From λ-calculus to universal algebra and back

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Abstract. We generalize to universal algebra concepts originating from λ-calculus and programming to prove a new result on the lattice λT of λ-theories, and a general theorem of pure universal algebra which can be seen as a meta version of the Stone Representation Theorem. In this paper we introduce the class of *Church algebras* (which includes all Boolean algebras, combinatory algebras, rings with unit and the term algebras of all λ-theories) to model the "if-then-else" instruction of programming. The interest of Church algebras is that each has a Boolean algebra of central elements, which play the role of the idempotent elements in rings. Central elements are the key tool to represent any Church algebra as a weak Boolean product of indecomposable Church algebras and to prove the meta representation theorem mentioned above. We generalize the notion of easy λ-term of lambda calculus to prove that any Church algebra with an "easy set" of cardinality n admits (at the top) a lattice interval of congruences isomorphic to the free Boolean algebra with n generators. This theorem has the following consequence for the lattice of λ-theories: for every recursively enumerable λ-theory ϕ and each n, there is a λ-theory $\phi_n \geq \phi$ such that $\{\psi : \psi \geq \phi_n\}$ "is" the Boolean lattice with 2^n elements.
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1 Introduction

Lambda theories are equational extensions of the untyped λ-calculus closed under derivation. They arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to some operational semantics of λ-calculus, as well as it may be induced by a λ-model, which is a particular *combinatory algebra* (CA, for short) [1, Sec. 5.2]. The set of λ-theories is naturally equipped with a structure of complete lattice (see [1, Ch. 4]), whose bottom element is the least λ-theory $\lambda\beta$, and whose top element is the inconsistent λ-theory. The lattice λT of λ-theories is a very rich and complex structure of cardinality $2^{\aleph_0}$ [1, 9].

The interest of a systematic study of the lattice λT of λ-theories grows out of several open problems on λ-calculus. For example, Selinger's order-incompleteness problem can be proved equivalent to the existence of a recursively enumerable (r.e., for short) λ-theory ϕ whose term algebra generates an n-permutable variety of algebras for some $n \geq 2$ (see [14] and the remark after [15, Thm. 3.4]).

Lipparini [8] has found out interesting non-trivial lattice identitites that hold in the congruence lattices of all algebras living in an n-permutable variety. The failure of some Lipparini's lattice identities in λT would imply that Selinger's problem has a negative answer.

Techniques of universal algebra were applied in [13, 9, 2] to study the structure of λT. In this paper we validate the inverse slogan: λ-calculus can be fruitfully applied to universal algebra. By generalizing to universal algebra concepts originating from λ-calculus and programming, we create a zigzag path from λ-calculus to universal algebra and back. All the algebraic properties we have shown in [10] for CAs, hold for a wider class of algebras, that we call *Church algebras*. Church algebras include, beside CAs, all BAs (Boolean algebras) and rings with unit, and model the "if-then-else" instruction by two constants $0, 1$ and a ternary term $q(x, y, z)$ satisfying the following identities:

$$q(1, x, y) = x; \quad q(0, x, y) = y.$$

The interest of Church algebras is that each has a BA of central elements, which can be used to represent a Church algebra as a weak Boolean product of directly indecomposable algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other non-trivial algebras). We generalize the notion of easy λ-term from λ-calculus and use central elements to prove that: (i) any Church algebra with an "easy set" of cardinality κ admits a congruence ϕ such that the lattice reduct of the free BA with κ generators embeds into the lattice interval $[\phi)$; (ii) If κ is finite, this embedding is an isomorphism. This theorem applies directly to all BAs and rings with units. For λT it has the following consequence: for every r.e. λ-theory ϕ and each natural number n, there is a λ-theory $\phi_n \geq \phi$ such that the lattice interval $[\phi_n)$ is the finite Boolean lattice with 2^n elements. It is the first time that it is found an interval of λT whose cardinality is not $1, 2$ or $2^{\aleph_0}$. We leave open the problem of whether the existence of such Boolean lattice intervals is inconsistent with the existence of an r.e. λ-theory ϕ whose term algebra generates an n-permutable variety of algebras for some $n \geq 2$. If yes, this would negatively solve Selinger's problem.

We also prove a meta version of Stone Representation Theorem that applies to all varieties of algebras and not only to the classic ones. Indeed, we show that any variety of algebras can be decomposed as a weak Boolean product of directly indecomposable subvarieties. This means that, given a variety $\mathcal{V}$, there exists a family of "directly indecomposable" subvarieties $\mathcal{V}_i$ $(i \in I)$ of $\mathcal{V}$ for which every algebra of $\mathcal{V}$ is isomorphic to a weak Boolean product of algebras of $\mathcal{V}_i$ $(i \in I)$.

2 Preliminaries

We will use the notation of Barendregt's classic work [1] for λ-calculus and combinatory logic, and the notation of McKenzie et al. [11] for universal algebra.

A lattice L is *bounded* if it has a top element 1 and a bottom element 0. $a \in L$ is an *atom* (*coatom*) if it is a minimal element in $L - \{0\}$ (maximal element in $L - \{1\}$). For $a \in L$, we set $L_a = \{b \in L - \{0\} : a \wedge b = 0\}$. L is called: *lower* *semicomplemented* if $L_a \neq \emptyset$ for all $a \neq 1$; *pseudocomplemented* if each L_a has a greatest element (called the *pseudocomplement* of a).

We write $[a)$ for $\{b : a \leq b \leq 1\}$ and $\mathcal{P}(X)$ for the powerset of a set X.

An *algebraic similarity type* Σ is constituted by a non-empty set of operator symbols together with a function assigning to each operator $f \in \Sigma$ a finite *arity*.

A *Σ-algebra* $\mathbf{A}$ is determined by a non-empty set A together with an operation $f^{\mathbf{A}} : A^n \to A$ for every $f \in \Sigma$ of arity n. $\mathbf{A}$ is *trivial* if $|A| = 1$.

A compatible equivalence relation ϕ on a Σ-algebra $\mathbf{A}$ is called a *congruence*. We often write $a\phi b$ or $a =_\phi b$ for $(a, b) \in \phi$. The set $\{b : a\phi b\}$ is denoted by $[a]_\phi$.

If $\phi \leq \psi$ are congruences on $\mathbf{A}$, then $\psi/\phi = \{([a]_\phi, [b]_\phi) : a\psi b\}$ is a congruence on the quotient $\mathbf{A}/\phi$. If $X \subseteq A \times A$, then we write $\theta(X)$ for the least congruence including X. We write $\theta(a, b)$ for $\theta(\{(a, b)\})$. If $a \in A$ and $Y \subseteq A$, then we write $\theta(a, Y)$ for $\theta(\{(a, b) : b \in Y\})$.

We denote by $\mathrm{Con}(\mathbf{A})$ the algebraic complete lattice of all congruences of $\mathbf{A}$, and by ∇ and Δ the top and the bottom element of $\mathrm{Con}(\mathbf{A})$. A congruence ϕ on $\mathbf{A}$ is called: *trivial* if it is equal to ∇ or Δ; *consistent* if $\phi \neq \nabla$; *compact* if $\phi = \theta(X)$ for some finite set $X \subseteq A \times A$.

An algebra $\mathbf{A}$ is *directly decomposable* if there exist two non-trivial algebras $\mathbf{B}, \mathbf{C}$ such that $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, otherwise it is called *directly indecomposable*.

An algebra $\mathbf{A}$ is a *subdirect product* of the algebras $(\mathbf{B}_i)_{i \in I}$, written $\mathbf{A} \leq \Pi_{i \in I}\mathbf{B}_i$, if there exists an embedding f of $\mathbf{A}$ into the direct product $\Pi_{i \in I}\mathbf{B}_i$ such that the projection $\pi_i \circ f : \mathbf{A} \to \mathbf{B}_i$ is onto for every $i \in I$.

A non-empty class $\mathcal{V}$ of algebras is a *variety* if it is closed under subalgebras, homomorphic images and direct products or, equivalently, if it is axiomatizable by a set of equations. A variety $\mathcal{V}'$ is a *subvariety* of the variety $\mathcal{V}$ if $\mathcal{V}' \subseteq \mathcal{V}$. We will denote by $\mathcal{V}(\mathbf{A})$ the variety generated by an algebra $\mathbf{A}$, i.e., $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ if every equation satisfied by $\mathbf{A}$ is also satisfied by $\mathbf{B}$.

Let $\mathcal{V}$ be a variety. We say that $\mathbf{A}$ is *the free $\mathcal{V}$-algebra over the set X of generators* iff $\mathbf{A} \in \mathcal{V}$, $\mathbf{A}$ is generated by X and for every $g : X \to \mathbf{B} \in \mathcal{V}$, there is a unique homomorphism $f : \mathbf{A} \to \mathbf{B}$ that extends g (i.e., $f(x) = g(x)$ for every $x \in X$). A free algebra in the class of all Σ-algebras is called *absolutely free*.

Given two congruences σ and τ on $\mathbf{A}$, we can form their relative product: $\tau \circ \sigma = \{(a, c) : \exists b \in A \; a\sigma b\tau c\}$.

Definition 1. *A congruence ϕ on an algebra $\mathbf{A}$ is a* factor congruence *if there exists another congruence ψ such that $\phi \wedge \psi = \Delta$ and $\phi \circ \psi = \nabla$. In this case we call (ϕ, ψ) a* pair of complementary factor congruences *or* cfc-pair*, for short.*

Under the hypotheses of the above definition the homomorphism $f : \mathbf{A} \to \mathbf{A}/\phi \times \mathbf{A}/\psi$ defined by $f(x) = ([x]_\phi, [x]_\psi)$ is an isomorphism. So, the existence of factor congruences is just another way of saying "this algebra is a direct product of simpler algebras".

The set of factor congruences of $\mathbf{A}$ is not, in general, a sublattice of $\mathrm{Con}(\mathbf{A})$. Δ and ∇ are the *trivial* factor congruences, corresponding to $\mathbf{A} \cong \mathbf{A} \times \mathbf{B}$, where $\mathbf{B}$ is a trivial algebra. An algebra $\mathbf{A}$ is directly indecomposable if, and only if, $\mathbf{A}$ has no non-trivial factor congruences.

Factor congruences can be characterized in terms of certain algebra homomorphisms called *decomposition operators* (see [11, Def. 4.32] for more details).

Definition 2. *A* decomposition operation *for an algebra* $\mathbf{A}$ *is an algebra homomorphism* $f : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ *such that* $f(x, x) = x$ *and* $f(f(x, y), z) = f(x, z) = f(x, f(y, z))$.

There exists a bijection between cfc-pairs and decomposition operations, and thus, between decomposition operations and factorizations like $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$.

Proposition 1. *[11, Thm. 4.33] Given a decomposition operator* f*, the relations* ϕ, ψ *defined by* $x \; \phi \; y$ *iff* $f(x, y) = y$; *and* $x \; \psi \; y$ *iff* $f(x, y) = x$, *form a cfc-pair. Conversely, given a cfc-pair* (ϕ, ψ)*, the map* f *defined by* $f(x, y) = u$ *iff* $x \; \phi \; u \; \psi \; y$*, is a decomposition operation (note that, for all* x *and* y*, there is just one element* u *satisfying* $x \; \phi \; u \; \psi \; y$*).*

The Boolean product construction allows us to transfer numerous fascinating properties of BAs into other varieties of algebras (see [4, Ch. IV]). We recall that a Boolean space is a compact, Hausdorff and totally disconnected topological space, and that *clopen* means "open and closed".

Definition 3. *A* weak Boolean product *of a family* $(\mathbf{A})_{i \in I}$ *of algebras is a subdirect product* $\mathbf{A} \leq \Pi_{i \in I} \mathbf{A}_i$*, where* I *can be endowed with a Boolean space topology such that: (i) the set* $\{i \in I : a_i = b_i\}$ *is open for all* $a, b \in A$*, and (ii) if* $a, b \in A$ *and* N *is a clopen subset of* I*, then the element* c*, defined by* $c_i = a_i$ *for every* $i \in N$ *and* $c_i = b_i$ *for every* $i \in I - N$*, belongs to* A*. A* Boolean product *is a weak Boolean product such that the set* $\{i \in I : a_i = b_i\}$ *is clopen for all* $a, b \in A$*.*

A λ-*theory* is any congruence (w.r.t. the binary operator of application and the lambda abstraction) on the set of λ-terms including (α)- and (β)-conversion (see [1, Ch. 2]). We use for λ-theories the same notational convention as for congruences. The set of all λ-theories is naturally equipped with a structure of complete lattice, hereafter denoted by λT, with meet defined as set theoretical intersection. The least element of λT is denoted by $\lambda\beta$, while the top element of λT is the inconsistent λ-theory ∇. The *term algebra of a* λ-*theory* ϕ, hereafter denoted by $\mathbf{\Lambda}_\phi$, has the equivalence classes of λ-terms modulo ϕ as elements, and the operations of application and of λ-abstractions as operations on these elements. The lattice λT of λ-theories is isomorphic to the congruence lattice of the term algebra $\mathbf{\Lambda}_{\lambda\beta}$ of the least λ-theory $\lambda\beta$, while the lattice interval $[\phi)$ is isomorphic to the congruence lattice of the term algebra $\mathbf{\Lambda}_\phi$.

The variety CA of *combinatory algebras* [1, Sec. 5.1] consists of algebras $\mathbf{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$, where $\cdot$ is a binary operation and $\mathbf{k}, \mathbf{s}$ are constants, satisfying $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ (as usual, the symbol "$\cdot$" is omitted and association is made on the left).

3 Church algebras

Many algebraic structures, such as CAs, BAs etc., have in common the fact that all are *Church algebras*. In this section we study the algebraic properties of this class of algebras. Applications are given in Section 5 and in Section 6.

Definition 4. *An algebra* $\mathbf{A}$ *is called* a Church algebra *if there are two constants* $0, 1 \in A$ *and a ternary term* $q(e, x, y)$ *such that* $q(1, x, y) = x$ *and* $q(0, x, y) = y$. *A variety* $\mathcal{V}$ *is called a* Church variety *if every algebra in* $\mathcal{V}$ *is a Church algebra with respect to the same term* $q(e, x, y)$ *and constants* $0, 1$.

Example 1. The following are easily checked to be Church algebras:

1. Combinatory algebras: $q(e, x, y) \equiv (e \cdot x) \cdot y$; $1 \equiv \mathbf{k}$; $0 \equiv \mathbf{sk}$
2. Boolean algebras: $q(e, x, y) \equiv (e \vee y) \wedge (e^- \vee x)$
3. Heyting algebras: $q(e, x, y) \equiv (e \vee y) \wedge ((e \to 0) \vee x)$
4. Rings with unit: $q(e, x, y) \equiv (y + e - ey)(1 - e + ex)$

Let $\mathbf{A} = (A, +, \cdot, 0, 1)$ be a commutative ring with unit. Every idempotent element $a \in A$ (i.e., satisfying $a \cdot a = a$) induces a cfc-pair $(\theta(1, a), \theta(a, 0))$. In other words, the ring $\mathbf{A}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\theta(1, a) \times \mathbf{A}/\theta(a, 0)$. $\mathbf{A}$ is directly indecomposable if 0 and 1 are the unique idempotent elements. Vaggione [17] generalized idempotent elements to any universal algebra whose top congruence ∇ is compact, and called them *central elements*. Central elements were used, among the other things, to investigate the closure of varieties of algebras under Boolean products. Here we give a new characterization based on decomposition operators (see Def. 2). Hereafter, we set $\theta_e \equiv \theta(1, e)$ and $\overline{\theta}_e \equiv \theta(e, 0)$.

Definition 5. *We say that an element* e *of a Church algebra* $\mathbf{A}$ *is* central*, and we write* $e \in \mathrm{Ce}(\mathbf{A})$*, if* θ_e *and* $\overline{\theta}_e$ *are a cfc-pair. A central element* e *is called* non-trivial *if* $e \neq 0, 1$.

We now show how to internally represent in a Church algebra factor congruences as central elements. We start with a lemma.

Lemma 1. *Let* $\mathbf{A}$ *be a Church algebra and* $e \in A$*. Then we have, for all* $x, y \in A$:

(a) $x\ \theta_e\ q(e, x, y)\ \overline{\theta}_e\ y$.
(b) $x \theta_e y$ *iff* $q(e, x, y)\ (\theta_e \wedge \overline{\theta}_e)\ y$.
(c) $x \overline{\theta}_e y$ *iff* $q(e, x, y)\ (\theta_e \wedge \overline{\theta}_e)\ x$.
(d) $\theta_e \circ \overline{\theta}_e = \overline{\theta}_e \circ \theta_e = \nabla$.

Proposition 2. *Let* $\mathbf{A}$ *be a Church* Σ*-algebra and* $e \in A$*. The following conditions are equivalent:*

(i) e *is central;*
(ii) $\theta_e \wedge \overline{\theta}_e = \Delta$*;*
(iii) For all x *and* y*,* $q(e, x, y)$ *is the unique element such that* $x\theta_e\ q(e, x, y)\ \overline{\theta}_e y$*;*
(iv) e *satisfies the following identities:*

1. $q(e, x, x) = x$.
2. $q(e, q(e, x, y), z) = q(e, x, z) = q(e, x, q(e, y, z))$.
3. $q(e, f(\overline{x}), f(\overline{y})) = f(q(e, x_1, y_1), \ldots, q(e, x_n, y_n))$*, for every* $f \in \Sigma$.
4. $e = q(e, 1, 0)$.

(v) The function f_e *defined by* $f_e(x, y) = q(e, x, y)$ *is a decomposition operator such that* $f_e(1, 0) = e$.

Corollary 1. *Let* $\mathbf{A}$ *be a Church algebra and* $e \in A$ *such that* $\theta_e \neq \nabla, \Delta$*. Then the equivalence class of* e *is a non-trivial central element in the algebra* $\mathbf{A}/\theta_e \wedge \overline{\theta}_e$*.*

Thus a Church algebra $\mathbf{A}$ is directly indecomposable iff $\mathrm{Ce}(\mathbf{A}) = \{0, 1\}$ iff $\theta_e \wedge \overline{\theta}_e \neq \Delta$ for all $e \neq 0, 1$.

Example 2. – All elements of a BA are central by Prop. 2(iv) and Example 1.
- An element is central in a commutative ring with unit iff it is idempotent. This characterization does not hold for non-commutative rings with unit.
- Let $\Omega \equiv (\lambda a.aa)(\lambda a.aa)$ be the usual looping term of λ-calculus. It is well-known that the λ-theories $\theta_\Omega = \theta(\Omega, \lambda ab.a)$ and $\overline{\theta}_\Omega = \theta(\Omega, \lambda ab.b)$ are consistent (see [1]). Then by Corollary 1 the term Ω is a non-trivial central element in the term algebra of $\theta_\Omega \wedge \overline{\theta}_\Omega$.

We now show that the partial ordering on the central elements, defined by:

$$e \leq d \text{ if, and only if, } \overline{\theta}_e \subseteq \overline{\theta}_d \tag{1}$$

is a Boolean ordering and that the meet, join and complementation operations are internally representable. 0 and 1 are respectively the bottom and top element of this ordering.

Theorem 1. *Let* $\mathbf{A}$ *be a Church algebra. The algebra* $(\mathrm{Ce}(\mathbf{A}), \wedge, \vee, ^-, 0, 1)$ *of central elements of* $\mathbf{A}$*, defined by* $e \wedge d = q(e, d, 0)$*,* $e \vee d = q(e, 1, d)$*,* $e^- = q(e, 0, 1)$*, is a* BA *isomorphic to the* BA *of factor congruences of* $\mathbf{A}$*.*

The Stone representation theorem for Church algebras is an easy corollary of Thm. 1 and of theorems by Comer [5] and by Vaggione [17].

Let $\mathbf{A}$ be a Church algebra. If I is a maximal ideal of the Boolean algebra $\mathrm{Ce}(\mathbf{A})$, then ϕ_I denotes the congruence on $\mathbf{A}$ defined by: $\phi_I = \cup_{e \in I} \overline{\theta}_e$. Moreover, X denotes the Boolean space of maximal ideals of $\mathrm{Ce}(\mathbf{A})$.

Theorem 2. (The Stone Representation Theorem) *Let* $\mathbf{A}$ *be a Church algebra. Then, for all* $I \in X$ *the quotient algebra* $\mathbf{A}/\phi_I$ *is directly indecomposable and the map* $f : A \to \Pi_{I \in X}(A/\phi_I)$*, defined by* $f(x) = ([x]_{\phi_I} : I \in X)$*, gives a weak Boolean product representation of* $\mathbf{A}$*.*

Note that, in general, Thm. 2 does not give a (non-weak) Boolean product representation. This was shown in [10] for combinatory algebras.

4 The main theorem

In λ-calculus there are *easy* λ*-terms*, i.e., terms that can be consistently equated with any other closed λ-term. In this section we generalize the notion of easiness to Church algebras to show that any Church algebra with an easy set of cardinality n admits a congruence ϕ such that the lattice interval of all congruences greater than ϕ is isomorphic to the free BA with n generators.

Definition 6. *Let* $\mathbf{A}$ *be a Church algebra. We say that a subset* X *of* A *is an* easy set *if, for every* $Y \subseteq X$*,* $\theta(1, Y) \vee \theta(0, X - Y) \neq \nabla$ *(by definition* $\theta(1, \emptyset) = \theta(0, \emptyset) = \Delta$*).*

We say that an element a is *easy* if $\{a\}$ is an easy set. Thus, a is easy if the congruences θ_a and $\overline{\theta}_a$ are both different from ∇.

Example 3. – A finite subset X of a BA is an easy set if it holds: (i) $\bigvee X \neq 1$; (ii) $\bigwedge X \neq 0$; (iii) for all $Y \subset X$, $\bigvee Y \not\geq \bigwedge(X - Y)$. Thus, for example, $\{\{1,2\},\{2,3\}\}$ is an easy set in the powerset of $\{1,2,3,4\}$.
– The term algebra of every r.e. λ-theory has a countable infinite easy set. This will be shown in Section 5.

The following three lemmas are used in the proof of the main theorem. Recall that a *semicongruence* on an algebra $\mathbf{A}$ is a reflexive compatible binary relation.

Lemma 2. *The semicongruences of a Church algebra permute with its factor congruences, i.e., $\phi \circ \psi = \psi \circ \phi$ for every semicongruence ϕ and factor congruence ψ.*

Lemma 3. *Let $\mathbf{A}$ be a Church algebra. Then the congruence lattice of $\mathbf{A}$ satisfies the Zipper condition, i.e., for all I and for all $\delta_i, \psi, \phi \in \mathrm{Con}(\mathbf{A})$ $(i \in I)$: if $\bigvee_{i \in I} \delta_i = \nabla$ and $\delta_i \wedge \psi = \phi$ $(i \in I)$, then $\psi = \phi$.*

Lemma 4. *Let $\mathbf{B}$ be a Church algebra and $\phi \in \mathrm{Con}(\mathbf{B})$. Then, $\mathbf{B}/\phi$ is also a Church algebra and the map $c_\phi : \mathrm{Ce}(\mathbf{B}) \to \mathrm{Ce}(\mathbf{B}/\phi)$, defined by $c_\phi(x) = [x]_\phi$ is a homomorphism of* BA*s.*

Theorem 3. *Let $\mathbf{A}$ be a Church algebra and X be an easy subset of A. Then there exists a congruence ϕ_X satisfying the following conditions:*

1. *The lattice reduct of the free* BA *with a set X of generators can be embedded into the lattice interval $[\phi_X)$;*
2. *If X has finite cardinality n, then the above embedding is an isomorphism and $[\phi_X)$ has 2^{2^n} elements.*

Proof. We start by defining ϕ_X. If we let $\delta_Y \equiv \theta(1, Y) \vee \theta(0, X - Y)$, for $Y \subseteq X$, then by the hypothesis of easiness we have that $\delta_Y \neq \nabla$. We consider an enumeration $(e_\gamma)_{\gamma < \kappa}$ of $A \times A$, where κ is the cardinal of $A \times A$. Define by transfinite induction an increasing sequence ψ_γ $(\gamma \leq \kappa)$ of congruences on $\mathbf{A}$:
- $\psi_0 = \cap_{Y \subseteq X}\ \delta_Y$.
- $\psi_{\gamma+1} = \psi_\gamma$ if $\psi_\gamma \vee \theta(e_\gamma) \vee \delta_Y = \nabla$ for some $Y \subseteq X$.
- $\psi_{\gamma+1} = \psi_\gamma \vee \theta(e_\gamma)$ otherwise.
- $\psi_\gamma = \cup_{\beta<\gamma} \psi_\beta$ for every limit ordinal $\gamma \leq \kappa$.

We define $\phi_X \equiv \psi_\kappa$. We now prove that the free BA with a set X of generators can be embedded into the interval $[\psi_\kappa)$.

Claim. $\psi_\gamma \vee \delta_Y \neq \nabla$ for all $\gamma \leq \kappa$ and $Y \subseteq X$.

Proof. By transfinite induction on γ. It is true for $\gamma = 0$ by definition of an easy set. If $\psi_\gamma = \cup_{\beta<\gamma} \psi_\beta$, then it follows from inductive hypothesis, because $(\cup_{\beta<\gamma} \psi_\beta) \vee \delta_Y = \cup_{\beta<\gamma}(\psi_\beta \vee \delta_Y)$. Finally, if $\psi_\gamma = \psi_{\gamma-1} \vee \theta(e_{\gamma-1})$ then by definition of ψ_γ we have $\psi_{\gamma-1} \vee \theta(e_{\gamma-1}) \vee \delta_Y = \psi_\gamma \vee \delta_Y \neq \nabla$ for all $Y \subseteq X$.

Let $\mathbf{A}_\gamma \equiv \mathbf{A}/\psi_\gamma$ and $[x]_\gamma \equiv [x]_{\psi_\gamma}$ $(x \in A)$.

Claim. $[x]_\gamma \in \mathrm{Ce}(\mathbf{A}_\gamma)$ for every $x \in X$ and $\gamma \leq \kappa$.

Proof. If we prove that $[x]_0$ is central in $\mathbf{A}_0$, then by $\psi_0 \leq \psi_\gamma$ and by Lemma 4 we get the same conclusion for γ. Since the element $x \in X$ is equivalent either to 1 or to 0 in each congruence δ_Y, then $[x]_{\delta_Y}$ is a trivial central element in the algebra $\mathbf{A}/\delta_Y$. Then $\langle [x]_{\delta_Y} : Y \subseteq X\rangle$ is central in the Cartesian product $\Pi_{Y\subseteq X}\mathbf{A}/\delta_Y$. Since $\psi_0 = \cap_{Y\subseteq X}\ \delta_Y$ then by [4, Lemma II.8.2] $\mathbf{A}_0 \equiv \mathbf{A}/\psi_0$ is a subdirect product of the algebras $\mathbf{A}/\delta_Y$, so that $\mathbf{A}_0$ can be embedded into $\Pi_{Y\subseteq X}\mathbf{A}/\delta_Y$. It follows that $[x]_0$ is central in $\mathbf{A}_0$.

Let $\mathbf{B}(X)$ be the free BA over the set X of generators and $f_\gamma : \mathbf{B}(X) \to \mathrm{Ce}(\mathbf{A}_\gamma)$ be the unique Boolean homomorphism satisfying $f_\gamma(x) = [x]_\gamma$.

Claim. $f_\gamma : \mathbf{B}(X) \to \mathrm{Ce}(\mathbf{A}_\gamma)$ is an embedding.

Proof. Let $Y \subseteq X$. By Claim 4 the algebra $\mathbf{A}/\psi_\gamma \vee \delta_Y$ is non-trivial, while by Lemma 4 there exists a Boolean homomorphism (denoted by h_Y in this proof) from $\mathrm{Ce}(\mathbf{A}_\gamma)$ into $\mathrm{Ce}(\mathbf{A}/\psi_\gamma \vee \delta_Y)$. Since $(x, 1) \in \psi_\gamma \vee \delta_Y$ for every $x \in Y$ and $(y, 0) \in \psi_\gamma \vee \delta_Y$ for every $y \in X - Y$, then the kernel of $h_Y \circ f_\gamma$ is an ultrafilter of $\mathbf{B}(X)$. By the arbitrariness of $Y \subseteq X$, every ultrafilter of $\mathbf{B}(X)$ can be the kernel of a suitable $h_Y \circ f_\gamma$. This is possible only if f_γ is an embedding.

This concludes the proof of (1). Recall that the lattice interval $[\psi_\kappa)$ of $\mathrm{Con}(\mathbf{A})$ is isomorphic to the congruence lattice $\mathrm{Con}(\mathbf{A}_\kappa)$. If $Y \subseteq X$, we denote by δ_Y^κ the congruence $(\psi_\kappa \vee \delta_Y)/\psi_\kappa \in \mathrm{Con}(\mathbf{A}_\kappa)$.

Claim. Let $\sigma \in \mathrm{Con}(\mathbf{A}_\kappa)$. If $\sigma \vee \delta_Y^\kappa \neq \nabla$ for all $Y \subseteq X$, then $\sigma = \Delta$.

Proof. Let $x, y \in A$ such that $(x, y) \notin \psi_\kappa$ and $([x]_\kappa, [y]_\kappa) \in \sigma$. From the hypothesis it follows that $\psi_\kappa \vee \delta_Y \vee \theta(x, y) \in \mathrm{Con}(\mathbf{A})$ is non-trivial for all $Y \subseteq X$. By definition of ψ_κ this last condition implies $(x, y) \in \psi_\kappa$. Contradiction.

Hereafter, we assume that X is a finite easy set of cardinality n.
We show that the set At_κ of atoms of $\mathrm{Con}(\mathbf{A}_\kappa)$ is not empty and has ∇ as join.

Claim. $\bigvee\{\beta \in At_\kappa : \beta \text{ is a factor congruence}\} = \nabla$.

Proof. Since X has cardinality n, then the free Boolean algebra $\mathbf{B}(X)$ is finite, atomic and has n generators. Let a be an atom of $\mathbf{B}(X)$ and let $f_\kappa(a) \in \mathbf{A}_\kappa$ be the central element determined by the embedding f_κ of Claim 4. Consider the factor congruence $\tau = \theta(f_\kappa(a), 0) \in \mathrm{Con}(\mathbf{A}_\kappa)$ associated with $f_\kappa(a)$. We claim that τ is an atom in $\mathrm{Con}(\mathbf{A}_\kappa)$. By the way of contradiction, assume that $\sigma \in \mathrm{Con}(\mathbf{A}_\kappa)$ is a non-trivial congruence which is strictly under τ. By Claim 4 and Lemma 4 we have a chain of Boolean homomorphisms:

$$\mathbf{B}(X) \xrightarrow{f_\kappa} \mathrm{Ce}(\mathbf{A}_\kappa) \xrightarrow{c_\sigma} \mathrm{Ce}(\mathbf{A}_\kappa/\sigma) \xrightarrow{c_{\tau/\sigma}} \mathrm{Ce}(\mathbf{A}_\kappa/\tau)$$

such that $c_\tau = c_{\tau/\sigma} \circ c_\sigma$. Since a is an atom of $\mathbf{B}(X)$ and $\tau = \theta(f_\kappa(a), 0)$, then the set $\{0, a\}$ is the Boolean ideal associated with the kernel of $c_\tau \circ f_\kappa$. If $c_\sigma(f_\kappa(a)) = 0$, then σ contains the pair $(f_\kappa(a), 0)$, i.e., $\sigma = \tau$. Then $c_\sigma(f_\kappa(a)) \neq 0$ and the map $c_\sigma \circ f_\kappa : \mathbf{B}(X) \to \mathrm{Ce}(\mathbf{A}_\kappa/\sigma)$ is an embedding. Then the images of the elements of X into $\mathrm{Ce}(\mathbf{A}_\kappa/\sigma)$ are distinct central elements, so that $\sigma \vee \delta^\kappa_Y \neq \nabla$ for all Y. By Claim 4 we get $\sigma = \Delta$, that contradicts the non-triviality of σ. Then τ is an atom. Finally, $\bigvee\{\beta \in At_\kappa : \beta \text{ is a factor congruence}\} = \nabla$ follows because the join of all atoms of $\mathbf{B}(X)$ is the top element.

Claim. The congruence lattice $\mathrm{Con}(\mathbf{A}_\kappa)$ is pseudocomplemented, complemented, atomic, and the coatoms form a finite irredundant decomposition of Δ.

Proof. The coatomic and complete lattice $\mathrm{Con}(\mathbf{A}_\kappa)$ satisfies the Zipper condition (by Lemma 3) and $\bigvee At_\kappa = \nabla$ (by Claim 4). Then by [6, Prop. 2] $\mathrm{Con}(\mathbf{A}_\kappa)$ is complemented, atomic and every coatom has a complement which is an atom. It is also pseudocomplemented by [6, Prop. 1]. Since the top element ∇ is compact, by [6, Prop. 3] we get that the coatoms form a finite irredundant decomposition of the least element.

Claim. Let $\xi \in \mathrm{Con}(\mathbf{A}_\kappa)$ be a non-trivial congruence and $\gamma = \bigvee\{\delta \in At_\kappa : \delta \leq \xi\}$. If $\beta \in At_\kappa$ is a factor congruence which is not under ξ, then $\xi \wedge (\beta \vee \gamma) = \gamma$.

Proof. We always have $\gamma \leq \xi \wedge (\beta \vee \gamma)$. We show the opposite direction. Let $(x, y) \in \xi \wedge (\beta \vee \gamma)$, i.e., $x\ \xi\ y$ and $x(\beta \vee \gamma)y$. We have to show that $x\gamma y$. Since β is a factor congruence, by Lemma 2 we have $\beta \vee \gamma = \beta \circ \gamma$. Then $x\ \beta\ z\ \gamma\ y$ for some z. Since $\gamma \leq \xi$ then $z\ \xi\ y$, that together with $x\ \xi\ y$ implies $x\ \xi\ z$. Then $x(\xi \wedge \beta)z$. Since β is an atom and $\beta \not\leq \xi$, we get $x = z$. This last equality and $z\gamma y$ imply $x\gamma y$. In other words, $\xi \wedge (\beta \vee \gamma) = \gamma$.

Claim. The congruence lattice $\mathrm{Con}(\mathbf{A}_\kappa)$ is a finite BA.

Proof. By Claim 4 $\mathrm{Con}(\mathbf{A}_\kappa)$ is complemented, atomic and pseudocomplemented. If we can show that each element $\xi \neq \nabla$ is a join of atoms, then $\mathrm{Con}(\mathbf{A}_\kappa)$ is isomorphic to the power set of At_κ. Let At_ξ be the set of atoms under ξ and $\gamma = \bigvee At_\xi$. We will show that $\gamma = \xi$ by applying the Zipper condition of Lemma 3. By Claim 4 and by the definition of γ we have: $\bigvee\{\nu : \xi \wedge \nu = \gamma\} \geq \bigvee\{\beta \vee \gamma : \beta \in At_\kappa, \beta \not\leq \xi,\ \beta \text{ is a factor congruence}\} \geq \bigvee\{\beta : \beta \in At_\kappa \text{ is a factor congruence}\}$. By Claim 4 this last element is equal to ∇, so that $\bigvee\{\nu : \xi \wedge \nu = \gamma\} = \nabla$. By the Zipper condition this entails $\xi = \gamma$.

Since $[\psi_\kappa) \cong \mathrm{Con}(\mathbf{A}_\kappa)$, then $[\psi_\kappa)$ is Boolean.

Claim. The Boolean lattice $[\psi_\kappa)$ has exactly 2^n atoms and 2^n coatoms.

Proof. Since $\psi_\kappa \vee \delta_Y \neq \nabla$ for every $Y \subseteq X$, $[\psi_\kappa)$ has at least 2^n coatoms. For every $Y \subseteq X$, let c_Y be a coatom including $\psi_\kappa \vee \delta_Y$. Assume now that there is a coatom ξ distinct from c_Y for every $Y \subseteq X$. Consider the intersection $\cap(Co_\kappa - \{\xi\})$, where Co_κ denotes the set of coatoms of $[\psi_\kappa)$. By Claim 4 and by $[\psi_\kappa) \cong \mathrm{Con}(\mathbf{A}_\kappa)$ we have that $\cap(Co_\kappa - \{\xi\}) \neq \psi_\kappa$, so that there is a pair $(a, b) \in \cap(Co_\kappa - \{\xi\}) - \psi_\kappa$. Since $\psi_\kappa \vee \delta_Y \vee \theta(a, b) \leq c_Y \neq \nabla$ for all $Y \subseteq X$, then $(a, b) \in \psi_\kappa$ by the inductive definition of ψ_κ. Contradiction. In conclusion, we have 2^n coatoms. A Boolean lattice has the same number of atoms and coatoms.

This concludes the proof of the main theorem.

The next proposition explains why the main theorem cannot be improved.

Proposition 3. *Let* $\mathbf{A}$ *be a Church algebra. Then there exists no congruence* ϕ *such that the interval sublattice* $[\phi)$ *is isomorphic to an infinite Boolean lattice.*

5 The lattice of λ-theories

The fact that the term algebra of every λ-theory ϕ is a Church algebra has the interesting consequence that the lattice λT admits (at the top) Boolean lattice intervals of cardinality 2^n for every n.

Berline and Salibra have noticed in [2] that there exists a countable infinite sequence of λ-terms that can be consistently equated to any other arbitrary infinite sequence of closed λ-terms. In the following lemma we generalize this result to any r.e. λ-theory.

Let ω be the set of natural numbers and Λ^o be the set of closed λ-terms. As a matter of notation, if $\mathcal{M} = \langle M_k \in \Lambda^o : k \in \omega\rangle$ and $\mathcal{N} = \langle N_k \in \Lambda^o : k \in \omega\rangle$ are infinite sequences, we write $(\mathcal{M}, \mathcal{N})$ for $\{(M_k, N_k) : k \in \omega\}$.

Lemma 5. *For every r.e.* λ*-theory* ϕ*, there exists an infinite sequence* $\mathcal{M} = \langle M_k \in \Lambda^o : k \in \omega\rangle$*, called* ϕ-easy sequence*, satisfying the following conditions:*

- $M_n \neq_\phi M_k$ *for every* $n \neq k$*;*
- *For all sequences* $\mathcal{N} = \langle N_k \in \Lambda^o : k \in \omega\rangle$ *the* λ*-theory generated by* $\phi \cup (\mathcal{M}, \mathcal{N})$ *is consistent.*

Theorem 4. *For every r.e.* λ*-theory* ϕ *and each natural number* n*, there is a* λ*-theory* $\phi_n \geq \phi$ *such that the lattice interval* $[\phi_n)$ *is isomorphic to the finite Boolean lattice with* 2^n *elements.*

6 Lattices of equational theories

We say that L is a *lattice of equational theories* iff L is isomorphic to the lattice $L(T)$ of all equational theories containing some equational theory T (or dually, the lattices of all subvarieties of some variety of algebras). Such lattices are algebraic and coatomic, possessing a compact top element; but stronger properties were not known before Lampe's discovery [7] that any lattice of equational theories obeys the Zipper condition (see Lemma 3).

In this section we show the existence of Boolean lattice intervals in the lattices of equational theories, and a meta version of the Stone representation theorem that holds for all varieties of algebras.

It is well known that a lattice of equational theories is isomorphic to a congruence lattice (see [4, 11]). Indeed, the lattice $L(T)$ of all equational theories containing T is isomorphic to the congruence lattice of the algebra $(\mathbf{F}_T, f)_{f\in \mathrm{End}(\mathbf{F}_T)}$, where $\mathbf{F}_T$ is the free algebra over a countable set of generators in the variety axiomatized by T, and $\mathrm{End}(\mathbf{F}_T)$ is the set of its endomorphisms.

We expand the algebra $(\mathbf{F}_T, f)_{f\in \mathrm{End}(\mathbf{F}_T)}$ (without changing the congruence lattice) by the operation q defined as follows (x_1, x_2 are two fixed variables) $q(t, s_1, s_2) = t[s_1/x_1, s_2/x_2]$, where $t[s_1/x_1, s_2/x_2]$ is the term obtained by substituting term s_i for variable x_i $(i = 0, 1)$ within t. The algebra $(\mathbf{F}_T, f, q)_{f\in \mathrm{End}(\mathbf{F}_T)}$ was defined, but not directly used, by Lampe in the proof of McKenzie Lemma in [7]. If we define $1 \equiv x_1$ and $0 \equiv x_2$, from the identities $q(x_1, s_1, s_2) = s_1$ and $q(x_2, s_1, s_2) = s_2$ we get that $(\mathbf{F}_T, f, q)_{f\in \mathrm{End}(\mathbf{F}_T)}$ is a Church algebra. It will be denoted by $\mathbf{C}_T$ and called hereafter the *Church algebra of* T.

In the following lemma we characterize the central elements of the Church algebra of an equational theory.

Lemma 6. *Let T be an equational theory and $\mathcal{V}$ be the variety of Σ-algebras axiomatized by T. Then the following conditions are equivalent, for every element $e \in \mathbf{C}_T$ and term $t(x_1, x_2) \in e$:*

(i) e is a central element of the Church algebra of T.
(ii) T contains the identities $t(x, x) = x$; $t(x, t(y, z)) = t(x, z) = t(t(x, y), z)$ and $t(f(\overline{x}), f(\overline{y})) = f(t(x_1, y_1), \dots, t(x_n, y_n))$, for $f \in \Sigma$.
(iii) For every $\mathbf{A} \in \mathcal{V}$, the function $t^{\mathbf{A}} : A \times A \to A$ is a decomposition operator.
(iv) $T = T_1 \cap T_2$, where T_i is the theory axiomatized (over T) by $t(x_1, x_2) = x_i$ $(i = 1, 2)$.

If the equivalence class of $t(x_1, x_2)$ is a central element of $\mathbf{C}_T$, then by Lemma 6(iii)-(iv) every algebra $\mathbf{A} \in \mathcal{V}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\phi \times \mathbf{A}/\overline{\phi}$, where $(\phi, \overline{\phi})$ is the cfc-pair associated with the decomposition operator $t^{\mathbf{A}}$, and the algebras $\mathbf{A}/\phi$ and $\mathbf{A}/\overline{\phi}$ satisfy respectively the equational theories T_1 and T_2. In such a case, we say that $\mathcal{V}$ is *decomposable* as a product of the two subvarieties axiomatized respectively by T_1 and T_2 (see [16]).

We say that a variety is *directly indecomposable* if the Church algebra of its equational theory is a directly indecomposable algebra.

Theorem 5. *Let T be an equational theory. Assume there exist n binary terms $t_0, \dots, t_{n-1}$ such that, for every function $k : n \to \{1, 2\}$, the theory axiomatized (over T) by $t_i(x_1, x_2) = x_{k(i)}$ $(i = 0, \dots, n-1)$ is consistent. Then there exists a theory $T' \geq T$ such that the lattice $L(T')$ of all equational theories extending T' is isomorphic to the free Boolean lattice with 2^{2^n} elements.*

The set of all factor congruences of an algebra does not constitute in general a sublattice of the congruence lattice. We now show that in every algebra there is a subset of factor congruences which always constitutes a Boolean sublattice of the congruence lattice.

We denote by $t_e^{\mathbf{A}}$ the decomposition operator associated with the central element e by Lemma 6(iii).

Lemma 7. *Let T be an equational theory and $\mathcal{V}$ be the variety axiomatized by T. For every algebra $\mathbf{A} \in \mathcal{V}$, the function $h : \mathrm{Ce}(\mathbf{C}_T) \to \mathrm{Con}(\mathbf{A})$, defined by $h(e) = \{(x, y) : t_e^{\mathbf{A}}(x, y) = x\}$, is a lattice homomorphism from the* BA *of central elements of $\mathbf{C}_T$ into the set of factor congruences of $\mathbf{A}$ such that $(h(e), h(e^-))$ is a cfc-pair for all $e \in \mathrm{Ce}(\mathbf{C}_T)$. The range of h constitutes a Boolean sublattice of* $\mathrm{Con}(\mathbf{A})$.

We say that a variety $\mathcal{V}$ is *decomposable as a weak Boolean product of directly indecomposable subvarieties* if there exists a family $\langle \mathcal{V}_i : i \in X \rangle$ of directly indecomposable subvarieties $\mathcal{V}_i$ of $\mathcal{V}$ such that every algebra $\mathbf{A} \in \mathcal{V}$ is isomorphic to a weak Boolean product $\Pi_{i \in X} \mathbf{B}_i$ of algebras $\mathbf{B}_i \in \mathcal{V}_i$.

Theorem 6. (Meta-Representation Theorem) *Every variety $\mathcal{V}$ of algebras is decomposable as a weak Boolean product of directly indecomposable subvarieties.*

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A Technical Appendix

This technical appendix is devoted to provide the proofs which are omitted in the paper.

Lemma 1. *Let* $\mathbf{A}$ *be a Church algebra and* $e \in A$*. Then we have, for all* $x, y \in A$*:*

(a) $x\ \theta_e\ q(e,x,y)\ \overline{\theta}_e\ y$.
(b) $x \theta_e y$ *iff* $q(e,x,y)\ (\theta_e \wedge \overline{\theta}_e)\ y$.
(c) $x \overline{\theta}_e y$ *iff* $q(e,x,y)\ (\theta_e \wedge \overline{\theta}_e)\ x$.
(d) $\theta_e \circ \overline{\theta}_e = \overline{\theta}_e \circ \theta_e = \nabla$.

Proof. (a) From $1\ \theta_e\ e\ \overline{\theta}_e\ 0$.
(b) By (a) we have $x\ \theta_e\ q(e,x,y)$. Then $x \theta_e y$ iff $q(e,x,y)\ \theta_e\ y$.
(c) Analogous to (b).
(d) By (a).

Proposition 2. *Let* $\mathbf{A}$ *be a Church* Σ*-algebra and* $e \in A$*. The following conditions are equivalent:*

(i) *e is central;*
(ii) $\theta_e \wedge \overline{\theta}_e = \Delta$*;*
(iii) *For all x and y,* $q(e,x,y)$ *is the unique element such that* $x \theta_e\ q(e,x,y)\ \overline{\theta}_e y$*;*
(iv) *e satisfies the following identities:*
1. $q(e,x,x) = x$.
2. $q(e, q(e,x,y), z) = q(e,x,z) = q(e,x,q(e,y,z))$.
3. $q(e, f(\overline{x}), f(\overline{y})) = f(q(e,x_1,y_1), \dots, q(e,x_n,y_n))$*, for every n-ary function symbol* $f \in \Sigma$.
4. $e = q(e,1,0)$.

(v) *The function* f_e *defined by* $f_e(x,y) = q(e,x,y)$ *is a decomposition operator such that* $f_e(1,0) = e$.

Proof. $(i) \Leftrightarrow (ii)$ From Lemma 1(d).

$(ii) \Rightarrow (iii)$ By Lemma 1(d) θ_e and $\overline{\theta}_e$ are a cfc-pair. Then the conclusion follows from Lemma 1(a).

$(iii) \Rightarrow (ii)$ First note that $q(e,x,x) = x$. If $x\ (\theta_e \wedge \overline{\theta}_e)\ y$ then $x \theta_e y \overline{\theta}_e x$, that is $y = q(e,x,x) = x$.

$(iv) \Leftrightarrow (v)$ By Prop. 1.

$(i) \Rightarrow (v)$ First we recall that (i) is equivalent to (iii). f_e is a decomposition operator because $(\theta_e, \overline{\theta}_e)$ is a cfc-pair and $q(e,x,y)$ is the unique element satisfying $x\ \theta_e\ q(e,x,y)\ \overline{\theta}_e\ y$. Moreover, $f_e(1,0) = q(e,1,0) = e$ follows from $1\ \theta_e\ e\ \overline{\theta}_e\ 0$.

$(v) \Rightarrow (i)$ Let $(\phi, \overline{\phi})$ be the cfc-pair associated with f_e. From Prop. 1 and from $f_e(1,0) = q(e,1,0) = e$ it follows that $1 \phi e \overline{\phi} 0$, so that $\theta_e, \overline{\theta}_e \subseteq \phi$. For the opposite direction, let $x \phi y$, which is equivalent to $q(e,x,y) = y$ by Prop. 1. Then by $1 \theta_e e$ we derive $x = q(1,x,y)\ \theta_e\ q(e,x,y) = y$. Similarly, for $\overline{\phi}$.

Theorem 1. *Let* $\mathbf{A}$ *be a Church algebra. The algebra* $(\mathrm{Ce}(\mathbf{A}), \wedge, \vee, ^-, 0, 1)$ *of central elements of* $\mathbf{A}$*, defined by*

$$e \wedge d = q(e, d, 0); \quad e \vee d = q(e, 1, d); \quad e^- = q(e, 0, 1),$$

is a Boolean algebra isomorphic to the Boolean algebra of factor congruences of $\mathbf{A}$.

Proof. If $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, then it is easy to check, by using the definition of a central element, that $\mathrm{Ce}(\mathbf{A}) = \mathrm{Ce}(\mathbf{B}) \times \mathrm{Ce}(\mathbf{C})$. In the terminology of universal algebra one says that $\mathbf{A}$ has no "skew factor congruences". From this and [3, Prop. 1.3] the factor congruences of $\mathbf{A}$ form a Boolean sublattice of the congruence lattice $\mathrm{Con}(\mathbf{A})$. It follows that the partial ordering on central elements, defined in (1) of Sec. 3, is a Boolean ordering. Then it is possible to show that, for all central elements e and d, the elements e^-, $e \wedge d$ and $e \vee d$ are central and are respectively associated with the cfc-pairs $(\overline{\theta}_e, \theta_e)$, $(\theta_e \vee \theta_d, \overline{\theta}_e \wedge \overline{\theta}_d)$ and $(\theta_e \wedge \theta_d, \overline{\theta}_e \vee \overline{\theta}_d)$.

We now check the details for e^-. Since e is central then $(\theta_e, \overline{\theta}_e)$ is a cfc-pair. The complement of $(\theta_e, \overline{\theta}_e)$ is the pair $(\overline{\theta}_e, \theta_e)$. We have that e^- is the unique element such that $0\ \theta_e\ e^-\ \overline{\theta}_e\ 1$. Then $1\ \overline{\theta}_e\ e^-\ \theta_e\ 0$ for the pair $(\overline{\theta}_e, \theta_e)$. This means that e^- is the central element associated with $(\overline{\theta}_e, \theta_e)$.

We now consider $e \vee d = q(e, 1, d)$. A similar reasoning work for $e \wedge d$. First of all, we show that $q(e, 1, d) = q(d, 1, e)$. By Lemma 1(a) we have that $1\ \theta_e\ q(e, 1, d)\ \overline{\theta}_e\ d$, while $1\ \theta_e\ q(d, 1, e)\ \overline{\theta}_e\ d$ can be obtained as follows:

$$\begin{aligned} 1 &= q(d, 1, 1) \text{ by Prop. 2(iv-1)},\\ q(d, 1, 1)\ &\theta_e\ q(d, 1, e) \text{ by } 1\ \theta_e\ e,\\ q(d, 1, e)\ &\overline{\theta}_e\ q(d, 1, 0) \text{ by } e\ \overline{\theta}_e\ 0,\\ q(d, 1, 0) &= d \qquad\quad \text{by Prop. 2(iv-4)}. \end{aligned}$$

Since there is a unique element c such that $1\ \theta_e\ c\ \overline{\theta}_e\ d$, then we have the conclusion $q(e, 1, d) = q(d, 1, e)$. We now show that $q(e, 1, d)$ is the central element associated with the factor congruence $\theta_e \wedge \theta_d$, i.e.,

$$1\ (\theta_e \wedge \theta_d)\ q(e, 1, d)\ (\overline{\theta}_e \vee \overline{\theta}_d)\ 0.$$

From $q(d, 1, e) = q(e, 1, d)$ we easily get that $1\ \theta_e\ q(e, 1, d)$ and $1\ \theta_d\ q(e, 1, d)$, that is, $1\ (\theta_e \wedge \theta_d)\ q(e, 1, d)$. Finally, Prop. 2(iv-4), we have: $q(e, 1, d)\ \overline{\theta}_e\ d = q(d, 1, 0)\ \overline{\theta}_d\ 0$, i.e., $q(e, 1, d)\ (\overline{\theta}_e \vee \overline{\theta}_d)\ 0$.

Theorem 2. (The Stone Representation Theorem) *Let* $\mathbf{A}$ *be a Church algebra. Then, for all* $I \in X$ *the quotient algebra* $\mathbf{A}/\phi_I$ *is directly indecomposable and the map*

$$f : A \to \Pi_{I \in X}(A/\phi_I),$$

defined by

$$f(x) = ([x]_{\phi_I} : I \in X),$$

gives a weak Boolean product representation of $\mathbf{A}$.

Proof. By the proof of Thm. 1 the factor congruences of **A** constitute a Boolean sublattice of Con(**A**). Then by Comer's generalization [5] of Stone representation theorem f gives a weak Boolean product representation of **A**. The quotient algebras $\mathbf{A}/\phi_I$ are directly indecomposable by [17, Thm. 8].

Lemma 2. *The semicongruences of a Church algebra permute with its factor congruences, i.e., $\phi \circ \psi = \psi \circ \phi$ for every semicongruence ϕ and factor congruence ψ.*

Proof. Let $\psi = \theta_e$ for a central element e and let $a\ \phi\ b\ \theta_e\ c$ for some b. We get the conclusion of the lemma if we show that $a\ \theta_e\ q(e,a,c)\ \phi\ c$. Notice that $a\ \theta_e\ q(e,a,c)$ is a consequence of Lemma 1(a). We now prove that $q(e,a,c)\ \phi\ c$. First we remark that by $b\ \theta_e\ c$ and by Prop. 2(iii) we have $q(e,b,c) = c$. From this last equality and from $a\ \phi\ b$ it follows the conclusion $q(e,a,c)\ \phi\ q(e,b,c) = c$.

Lemma 3. *Let* **A** *be a Church algebra. Then the congruence lattice of* **A** *satisfies the Zipper condition, i.e., for all I and for all $\delta_i, \psi, \phi \in \mathrm{Con}(\mathbf{A})$ $(i \in I)$:*

$$\text{If } \bigvee_{i \in I} \delta_i = \nabla \text{ and } \delta_i \wedge \psi = \phi \ (i \in I), \text{ then } \psi = \phi.$$

Proof. By [7], where it is shown that the congruence lattice of every $0,1$-algebra (i.e., an algebra having a binary term with a right unit and a right zero) satisfies the Zipper condition.

Lemma 4. *Let* **B** *be a Church algebra and $\phi \in \mathrm{Con}(\mathbf{B})$. Then, $\mathbf{B}/\phi$ is also a Church algebra and the map $c_\phi : \mathrm{Ce}(\mathbf{B}) \to \mathrm{Ce}(\mathbf{B}/\phi)$, defined by*

$$c_\phi(x) = [x]_\phi$$

is a homomorphism of Boolean algebras.

Proof. It is not difficult to show that c_ϕ is a homomorphism with respect to the Boolean operations defined in Thm. 1.

Proposition 3. *Let* **A** *be a Church algebra. Then there exists no congruence ϕ such that the interval sublattice $[\phi)$ is isomorphic to an infinite Boolean lattice.*

Proof. From [6, Prop. 4], where it is shown that a complete coatomic Boolean lattice satisfying the Zipper condition, and whose top element is compact, is finite.

Lemma 5. *For every r.e. λ-theory ϕ, there exists an infinite sequence $\mathcal{M} = \langle M_k \in \Lambda^o : k \in \omega \rangle$, called ϕ-*easy sequence*, satisfying the following conditions:*

- *$M_n \neq_\phi M_k$ for every $n \neq k$;*
- *For all sequences $\mathcal{N} = \langle N_k \in \Lambda^o : k \in \omega \rangle$ the λ-theory generated by $\phi \cup (\mathcal{M}, \mathcal{N})$ is consistent.*

Proof. By [1, Prop. 17.1.9] there is a *ϕ-easy λ-term*, i.e., a term P such that the λ-theory generated by $\phi \cup \{P = Q\}$ is consistent for every closed term Q. We define $M_n \equiv P\underline{n}$, where $\underline{n}$ is the Church numeral of n (see [1, Def. 6.4.4]). Let $\mathcal{N} = \langle N_k \in \Lambda^o : k \in \omega\rangle$ be an arbitrary sequence. By compactness we get the conclusion of the lemma if the λ-theory generated by $\phi \cup \{(M_i, N_i) : i \leq n\}$ is consistent for every natural number n. Fix n. It is routine to find a λ-term R such that $R\underline{i} =_{\lambda\beta} N_i$ for all $i \leq n$. Since P is a ϕ-easy λ-term, then the λ-theory ψ generated by $\phi \cup \{P = R\}$ is consistent and $P\underline{i} =_\psi N_i$ (i.e., $M_i =_\psi N_i$) for all $i \leq n$.

Theorem 4. *For every r.e. λ-theory ϕ and each natural number n, there is a λ-theory $\phi_n \geq \phi$ such that the lattice interval $[\phi_n)$ is isomorphic to the finite Boolean lattice with 2^n elements.*

Proof. The term algebra of ϕ is a Church algebra by Example 1. Let $\mathcal{M} = \langle M_k \in \Lambda^o : k \in \omega\rangle$ be the ϕ-easy sequence of Lemma 5. Then the set $\{[M_k]_\phi : k \in \omega\}$ is a countable infinite easy subset of the term algebra of ϕ. From Thm. 3 there exists a congruence ψ_n such that $\psi_n \geq \phi$ and $[\psi_n)$ is isomorphic to the free Boolean algebra with 2^{2^n} elements. The congruence ϕ_n of the theorem can be defined by using ψ_n and the following facts: (a) Every filter of a finite Boolean algebra is a Boolean lattice; (b) The free Boolean algebra with 2^{2^n} elements has filters of arbitrary cardinality 2^k $(k \leq 2^n)$.

Lemma 6. *Let T be an equational theory and $\mathcal{V}$ be the variety of Σ-algebras axiomatized by T. Then the following conditions are equivalent, for every element $e \in \mathbf{C}_T$ and term $t(x_1, x_2) \in e$:*

- *(i) e is a central element of the Church algebra of T.*
- *(ii) T contains the identities $t(x, x) = x$; $\quad t(x, t(y, z)) = t(x, z) = t(t(x, y), z)$ and $t(f(\overline{x}), f(\overline{y})) = f(t(x_1, y_1), \ldots, t(x_n, y_n))$, for $f \in \Sigma$.*
- *(iii) For every $\mathbf{A} \in \mathcal{V}$, the function $t^{\mathbf{A}} : A \times A \to A$ is a decomposition operator.*
- *(iv) $T = T_1 \cap T_2$, where T_i is the theory axiomatized (over T) by $t(x_1, x_2) = x_i$ $(i = 1, 2)$.*

Proof. (i) $\Leftrightarrow$ (ii) By the identities in Prop. 2(iv) characterizing the central elements and by the definition of the term $q(e, x, y)$ in the Church algebra $\mathbf{C}_T$ of T. For example, the identity $x = q(e, x, x)$ becomes $t[x/x_1, x/x_2] = t(x, x) = x$ for the term $t(x_1, x_2) \in e$.

(i) $\Leftrightarrow$ (iii) By the equivalence of Prop. 2(i) and Prop. 2(v).

(i) $\Leftrightarrow$ (iv) By Prop. 2 we have that e is central iff $\theta_e \wedge \overline{\theta}_e = \Delta$, where the congruence θ_e is generated by the pair $(e, 1)$ and the congruence $\overline{\theta}_e$ is generated by the pair $(e, 0)$. Since $1 \equiv x_1$ and $0 \equiv x_2$ in the Church algebra of T and $t(x_1, x_2) \in e$, then the pair $(e, 1)$ represents the identity $t(x_1, x_2) = x_1$ and the pair $(e, 0)$ the identity $t(x_1, x_2) = x_2$.

Theorem 5. *Let T be an equational theory. Assume there exist n binary terms $t_0, \ldots, t_{n-1}$ such that, for every function $k : n \to \{1, 2\}$, the theory axiomatized (over T) by $t_i(x_1, x_2) = x_{k(i)}$ $(i = 0, \ldots, n-1)$ is consistent. Then there exists* *a theory $T' \geq T$ such that the lattice $L(T')$ of all equational theories extending T' is isomorphic to the free Boolean lattice with 2^{2^n} elements.*

Proof. The equivalence classes of the terms $t_0, \dots, t_{n-1}$ constitute a finite easy set in the Church algebra of T. The conclusion follows from Thm. 3.

Lemma 7. *Let T be an equational theory and $\mathcal{V}$ be the variety axiomatized by T. For every algebra $\mathbf{A} \in \mathcal{V}$, the function $h : \mathrm{Ce}(\mathbf{C}_T) \to \mathrm{Con}(\mathbf{A})$, defined by $h(e) = \{(x, y) : t_e^{\mathbf{A}}(x, y) = x\}$, is a lattice homomorphism from the* BA *of central elements of $\mathbf{C}_T$ into the set of factor congruences of $\mathbf{A}$ such that $(h(e), h(e^-))$ is a cfc-pair for all $e \in \mathrm{Ce}(\mathbf{C}_T)$. The range of h constitutes a Boolean sublattice of $\mathrm{Con}(\mathbf{A})$.*

Proof. We only show that h is a homomorphism with respect to the join operator. Recall from Thm. 1 that $e \vee d = q(e, 1, d)$ and that in the Church algebra of T the term q is the substitution operator. Then we obtain $t_{e\vee d}(x, y) = t_e(x, t_d(x, y)) = t_d(x, t_e(x, y))$. We have $(x, y) \in h(e \vee d) \Leftrightarrow t_{e\vee d}(x, y) = x \Leftrightarrow t_e(x, t_d(x, y)) = x \Leftrightarrow (x, t_d(x, y)) \in h(e) \Leftrightarrow (x, y) \in h(d) \circ h(e)$, because $(t_d(x, y), y) \in h(d)$ holds from property $t_d(t_d(x, y), y) = t_d(x, y)$ of decomposition operators. We get $h(e \vee d) \subseteq h(e) \vee h(d)$. For the opposite it is sufficient to check $h(e), h(d) \subseteq h(e \vee d)$. Let $(x, y) \in h(e)$, i.e., $t_e(x, y) = x$. Then $t_d(x, t_e(x, y)) = x$, so that $t_{e\vee d}(x, y) = x$. A similar reasoning works for $h(d)$.

Theorem 6. (Meta-Representation Theorem) *Every variety $\mathcal{V}$ of algebras is decomposable as a weak Boolean product of directly indecomposable subvarieties.*

Proof. Let T be the equational theory of $\mathcal{V}$ and $\mathbf{C}_T$ be the Church algebra of T. By Thm. 2 we can represent $\mathbf{C}_T$ as a weak Boolean product $f : \mathbf{C}_T \to \Pi_{I\in X}(\mathbf{C}_T/\phi_I)$, where X is the Stone space of the Boolean algebra $\mathrm{Ce}(\mathbf{C}_T)$ of central elements of $\mathbf{C}_T$, $I \in X$ ranges over the maximal ideals of $\mathrm{Ce}(\mathbf{C}_T)$, $\phi_I = \cup_{e\in I}\overline{\theta}_e$, and $\overline{\theta}_e$ is the factor congruence associated with the central element $e \in I$. Since the lattice $L(T)$ of the equational theories extending T is isomorphic to the congruence lattice of $\mathbf{C}_T$, the congruence ϕ_I corresponds to an equational theory, say T_I. The Church algebra of T_I is isomorphic to $\mathbf{C}_T/\phi_I$, so that it is directly indecomposable. Then by Lemma 6 the variety $\mathcal{V}_I$ axiomatized by T_I is directly indecomposable.

Let $\mathbf{A} \in \mathcal{V}$ and $h : \mathrm{Ce}(\mathbf{C}_T) \to \mathrm{Con}(\mathbf{A})$ be the lattice homomorphism defined in Lemma 7. For every maximal ideal I of $\mathrm{Ce}(\mathbf{C}_T)$, consider the congruence $\phi_I^{\mathbf{A}} = \cup_{e\in I} h(e)$. The map $f : \mathbf{A} \to \Pi_{I\in X}(\mathbf{A}/\phi_I^{\mathbf{A}})$ defined by $f(x) = ([x]_{\phi_I^{\mathbf{A}}} : I \in X)$, determines a weak Boolean representation of $\mathbf{A}$, where $\mathbf{A}/\phi_I^{\mathbf{A}} \in \mathcal{V}_I$. The algebra $\mathbf{A}/\phi_I^{\mathbf{A}}$ may be directly decomposable also if it belongs to the directly indecomposable variety $\mathcal{V}_I$.