted by λT, with meet defined as set theoretical intersection. The join of two lambda theories T and S is the least equivalence relation including $T \cup S$. It is clear that $\lambda\beta$ is the least element of $\lambda\mathcal{T}$, while the inconsistent lambda theory $\Lambda \times \Lambda$ is the top element of $\lambda\mathcal{T}$.

The lambda theory generated (or axiomatized) by a set of equations is the least lambda theory containing it. As a matter of notation, $T \vdash M = N$ stands for $M = N \in T$; this is also written as $M =_T N$. A lambda theory T is consistent if there exists at least an equation $M = N$ such that $T \not\vdash M = N$.

Solvable λ-terms can be characterized as follows: a λ-term M is solvable if, and only if, it has a *head normal form*, that is, $M =_{\lambda\beta} \lambda x_1 \dots x_n.yM_1 \dots M_k$ for some $n, k \geq 0$ and λ-terms $M_1, \dots, M_k$. $M \in \Lambda$ is *unsolvable* if it is not solvable.

$\mathcal{H}$ is the lambda theory generated by equating all the unsolvable λ-terms, while $\mathcal{H}^*$ is the unique maximal consistent lambda theory such that $\mathcal{H} \subseteq \mathcal{H}^*$. A lambda theory T is called *semisensible* [3, Def. 4.1.7(iii)] if $T \not\vdash M = N$ whenever M is solvable and N is unsolvable. T is semisensible iff $T \subseteq \mathcal{H}^*$. A lambda theory T is *sensible* if $\mathcal{H} \subseteq T$ (see Section 10.2 and Section 16.2 in [3]).

2.2. Combinatory algebra

An *applicative structure* is an algebra with a distinguished 2-ary function symbol which we call *application*. We may write it infix as $s \cdot t$, or even drop it entirely and write st. As usual, application associates to the left; stu means $(st)u$.

Schönfinkel and Curry discovered that a particularly simple applicative structure has tremendous expressive power [27, 14]: a *combinatory algebra* $\mathbf{C}$ is an applicative structure for a signature with two constants $\mathbf{k}$ and $\mathbf{s}$, such that $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ for all x, y, and z. See elsewhere [14] for a full treatment.

Call $\mathbf{k}$ and $\mathbf{s}$ the *basic combinators*. In the equational language of combinatory algebras the derived combinators $\mathbf{i}$ and $\mathbf{1}$ are defined as $\mathbf{i} \equiv \mathbf{skk}$ and $\mathbf{1} \equiv \mathbf{s}(\mathbf{ki})$. It is not hard to verify that every combinatory algebra satisfies the identities $\mathbf{i}x = x$ and $\mathbf{1}xy = xy$.

We say that $c \in \mathbf{C}$ *represents* a function $f : \mathbf{C} \to \mathbf{C}$ (and that f is *representable*) if $cz = f(z)$ for all $z \in \mathbf{C}$. Call $c, d \in \mathbf{C}$ *extensionally equal* when they represent the same function in $\mathbf{C}$. For example c and $\mathbf{1}c$ are always extensionally equal. (We use $\mathbf{1}$ below to select a canonical representative inside a class of extensionally equivalent elements.)

For each variable x we define a transformation λx^* of the set of combinatory terms as follows: $\lambda x^*.x = \mathbf{i}$. Let t be a combinatory term different from x. If x does not occur in t, define $\lambda x^*.t = \mathbf{k}t$. Otherwise, t must be of the form rs where s and r are combinatory terms, at least one of which contains x; in this case define $\lambda x^*.t = \mathbf{s}(\lambda x^*.r)(\lambda x^*.s)$. It is well known that x does not occur in $\lambda x^*.t$ and that, for every combinatory algebra $\mathbf{C}$ and combinatory term u, we have:

$$\mathbf{C} \models (\lambda x^*.t)u = t[x := u],$$

where the combinatory term $t[x := u]$ is obtained by substituting u for x in t.

2.3. Lambda model

The axioms of an elementary subclass of combinatory algebras, called λ-*models* or models of the lambda calculus, were expressly chosen to make coherent the definition of interpretation of λ-terms (see [3, Def. 5.2.7]). Let $\mathbf{C}$ be a λ-model and let $\bar{c}$ be a new symbol for each $c \in C$. Extend the language of the lambda calculus by adding $\bar{c}$ as a new constant symbol for each $c \in C$. Let $\Lambda^o(C)$ be the set of closed λ-terms with constants from C. The interpretation of terms in $\Lambda^o(C)$ with elements of C can be defined by induction as follows (for all $M, N \in \Lambda^o(C)$ and $c \in C$):

$$|\bar{c}| = c;\ |(MN)| = |M|\,|N|;\ |\lambda x.M| = \mathbf{1}m,$$

where $m \in C$ is any element representing the following function $f : C \to C$:

$$f(c) = |M[x := \bar{c}]|, \quad \text{for all } c \in C.$$

The *Meyer-Scott axiom* is the most important axiom in the definition of a λ-model. In the first-order language of combinatory algebras it takes the following form

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow \mathbf{1}x = \mathbf{1}y).$$

The combinator $\mathbf{1}$ becomes an inner choice operator, that makes coherent the interpretation of an abstraction λ-term.

Each λ-model $\mathcal{M}$ induces a lambda theory, denoted here by $Th(\mathcal{M})$, and called *the equational theory of* $\mathcal{M}$. Thus, $M = N \in Th(\mathcal{M})$ if, and only if, M, N have the same interpretation in $\mathcal{M}$.

The *term model* $\mathcal{M}_T$ of a lambda theory T (see [3, Def. 5.2.11]) consists of the set of the equivalence classes of λ-terms modulo the lambda theory T together with the operation of application on the equivalence classes. By [3, Cor. 5.2.13(ii)] $\mathcal{M}_T$ is a λ-model which induces the lambda theory T.

We define various notions of representability of theories in classes of models.

Definition 1 *Given a lambda theory T,*

- *A λ-model $\mathcal{M}$ is* a model *of T if $T \subseteq Th(\mathcal{M})$.*
- *A λ-model $\mathcal{M}$* represents (or induces) *T if $T = Th(\mathcal{M})$.*

Definition 2 *Given a class $\mathbb{C}$ of λ-models and a lambda theory T,*

1. *$\mathbb{C}$* represents *T if there is some $\mathcal{M} \in \mathbb{C}$ representing T.*
2. *$\mathbb{C}$* omits *T if there is no $\mathcal{M} \in \mathbb{C}$ representing T.*
3. *$\mathbb{C}$ is* complete *for the set $\mathcal{S} \subseteq \lambda T$ of lambda theories if $\mathbb{C}$ represents all elements of $\mathcal{S}$.*
4. *$\mathbb{C}$ is* incomplete *if it omits a consistent lambda theory.*

2.4. Algebra

A congruence θ on an algebra $\mathbf{A}$ is an equivalence relation which is compatible with respect to the basic operation of the algebra. Write Con$\mathbf{A}$ for the set of congruences of $\mathbf{A}$. This has a natural complete lattice structure by inclusion of sets (considering θ as a subset of $A \times A$, so the meet is just set-intersection).

θ is *trivial* if it is the top or bottom element in the natural inclusion ordering; write these $\nabla^{\mathbf{A}}$ (equal to $A \times A$) and $\Delta^{\mathbf{A}}$ (equal to $\{(a,a) \mid a \in A\}$) respectively. Also, given $a, b \in A$ write $\theta(a,b)$ for the least congruence relating a and b.

Given two congruences σ and τ on the algebra $\mathbf{A}$, we can form the *relative product*:

$$\tau \circ \sigma = \{(a,c) \mid a\sigma b\tau c, \text{ for some } b \in A\}.$$

This is a compatible relation on $\mathbf{A}$, but not necessarily a congruence.

An algebra $\mathbf{A}$ is *simple* when its only congruences are $\Delta^{\mathbf{A}}$ and $\nabla^{\mathbf{A}}$.

An algebra $\mathbf{A}$ is a *subdirect product* of the algebras $(\mathbf{B}_i : i \in I)$ if there exists an embedding f of $\mathbf{A}$ into the direct product $\Pi_{i\in I}\mathbf{B}_i$ such that the projection $\pi_i \circ f : \mathbf{A} \to \mathbf{B}_i$ is onto for every $i \in I$. We write $\mathbf{A} \leq \Pi_{i\in I}\mathbf{B}_i$ if $\mathbf{A}$ is a subdirect product of the algebras $(\mathbf{B}_i : i \in I)$.

Call a nonempty class K of algebras of the same similarity type a *variety* if it is closed under subalgebras, homomorphic images and direct products. By Birkhoff's theorem (see [21]) a class of algebras is a variety if, and only if, it is an equational class (that is, it is axiomatized by a set of equations).

2.5. Factor congruence

Call θ a *factor congruence* when there exists another congruence $\overline{\theta}$ such that $\theta \cap \overline{\theta} = \Delta^{\mathbf{A}}$ and $\nabla^{\mathbf{A}} = \theta \circ \overline{\theta}$. In this case call θ and $\overline{\theta}$ a *pair of complementary factor congruences*.

It is easy to see that $\mathbf{A}$ has a pair $(\theta, \overline{\theta})$ of complementary factor congruences precisely when it is isomorphic to $\mathbf{B} \times \mathbf{C}$ (with $\mathbf{B}$ isomorphic to $\mathbf{A}/\theta$ and $\mathbf{C}$ isomorphic to $\mathbf{A}/\overline{\theta}$).

So factor congruences are another way of saying 'this algebra is a direct product of simpler algebras'.

The set of factor congruences of $\mathbf{A}$ is not, in general, a sublattice of $\mathrm{Con}\,\mathbf{A}$. $\Delta^{\mathbf{A}}$ and $\nabla^{\mathbf{A}}$ are the *trivial* factor congruences, corresponding to $\mathbf{A} \cong 1 \times \mathbf{A}$; of course, 1 is isomorphic to $\mathbf{A}/\nabla^{\mathbf{A}}$ and $\mathbf{A}$ is isomorphic to $\mathbf{A}/\Delta^{\mathbf{A}}$.

Call $\mathbf{A}$ *directly indecomposable* when FC$\mathbf{A}$ has two elements ($\Delta^{\mathbf{A}}$ and $\nabla^{\mathbf{A}}$).

Clearly, a simple algebra is directly indecomposable, though there are algebras which are directly indecomposable but not simple (they just have congruences which do not split the algebra up neatly as a Cartesian product).

It is useful to characterize factor congruences in terms of algebra homomorphisms satisfying certain equalities (the next step will be to express the equalities in the equational language of the algebra itself).

A decomposition operation (see [21, Def. 4.32]) for an algebra $\mathbf{A}$ is an algebra homomorphism $f : \mathbf{A}^2 \to \mathbf{A}$ such that

$$f(x,x) = x; \quad f(f(x,y),z) = f(x,z) = f(x,f(y,z)).$$

By [21, Thm. 4.33] there exists a bijective correspondence between pairs of complementary factor congruences and decomposition operations, and thus between decomposition operations and factorizations $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$.

By this intuition we see that the binary relations θ and $\overline{\theta}$ defined by

$$x\;\theta\;y \;\text{ iff }\; f(x,y) = y; \quad x\;\overline{\theta}\;y \;\text{ iff }\; f(x,y) = x,$$

are a pair of complementary factor congruences, and conversely we see that for every pair θ and $\overline{\theta}$ of complementary factor congruences, the map f defined by

$$f(x,y) = u \;\text{ iff }\; x\;\theta\;u\;\overline{\theta}\;y, \tag{1}$$

is a decomposition operation. Notice that for any x and y there is just one element u such that $x\;\theta\;u\;\overline{\theta}\;y$.

The reader can easily verify these facts for themselves, or find proofs elsewhere [21].

An algebra has *Boolean factor congruences* if the factor congruences form a Boolean sublattice of the congruence lattice. Most known examples of varieties in which all algebras have Boolean factor congruences are those with *factorable congruences*, that is, varieties in which every congruence θ on $\mathbf{A} \times \mathbf{B}$ is a product congruence $\theta_1 \times \theta_2$ of two congruences $\theta_1 \in \mathrm{Con}\,\mathbf{A}$ and $\theta_2 \in \mathrm{Con}\,\mathbf{B}$. Recall that $(b,c)\;\theta_1 \times \theta_2\;(b',c')$ iff $b\;\theta_1\;b'$ and $c\;\theta_2\;c'$.

2.6. Boolean product

The Boolean product construction (see [12, Chapter IV]) provides a method for translating numerous fascinating properties of Boolean algebras into other varieties of algebras. Actually the construction that we call "Boolean product" has been known for several years as "the algebra of global sections of sheaves of algebras over Boolean spaces" (see [13, 19]); however the definition of the latter was unnecessarily involved. We recall that a Boolean space is a compact, Hausdorff and totally disconnected topological space.

A *weak Boolean product* of an indexed family $(\mathbf{A}_i : i \in I)$ of algebras is a subdirect product $\mathbf{A} \leq \Pi_{i\in I}\mathbf{A}_i$, where I can be endowed with a Boolean space topology so that

(i) the set $\{i \in I : a_i = b_i\}$ is open for all $a, b \in A$, and

(ii) if $a, b \in A$ and N is a clopen subset of I, then the element c, defined by $c_i = a_i$ for every $i \in N$ and $c_i = b_i$ for every $i \in I - N$, belongs to A.

A *Boolean product* is just a weak Boolean product such that the set $\{i \in I : a_i = b_i\}$ is clopen for all $a, b \in A$.

3. The Stone representation theorem for combinatory algebras

The axioms characterizing the variety of combinatory algebras are suggested by an analysis of recursive processes, not by logic (as for Boolean algebras and Heyting algebras) or by algebra (as for groups and rings). Combinatory algebras are never commutative, associative, finite and recursive, so that there is a common belief that these algebras are algebraically pathological.

On the contrary, in this section we show that combinatory algebras satisfy interesting algebraic properties: the Stone representation theorem for Boolean algebras admits a generalization to combinatory algebras.

3.1. Stone and Peirce

The Stone representation theorem for Boolean rings (the observation that Boolean algebras could be regarded as rings is due to Stone) admits a generalization, due to Peirce, to commutative rings with unit (see [22] and [19, Chapter V]). To make the reader familiar with the argument, we give in this subsection an outline of Peirce construction.

Let $\mathbf{A} = (A, +, \cdot, 0, 1)$ be a commutative ring with unit, and let $E(A) = \{a \in A : a \cdot a = a\}$ be the set of idempotent elements of A. We define a structure of Boolean algebra on $E(A)$ as follows, for all $a, b \in E(A)$:

- $a \wedge b = a \cdot b$;
- $a \vee b = a + b - (a \cdot b)$;
- $a^- = 1 - a$.

Then it is possible to show that every idempotent element $a \neq 0, 1$ defines a pair $\theta(a, 1)$, $\theta(a, 0)$ of nontrivial complementary factor congruences, where $\theta(a, 1)$ is the least congruence containing the pair $(a, 1)$ and similarly for the other congruence $\theta(a, 0)$. In other words, the ring $\mathbf{A}$ can be decomposed in a non trivial way as $\mathbf{A} = \mathbf{A}/\theta(a, 1) \times \mathbf{A}/\theta(a, 0)$. If $E(A) = \{0, 1\}$, then A is directly indecomposable. Then the Peirce theorem for commutative rings with unit can be stated as follows: every commutative ring with unit is isomorphic to a Boolean product of directly indecomposable rings. If $\mathbf{A}$ is a Boolean ring, then we get the Stone representation theorem for Boolean algebras, because the ring of truth values is the unique directly indecomposable Boolean ring.

The remaining part of this section is devoted to the proof of the representation theorem for combinatory algebras.

3.2. The Boolean algebra of central elements

Combinatory logic and lambda calculus internalise many important things (computability theory, for example). 'To be directly decomposable' is another internalisable property of these formalisms, as it will be shown in this subsection.

As a matter of notation, let **t** $\equiv \lambda xy^*.x$ and **f** $\equiv \lambda xy^*.y$, where λxy^* is defined in Section 2.2.

The combinators **t** and **f** correspond to the constants 0 and 1 in a commutative ring with unit, while, as it will be shown below, the so-called central elements of a combinatory algebra correspond to idempotent elements in a ring. Central elements in universal algebra were introduced by Vaggione in [31] and were used, among the other things, to investigate the closure of varieties of algebras under Boolean products.

Definition 3 *Let* $\mathbf{A}$ *be a combinatory algebra. We say an element* $e \in A$ *is* central *when it satisfies the following equations, for all* $x, y, z, t \in A$*:*

(i) $exx = x$.

(ii) $e(exy)z = exz = ex(eyz)$.

(iii) $e(xy)(zt) = exz(eyt)$.

(iv) $e = e\mathbf{tf}$.

The set of central elements of $\mathbf{A}$ will be denoted by $E(\mathbf{A})$.

Every combinatory algebra admits at least two central elements, namely the combinators **t** and **f**. Now we show that central elements, as idempotent elements in a ring, decompose a combinatory algebra $\mathbf{A}$ as a Cartesian product: if $e \in E(\mathbf{A})$, then $\mathbf{A} = \mathbf{A}/\theta(e, \mathbf{t}) \times \mathbf{A}/\theta(e, \mathbf{f})$. This will be shown in the next proposition via decomposition operators. The use of decomposition operators to characterize central elements is new.

Fix some combinatory algebra.

Proposition 4 *There is a (natural) bijective correspondence between central elements and decomposition operators.*

Proof. Given a central element e we obtain a decomposition operator by taking $f_e(x, y) = exy$. It is a simple exercise to show that axioms (i)-(iii) of a central element make f_e a decomposition operator.

Conversely, given a decomposition operator f, we have to show that the element $f(\mathbf{t}, \mathbf{f})$ is central. From Section 2.5 we have that $f(\mathbf{t}, \mathbf{f})$ is the unique element u satisfying $\mathbf{t}\ \theta\ u\ \overline{\theta}\ \mathbf{f}$, where θ and $\overline{\theta}$ are the pair of complementary factor congruences associated with the decomposition operator f. Then, from the property of congruence of θ and $\overline{\theta}$ it follows, for all x, y:

$$\mathbf{t}xy\ \theta\ f(\mathbf{t}, \mathbf{f})xy\ \overline{\theta}\ \mathbf{f}xy,$$

that implies

$$x\ \theta\ f(\mathbf{t}, \mathbf{f})xy\ \overline{\theta}\ y.$$

Since by definition $f(x, y)$ is the unique element satisfying

$$x\ \theta\ f(x, y)\ \overline{\theta}\ y,$$

then we obtain

$$f(x, y) = f(\mathbf{t}, \mathbf{f})xy \tag{2}$$

Finally, the identities defining f as decomposition operator make $f(\mathbf{t}, \mathbf{f})$ a central element.

It is easy to verify that these correspondences form the two sides of a bijection. If e is central, then the central element $f_e(\mathbf{t}, \mathbf{f})$ is equal to e, because $f_e(\mathbf{t}, \mathbf{f}) = e\mathbf{t}\mathbf{f} = e$ by (iv). If f is a decomposition operator, then by (2) we have that $f_{f(\mathbf{t},\mathbf{f})}(x, y) = f(\mathbf{t}, \mathbf{f})xy = f(x, y)$ for all x, y. ■

For every central element e, we denote respectively by f_e and by $(\theta_e, \overline{\theta}_e)$ the decomposition operator and the pair of complementary factor congruences determined by e.

Corollary 5 *If e is central, then we have:*

1. *$x\ \theta_e\ exy\ \overline{\theta}_e\ y$;*
2. *$x\ \theta_e\ y$ iff $exy = y$; $x\ \overline{\theta}_e\ y$ iff $exy = x$.*

The congruence θ_e is generated by the pair $(e, \mathbf{t})$ (i.e., $\theta_e = \theta(e, \mathbf{t})$), while the congruence $\overline{\theta}_e$ by the pair $(e, \mathbf{f})$ (i.e., $\overline{\theta}_e = \theta(e, \mathbf{f})$).

We now show that the partial ordering over central elements, defined by

$$d \leq e \text{ iff } \theta_d \subseteq \theta_e \tag{3}$$

is a Boolean ordering. The combinators $\mathbf{t}$ and $\mathbf{f}$ are respectively the bottom element and the top element of this ordering, while the combinators $\lambda xy^*.x\mathbf{t}y$ and $\lambda x^*.x\mathbf{f}\mathbf{t}$ represent respectively the meet operation and the complementation.

Theorem 6 *Let $\mathbf{A}$ be a combinatory algebra. Then the algebra $\mathbf{E}(\mathbf{A}) = (E(\mathbf{A}), \wedge, ^-)$ of central elements of $\mathbf{A}$, defined by*

$$e \wedge d = e\mathbf{t}d; \quad e^- = e\mathbf{f}\mathbf{t},$$

is a Boolean algebra.

Proof. We first show that the factor congruences of $\mathbf{A}$ form a Boolean sublattice of the congruence lattice Con$\mathbf{A}$. Let $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ be a decomposition of $\mathbf{A}$ as the direct product of two combinatory algebras $\mathbf{B}$ and $\mathbf{C}$. It is easy to show, by using the equations defining central elements, that $E(\mathbf{A}) = E(\mathbf{B}) \times E(\mathbf{C})$, i.e., every central element of $\mathbf{A}$ can be decomposed as a pair of central elements of $\mathbf{B}$ and $\mathbf{C}$. In the terminology of universal algebra this means that $\mathbf{A}$ has no 'skew factor congruences'. We get the conclusion from [9, Prop. 1.3], where it is shown that an algebra $\mathbf{A}$ has no skew factor congruences if, and only if, the factor congruences of $\mathbf{A}$ form a Boolean sublattice of the congruence lattice Con$\mathbf{A}$.

It follows that the partial ordering on central elements, defined in (3), is a Boolean ordering. We have to show now that, for all central elements d, e, the elements $e^- = e\mathbf{f}\mathbf{t}$ and $e \wedge d = e\mathbf{t}d$ are central and are respectively associated with the pairs $(\overline{\theta}_e, \theta_e)$ and $(\theta_e \cap \theta_d, \overline{\theta}_e \vee \overline{\theta}_d)$ of complementary factor congruences (recall that e is associated with the pair $(\theta_e, \overline{\theta}_e)$).

We check the details for $e\mathbf{f}\mathbf{t}$. By Cor. 5(1) we have that $e\mathbf{f}\mathbf{t}$ is the unique element u such that $\mathbf{t}\ \overline{\theta}_e\ u\ \theta_e\ \mathbf{f}$. By (1) in Section 2.5 this means that $e\mathbf{f}\mathbf{t} = g(\mathbf{t}, \mathbf{f})$ for the decomposition operator g associated with the pair $(\overline{\theta}_e, \theta_e)$ of complementary factor congruences. We have the conclusion that $e\mathbf{f}\mathbf{t}$ is central associated with the pair $(\overline{\theta}_e, \theta_e)$ as in the proof of Prop. 4.

We now consider $e \wedge d = e\mathbf{t}d$. First of all, we show that $e\mathbf{t}d = d\mathbf{t}e$. By Cor. 5(1) we have that $\mathbf{t}\ \theta_e\ e\mathbf{t}d\ \overline{\theta}_e\ d$, while $\mathbf{t}\ \theta_e\ d\mathbf{t}e\ \overline{\theta}_e\ d$ can be obtained as follows: $\mathbf{t} =$ (by Def. 3(i)) $d\mathbf{t}\mathbf{t}\ \theta_e$ (by $\mathbf{t}\theta_e e$) $d\mathbf{t}e\ \overline{\theta}_e$ (by $e\overline{\theta}_e\mathbf{f}$) $d\mathbf{t}\mathbf{f} =$ (by Def. 3(iv)) d. Since there is a unique element u such that $\mathbf{t}\ \theta_e\ u\ \overline{\theta}_e\ d$, then we have the conclusion $d\mathbf{t}e = e\mathbf{t}d$. We now show that $e\mathbf{t}d$ is the central element associated with the factor congruence $\theta_e \cap \theta_d$, i.e.,

$$\mathbf{t}\ (\theta_e \cap \theta_d)\ e\mathbf{t}d\ (\overline{\theta}_e \vee \overline{\theta}_d)\ \mathbf{f}.$$

From $d\mathbf{t}e = e\mathbf{t}d$ we easily get that $\mathbf{t}\ \theta_e\ e\mathbf{t}d$ and $\mathbf{t}\ \theta_d\ e\mathbf{t}d$, that is, $\mathbf{t}\ (\theta_e \cap \theta_d)\ e\mathbf{t}d$. Finally, by Cor. 5 we have: $e\mathbf{t}d\ \overline{\theta}_e\ d = d\mathbf{t}\mathbf{f}\ \overline{\theta}_d\ \mathbf{f}$, i.e., $e\mathbf{t}d\ (\overline{\theta}_e \vee \overline{\theta}_d)\ \mathbf{f}$. ■

We now provide the promised representation theorem. If I is a maximal ideal of the Boolean algebra $E(\mathbf{A})$, then $\cup I$ denotes the congruence on $\mathbf{A}$ defined by

$$x\ (\cup I)\ y \text{ iff } x\ \theta_e\ y \text{ for some } e \in I.$$

By a *Peirce variety* (see [30]) we mean a variety of algebras for which there are two constants $0, 1$ and a term $u(x,y,z,v)$ such that the following identities hold: $u(x,y,0,1) = x$ and $u(x,y,1,0) = y$.

Theorem 7 (Representation Theorem) *Let* $\mathbf{A}$ *be a combinatory algebra and* X *be the Boolean space of maximal ideals of the Boolean algebra* $\mathbf{E}(\mathbf{A})$ *of central elements. Then the map*

$$f : A \to \Pi_{I\in X}(A/\cup I),$$

defined by

$$f(x) = (x/\cup I : I \in X),$$

gives a **weak** *Boolean product representation of* $\mathbf{A}$*, where the quotient algebras* $\mathbf{A}/\cup I$ *are directly indecomposable.*

Proof. By Thm. 6 the set of factor congruences of $\mathbf{A}$ constitutes a Boolean sublattice of Con$\mathbf{A}$. Then by [13] f gives a weak Boolean product representation of $\mathbf{A}$. The quotient algebras $\mathbf{A}/\cup I$ are directly indecomposable by [30, Thm. 8], because the variety of combinatory algebras is a Peirce variety if we define $1 \equiv \mathbf{t}$, $0 \equiv \mathbf{f}$ and $u = \lambda xyzv^*.zyx$. ■

The map f of the above theorem does not give in general a Boolean product representation. This follows from two results due to Vaggione [31] and to Plotkin-Simpson [29]. Vaggione has shown that, if a variety has factorable congruences (i.e., every congruence in a product is a product of congruences) and every member of the variety can be represented as a Boolean product of directly indecomposable algebras, then the variety is a discriminator variety (see [12] for the terminology). Discriminator varieties satisfy very strong algebraic properties, in particular they are congruence permutable (i.e., the join of two congruences is just their composition). Plotkin and Simpson have shown that the property of having permutable congruences is inconsistent with combinatory logic.

4. The algebraic incompleteness of lambda calculus

The representation theorem of combinatory algebras can be roughly summarized as follows: the directly indecomposable combinatory algebras are the 'building blocks' in the variety of combinatory algebras. Then it is natural to investigate the class of models of lambda calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short). In this section we show that the indecomposable semantics includes: (i) the continuous semantics; (ii) the stable and strongly stable semantics restricted to models whose underlying domain is algebraic; (iii) the term models of all semisensible lambda theories. However, in one of the main results of the paper we give a proof, based on central elements, that the indecomposable semantics is incomplete, and this incompleteness is as wide as possible: for every recursively enumerable lambda theory T, there is a continuum of lambda theories including T which are omitted by the indecomposable semantics. In last result of the paper we show that the set of lambda theories induced by each of the known semantics is not closed under finite intersection, so that it cannot constitute a sublattice of the lattice of lambda theories.

4.1. Internalising 'indecomposable'

We have shown how to internally represent a factor congruence as a central element. Now we show how to represent the logical assertion that the only factor congruences of a combinatory algebra are trivial.

We recall that an algebra $\mathbf{A}$ is *directly indecomposable* when it is not trivial and it is not isomorphic to a product of two nontrivial algebras (i.e., there is not a pair of nontrivial complementary factor congruences). A combinatory algebra $\mathbf{A}$ is directly indecomposable if $E(\mathbf{A}) = \{\mathbf{t}, \mathbf{f}\}$.

For two combinatory terms t and u, define the pair $[t,u] \equiv \lambda z^*.ztu$ and, for every sequence $t_1,\dots,t_n$, define $[t_1,\dots,t_n] \equiv [t_1,[t_2,\dots,t_n]]$.

Define the following combinatory terms:

- $\mathbf{Z} \equiv \lambda e^*.[\lambda x^*.exx, \lambda xyz^*.e(exy)z, \lambda xyz^*.exz,$ $\lambda xyzu^*.e(xy)(zu), e\mathbf{tf}]$;
- $\mathbf{U} \equiv \lambda e^*.[\lambda x^*.x, \lambda xyz^*.exz, \lambda xyz^*.ex(eyz),$ $\lambda xyzu^*.exz(eyu), e]$.

Lemma 8 *The class* CA_{DI} *of the directly indecomposable combinatory algebras is a universal class (i.e., it is an elementary class which can be axiomatized by universal sentences).*

Proof. By Def. 3 we have that e is central if, and only if, the equation $\mathbf{Z}e = \mathbf{U}e$ holds. Then the class CA_{DI} is axiomatized by the following universal formula ϕ:

$$\phi \equiv \forall e((\mathbf{Z}e = \mathbf{U}e \to e = \mathbf{t} \vee e = \mathbf{f}) \wedge \neg(\mathbf{t} = \mathbf{f})).$$

■

Corollary 9 *The class* CA_{DI} *of the directly indecomposable combinatory algebras is closed under subalgebras and ultraproducts.*

4.2. Algebraic incompleteness

The closure of the class of directly indecomposable combinatory algebras under subalgebras is the key trick in the proof of the algebraic incompleteness theorem.

Theorem 10 (The algebraic Incompleteness Theorem) *The indecomposable semantics is incomplete.*

Proof. Let $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ be the usual looping term of lambda calculus. Consider two arbitrary consistent lambda theories $\mathcal{T}$ and $\mathcal{S}$ satisfying the following conditions:

$$\mathcal{T} \vdash \Omega = \boldsymbol{T}; \quad \mathcal{S} \vdash \Omega = \boldsymbol{F}.$$

$\mathcal{T}$ and $\mathcal{S}$ exist because Ω is an easy term (see [3, Prop. 15.3.9]), i.e., it can be equated consistently with any other closed term. It is a simple exercise to verify that the lambda theory $\mathcal{T} \cap \mathcal{S}$ contains all equations (i)-(iv) of Def. 3 for $e = \Omega$, making the equivalence class of Ω a nontrivial central element in the term model of $\mathcal{T} \cap \mathcal{S}$.

Assume, by the way of contradiction, that the semantics given in terms of directly indecomposable λ-models is complete, so that there is a directly indecomposable λ-model $\mathbf{A}$ such that $Th(\mathbf{A}) = \mathcal{T} \cap \mathcal{S}$. Since $\mathbf{A}$ is directly indecomposable, then $\mathbf{A}$ satisfies the universal formula ϕ defined in the proof of Lemma 8. Since ϕ is universal, every subalgebra of $\mathbf{A}$ satisfies ϕ. In particular, the term model of $\mathcal{T} \cap \mathcal{S}$ satisfies ϕ, so that it is directly indecomposable. This is a contradiction, because the term model of $\mathcal{T} \cap \mathcal{S}$ admits Ω as a nontrivial central element. ■

We have shown that theories exist with no indecomposable models, so that any class of models which excludes decomposable models cannot be complete.

A lambda theory is semisensible if it does not equate a solvable and an unsolvable lambda term. The most important lambda theories are semisensible: for example, the minimal lambda theory $\lambda\beta$ and the minimal extensional lambda theory $\lambda\eta$.

In the following theorem we show that, although the class of directly indecomposable λ-models is incomplete, it is so wide to include all term models of the semisensible lambda theories.

Theorem 11 *The indecomposable semantics is complete for the set of semisensible lambda theories.*

Proof. We recall from Section 2.3 that the term model of a lambda theory is a model of lambda calculus. Then the conclusion of the theorem follows if we show that the term model of every semisensible lambda theory is directly indecomposable. Assume, by the way of contradiction, that there exists a semisensible lambda theory $\mathcal{T}$ such that the term model $\mathcal{M}_{\mathcal{T}}$ of $\mathcal{T}$ admits a nontrivial central element e. From the identity $exx = x$ (see Def. 3) and from the hypothesis on $\mathcal{T}$ it follows that e is a solvable λ-term. Since the congruences $\theta_e = \theta(e, \boldsymbol{T})$ and $\overline{\theta}_e = \theta(e, \boldsymbol{F})$ on the term model of $\mathcal{T}$ are nontrivial, then the lambda theories $\mathcal{T}_1$ and $\mathcal{T}_2$, generated respectively by $\mathcal{T} \cup \{\boldsymbol{F} = e\}$ and $\mathcal{T} \cup \{\boldsymbol{T} = e\}$, are consistent. By [3, Lemma 10.4.1(i)] it is consistent to equate two solvable λ-terms only if they are equivalent according to [3, Def. 10.2.9]. Then the λ-term e should be equivalent to $\boldsymbol{F}$ and $\boldsymbol{T}$. By Remark 10.2.20(ii) in [3] this is possible only if the head variable of exy, where x and y are distinct variables, is equal to x and to y. This is a contradiction. In conclusion, our hypothesis that there is a semisensible lambda theory $\mathcal{T}$, whose term model has a nontrivial central element, is contradictory. ■

4.3. Continuous, stable and strongly stable semantics

We recall that an algebra is *simple* when it has just two congruences (so that every simple algebra is directly indecomposable).

In the next two theorems we give simple proofs of incompleteness for the classic semantics of lambda calculus.

Theorem 12 (Honsell-Ronchi della Rocca [16]) *The semantics of lambda calculus given in terms of continuous models is incomplete.*

Proof. Let $\mathcal{M}$ be a continuous model of lambda calculus. The function g, defined by: $g(x) = c$ if $x \not\leq b$ and $g(x) = \bot$ otherwise, is Scott continuous for every arbitrary element c. We now show that $\mathcal{M}$ is simple as a combinatory algebra. Let θ be a congruence on $\mathcal{M}$ such that $a\ \theta\ b$ with $a \neq b$. We have $a \not\leq b$ or $b \not\leq a$. Suppose that we are in the first case. Since the continuous function g is representable in the model, then we have: $\bot = g(a)\ \theta\ g(b) = c$. By the arbitrariness of c we get that θ is trivial, so that $\mathcal{M}$ is simple. The conclusion of the theorem follows from the algebraic incompleteness theorem (see Thm. 10), because every simple combinatory algebra is directly indecomposable. ■

The continuous function g of the above proof is neither stable nor strongly stable (see [5] for a full treatment of stable and strongly stable semantics).

Theorem 13 (Gouy-Bastonero [15, 4]; Salibra [24, 25]) *The semantics of lambda calculus given in terms of stable or strongly stable models, and whose underlying domain is algebraic, is incomplete.*

Proof. Let $\mathcal{M}$ be a (strongly) stable model of lambda calculus. Take $a, b \in \mathcal{M}$ such that $a \neq b$. We have $a \not\leq b$ or $b \not\leq a$. Suppose that we are in the first case. Then there is a compact element d of $\mathcal{M}$ such that $d \leq a$ and $d \not\leq b$. The step function f defined by : $f(x) = c$ if $x \geq d$ and $f(x) = \bot$ otherwise, is stable, and strongly stable for every element c. This function f can be used to show that every congruence on $\mathcal{M}$ is trivial as in the proof of Thm. 12. Then the conclusion is again a consequence of the algebraic incompleteness theorem. ■

We do not know whether the stable and strongly stable models, whose underlying domains are *not* algebraic, are directly indecomposable as combinatory algebras.

Given a class $\mathbb{C}$ of λ-models, we denote by $\lambda\mathbb{C}$ the set of lambda theories which are representable in $\mathbb{C}$ (see Section 2.3). It is unknown, in general, whether $\lambda\mathbb{C}$ is a lattice with respect to the inclusion ordering of sets and whether $\lambda\mathbb{C}$ is a sublattice of the lattice λT of lambda theories. In the remaining part of this subsection we show for each of the classic semantics of lambda calculus that the set $\lambda\mathbb{C}$ is not closed under finite intersection, so that it is not a sublattice of the lattice λT of lambda theories.

Theorem 14 *Let $\mathbb{C}$ be a class of directly indecomposable models of lambda calculus. If there are two consistent lambda theories $\mathcal{T}, \mathcal{S} \in \lambda\mathbb{C}$ such that*

$$\mathcal{T} \vdash \Omega = \boldsymbol{T}; \quad \mathcal{S} \vdash \Omega = \boldsymbol{F},$$

then $\lambda\mathbb{C}$ is not closed under finite intersection, so it is not a sublattice of λT.

Proof. The term model of $\mathcal{T} \cap \mathcal{S}$ admits a nontrivial central element Ω, so that it is directly decomposable. It follows that $\mathcal{T} \cap \mathcal{S} \notin \lambda\mathbb{C}$. ■

We recall that the graph λ-models (see, for example, [6, 11]) and the filter λ-models (see, for example, [2]) are classes of models within the continuous semantics.

Corollary 15 *Let $\mathbb{C}$ be one of the following semantics: graph semantics, filter semantics, continuous semantics and stable semantics (this last semantics restricted to models whose underlying domain is algebraic). Then $\lambda\mathbb{C}$ is not a sublattice of λT.*

Proof. Semantic proofs that Ω is an easy term were given in each of the semantics specified in the statement of the theorem (see [5]). Then the conclusion follows from Thm. 14, because the models in each of these semantics are directly indecomposable as combinatory algebras. ■

4.4. Concerning the number of decomposable models

We have shown that lambda theories exist with no indecomposable models. Now we can ask *how many* such theories there are. Is there some sense in which 'most' theories have an indecomposable model?

On the contrary, in this section we shall see that it is the directly indecomposable models which are the exception.

First of all we need some results about theories.

The proof of following lemma is similar to that of [3, Prop. 17.1.9], where the case $k = 1$ (due to Visser) is shown, and it is omitted.

Lemma 16 *Suppose $\mathcal{T}$ is a recursively enumerable (*r.e.*) lambda theory and fix arbitrary terms M_i, N_i for $1 \leq i \leq k$ which are not provably equal in $\mathcal{T}$, that is, such that $\mathcal{T} \nvdash M_i = N_i$ for all i. Then there exists a term M such that*

$$\mathcal{T} \cup \{M = P\} \nvdash M_i = N_i, \text{ for all } i \text{ and all closed terms } P.$$

Then the following theorem is a corollary of the algebraic incompleteness theorem.

Theorem 17 *Let $\mathcal{T}$ be an r.e. lambda theory. Then, the interval $[\mathcal{T}) = \{S : \mathcal{T} \subseteq S\}$ contains a subinterval $[\mathcal{S}_1, \mathcal{S}_2] = \{S : \mathcal{S}_1 \subseteq S \subseteq \mathcal{S}_2\}$, constituted by a continuum of lambda theories, satisfying the following conditions:*

- *$\mathcal{S}_1$ and $\mathcal{S}_2$ are r.e. lambda theories;*
- *Every $S \in [\mathcal{S}_1, \mathcal{S}_2]$ is omitted by the indecomposable semantics (in particular, S is omitted by the continuous, stable and strongly stable semantics).*

Proof. The proof is divided into claims.

We first construct $\mathcal{S}_1$. We recall that a λ-term Q is $\mathcal{T}$*-easy* when, for every fixed closed λ-term P, the lambda theory generated by $\mathcal{T} \cup \{Q = P\}$ is consistent.

Claim 18 *There exists a $\mathcal{T}$-easy λ-term Q.*

By Lemma 16.

Claim 19 *$\mathcal{T} \nvdash Q = \boldsymbol{T}$ and $\mathcal{T} \nvdash Q = \boldsymbol{F}$.*

Trivial, because Q is $\mathcal{T}$-easy.

Let $\mathcal{S}_1 = \mathcal{T}_1 \cap \mathcal{T}_2$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are the consistent lambda theories generated respectively by $\mathcal{T} \cup \{Q = \boldsymbol{T}\}$ and $\mathcal{T} \cup \{Q = \boldsymbol{F}\}$.

Claim 20 *The lambda theory $\mathcal{S}_1$ is r.e. and contains $\mathcal{T}$.*

$\mathcal{S}_1$ is r.e., because it is intersection of two r.e. lambda theories. The other property follows from $\mathcal{T} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{S}_1$.

Claim 21 *The term model of $\mathcal{S}_1$ has a non trivial central element e.*

Let $e = [Q]_{\mathcal{S}_1}$ be the equivalence class of the lambda term Q. It is easy to show that e satisfies the equation of Def. 3. Moreover, e is not trivial because $\mathcal{S}_1 \nvdash Q = \boldsymbol{T}$ and $\mathcal{S}_1 \nvdash Q = \boldsymbol{F}$.

We now define the lambda theory $\mathcal{S}_2$.

Claim 22 *There exists an r.e. lambda theory $\mathcal{S}_2$, which is a proper extension of $\mathcal{S}_1$, such that $\mathcal{S}_2 \nvdash Q = \boldsymbol{T}$ and $\mathcal{S}_2 \nvdash Q = \boldsymbol{F}$.*

We apply Lemma 16 to the lambda theory $\mathcal{S}_1$ and to the equations $Q = \boldsymbol{T}$ and $Q = \boldsymbol{F}$. We get a $\mathcal{S}_1$-easy term R such that $\mathcal{S}_1 \cup \{R = P\} \nvdash Q = \boldsymbol{T}$ and $\mathcal{S}_1 \cup \{R = P\} \nvdash Q = \boldsymbol{F}$, for all lambda terms P. Let $\mathcal{S}_2 = \mathcal{S}_1 \cup \{R = \lambda x.x\}$. $\mathcal{S}_2$ is a proper extension of $\mathcal{S}_1$ because otherwise R would not be a $\mathcal{S}_1$-easy term.

Claim 23 *The equivalence class of Q is a non trivial central element of the term model of $\mathcal{S}_2$.*

The term model $\mathcal{M}_{\mathcal{S}_2}$ of $\mathcal{S}_2$ is a homomorphic image of the term model $\mathcal{M}_{\mathcal{S}_1}$ of $\mathcal{S}_1$. Then, every equation satisfied by $\mathcal{M}_{\mathcal{S}_1}$ is also satisfied by $\mathcal{M}_{\mathcal{S}_2}$. In particular, the equations characterizing Q as a central element. Finally, $[Q]_{\mathcal{S}_2}$ is nontrivial as a central element because $\mathcal{S}_2 \not\vdash Q = \boldsymbol{T}$ and $\mathcal{S}_2 \not\vdash Q = \boldsymbol{F}$.

Claim 24 *For every lambda theory U such that $\mathcal{S}_1 \subseteq U \subseteq \mathcal{S}_2$ the equivalence class of Q is non trivial central element of the term model of U.*

We get the conclusion of the theorem because the interval $[\mathcal{S}_1, \mathcal{S}_2]$ has a continuum of elements (see [3, Cor. 17.1.11]). ∎

References

[1] S. Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51:1–77, 1991.

[2] F. Alessi, M. Dezani, F. Honsell. Filter models and easy terms. ICTCS'01, LNCS 2202, Springer-Verlag, 17–37, 2001.

[3] H.P. Barendregt. *The lambda calculus: Its syntax and semantics*. North-Holland Publishing Co., Amsterdam, 1984.

[4] O. Bastonero, X. Gouy. Strong stability and the incompleteness of stable models of λ-calculus. *Annals of Pure and Applied Logic*, 100:247–277, 1999.

[5] C. Berline. From computation to foundations via functions and application: The λ-calculus and its webbed models. *Theoretical Computer Science*, 249:81–161, 2000.

[6] C. Berline. Graph models of λ-calculus at work, and variations. *Math. Struct. in Comp. Science*, to appear.

[7] C. Berline, A. Salibra. Easiness in graph models, *Theoretical Computer Science*, 2006 (to appear).

[8] G. Berry. Stable models of typed lambda-calculi. In *Proc. 5th Int. Coll. on Automata, Languages and Programming*, LNCS 62, Springer-Verlag, Berlin, 1978.

[9] Bigelow D. and S. Burris, Boolean algebras of factor congruences, *Acta Sci. Math.* 54:11–20, 1990.

[10] A. Bucciarelli, T. Ehrhard. Sequentiality and strong stability. *Sixth Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 138–145, 1991.

[11] A. Bucciarelli, A. Salibra. Sensible graph theories of lambda calculus, In: *19th Annual IEEE Symposium on Logic in Computer Science* (LICS'04), IEEE Computer Society Publications, 276–285, 2004.

[12] S. Burris, H.P. Sankappanavar. *A course in universal algebra*, Springer-Verlag, Berlin, 1981.

[13] S. Comer. Representations by algebras of sections over Boolean spaces. *Pacific J. Math.*, 38:29–38, 1971.

[14] H.B. Curry, R. Feys. *Combinatory Logic*, Vol. I, North-Holland Publishing Co., Amsterdam, 1958.

[15] X. Gouy. Etude des théories équationnelles et des propriétés algébriques des modéles stables du λ-calcul. Thèse, Université de Paris 7, 1995.

[16] F. Honsell, S. Ronchi della Rocca. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *Journal of Computer and System Sciences* 45:49–75, 1992.

[17] R. Kerth. Isomorphism and equational equivalence of continuous lambda models. *Studia Logica* 61:403–415, 1998.

[18] R. Kerth. On the construction of stable models of λ-calculus. *Theoretical Computer Science* 269:23–46, 2001.

[19] P.T. Johnstone. *Stone spaces*, Cambridge University Press, 1982.

[20] S. Lusin, A. Salibra. The lattice of lambda theories. *Journal of Logic and Computation*, 14:373–394, 2004.

[21] R.N. McKenzie, G.F. McNulty, W.F. Taylor. *Algebras, Lattices, Varieties, Volume I*, Wadsworth Brooks, Monterey, California, 1987.

[22] R.S. Peirce. *Modules over commutative regular rings*. Memoirs Amer. Math. Soc. 70.

[23] G.D. Plotkin. Set-theoretical and other elementary models of the λ-calculus. *Theoretical Computer Science* 121:351–409, 1993.

[24] A. Salibra. A continuum of theories of lambda calculus without semantics. *16th Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 334–343, 2001.

[25] A. Salibra. Topological incompleteness and order incompleteness of the lambda calculus. (LICS'01 Special Issue). *ACM Transactions on Computational Logic* 4:379–401, 2003.

[26] A. Salibra. On the algebraic models of lambda calculus, *Theoretical Computer Science* 249:197–240, 2000.

[27] M. Schönfinkel. Uber die bausteine der Mathematischen Logik, *Mathematischen Annalen* (english translation in J. van Heijenoort ed.'s book"From Frege to Gödel, a source book in Mathematical Logic, 1879-1931", Harvard University Press, 1967), **92** (1924), 305–316.

[28] D.S. Scott. Continuous lattices. In *Toposes, Algebraic geometry and Logic*. LNM 274, Springer-Verlag, Berlin, 1972.

[29] P. Selinger. Order-incompleteness and finite lambda reduction models. *Theoretical Computer Science* 309:43–63, 2003.

[30] D. Vaggione. Varieties in which the Pierce stalks are directly indecomposable, *Journal of Algebra* 184:424–434, 1996.

[31] D. Vaggione. $\mathcal{V}$ with factorable congruences and $\mathcal{V} = I\Gamma^a(\mathcal{V}_{DI})$ imply $\mathcal{V}$ is a discriminator variety. *Acta Sci. Math.* 62:359–368, 1996.