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Models and theories of λ-calculus

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To Alonzo Church,
for his greatest invention.

Abstract

A quarter of century after Barendregt's main book, a wealth of interesting problems about models and theories of the untyped λ-calculus are still open. In this thesis we will be mainly interested in the main semantics of λ-calculus (i.e., the Scott-continuous, the stable, and the strongly stable semantics) but we will also define and study two new kinds of semantics: *the relational* and *the indecomposable semantics*.

Models of the untyped λ-calculus may be defined either as reflexive objects in Cartesian closed categories (*categorical models*) or as combinatory algebras satisfying the five axioms of Curry and the Meyer-Scott axiom (*λ-models*).

Concerning categorical models we will see that each of them can be presented as a λ-model, even when the underlying category does not have enough points, and we will provide sufficient conditions for categorical models living in arbitrary cpo-enriched Cartesian closed categories to have $\mathcal{H}^*$ as equational theory. We will build a categorical model living in a non-concrete Cartesian closed category of sets and relations (*relational semantics*) which satisfies these conditions, and we will prove that the associated λ-model enjoys some algebraic properties which make it suitable for modelling non-deterministic extensions of λ-calculus.

Concerning combinatory algebras, we will prove that they satisfy a generalization of Stone representation theorem stating that every combinatory algebra is isomorphic to a weak Boolean product of directly indecomposable combinatory algebras. We will investigate the semantics of λ-calculus whose models are directly indecomposable as combinatory algebras (the *indecomposable semantics*) and we will show that this semantics is large enough to include all the main semantics and all the term models of semi-sensible λ-theories, and that it is however largely incomplete.

Finally, we will investigate the problem of whether there exists a non-syntactical model of λ-calculus belonging to the main semantics which has an r.e. (recursively enumerable) order or equational theory. This is a natural generalization of Honsell-Ronchi Della Rocca's longstanding open problem about the existence of a Scott-continuous model of λ_β or $\lambda_{\beta\eta}$. Then, we introduce an appropriate notion of *effective model of λ-calculus*, which covers in particular all the models individually introduced in the literature, and we prove that no order theory of an effective model can be r.e.; from this it follows that its equational theory cannot be λ_β or $\lambda_{\beta\eta}$. Then, we show that no effective model living in the stable or strongly stable semantics has an r.e. equational theory. Concerning Scott-continuous semantics, we prove that no order theory of a graph model can be r.e. and that many effective graph models do not have an r.e. equational theory.

Résumé

Dans cette thèse on s'intéresse surtout aux sémantiques principales du λ-calcul (c'est-à-dire la sémantique continue de Scott, la sémantique stable, et la sémantique fortement stable) mais on introduit et étudie aussi deux nouvelles sémantiques: *la sémantique relationnelle* et *la sémantique indécomposable*.

Les modèles du λ-calcul pur peuvent être définis soit comme des objets réflexifs dans des catégories Cartésiennes fermées (*modèles catégoriques*) soit comme des algèbres combinatoires satisfaisant les cinq axiomes de Curry et l'axiome de Meyer-Scott (λ-*modèles*).

En ce qui concerne les modèles catégoriques, on montre que tout modèle catégorique peut être présenté comme un λ-modèle, même si la ccc (catégorie Cartésienne fermée) sous-jacente n'a pas assez de points, et on donne des conditions suffisantes pour qu'un modèle catégorique vivant dans une ccc "cpo-enriched" arbitraire ait $\mathcal{H}^*$ pour théorie équationnelle. On construit un modèle catégorique qui vit dans une ccc d'ensembles et relations (*sémantique relationnelle*) et qui satisfait ces conditions. De plus, on montre que le λ-modèle associé possède des propriétés algébriques qui le rendent apte à modéliser des extensions non-déterministes du λ-calcul.

En ce qui concerne les algèbres combinatoires, on montre qu'elles satisfont une généralisation du Théorème de Représentation de Stone qui dit que toute algèbre combinatoire est isomorphe à un produit Booléen faible d'algèbres combinatoires directement indécomposables. On étudie la sémantique du λ-calcul dont les modèles sont directement indécomposable comme algèbres combinatoires (*sémantique indécomposable*); on prouve en particulier que cette sémantique est assez générale pour inclure d'une part les trois sémantiques principales et d'autre part les modèles de termes de toutes les λ-théories semi-sensibles. Par contre, on montre aussi qu'elle est largement incomplète.

Finalement, on étudie la question de l'existence d'un modèle non-syntaxique du λ-calcul appartenant aux sémantiques principales et ayant une théorie équationnelle ou inéquationnelle r.e. (récursivement énumérable). Cette question est une généralisation naturelle du problème de Honsell et Ronchi Della Rocca (ouvert depuis plus que vingt ans) concernant l'existence d'un modèle continu de λ_β ou $\lambda_{\beta\eta}$. On introduit une notion adéquate de *modèles effectifs du λ-calcul*, qui couvre en particulier tous les modèles qui ont été introduits individuellement en littérature, et on prouve que la théorie inéquationnelle d'un modèle effectif n'est jamais r.e.; en

conséquence sa théorie équationnelle ne peut pas être λ_β ou $\lambda_{\beta\eta}$. On montre aussi que la théorie équationnelle d'un modèle effectif vivant dans la sémantique stable ou fortement stable n'est jamais r.e. En ce qui concerne la sémantique continue de Scott, on démontre que la théorie inéquationnelle d'un modèle de graphe n'est jamais r.e. et qu'il existe beaucoup de modèles de graphes effectifs qui ont une théorie équationnelle qui n'est pas r.e.

Sommario

In questa tesi ci interessiamo soprattutto alle semantiche principali del λ-calcolo (ovvero la semantica continua di Scott, la semantica stabile, e la semantica fortemente stabile) ma introduciamo e studiamo anche due nuove semantiche: *la semantica relazionale* e *la semantica indecomponibile*.

I modelli del λ-calcolo puro possono essere definiti o come oggetti riflessivi in categorie Cartesiane chiuse (*modelli categorici*) oppure come algebre combinatorie che soddisfano i cinque assiomi di Curry e l'assioma di Meyer-Scott (λ-*modelli*).

Per quanto concerne i modelli categorici, mostriamo che tutti i modelli categorici possono essere presentati come un λ-modello, anche se la ccc (categoria Cartesiana chiusa) soggiacente non ha abbastanza punti, e forniamo delle condizioni sufficienti a garantire che un modello categorico che vive in una categoria "cpo-enriched" arbitraria abbia $\mathcal{H}^*$ come teoria equazionale. Costruiamo un modello categorico che vive in una ccc di insiemi e relazioni (*semantica relazionale*) e che soddisfa queste condizioni. Inoltre, mostriamo che il λ-modello associato possiede delle proprietà algebriche che lo rendono adatto a modellare delle estensioni non-deterministiche del λ-calcolo.

Per quanto concerne le algebre combinatorie, mostriamo che soddisfano una generalizzazione del Teorema di Rappresentazione di Stone che dice che ogni algebra combinatoria è isomorfa a un prodotto Booleano debole di algebre combinatorie direttamente indecomponibili. Quindi, studiamo la semantica del λ-calcolo i cui modelli sono direttamente indecomponibili come algebre combinatorie (*semantica indecomponibile*) e dimostriamo che questa semantica è abbastanza generale da includere sia le tre semantiche principali sia i modelli dei termini di tutte le λ-teorie semi-sensibili. Ciò nonostante, questa semantica risulta essere anche ampiamente incompleta.

In seguito, investighiamo il problema dell'esistenza di un modello non sintattico del λ-calcolo che appartenga alle semantiche principali e che abbia una teoria equazionale o disequazionale r.e. (ricorsivamente enumerabile). Questo problema è una generalizzazione naturale del problema di Honsell e Ronchi Della Rocca (aperto da più di vent'anni) concernente l'esistenza di un modello continuo di λ_β o $\lambda_{\beta\eta}$. Introduciamo una nozione adeguata di *modello effettivo* del λ-calcolo, che copre in particolare tutti i modelli che sono stati introdotti individualmente in letteratura, e dimostriamo che la teoria disequazionale di un modello effettivo non è mai r.e.;

come conseguenza otteniamo che la sua teoria equazionale non può essere λ_β o $\lambda_{\beta\eta}$. Dimostriamo anche che la teoria equazionale di un modello effettivo che vive nella semantica stabile o fortemente stabile non è mai r.e. Per quanto concerne la semantica continua di Scott, dimostriamo che la teoria equazionale di un modello di grafo non è mai r.e. e che esistono molti modelli di grafo effettivi che hanno una teoria equazionale che non è r.e.

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I dedicate this thesis to the memory of Alonzo Church.

Giulio Manzonetto
Paris, January 2008

Contents

Preface

The work presented in this thesis is partially based on some previously published papers, thus much credit for the technical contents and the underlying ideas goes to my co-authors.

In more detail, the construction described in the first part of Chapter 2 for turning any categorical model into a λ-model, as well as the definition of the relational model of λ-calculus described in Chapter 3, are the fruits of joint research with Antonio Bucciarelli and Thomas Ehrhard. This material appeared in the Proceedings of the 16$^{\text{th}}$ EACSL Annual Conference on Computer Science and Logic (CSL'07) [27]. However, our presentation in Chapter 2 of the above mentioned construction is slightly different; in particular, the proof of Lemma 2.2.5 given in [27] relied unnecessarily on Lemma 2.2.9 (for constraints of space), whilst here we give the correct argumentation. The proof of the fact that the equational theory of every extensional well stratifiable $\perp$-model living in arbitrary cpo-enriched Cartesian closed categories is $\mathcal{H}^*$ (and that this is true for the relational model of Chapter 3) is new.

The contents of Chapter 4 were obtained as the result of joint research with my supervisor Antonino Salibra and appeared in the proceedings of the 21$^{\text{st}}$ Annual IEEE Symposium on Logic in Computer Science (LICS'06) [78]. At the end of the chapter, a new result about the size of the incompleteness of the indecomposable semantics (Theorem 4.4.16) is presented.

The contents of Chapter 5 and 6 were obtained as the result of joint research with my supervisors Antonino Salibra and Chantal Berline. A preliminary version of this work appeared in the Proceedings of the 16$^{\text{th}}$ EACSL Annual Conference on Computer Science and Logic (CSL'07) [15]. In these two chapters there are more proofs, explanations and examples. Chapter 6 contains also some deeper results (Theorem 6.4.11 and its corollaries).

The following figure roughly indicates the interdependence of the chapters. Since all chapters depend on Chapter 1, which contains the common preliminaries, it has been splitted into subsections.

$$\begin{tikzcd}
 & & 1.3 \arrow[r] & 2 \arrow[r] & 3 \\
1.1 \arrow[r] & 1.2 \arrow[ur] \arrow[dr] & & 4 & \\
 & & 1.4 \arrow[r] \arrow[ur] & 5 \arrow[r] & 6
\end{tikzcd}$$

It turns out that there are three "independent" paths corresponding to three main topics:

1) In Chapter 2 and 3 we set the mathematical framework for dealing with models living in non-concrete semantics and we build an example of a model living in a Cartesian closed category of sets and relations which does not have enough points;

2) In Chapter 4 we generalize the Stone representation theorem to combinatory algebras and we investigate the semantics of λ-calculus given in terms of indecomposable λ-models;

3) In Chapter 5 we provide some mathematical tools for studying the class of graph models, and in Chapter 6 we investigate, using techniques of recursion theory, the question of whether a model belonging to the Scott-continuous semantics, or one of its refinements, can have a recursively enumerable order or equational theory.

We have chosen to discuss these subjects in this order, because we prefer to move - globally - from the completely abstract notion of *categorical interpretation*, which is needed to work in non-concrete categories, towards the more concrete interpretation function used in the main semantics.

However, the reader interested only in, say, results about models living in Scott-continuous semantics or in one of its refinements, can perform a *local* reading skipping Section 1.3 and using the *ad hoc* definitions of Section 1.4.

Introduction

The untyped λ-calculus was introduced around 1930 by Church [33, 34] as the kernel of an investigation in the formal foundations of mathematics and logic. Although λ-calculus is a very basic language (which is sufficient, however, to represent all the computable functions), the analysis of its models and theories constitutes a very complex subject of study.

In this discussion we (mainly) use techniques of category theory, universal algebra and recursion theory to shed new light on the known semantics of λ-calculus and on the λ-theories which can be represented in these semantics.

Models and theories of λ-calculus

λ-theories are, by definition, equational extensions of the untyped λ-calculus which are closed under derivation; more precisely: a *λ-theory* is a congruence on λ-terms which contains $\alpha\beta$-conversion (λ_β); *extensional* λ-theories are those which contain $\alpha\beta\eta$-conversion ($\lambda_{\beta\eta}$). λ-theories arise by syntactical or semantic considerations. Indeed, a λ-theory may correspond to a possible operational (observational) semantics of λ-calculus, as well as it may be induced by a model of λ-calculus through the kernel congruence relation of the interpretation function. Although researchers have, till recently, mainly focused their interest on a limited number of them, the set of λ-theories constitutes a very rich, interesting and complex mathematical structure (see [8, 13, 14]), whose cardinality is $2^{\aleph_0}$.

Models of λ-calculus. In 1969, Scott found the first model of λ-calculus ($\mathscr{D}_\infty$) in the category of complete lattices and Scott continuous functions. The question of *what is a model of λ-calculus* has been investigated by several researchers, but only at the end of the seventies they were able to provide general definitions. Throughout this thesis we will work mainly with two notions of model of λ-calculus: the former is category-theoretic and the latter is algebraic. From the categorical point of view, a model of λ-calculus is a reflexive object in a Cartesian closed category (*categorical model*). From the algebraic point of view, it is a combinatory algebra satisfying the five axioms of Curry, and the Meyer-Scott axiom (*λ-model*).

The main semantics. After Scott's $\mathscr{D}_\infty$, a large number of mathematical models of λ-calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [8, 13, 89]. The Scott-continuous semantics [95] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics (Berry [17]) and the strongly stable semantics (Bucciarelli-Ehrhard [25]) are refinements of the Scott-continuous semantics, introduced to capture the notion of "sequential" Scott continuous function. By "the main semantics" we will understand one of these and, for brevity, we will respectively call the models living inside: continuous, stable and strongly stable models. In each of these semantics all the models come equipped with a partial order, and some of them, called *webbed models*, are built from lower level structures called "webs". The simplest class of webbed models is the class of *graph models*, which was isolated in the seventies by Plotkin, Scott and Engeler within the Scott-continuous semantics.

Non-concrete semantics of λ-calculus

In this thesis we will be mainly interested in models of λ-calculus living in the main semantics. However, in Chapter 3 we will build and study a categorical model $\mathscr{D}$ living in a Cartesian closed category of sets and relations. It turns out that $\mathscr{D}$ has not enough points.

If we choose as definition of model of λ-calculus the notion of λ-model, we could be reluctant to consider $\mathscr{D}$ as a real model, since the only known construction for turning a categorical model into a λ-model asks for reflexive objects having enough points (see, e.g., [8, Ch. 5]). In Chapter 2 we will see that this does not constitute a problem: we will indeed give an alternative construction which works in greater generality and allows us to present *any* categorical model as a λ-model. We will also provide sufficient conditions for categorical models living in arbitrary cpo-enriched Cartesian closed categories to have $\mathcal{H}^*$ as equational theory, where $\mathcal{H}^*$ is the maximal consistent sensible λ-theory, and we will see that our model $\mathscr{D}$ fulfils these conditions.

The last part of Chapter 3 is devoted to prove that the λ-model associated with $\mathscr{D}$ by our construction satisfies interesting algebraic properties, which make it suitable for modelling non-deterministic extensions of λ-calculus.

The indecomposable semantics

In Chapter 4 we will show that the Stone representation theorem for Boolean algebras admits a generalization to combinatory algebras: any combinatory algebra is isomorphic to a weak Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two non-trivial algebras).

We will investigate the semantics of λ-calculus given in terms of models which are directly indecomposable as combinatory algebras (and called the *indecomposable semantics*, for short). We will prove that the indecomposable semantics encompasses all the main semantics, as well as the term models of all semi-sensible λ-theories.

However, the indecomposable semantics is incomplete, and this incompleteness is as large as possible, indeed we will see that: (i) there exists a continuum of pairwise incompatible λ-theories which are omitted by the indecomposable semantics; (ii) for every recursively enumerable (r.e., for short) λ-theory $\mathcal{T}$ there are $2^{\aleph_0}$ λ-theories including $\mathcal{T}$ and forming an interval, which are omitted by the indecomposable semantics. In particular, this gives a new and elegant uniform proof of the fact that the Scott-continuous, the stable, and the strongly stable semantics are (largely) incomplete.

Finally, we will show that each of the main semantics represents a set of λ-theories which is not closed under finite intersection, so that it cannot constitute a sublattice of the lattice of all λ-theories.

A longstanding open problem, and developments

The question of the existence of a continuous model or, more generally, of a non-syntactical model of λ_β (or $\lambda_{\beta\eta}$) was proposed by Honsell and Ronchi Della Rocca in the eighties. It is still open, but generated a wealth of interesting research and results (see, e.g., [13, 14]).

In 1995, Di Gianantonio, Honsell and Plotkin succeeded to build an extensional model of $\lambda_{\beta\eta}$ living in some "weakly continuous" semantics [46]. However, the construction of this model starts from the term model of $\lambda_{\beta\eta}$, and hence it cannot be seen as having a purely non-syntactical presentation. Furthermore, the problem of whether there exists a model of λ_β or $\lambda_{\beta\eta}$ *living in one of the main semantics* remains completely open. Nevertheless, the authors showed in the same paper that the set of λ-theories of extensional continuous models had a least element.

Hence, it became natural to ask whether, given a (uniform) class of models of λ-calculus, there is a minimum λ-theory represented in it; a question which was raised in [13]. Bucciarelli and Salibra showed [29, 30] that the answer is also positive for the class of graph models, and that the least graph theory (i.e., theory of a graph model) is different from λ_β. At the moment the problem remains open for the other classes.

(Concrete) Effective models versus r.e. λ-theories

In Chapter 6 we will investigate the problem of whether the equational theory of a non-syntactical model of λ-calculus can be r.e. (note that this is a generalization of Honsell-Ronchi Della Rocca's open question since λ_β and $\lambda_{\beta\eta}$ are r.e.). As far as we know, this problem was first raised in [14], where it is conjectured that no graph model can have an r.e. equational theory, but we expect that this could indeed be true for all models living in the main semantics. Since all these models are partially ordered, and since their equational theory is easily expressible from their order theory, we will also address the analogue problem for order theories.

Moreover, we find it natural to concentrate on models with built-in effectivity properties. It seems indeed reasonable to think that, if these models do not even succeed to have an r.e. theory, then the other ones have no chance to succeed, and we will give deeper arguments in Section 6.1. Hence, starting from the known notion of an effective domain, we introduce an appropriate notion of an *effective model of* λ*-calculus* which covers in particular all the models individually introduced in the literature.

We prove that the order theory of an effective model is never r.e.; from this it follows that its equational theory cannot be λ_β or $\lambda_{\beta\eta}$. Then, we show that no effective model living in the stable or in the strongly stable semantics has an r.e. equational theory. Concerning Scott-continuous semantics, we investigate the class of effective graph models, and make it precise when the results hold for wide classes of effective webbed models.

Graph models: a case of study

In order to attack difficult open problems as those stated above, it is often convenient to focus the attention, first, on the class of graph models. Indeed, for all webbed models it is possible to infer properties of the models by analyzing the structure of their web, and graph models have the simplest kind of web. Moreover, the class of graph models is very rich, since it represents $2^{\aleph_0}$ pairwise distinct (non extensional) λ-theories.

In Chapter 5 we will recall a free completion process for building (the web of) a graph model starting from a "partial web", and we will develop some mathematical tools for studying the framework of partial webs. These tools will be fruitfully used to prove that there exists a minimum graph theory and that graph models enjoy a kind of Löwenheim Skolem theorem: every equational/order graph theory is the theory of a graph model having a countable web.

In the last part of Chapter 6 we will investigate the question of whether a graph theory can be r.e. We will prove that no order graph theory can be r.e., and that there exists an effective graph model whose equational/order theory is the minimum one. We will also characterize various classes of effective graph models whose equational theory cannot be r.e. and we will mention when the results can be extended to some other classes of webbed models.

Notations

Set theory

$\mathbb{N}$	set of natural numbers	2
$\mathrm{card}(X)$	cardinality of X	2
$\mathscr{P}(X)$	collection of all subsets of X (powerset of X)	2
$\mathcal{P}_{\mathrm{f}}(X), X^*$	collection of all finite subsets of X	2
$X \subseteq_{\mathrm{f}} Y$	X is a finite subset of Y	2
$\mathrm{dom}(f)$	domain of f	2
$\mathrm{rg}(f)$	range of f	2
$\mathrm{graph}(f)$	graph of f	2
$f^+(X)$	image of X via f	2
$f^-(X)$	inverse image of X via f	2
f^{-1}	partial inverse of an injective f	2
$tr(f)$	trace of f	15
$Tr(f)$	trace of f on prime algebraic domains	16
Tr_s	stable trace of f	18
$f \cap g$	$\mathrm{graph}(f \cap g) = \mathrm{graph}(f) \cap \mathrm{graph}(g)$	2
$f \cup g$	$\mathrm{graph}(f \cup g) = \mathrm{graph}(f) \cup \mathrm{graph}(g)$	2
$\mathrm{Aut}(\mathcal{S})$	group of automorphisms of $\mathcal{S}$	2
$[]$	empty multiset	2
$m = [a_1, a_2, \ldots]$	multiset whose elements are $a_1, a_2, \ldots$	2
$m_1 \uplus m_2$	multiset union of m_1, m_2	2
$\mathcal{M}_{\mathrm{f}}(X)$	set of finite multisets over X	2
$\sigma = (m_1, m_2, \ldots)$	$\mathbb{N}$-indexed sequence of multisets	2
σ_i	i-th element of σ	2
$\mathcal{M}_{\mathrm{f}}(X)^{(\omega)}$	set of quasi-finite $\mathbb{N}$-indexed sequences of $\mathcal{M}_{\mathrm{f}}(X)$	2
$*$	unique inhabitant of $\mathcal{M}_{\mathrm{f}}(\emptyset)^{(\omega)}$	2

Recursion theory

φ_n	partial recursive function of index n	2
$\mathcal{W}_n$	domain of φ_n	2
E^c	complement of E w.r.t. $\mathbb{N}$ (i.e., $\mathbb{N} - E$)	3
$\#_*(-)$	encoding of $\mathbb{N}^*$	3
$\#(-,-)$	encoding of $\mathbb{N} \times \mathbb{N}$	3
$\#\langle -,- \rangle$	encoding of $\mathbb{N}^* \times \mathbb{N}$	3

Algebras

$\mathscr{A}, \mathscr{B}, \mathscr{C}$	algebras	50
$\mathrm{Con}(\mathscr{A})$	set of congruences of $\mathscr{A}$	50
$\nabla^{\mathscr{A}}$	top element of $\mathrm{Con}(\mathscr{A})$	50
$\Delta^{\mathscr{A}}$	bottom element of $\mathrm{Con}(\mathscr{A})$	50
$\vartheta(a,b)$	least congruence relating a and b	50
$\vartheta_1 \times \vartheta_2$	product congruence of ϑ_1, ϑ_2	50
$\mathscr{A} \times \mathscr{B}$	direct product of $\mathscr{A}, \mathscr{B}$	50
$\mathscr{A} \cong \mathscr{B}$	$\mathscr{A}$ is isomorphic to $\mathscr{B}$	50
$\mathscr{A} \leq \Pi_{i\in I}\mathscr{B}_i$	subdirect product	50
$\mathrm{IE}(\mathscr{A})$	set of idempotent elements of $\mathscr{A}$	54
$\mathrm{CE}(\mathscr{A})$	set of central elements of $\mathscr{A}$	55
f_e	decomposition operator determined by e	55
$(\vartheta_e, \overline{\vartheta}_e)$	complementary factor congruences determined by e	56

Domain theory and orders

$\mathcal{D}$	partially ordered set	3
$\sqsubseteq_{\mathcal{D}}$	partial order on $\mathcal{D}$	3
$\bot_{\mathcal{D}}$	bottom element of $\mathcal{D}$	3
$\bigsqcup A$	least upper bound of A	3
$u \sqcup v$	least upper bound of $\{u,v\}$	3
$u \sqcap v$	gratest lower bound of $\{u,v\}$	4
$\mathcal{K}(\mathcal{D})$	set of compact elements of $\mathcal{D}$	3
$\mathcal{P}r(\mathcal{D})$	set of prime elements of $\mathcal{D}$	4
$\mathfrak{S}(D)$	set of all initial segments of $(D,\preceq)$	16
$\mathfrak{S}_{\mathrm{coh}}(D)$	set of all coherent initial segments of $(D,\preceq,\frown)$	16
$\mathfrak{F}(D)$	set of all filters of $(D,\preceq)$	16
$D_\bot$	flat domain built on a set D	4
$[s,s']$	closed interval between s,s'	4
$]s,s'[$	open interval between s,s'	4

Category theory

$\mathbf{C}$	locally small Cartesian closed category	4
$\mathbf{C}(A,B)$	set of morphisms from A to B	4
$\sqsubseteq_{(A,B)}$	partial order on $\mathbf{C}(A,B)$	5
$\bot_{(A,B)}$	least element of $\mathbf{C}(A,B)$	5
CPO	category of cpo's	38
ED	category of effective domains	82
EDID	category of effective DI-domains	83
$\mathbf{EDID}^{\mathbf{coh}}$	category of effective DI-domains with coherences	83
Rel	category of sets and relations	40
MRel	Kleisly category of **Rel** using $\mathcal{M}_{\mathrm{f}}(-)$	40
$A \times B$	categorical product	5

$A \& B$	categorical product in **MRel**	40
π_1, π_2	projections	5
$\langle f, g\rangle$	mediating arrow of a product	5
$f \times g$	arrow between product objects	5
$[A \Rightarrow B]$	exponential object	5
ev	evaluation morphism	5
Λ	Curry's morphism	5
$\mathbb{1}$	terminal object	5
$!_A$	unique morphism in $\mathbf{C}(A, \mathbb{1})$	5
$\mathscr{U} = (U, \mathrm{Ap}, \lambda)$	reflexive object of a ccc	9
$\|M\|_I$	categorical interpretation of M in I	10
$\Pi_{i\in I} A_i$	I-indexed categorical product of $(A_i)_{i\in I}$	5
A^I	I-indexed categorical product of copies of A	5
π^I_x	projection on the x-th component of A^I	5
Π^I_J	shorthand for $\langle \pi^I_x\rangle_{x\in J}$	5
$\mathrm{Ad}(f)$	set of adequate pairs of f	21
$\mathrm{dom}_\mathrm{f}(f)$	finitary domain of f	21
$\mathbf{C}_\mathrm{f}(U^{\mathrm{Var}}, U),\ A_{\mathscr{U}}$	set of finitary morphisms in $\mathbf{C}(U^{\mathrm{Var}}, U)$	21
$\|M\|_{\mathrm{Var}}$	categorical interpretation of M in Var	28
$M \sqsubseteq_{\mathscr{U}} N$	means $\|M\|_{\mathrm{Var}} \sqsubseteq \|N\|_{\mathrm{Var}}$	31
$M =_{\mathscr{U}} N$	means $\|M\|_{\mathrm{Var}} = \|N\|_{\mathrm{Var}}$	31

λ-calculus

Var	set of variables of λ-calculus	6
Λ	set of λ terms	6
Λ^o	set of closed λ-terms	6
$\Lambda(\mathcal{D})$	set of λ-terms with parameters in $\mathcal{D}$	14
$\Lambda^o(\mathcal{D})$	set of closed λ-terms with parameters in $\mathcal{D}$	14
$\Lambda_{\mathcal{T}-easy}$	set of $\mathcal{T}$-easy λ-terms	9
$\mathrm{FV}(M)$	set of free variables of M	6
$\vec{x}$	sequence of variables $(x_1, \ldots, x_n)$	43
$\mathbf{I}$	$\equiv \lambda x.x$	6
$\mathbf{1}$	$\equiv \lambda xy.xy$	6
$\mathbf{T}, \mathbf{K}$	$\equiv \lambda xy.x$	6
$\mathbf{F}$	$\equiv \lambda xy.y$	6
δ	$\equiv \lambda x.xx$	6
Ω	$\equiv \delta\delta \equiv (\lambda x.xx)(\lambda x.xx)$	6
Ω_3	$\equiv (\lambda x.xxx)(\lambda x.xxx)$	17
$\mathcal{U}$	set of all unsolvable λ-terms	7
$C[\xi_1, \ldots, \xi_n]$	context having $\xi_1, \ldots, \xi_n$ as algebraic variables	6
$C[-]$	context having just one algebraic variable	6
$M \to_R N$, $(M \twoheadrightarrow_R N)$	M R-reduces to N (in 0 or several steps)	6

Labelled $\lambda\perp$-calculus

$\underline{\perp}$	constant indicating divergence	30
c_k	label	30
$\Lambda_{\underline{\perp}}^{lab}$	set of λ-terms with possible occurrences of $\underline{\perp}$ and c_k's	30
$\Lambda_{\underline{\perp}}$	set of λ-terms with possible occurrences of $\underline{\perp}$	31
ω-reduction	$\underline{\perp}M \to_\omega \underline{\perp}$, $\lambda x.\underline{\perp} \to_\omega \underline{\perp}$	31
γ-reduction	$c_0(\lambda x.M)N \to_\gamma c_0(M[\underline{\perp}/x])$, $c_{k+1}(\lambda x.M)N \to_\gamma c_k(M[c_kN/x])$	32
ϵ-reduction	$c_k\underline{\perp} \to_\epsilon \underline{\perp}$, $c_k(c_nM) \to_\epsilon c_{\min(k,m)}M$	32
L	complete labelling	33
$L_1 \sqsubseteq_{lab} L_2$	$L_1(N) \leq L_2(N)$ for all subterms N of M	34
M^L	completely labelled $\lambda\perp$-term corresponding to L	33
$\overline{M}$	term obtained from M by erasing all labels	35
$M^{[k]}$	unique $\beta\omega$-normal form such that $\mathrm{BT}(M^{[k]}) = \mathrm{BT}^k(M)$	36
$\mathcal{A}(M)$	set of all direct approximants of M	34

Böhm trees

$\mathrm{BT}(M)$	Böhm tree of M	7
$\mathrm{BT}^k(M)$	Böhm tree of M pruned at depth k	36
BT	set of Böhm trees	7
$\subseteq_{BT}$	partial order on Böhm trees	7
$M \sqsubseteq_{BT} N$	$\mathrm{BT}(M) \subseteq_{BT} \mathrm{BT}(N)$	7
$M \sqsubseteq_{\eta,\infty} N$	$\mathrm{BT}(N)$ is a (possibly infinite) η-expansion of $\mathrm{BT}(M)$	7
$M \precsim_\eta N$	$\exists M', N'$ such that $M \sqsubseteq_{\eta,\infty} M' \sqsubseteq_{BT} N' \sqsupseteq_{\eta,\infty} N$	8
$M =_\eta N$	$M \precsim_\eta N \precsim_\eta M$	8

Lambda theories

$\lambda\mathscr{T}$	lattice of λ-theories	8
λ_β	least λ-theory	6
$\lambda_{\beta\eta}$	least extensional λ-theory	6
$M =_\mathcal{T} N$,	stands for $M = N \in \mathcal{T}$	8
$\mathcal{T} \vdash M = N$	stands for $M = N \in \mathcal{T}$	8
$[M]_\mathcal{T}$	$\mathcal{T}$-equivalence class of M	8
$V/\mathcal{T}$	quotient space of V modulo $\mathcal{T}$	8
$\mathcal{H}$	minimum sensible λ-theory	9
$\mathcal{H}^*$	maximal consistent sensible λ-theory	9
$\mathcal{B}_\mathcal{T}$	λ-theory $\{M = N : \mathrm{BT}(M) = \mathrm{BT}(N)\}$	9
$\lambda\mathbb{C}$	set of λ-theories represented in a class $\mathbb{C}$ of λ-models	13

Combinatory algebras and λ-models

$\mathbb{C}$	a class of λ-models	13
$\mathscr{C}$	a combinatory algebra	11
$\mathscr{M}_{\mathcal{T}}$	the term model of $\mathcal{T}$	13
$\mathscr{D}_\infty$	Scott's model	16
$\mathbf{k}, \mathbf{s}$	basic combinators: $\mathbf{k}xy = x,\ \mathbf{s}xyz = xz(yz)$	10
$\mathbf{i}$	$\equiv \mathbf{skk}$	10
ε	$\equiv \mathbf{s}(\mathbf{ki})$	10
$\mathbf{t}$	$\equiv \mathbf{k}$	10
$\mathbf{f}$	$\equiv \mathbf{ki}$	10
$\lambda^* x$	abstraction for combinatory terms in combinatory logic	11
$[t, u]$	pair of combinators	11
Env_C	set of environments with values in C	12
$[\![M]\!]_\rho$	interpretation of M in a (environment) λ-model	12
Ψ	morphism of λ-models	12,15
$\mathscr{A}_{\mathscr{U}}$	applicative structure associated with $\mathscr{U}$	22
$\mathscr{S}_{\mathscr{U}}$	environment λ-model associated with $\mathscr{U}$	23
$\mathscr{C}_{\mathscr{U}}$	λ-model associated with $\mathscr{U}$	26

Graph models and partial pairs

$\mathscr{G}$	graph model with web $\mathcal{G}$	17
$\mathcal{G} = (G, i_{\mathcal{G}})$	total pair, also called "web"	17
$\mathscr{E}$	Engeler's graph model	16
$\mathscr{P}_\omega$	Plotkin's graph model	16
$\mathcal{A} = (A, j_{\mathcal{A}})$	partial pair	66
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	partial pairs	66
$\overline{\mathcal{A}} = (\overline{A}, i_{\bar{A}})$	the total pair generated by $\mathcal{A}$	68
$\mathscr{G}_{\mathcal{A}}$	the graph model with web $\overline{\mathcal{A}}$	69
$\mathcal{A} \sqsubseteq \mathcal{B}$	$\mathcal{A}$ is a subpair of $\mathcal{B}$	66
$\lvert M\rvert^{\mathscr{G}}_\rho$	interpretation of M in $\mathscr{G}$	17
$\lvert M\rvert^{\mathcal{A}}_\rho$	partial interpretation of M w.r.t. $\mathcal{A}$	66
$\sigma = \rho \cap C$	$\sigma(x) = \rho(x) \cap C$ for all $x \in \mathrm{Var}$	66
$\rho \subseteq \sigma$	$\rho(x) \subseteq \sigma(x)$ for all $x \in \mathrm{Var}$	66
$\mathrm{Hom}(\mathcal{A}, \mathcal{B})$	set of morphisms from $\mathcal{A}$ to $\mathcal{B}$	67
$\mathrm{Iso}(\mathcal{A}, \mathcal{B})$	set of isomorphisms from $\mathcal{A}$ to $\mathcal{B}$	68
$\mathrm{Aut}(\mathcal{A})$	group of automorphisms of $\mathcal{A}$	68
$\theta : \mathcal{A} \to \mathcal{B}$	means $\theta \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$	68
$\mathcal{A} \lhd \mathcal{B}$	$\mathcal{A}$ is a retract of $\mathcal{B}$	71
$(e, \pi) : \mathcal{A} \lhd \mathcal{B}$	$e \in \mathrm{Hom}(\mathcal{A}, \mathcal{B}), \pi \in \mathrm{Hom}(\mathcal{B}, \mathcal{A})$ and $\pi \circ e = \mathrm{Id}_A$	71
$\mathscr{G} \lhd \mathscr{G}'$	if $\mathcal{G}$ is a retract of $\mathcal{G}'$	71

Effective models and effective domains

$\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$	categorical model based on the effective domain $\mathcal{D}$	85
d_k	k-th compact element of $\mathcal{D}$	80
$\widehat{v}$	set of compact elements below v	80
$\mathcal{D}^{r.e.}$	set of r.e. elements of $\mathcal{D}$	81
$\mathcal{D}^{dec}$	set of decidable elements of $\mathcal{D}$	81
$\zeta^{\mathcal{D}}$	an adequate numeration of $\mathcal{D}^{r.e.}$	82
$\lvert M\rvert, \lvert M\rvert^{\mathscr{M}}$	interpretation of M in $\mathscr{M}$	14
$M^-, (M^-)_{\mathscr{M}}$	$= \{N \in \Lambda^o : \lvert N\rvert^{\mathscr{M}} \sqsubseteq_{\mathcal{D}} \lvert M\rvert^{\mathscr{M}}\}$	86
$\perp_\omega^-$	$= \cup_{n\in\mathbb{N}} \perp_n^-$, where $\perp_n \equiv \lambda x_1 \dots x_n.\perp_{\mathcal{D}}$	86
Λ_E^o	$= \{N \in \Lambda^o : \widehat{\lvert N\rvert} \subseteq E\}$	87
ν_Λ	effective bijective numeration of Λ	79
ν_{Var}	effective bijective numeration of Var	85

1 Preliminaries

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.
(Alfred North Whitehead)

Since this thesis spans several fields (logic, category theory, algebra and recursion theory), which may each have their own vocabularies, it may be useful to recall some basic terminology. We will generally use the notation of Barendregt's classic work [8] for λ-calculus and combinatory logic and that of Burris and Sankappanavar [31] for universal algebra. Our main reference for category theory is [5], for recursion theory is [81] and for domain theory is [4]. Occasionally, some elementary notions of topology [64] are needed.

This chapter is organized as follows: in Section 1.1 we introduce the notations we will use for set theory, recursion theory, domain theory and category theory; in Section 1.2 we briefly recall the syntax of λ-calculus and the definition of λ-theory; Section 1.3 is devoted to introduce the main notions of model of λ-calculus and to analyze the relations between them; in Section 1.4 we present the "main semantics" of λ-calculus, namely, the Scott-continuous semantics and its refinements: the stable and the strongly stable semantics.

Finally, we briefly describe the classes of webbed models, living in these semantics, which are interesting for our purposes: 1. for the Scott-continuous semantics: the classes of K-models (introduced by Krivine in [72]), of pcs-models (see [13]), and filter models[1] [37]; 2. for the stable semantics: Girard's reflexive coherences (G-models); 3. for the strongly stable semantics: Ehrhard's reflexive hypercoherences (H-models). The terminology of K-, G-, H- models, that we will use freely here, as well as the informal terminology of "webbed models", was introduced in [13].

[1] We will investigate the class of *Scott's information systems* [94] (a class of continuous webbed models containing the K-, pcs- and filter models) in further works, since here we want to keep technicalities at the lowest possible level.

1.1 Generalities

1.1.1 Sets, functions and groups of automorphisms

We will denote by $\mathbb{N}$ the set of natural numbers. Given a set X, we denote by $\mathscr{P}(X)$ (resp. $\mathcal{P}_{\mathrm{f}}(X)$) the collection of all subsets (resp. finite subsets) of X and by $\mathrm{card}(X)$ the cardinality of X. A less heavy notation for $\mathcal{P}_{\mathrm{f}}(X)$ will be X^*. Finally, we let $X \subseteq_{\mathrm{f}} Y$ mean that X is a finite subset of Y.

For any function f we write $\mathrm{dom}(f)$ for the domain of f, $\mathrm{rg}(f)$ for its range, $\mathrm{graph}(f)$ for its graph, and $f\restriction_X$ for its restriction to a subset $X \subseteq \mathrm{dom}(f)$. We define the *image* and the *inverse image* of X via f respectively as $f^+(X) = \{f(x) : x \in X\}$ and $f^-(X) = \{x : f(x) \in X\}$. The *partial inverse* of an injective function f, denoted by f^{-1}, is defined as follows: $\mathrm{dom}(f^{-1}) = \mathrm{rg}(f)$ and $f^{-1}(x) = y$ if $f(y) = x$.

Let f, g be two partial functions, then: f and g are called *compatible* if $f(x) = g(x)$ for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$; we write $f \cap g$ for the function whose graph is $\mathrm{graph}(f) \cap \mathrm{graph}(g)$; if f, g are compatible, then $f \cup g$ denotes the function whose graph is $\mathrm{graph}(f) \cup \mathrm{graph}(g)$.

Given any mathematical structure $\mathcal{S}$ having a carrier set S, we denote by $\mathrm{Aut}(\mathcal{S})$ the *group of automorphisms* of $\mathcal{S}$. For all $s \in S$ the *orbit* $O(s)$ (with respect to $\mathrm{Aut}(\mathcal{S})$) is defined by $O(s) = \{\theta(s) : \theta \in \mathrm{Aut}(\mathcal{S})\}$. A structure $\mathcal{S}$ is *finite modulo* $\mathrm{Aut}(\mathcal{S})$ if the number of orbits of $\mathcal{S}$ (with respect to $\mathrm{Aut}(\mathcal{S})$) is finite.

1.1.2 Multisets and sequences

Let X be a set. A *multiset* m over X can be defined as an unordered list $m = [a_1, a_2, \ldots]$ with repetitions such that every a_i belongs to X. In particular, $[]$ denotes the empty multiset. For each $a \in X$ the *multiplicity of a in m* is the number of occurrences of a in m. If m is a multiset over X, then its *support* is the set of elements of X belonging to m. A multiset m is called *finite* if it is a finite list. The set of all finite multisets over X will be denoted by $\mathcal{M}_{\mathrm{f}}(X)$.

Given two multisets $m_1 = [a_1, a_2, \ldots]$ and $m_2 = [b_1, b_2, \ldots]$ the *multiset union* of m_1, m_2 is defined by $m_1 \uplus m_2 = [a_1, b_1, a_2, b_2, \ldots]$.

An $\mathbb{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all but a finite number of indices i. We write σ_i for the i-th element of σ. If X is a set, we denote by $\mathcal{M}_{\mathrm{f}}(X)^{(\omega)}$ the set of all quasi-finite $\mathbb{N}$-indexed sequences of finite multisets over X. We write $*$ for the $\mathbb{N}$-indexed family of empty multisets, in other words $*$ is the only inhabitant of $\mathcal{M}_{\mathrm{f}}(\emptyset)^{(\omega)}$.

1.1.3 Recursion theory

We write $\varphi_n : \mathbb{N} \to \mathbb{N}$ for the partial recursive function of index n and we indicate by $\mathcal{W}_n$ the domain of φ_n.

A set $E \subseteq \mathbb{N}$ is *recursively enumerable* (*r.e.*, for short) if it is the domain of a partial recursive function. The complement E^c of an r.e. set E is called *co-r.e.*

If both E and E^c are r.e., then E is called *decidable*. Note that the collection of all r.e. (co-r.e.) sets is closed under finite union and finite intersection.

We say that ν is an *encoding* of a countable set X if $\nu : X \to \mathbb{N}$ is bijective. A *numeration* of a set X is a function $\nu_X : \mathbb{N} \to X$ which is total and onto. Note that (the inverse of) an encoding is a special case of numeration.

Given two numerations ν_X and ν_Y of X and Y respectively, we say that a partial recursive function φ *tracks* $f : X \to Y$ with respect to ν_X, ν_Y if the following diagram commutes:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\varphi"] \arrow[d, "\nu_X"'] & \mathbb{N} \arrow[d, "\nu_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

A partial function $f : X \to Y$ is said *computable* (with respect to ν_X, ν_Y) if there exists a partial recursive function φ tracking f with respect to ν_X, ν_Y. A set $Y \subseteq X$ is *r.e.* (resp. *co-r.e.*) *with respect to* ν_X if the set $\nu_X^-(Y)$ is r.e. (resp. co-r.e.).

Hereafter we suppose that a computable encoding $\#(-,-) : \mathbb{N}^2 \to \mathbb{N}$ for the pairs has been fixed. Moreover, we fix an encoding $\#_* : \mathbb{N}^* \to \mathbb{N}$ which is effective in the sense that the relations $m \in \#_*^{-1}(n)$ and $m = \mathrm{card}(\#_*^{-1}(n))$ are decidable in (m, n). Finally, we define a map $\#\langle -,- \rangle : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}$ as follows: $\#\langle a, n \rangle = \#(\#_*(a), n)$.

We recall here a basic property of recursion theory which will be often used in Chapter 6.

Remark 1.1.1. *The property of being r.e. is preserved under (images and) inverse images via partial recursive functions.*

1.1.4 Partial orderings and cpo's

Let $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ be a *partially ordered set* (*poset*, for short). When there is no ambiguity we write $\mathcal{D}$ instead of $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$. Two elements u and v of $\mathcal{D}$ are: *comparable* if either $u \sqsubseteq_{\mathcal{D}} v$ or $v \sqsubseteq_{\mathcal{D}} u$; *incomparable* if they are not comparable; *compatible* if they have an upper bound, i.e., if there exists z such that $u \sqsubseteq_{\mathcal{D}} z$ and $v \sqsubseteq_{\mathcal{D}} z$.

Let $A \subseteq \mathcal{D}$ be a set. A is *upward* (resp. *downward*) *closed* if $v \in A$ and $v \sqsubseteq_{\mathcal{D}} u$ (resp. $u \sqsubseteq_{\mathcal{D}} v$) imply $u \in A$. The set A is *directed* if, for all $u, v \in A$, there exists $z \in A$ such that $u \sqsubseteq_{\mathcal{D}} z$ and $v \sqsubseteq_{\mathcal{D}} z$.

A poset $\mathcal{D}$ is a *complete partial order* (*cpo*, for short) if it has a least element $\perp_{\mathcal{D}}$ and every directed set $A \subseteq \mathcal{D}$ admits a least upper bound $\bigsqcup A$. A cpo is *bounded complete* if $u \sqcup v$ exists, for all compatible u, v.

An element $d \in \mathcal{D}$ is called *compact* if for every directed $A \subseteq \mathcal{D}$ we have that $d \sqsubseteq_{\mathcal{D}} \bigsqcup A$ implies $d \sqsubseteq_{\mathcal{D}} v$ for some $v \in A$. We write $\mathcal{K}(\mathcal{D})$ for the collection of compact elements of $\mathcal{D}$. A compact element $p \neq \perp_{\mathcal{D}}$ of $\mathcal{D}$ is *prime* if, for all compatible $u, v \in \mathcal{D}$, we have that $p \sqsubseteq_{\mathcal{D}} u \sqcup v$ implies $p \sqsubseteq_{\mathcal{D}} u$ or $p \sqsubseteq_{\mathcal{D}} v$. We denote by $\mathcal{P}r(\mathcal{D})$ the set of prime elements of $\mathcal{D}$.

A cpo $\mathcal{D}$ is *algebraic* if for every $u \in \mathcal{D}$ the set $\{d \in \mathcal{K}(\mathcal{D}) : d \sqsubseteq_{\mathcal{D}} u\}$ is directed and u is its least upper bound. A cpo $\mathcal{D}$ is *prime algebraic* if for all $u \in \mathcal{D}$ we have $u = \bigsqcup\{p \in \mathcal{P}r(\mathcal{D}) : p \sqsubseteq_{\mathcal{D}} u\}$. An algebraic cpo $\mathcal{D}$ is called *ω-algebraic* if the set of its compact elements is countable.

A bounded complete algebraic cpo is called a *Scott domain*.

Example 1.1.2. *The simplest examples of prime algebraic Scott domains are the flat domains and the powerset domains. If D is a set and $\perp$ an element not belonging to D, the* flat domain *$D_\perp$ is, by definition, the poset $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ such that $\mathcal{D} = D \cup \{\perp\}$ and for all $u, v \in \mathcal{D}$ we have $u \sqsubseteq_{\mathcal{D}} v$ if, and only if, $u = \perp$ or $u = v$. All elements of $\mathcal{D} - \{\perp\}$ are prime. Concerning the full* powerset domain *$(\mathscr{P}(D), \subseteq)$, the compact elements are the finite subsets of D and the prime elements are the singleton sets. If D is countable, then $D_\perp$ and $\mathscr{P}(D)$ are ω-algebraic (and prime).*

1.1.5 Lattices

A *lattice* is a poset $\mathcal{S} = (S, \sqsubseteq)$ such that any two elements $s, s' \in S$ have a least upper bound $s \sqcup s'$ and a greatest lower bound $s \sqcap s'$ which are respectively called, in this context, *join* and *meet*. Then, $\sqsubseteq$ is definable from the meet or the join. A lattice is *complete* if any $A \subseteq S$ has a least upper bound (then all A's have also a greatest lower bound); in particular a complete lattice has a top and a bottom element.

The interval notation will have the obvious meaning; for example, given $s, s' \in S$, we let $[s, s'] = \{s'' \in S : s \sqsubseteq s'' \sqsubseteq s'\}$ and $[s, s'[= [s, s'] - \{s'\}$.

Given a lattice $\mathcal{S}$, and $S' \subseteq S$ we recall that: S' is a *chain* of $\mathcal{S}$ if it is totally ordered by $\sqsubseteq$, and S' is *discrete* in case its elements are pairwise incomparable. S' is *dense* in $\mathcal{S}$ if $\mathrm{card}(S') \geq 2$ and for all distinct $s, s' \in S'$ with $s \sqsubseteq s'$ we have that $]s, s'[\cap S'$ is non-empty, and $\mathcal{S}$ itself is a *dense* lattice if S is dense in $\mathcal{S}$. Finally, we will call S' an *antichain* of $\mathcal{S}$ if it does not contain the top element and the only possible common upper bound of two distinct $s, s' \in S'$ is the top element.

Following the terminology of [14], we say that a lattice $\mathcal{S}$ is *c-high* (resp. *c-wide*, *c-broad*), where c is a cardinal, if $\mathcal{S}$ has a chain (resp. a discrete subset, an antichain) of cardinality c. In particular, if $\mathcal{S}$ is c-broad, then it is also c-wide. Moreover, if $\mathcal{S}$ is complete and the set S is dense in $\mathcal{S}$, then $\mathcal{S}$ is $2^{\aleph_0}$-high.

1.1.6 Cartesian closed categories

In the following, $\mathbf{C}$ is a locally small[2] *Cartesian closed category* (*ccc*, for short) and A, B, C are arbitrary objects of $\mathbf{C}$.

[2] This means that $\mathbf{C}(A, B)$ is a set (called *homset*) for all objects A, B.

We denote by $A \times B$ the *categorical product* of A and B, and by $\pi_1 \in \mathbf{C}(A \times B, A)$, $\pi_2 \in \mathbf{C}(A \times B, B)$ the associated *projections*. Given a pair of arrows $f \in \mathbf{C}(C, A)$ and $g \in \mathbf{C}(C, B)$, $\langle f, g\rangle \in \mathbf{C}(C, A \times B)$ is the unique arrow such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$.

We will write $[A \Rightarrow B]$ for the *exponential object* and $ev_{AB} \in \mathbf{C}([A \Rightarrow B] \times A, B)$ for the *evaluation morphism* relative to A, B. Whenever A, B are clear from the context we will simply call it ev.

Moreover, for all objects A, B, C and arrow $f \in \mathbf{C}(C \times A, B)$ we denote by $\Lambda(f) \in \mathbf{C}(C, [A \Rightarrow B])$ the unique morphism such that $ev_{AB} \circ \langle \Lambda(f) \circ \pi_1, \pi_2\rangle = f$. Finally, $\mathbb{1}$ is the *terminal object* and $!_A$ is the only morphism in $\mathbf{C}(A, \mathbb{1})$.

We recall that in every ccc the following equalities hold:

$$\begin{array}{llrr}
\text{(pair)} & \langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle & \Lambda(f) \circ g = \Lambda(f \circ (g \times \mathrm{Id})) & \text{(Curry)} \\
\text{(beta)} & ev \circ \langle \Lambda(f), g\rangle = f \circ \langle \mathrm{Id}, g\rangle & \Lambda(ev) = \mathrm{Id} & \text{(Id-Curry)}
\end{array}$$

where $f_1 \times f_2$ is the *product map* defined by $\langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle$.

Given a set I and a family $(A_i)_{i\in I}$ of objects of $\mathbf{C}$, we denote the *I-indexed product* of $(A_i)_{i\in I}$ by $\Pi_{i\in I} A_i$. If the object $\Pi_{i\in I} A_i$ exists in the category $\mathbf{C}$ for all families $(A_i)_{i\in I}$ such that $\mathrm{card}(I) \leq \aleph_0$, then we say that $\mathbf{C}$ *has countable products*.

Let us fix now an object A. For all sets I, we write A^I for the I-indexed product of an adequate number of copies of A, $\pi_i^I \in \mathbf{C}(A^I, A)$ for the projection on the i-th component, and Π_J^I, where $J \subseteq I$, for $\langle \pi_i^I\rangle_{i\in J} \in \mathbf{C}(A^I, A^J)$.

Remark 1.1.3. *For all sets I, J such that $I \subseteq J$ we have:*

(i) $\pi_i^J = \pi_i^I \circ \Pi_I^J$ *for all* $i \in I$,

(ii) $\Pi_{I\cup\{i\}}^{J\cup\{i\}} = \Pi_I^J \times \mathrm{Id}$ *for all* $i \notin J \cup I$.

We say that the ccc $\mathbf{C}$ *has enough points* if, for all objects A, B and morphisms $f, g \in \mathbf{C}(A, B)$, whenever $f \neq g$, there exists a morphism $h \in \mathbf{C}(\mathbb{1}, A)$ such that $f \circ h \neq g \circ h$. Similarly, an object A *has enough points* if the above property holds for all $f, g \in \mathbf{C}(A, A)$.

The ccc $\mathbf{C}$ is *cpo-enriched* if every homset is a cpo $(\mathbf{C}(A, B), \sqsubseteq_{(A,B)}, \perp_{(A,B)})$, composition is continuous (see Subsection 1.4.2 later on), pairing and currying are monotonic, and the following strictness conditions hold:

$$\text{(l-strict)} \quad \perp \circ f = \perp, \qquad ev \circ \langle \perp, f\rangle = \perp \quad \text{(ev-strict)}.$$

Lemma 1.1.4. *[4, Lemma 6.1.3] In a cpo-enriched ccc pairing and currying are continuous.*

1.2 The untyped λ-calculus

1.2.1 λ-terms

The two primitive notions of the λ-calculus are *application*, the operation of applying a function to an argument, and *lambda abstraction*, the process of forming a function from the "expression" defining it.

The set Λ of λ-terms over a countable set Var of variables is constructed as usual: every variable is a λ-term; if M and N are λ-terms, then so are (MN) and $\lambda x.M$ for each variable x. Concerning specific λ-terms we set:

$$\mathbf{I} \equiv \lambda x.x, \quad \mathbf{1} \equiv \lambda xy.xy, \quad \mathbf{T} \equiv \lambda xy.x, \quad \mathbf{F} \equiv \lambda xy.y,$$
$$\mathbf{S} \equiv \lambda xyz.xz(yz), \quad \delta \equiv (\lambda x.xx), \quad \Omega \equiv \delta\delta,$$

the symbol $\equiv$ denotes syntactical equality. A more traditional notation for $\mathbf{T}$, when it is not viewed as a boolean, is $\mathbf{K}$.

An occurrence of a variable x in a λ-term is *bound* if it lies within the scope of a lambda abstraction λx; otherwise it is called *free*. The set of free variables of M is denoted by $\mathrm{FV}(M)$. A λ-term without free variables is said to be *closed*. The set of closed λ-terms will be denoted by Λ^o.

We denote by $M[N/x]$ the result of substituting the λ-term N for all free occurrences of x in M subject to the usual proviso about renaming bound variables in M to avoid capture of free variables in N.

The basic axioms of λ-calculus are the following (here M and N are arbitrary λ-terms and x, y are variables):

(α) $\lambda x.M = \lambda y.M[y/x]$, for any variable y that does not occur free in M;

(β) $(\lambda x.M)N = M[N/x]$.

The rules for deriving equations from instances of (α) and (β) are the usual ones from equational calculus asserting that equality is a congruence for application and abstraction.

Extensional λ-calculus adds another axiom, which equates all the λ-terms having the same extensional behaviour:

(η) $\lambda x.Mx = M$, where x does not occur free in M.

If two λ-terms are provably equal using the rule (α) we say that they are *α-convertible* or *α-equivalent* (and similarly for (β) and (η)). Throughout this thesis we will identify all α-convertible λ-terms. We will denote β-conversion by λ_β and $\beta\eta$-conversion by $\lambda_{\beta\eta}$. By applying the rules (β) and (η) only from left to right we obtain, respectively, the β-, η-reduction. In general, given an R-reduction rule, we write $M \to_R N$ (resp. $M \twoheadrightarrow_R N$) if M reduces to N in one step (resp. zero or several steps) of R-reduction.

Contexts are terms with some occurrences of algebraic variables (also called "*holes*"), denoted by ξ_i. A *context* is inductively defined as follows: ξ_i is a context, x is a context for every variable x, if C_1 and C_2 are contexts then so are C_1C_2 and $\lambda x.C_1$ for each variable x. If $M_1, \dots, M_n$ are λ-terms we will write $C[M_1, \dots, M_n]$ for the context $C[\xi_1, \dots, \xi_n]$ where all the occurrences of ξ_i have been simultaneously replaced by M_i. When $n = 1$ we simply write $C[-]$ instead of $C[\xi_1]$.

A λ-term M is a *head normal form* (*hnf*, for short) if $M \equiv \lambda x_1 \dots x_n.yM_1 \cdots M_k$ for some $n, k \geq 0$. The *principal hnf* of a λ-term M is the hnf obtained from M by head reduction [8, Def. 8.3.10]. Given two hnf's $M \equiv \lambda x_1 \dots x_n.yM_1 \cdots M_k$ and $N \equiv \lambda x_1 \dots x_{n'}.y'N_1 \cdots N_{k'}$ we say that they are *equivalent* if y is free/bound in M whenever y' is in N, $y \equiv y'$ and $k - n = k' - n'$.

A λ-term M is called *solvable* if it is β-convertible to a hnf, otherwise M is called *unsolvable*. We will denote by $\mathcal{U}$ the set of all unsolvable λ-terms.

Two closed λ-terms M and N are *separable* if there exists $S \in \Lambda^o$ such that $SM = \mathbf{T}$ and $SN = \mathbf{F}$. Otherwise they are *inseparable*. There exist simple criteria implying separability or inseparability.

Proposition 1.2.1. *(Böhm) [8, Lemma 10.4.1, Thm. 10.4.2]*

(i) Two hnf's are separable or equivalent (as hnf's),

(ii) two normal λ-terms are separable or η-equivalent.

1.2.2 Böhm trees

The *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M is a finite or infinite labelled tree. If M is unsolvable, then $\mathrm{BT}(M) = \perp$, that is, $\mathrm{BT}(M)$ is a tree with a unique node labelled by $\perp$. If M is solvable and $\lambda x_1 \dots x_n.yM_1 \cdots M_k$ is its principal hnf, then:

$$\mathrm{BT}(M) = \quad \begin{array}{ccc} & \lambda x_1 \dots x_n.y & \\ \swarrow & & \searrow \\ \mathrm{BT}(M_1) & \cdots & \mathrm{BT}(M_k) \end{array}$$

We call BT the set of all Böhm trees. Given $t, t' \in BT$ we define $t \subseteq_{BT} t'$ if, and only if, t results from t' by cutting off some subtrees. It is easy to verify that $(BT, \subseteq_{BT})$ is an ω-algebraic cpo. The relation $\subseteq_{BT}$ is transferred on λ-terms by setting $M \sqsubseteq_{BT} N$ if $\mathrm{BT}(M) \subseteq_{BT} \mathrm{BT}(N)$.

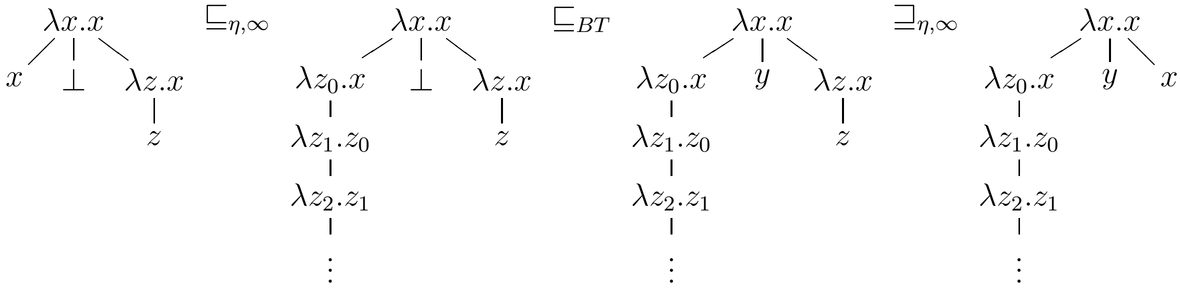
Moreover, we write $M \sqsubseteq_{\eta,\infty} N$ if $\mathrm{BT}(N)$ is a (possibly infinite) η-expansion of $\mathrm{BT}(M)$ (see [8, Def. 10.2.10]). For example, let us consider $J \equiv \Theta(\lambda jxy.x(jy))$, where Θ is Turing's fixpoint combinator [104]. Then $x \sqsubseteq_{\eta,\infty} Jx$ (see [8, Ex. 10.2.9]), since

$$\begin{aligned} Jx &=_{\lambda_\beta} \lambda z_0.x(Jz_0) =_{\lambda_\beta} \lambda z_0.x(\lambda z_1.z_0(Jz_1)) \\ &=_{\lambda_\beta} \lambda z_0.x(\lambda z_1.z_0(\lambda z_2.z_1(Jz_2))) =_{\lambda_\beta} \dots \end{aligned}$$

Thus, the Böhm tree of Jx is the following:

$$\mathrm{BT}(Jx) = \quad \begin{array}{c} \lambda z_0.x \\ | \\ \lambda z_1.z_0 \\ | \\ \lambda z_2.z_1 \\ | \\ \vdots \end{array}$$

Using $\sqsubseteq_{\eta,\infty}$, we can define another relation on λ-terms which will be useful in Subsection 2.3.8. For all $M, N \in \Lambda$ we set $M \precsim_\eta N$ if there exist M', N' such that $M \sqsubseteq_{\eta,\infty} M' \sqsubseteq_{BT} N' \sqsupseteq_{\eta,\infty} N$. Let us provide an example of this situation:



Finally, we write $M \simeq_\eta N$ for $M \precsim_\eta N \precsim_\eta M$. In the next section we will see an alternative characterization of $\simeq_\eta$ in terms of a λ-theory called $\mathcal{H}^*$.

1.2.3 The lattice of λ-theories

A λ-*theory* is a congruence on Λ, with respect to the operators of lambda abstraction and application, which contains λ_β; it can also be seen as a (special) set of equations between λ-terms. A λ-theory $\mathcal{T}$ is *consistent* if $\mathcal{T} \neq \Lambda \times \Lambda$, and *extensional* if it contains the equation $\mathbf{I} = \mathbf{1}$ or, equivalently, if $\lambda_{\beta\eta} \subseteq \mathcal{T}$.

The set of all λ-theories is naturally equipped with a structure of complete lattice, hereafter denoted by $\lambda\mathscr{T}$, with meet defined as set-theoretical intersection. The join of two λ-theories $\mathcal{T}$ and $\mathcal{S}$ is the least equivalence relation including $\mathcal{T} \cup \mathcal{S}$. It is clear that λ_β is the least element of $\lambda\mathscr{T}$, while the unique inconsistent λ-theory $\Lambda \times \Lambda$ is the top element of $\lambda\mathscr{T}$. Moreover, $\lambda_{\beta\eta}$ is the least extensional λ-theory. It is well known that $\lambda\mathscr{T}$ is $2^{\aleph_0}$-high and $2^{\aleph_0}$-broad (hence, $2^{\aleph_0}$-wide) [8, Ch. 16.3, 17.1].

Two λ-theories $\mathcal{T}, \mathcal{T}'$ are *incompatible* if their join is the inconsistent λ-theory. Hence an *antichain* of $\lambda\mathscr{T}$ is a set of λ-theories which are pairwise incompatible.

The λ-theory *generated (or axiomatized) by* a set of equations is the least λ-theory containing it.

As a matter of notation, both $\mathcal{T} \vdash M = N$ and $M =_{\mathcal{T}} N$ stand for $M = N \in \mathcal{T}$, $[M]_{\mathcal{T}}$ denotes the $\mathcal{T}$-equivalence class of M and if $V \subseteq \Lambda$ we write $V/\mathcal{T}$ for the quotient set of V modulo $\mathcal{T}$.

Given a λ-theory $\mathcal{T}$, we say that $\mathcal{T}$ is:

- *recursively enumerable* (*r.e.*, for short) if the set of Gödel numbers of all pairs of $\mathcal{T}$-equivalent λ-terms is r.e.,

- *semi-sensible* if it contains no equations of the form $S = U$ where S is solvable and U unsolvable,

- *sensible* if it contains all the equations between unsolvable λ-terms.

The λ-theory $\mathcal{H}$, generated by equating all the unsolvable λ-terms, is consistent by [8, Thm. 16.1.3]. $\mathcal{H}$ admits a unique maximal consistent extension [8, Thm. 16.2.6] $\mathcal{H}^*$, which is an extensional λ-theory and can be characterized as follows:

$$M =_{\mathcal{H}^*} N \iff M \simeq_\eta N \qquad \text{see [8, Thm. 16.2.7].}$$

A λ-theory $\mathcal{T}$ is semi-sensible if, and only if, $\mathcal{T} \subseteq \mathcal{H}^*$ and it is sensible if, and only if, $\mathcal{H} \subseteq \mathcal{T}$ (see Section 10.2 and Section 16.2 in [8]). Consistent sensible λ-theories are semi-sensible and never r.e. [8, Sec. 17.1].

The λ-theory $\mathcal{B}_\mathcal{T}$, generated by equating all λ-terms with the same Böhm tree, is sensible, non-extensional and distinct from $\mathcal{H}$ and $\mathcal{H}^*$, so that $\mathcal{H} \subsetneq \mathcal{B}_\mathcal{T} \subsetneq \mathcal{H}^*$.

We recall here some results about the "size" of various subsets of $\lambda\mathscr{T}$ which have been shown either in [8] or in [16].

Theorem 1.2.2. *(Barendregt [8, Ch. 16.3, 17.1])*

(i) The set of all r.e. λ-theories is dense in $\lambda\mathscr{T}$, so $\lambda\mathscr{T}$ is $2^{\aleph_0}$-high, more generally:

(i') If $\mathcal{T}, \mathcal{S}$ are r.e. λ-theories, then the interval $[\mathcal{T}, \mathcal{S}]$ is $2^{\aleph_0}$-high.

(ii) The set of all sensible λ-theories is $2^{\aleph_0}$-high and $2^{\aleph_0}$-wide.

Theorem 1.2.3. *(Berline and Salibra [16]) There exists an r.e. λ-theory $\mathcal{T}$ such that $[\mathcal{T}[= \{\mathcal{S} : \mathcal{T} \subseteq \mathcal{S}\}$ is moreover $2^{\aleph_0}$-broad.*

For proving Theorem 1.2.2 the notion of $\mathcal{T}$-easy term is useful.

A λ-term U is *$\mathcal{T}$-easy* when, for every fixed closed λ-term M, the λ-theory generated by $\mathcal{T} \cup \{U = M\}$ is consistent [8, Prop. 15.3.9]. If U is λ_β-easy, then we simply say that U is *easy*.

As a matter of notation, we denote by $\Lambda_{\mathcal{T}-easy}$ the set of all $\mathcal{T}$-easy terms. It is clear that $\Lambda_{\mathcal{T}-easy} \subseteq \mathcal{U}$. Moreover, if $\mathcal{T}$ is r.e. then $\Lambda_{\mathcal{T}-easy} \neq \emptyset$ by [8, Prop. 17.1.9].

1.3 Models of the untyped λ-calculus

For our purposes it will be convenient to work mainly with two notions of model of λ-calculus. The former is connected with category theory (*categorical models* [8, Sec. 5.5]) and the latter is related to combinatory algebras (λ-*models* [8, Sec. 5.2]). Sometimes, we will also use a third notion of model (*environment λ-models*), which is an alternative description of λ-models and is convenient when dealing with the interpretation of λ-terms in a λ-model. The notions of categorical model and of λ-model are also tightly linked as we will see in Subsection 1.3.5 and in Section 2.2.

1.3.1 Categorical models

A *categorical model* of λ-calculus is a *reflexive object* of a Cartesian closed category $\mathbf{C}$, i.e., a triple $\mathscr{U} = (U, \mathrm{Ap}, \lambda)$ such that U is an object of $\mathbf{C}$, and $\lambda \in \mathbf{C}([U \Rightarrow U], U)$

and $\mathrm{Ap} \in \mathbf{C}(U, [U \Rightarrow U])$ satisfy $\mathrm{Ap} \circ \lambda = \mathrm{Id}_{[U \Rightarrow U]}$. In this case we write $[U \Rightarrow U] \lhd U$ and we say that (Ap, λ) is a *retraction pair*. When moreover $\lambda \circ \mathrm{Ap} = \mathrm{Id}_U$, the model $\mathscr{U}$ is called *extensional*.

Given a λ-term M and a subset $I \subseteq \mathrm{Var}$, we say that I is *adequate for* M if I contains all the free variables of M. We simply say that I is *adequate* whenever M is clear from the context.

Let $\mathscr{U} = (U, \mathrm{Ap}, \lambda)$ be a categorical model. For all $M \in \Lambda$ and for all adequate $I \subseteq_{\mathrm{f}} \mathrm{Var}$, the *interpretation of* M (in I) is a morphism $|M|_I \in \mathbf{C}(U^I, U)$ defined by structural induction on M as follows:

- If $M \equiv x$, then $|x|_I = \pi^I_x$;
- If $M \equiv NP$, then by inductive hypothesis we have defined $|N|_I, |P|_I \in \mathbf{C}(U^I, U)$. Hence, we set $|NP|_I = ev \circ \langle \mathrm{Ap} \circ |N|_I, |P|_I \rangle \in \mathbf{C}(U^I, U)$;
- If $M \equiv \lambda x.N$, by inductive hypothesis we have defined $|N|_{I \cup \{x\}} \in \mathbf{C}(U^{I \cup \{x\}}, U)$ where we suppose that x does not belong to I. Thus, we set $|\lambda x.N|_I = \lambda \circ \Lambda(|N|_{I \cup \{x\}})$;

We refer to [8, Ch. 5] for the proof of the soundness of this definition and of the fact that, if M and N are β-equivalent, then $|M|_I = |N|_I$ for every $I \subseteq_{\mathrm{f}} \mathrm{Var}$ adequate for M and N. The reflexive object $\mathscr{U}$ is extensional, if and only if, $|M|_I = |N|_I$ holds whenever M and N are $\beta\eta$-equivalent.

When $\mathbf{C}$ is a *concrete*[3] category having enough points, the categorical interpretation admits a simpler presentation, which will be recalled in Subsection 1.4.1.

1.3.2 Combinatory logic and combinatory algebras

Combinatory logic is a formalism for writing expressions which denote functions. Combinators are designed to perform the same tasks as λ-terms, but without using bound variables. Schönfinkel and Curry discovered that a formal system of combinators, having almost the same expressive power of the λ-calculus, can be based on only two primitive combinators $\underline{\mathbf{k}}$ and $\underline{\mathbf{s}}$ [93, 40].

The terms of combinatory logic, namely *combinatory terms*, are defined by induction as follows: every variable x is a combinatory term; the constants $\underline{\mathbf{k}}$ and $\underline{\mathbf{s}}$ are combinatory terms; if A, B are combinatory terms, then also AB is a combinatory term. The constants $\underline{\mathbf{k}}$ and $\underline{\mathbf{s}}$ are called *basic combinators*; the derived combinators $\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{t}, \mathbf{f}$ are defined respectively as $\mathbf{i} \equiv \underline{\mathbf{skk}}$, $\boldsymbol{\varepsilon} \equiv \underline{\mathbf{s}}(\underline{\mathbf{ki}})$, $\mathbf{t} \equiv \underline{\mathbf{k}}$ and $\mathbf{f} \equiv \underline{\mathbf{ki}}$.

An *applicative structure* is an algebra with a binary operation $\cdot$ which we call *application*. We may write it infix as $s \cdot t$, or even drop it entirely and write st.

[3] Roughly speaking, a *concrete category* is a category whose objects are sets (possibly carrying some additional structure) and whose morphisms are (special) functions.

As usual, application associates to the left; hence xyz means $(xy)z$. An applicative structure $\mathscr{A} = (A, \cdot)$ is *extensional* if the following axiom holds:

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow x = y).$$

A *combinatory algebra* is an algebra $\mathscr{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ where $(C, \cdot)$ is an applicative structure and $\mathbf{k}, \mathbf{s}$ are two distinguished elements of C such that $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$ for all $x, y, z \in C$. Combinatory terms can be interpreted into combinatory algebras; in particular $\underline{\mathbf{k}}$ and $\underline{\mathbf{s}}$ are interpreted by $\mathbf{k}$ and $\mathbf{s}$, respectively. Hence hereafter, to lighten the notation, we omit the underlinings and write $\mathbf{k}, \mathbf{s}$ also to denote the constants. It is easy to check that every combinatory algebra satisfies the identities $\mathbf{i}x = x$, $\boldsymbol{\varepsilon}xy = xy$, $\mathbf{t}xy = x$ and $\mathbf{f}xy = y$. See [40], for a full treatment.

Let $\mathscr{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ and $\mathscr{C}' = (C', \cdot', \mathbf{k}', \mathbf{s}')$ be two combinatory algebras. A function $\Psi : C \to C'$ is a *morphism* from $\mathscr{C}$ to $\mathscr{C}'$ if $\Psi(u \cdot v) = \Psi(u) \cdot' \Psi(v)$ and $\Psi(\mathbf{k}) = \mathbf{k}'$, $\Psi(\mathbf{s}) = \mathbf{s}'$; it is an *isomorphism* if Ψ is, moreover, a bijection.

We say that $c \in C$ *represents* the function $f : C \to C$ (and that f is *representable* by c) if $cz = f(z)$ for all $z \in C$. We will call $c, d \in C$ *extensionally equal* when they represent the same function in $\mathscr{C}$. For example c and $\boldsymbol{\varepsilon}c$ are always extensionally equal.

For each variable x one can perform a transformation λ^*x of combinatory terms as follows: $\lambda^*x.x = \mathbf{i}$. Let t be a combinatory term different from x. If x does not occur in t, we define $\lambda^*x.t = \mathbf{k}t$. Otherwise, t is of the form rs where r and s are combinatory terms, at least one of which contains x; in this case we define $\lambda^*x.t = \mathbf{s}(\lambda^*x.r)(\lambda^*x.s)$. It is well known that x does not occur in $\lambda^*x.t$ and that, for every combinatory algebra $\mathscr{C}$ and combinatory term u, we have:

$$\mathscr{C} \models (\lambda^*x.t)u = t[u/x],$$

where the combinatory term $t[u/x]$ is obtained by substituting u for x in t. With the help of λ^* it is possible to translate any λ-term into a combinatory term and then to interpret it in any combinatory algebra $\mathscr{C}$.

If t is a combinatory term and $x_1, x_2 \dots, x_n$ (with $n \geq 2$) are variables then $\lambda^*x_1x_2 \dots x_n.t$ is defined by induction as follows: $\lambda^*x_1x_2 \dots x_n.t \equiv \lambda^*x_1.(\lambda^*x_2 \dots x_n.t)$.

For two combinatory terms t and u, we define the *pair* $[t, u] \equiv \lambda^*z.ztu$ and, for every sequence $t_1, \dots, t_n$ (with $n \geq 3$), we define $[t_1, \dots, t_n] \equiv [t_1, [t_2, \dots, t_n]]$.

1.3.3 λ-models

The axioms of the subclass of combinatory algebras which define λ-*models* were expressly chosen to make coherent the definition of the interpretation of λ-terms (see [8, Def. 5.2.7]).

A combinatory algebra $\mathscr{C}$ satisfying the five combinatory axioms of Curry [8, Thm. 5.2.5] is called a λ-*algebra*; $\mathscr{C}$ is a λ-*model* if, moreover, it satisfies the *Meyer-Scott axiom* (also known as "weak extensionality"):

$$\forall x \forall y (\forall z (xz = yz) \Rightarrow \boldsymbol{\varepsilon}x = \boldsymbol{\varepsilon}y).$$

The combinator $\boldsymbol{\varepsilon}$ becomes an inner choice operator, that makes coherent the interpretation of a lambda abstraction. Indeed, given any c, the element $\boldsymbol{\varepsilon}c$ is in the same equivalence class as c with respect to extensional equality; and, by the Meyer-Scott axiom, $\boldsymbol{\varepsilon}c = \boldsymbol{\varepsilon}d$ for every d extensionally equal to c.

Given two λ-models $\mathscr{C}, \mathscr{C}'$, a function $\Psi : C \to C'$ is a *morphism* from $\mathscr{C}$ to $\mathscr{C}'$ if it is a morphism of combinatory algebras; Ψ is an *isomorphism* if it is, furthermore, bijective.

1.3.4 Environment λ-models

The first-order definition of λ-models has the advantage that it gives a model theoretic status to the models of λ-calculus, but it has the disadvantage that the interpretation of λ-terms is awkward and difficult to handle in practice.

For this reason it will be sometimes convenient to view λ-models as "environment λ-models[4]" which have been introduced by Hindley and Longo in [56].

Given a set C, an *environment* with values in C is a total function $\rho : \mathrm{Var} \to C$, where Var is the set of variables of λ-calculus. For every $x \in \mathrm{Var}$ and $c \in C$ we denote by $\rho[x := c]$ the environment ρ' which coincides with ρ, except on x, where ρ' takes the value c. We let Env_C be the set of environments with values in C.

An *environment λ-model* is a pair $\mathscr{S} = (\mathscr{A}, [\![-]\!])$ where, $\mathscr{A}$ is an applicative structure and the map $[\![-]\!] : \Lambda \times Env_A \to A$ satisfies the following conditions:

(i) $[\![x]\!]_\rho = \rho(x)$,

(ii) $[\![PQ]\!]_\rho = [\![P]\!]_\rho \cdot [\![Q]\!]_\rho$,

(iii) $[\![\lambda x.P]\!]_\rho \cdot a = [\![P]\!]_{\rho[x:=a]}$,

(iv) $\rho\restriction_{\mathrm{FV}(M)} = \rho'\restriction_{\mathrm{FV}(M)} \Rightarrow [\![M]\!]_\rho = [\![M]\!]_{\rho'}$,

(v) $\forall a \in A,\ [\![M]\!]_{\rho[x:=a]} = [\![N]\!]_{\rho[x:=a]} \Rightarrow [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$.

Barendregt has proved in [8, Thm. 5.3.6] that the category of environment λ-algebras (i.e., environment λ-models possibly not satisfying the condition (v) above), and that of λ-algebras are isomorphic. Moreover, environment λ-models correspond exactly to λ-models under this isomorphism (and also the corresponding notions of interpretation coincide). Hence, in the following, we will also use $[\![-]\!]$ for the interpretation of λ-terms in a λ-model.

1.3.5 Equivalence between categorical models and λ-models

The notions of λ-model and of categorical model are equivalent in the following sense (see, e.g., [8, Ch. 5] and [5, Sec. 9.5]). Given a λ-model $\mathscr{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ we may build a ccc in which C is a reflexive object having enough points; conversely if

[4] They are also known in the literature as *syntactical λ-models*.

$\mathscr{U} = (U, \mathrm{Ap}, \lambda)$ is a reflexive object having enough points in a ccc $\mathbf{C}$, then the set $\mathbf{C}(\mathbb{1}, U)$ can be endowed with a structure of λ-model.

If the reflexive object $\mathscr{U}$ does not have enough points then, in general, $\mathbf{C}(\mathbb{1}, U)$ cannot be turned into a λ-model, but only into a λ-algebra. In Section 2.2 we will show that this apparent mismatch can be avoided by changing $\mathbf{C}(\mathbb{1}, U)$ for another set of morphisms with codomain U.

It is of course important that our construction, as well as the classical one, preserves the equalities between (the denotations of) the λ-terms, in the sense that two λ-terms having the same interpretation in a categorical model will have the same interpretation in the associated λ-model and *vice versa*.

1.3.6 λ-theories of λ-models

Given a λ-model $\mathscr{C}$, the *equational theory of* $\mathscr{C}$ is defined by:

$$\mathrm{Th}(\mathscr{C}) = \{M = N : [\![M]\!]_\rho = [\![N]\!]_\rho \text{ for all } \rho \in Env_C\}.$$

The λ-model $\mathscr{C}$ is called *trivial* if $\mathrm{card}(C) = 1$, which is equivalent to saying that $\mathrm{Th}(\mathscr{C})$ is inconsistent, and is called *sensible* (resp. *semi-sensible*) if $\mathrm{Th}(\mathscr{C})$ is. In the sequel we will of course only be interested in non-trivial λ-models.

The *term model* $\mathscr{M}_\mathcal{T}$ of a λ-theory $\mathcal{T}$ (viewed as a λ-model) consists of the set of the equivalence classes of λ-terms modulo $\mathcal{T}$ together with the operation of application on the equivalence classes (see [8, Def. 5.2.11]) and the obvious interpretations of $\mathbf{k}$ and $\mathbf{s}$. By [8, Cor. 5.2.13(ii)] $\mathscr{M}_\mathcal{T}$ is a λ-model which represents the λ-theory $\mathcal{T}$. In the following, we will say that a model of λ-calculus is *syntactical* if its construction depends on the syntax of λ-calculus; in particular, all term models are syntactical.

We define various notions of representability of λ-theories in classes of models.

Definition 1.3.1. *Given a class $\mathbb{C}$ of λ-models and a λ-theory $\mathcal{T}$, we say that:*

(i) $\mathbb{C}$ represents $\mathcal{T}$ *if there is some $\mathscr{C} \in \mathbb{C}$ such that* $\mathrm{Th}(\mathscr{C}) = \mathcal{T}$.

(ii) $\mathbb{C}$ omits $\mathcal{T}$ *if $\mathbb{C}$ does not represent $\mathcal{T}$.*

(iii) $\mathbb{C}$ *is* complete *for $S \subseteq \lambda\mathscr{T}$ if $\mathbb{C}$ represents all the elements of S.*

(iv) $\mathbb{C}$ *is* incomplete *if it omits a consistent λ-theory.*

We will denote by $\lambda\mathbb{C}$ the set of λ-theories which are represented in $\mathbb{C}$.

A *partially ordered λ-model* is a pair $(\mathscr{C}, \sqsubseteq)$ where $\mathscr{C}$ is a λ-model and $\sqsubseteq$ is a partial order on C which makes the application operator of $\mathscr{C}$ monotonic.

Every partially ordered λ-model $(\mathscr{C}, \sqsubseteq)$ induces not only an equational theory but also an order theory, called the *order theory of* $\mathscr{C}$, which is defined as follows:

$$\mathrm{Th}_\sqsubseteq(\mathscr{C}) = \{M \sqsubseteq N : [\![M]\!]_\rho \sqsubseteq [\![N]\!]_\rho \text{ for all } \rho \in Env_C\}.$$

Since the equational theories are the most frequently considered, we will often call them simply "λ-theories".

1.4 The main semantics

The models of λ-calculus are classified into "semantics", according to the nature of the morphisms of their underlying categories. In this section we give a sketchy presentation of the three "main semantics": the Scott-continuous semantics [95], the stable semantics [17, 18, 51] and the strongly stable semantics [25]; the last two are strengthenings of the Scott-continuous semantics. No more details on these semantics than what is stated in this section should be necessary for reading this thesis and, if necessary, [13] contains a more detailed presentation.

We will call the models living inside these semantics: *continuous*[5], *stable* and *strongly stable* models. All these semantics are structurally and equationally rich: in particular, in each of them it is possible to build up $2^{\aleph_0}$ models having pairwise distinct, and even incomparable, λ-theories.

1.4.1 Simplifications when working in the main semantics

The main semantics correspond to Cartesian closed categories whose objects are cpo's, possibly satisfying some constraints, and morphisms are (special) monotonous functions between these cpo's. Under these hypotheses the λ-terms can be interpreted in a categorical model $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$ as elements of $\mathcal{D}$, instead of morphisms, by using environments with values in $\mathcal{D}$.

Note that $Env_{\mathcal{D}}$, ordered pointwise, is a cpo whose bottom element is the environment $\rho_\perp$ mapping everybody to $\perp_{\mathcal{D}}$. Note also that $\rho \in Env_{\mathcal{D}}$ is compact if, and only if, $\mathrm{rg}(\rho) \subseteq \mathcal{K}(\mathcal{D})$ and $\rho(x) \neq \perp_{\mathcal{D}}$ only for a finite number of $x \in \mathrm{Var}$.

The interpretation $|M| : Env_{\mathcal{D}} \to \mathcal{D}$ of a λ-term M in $\mathcal{D}$ (relatively to $\mathscr{M}$) is defined by structural induction on M, as follows:

- $|x|_\rho = \rho(x)$,
- $|MN|_\rho = \mathrm{Ap}(|M|_\rho)(|N|_\rho)$,
- $|\lambda x.M|_\rho = \lambda(d \in \mathcal{D} \mapsto |M|_{\rho[x:=d]})$.

This interpretation function generalizes to terms with parameters in $\mathcal{D}$ (where an element of $\mathcal{D}$ is interpreted by itself) and to $\Lambda_\perp$ by setting $|\perp|_\rho = \perp_{\mathcal{D}}$ for all $\rho \in Env_{\mathcal{D}}$. The set of all open (resp. closed) terms with parameters in $\mathcal{D}$ is denoted by $\Lambda(\mathcal{D})$ (resp. $\Lambda^o(\mathcal{D})$). If $M \in \Lambda^o(\mathcal{D})$ we write $|M|$ instead of $|M|_\rho$ since, clearly, $|M|_\rho$ only depends on the value of ρ on the free variables of M; in particular $|M| = |M|_{\rho_\perp}$. In case of ambiguity we denote by $|M|^{\mathscr{M}}$ the interpretation of $M \in \Lambda^o(\mathcal{D})$ in $\mathscr{M}$.

In this framework, the equivalence claimed in Subsection 1.3.5 between the categorical and the algebraic notions of model of λ-calculus becomes very simple.

[5] Note that, for many authors, a "continuous model" is a model satisfying a (weak) Approximation Theorem (see, e.g., [8, Sec. 19.3]). Here, it is just a shorthand for "a model living in the Scott-continuous semantics".

Every categorical model $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$ can be viewed as the combinatory algebra $\mathscr{C} = (\mathcal{D}, \cdot, \mathbf{k}, \mathbf{s})$ where $\cdot$ is an alternative notation for Ap and $\mathbf{k}, \mathbf{s}$ are, respectively, the interpretation of $\mathbf{K}, \mathbf{S}$ in $\mathscr{M}$. It is also easy to check that the interpretations of λ-terms in $\mathscr{M}$ and in $\mathscr{C}$ coincide, i.e., $|M|_\rho = [\![M]\!]_\rho$ for all $M \in \Lambda$ and $\rho \in Env_{\mathcal{D}}$.

Convention 1.4.1. *In this context, no confusion can arise if we call the categorical models simply "models" and we keep "λ-models" for the combinatory algebras.*

Moreover, $\mathcal{D}$ is a cpo where $\sqsubseteq_{\mathcal{D}}$ is compatible with application and abstraction, hence every $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$ is a partially ordered model. We will prove in Section 4.4.3 that all the λ-models living in the main semantics are *simple* (combinatory) *algebras*. Hence all the morphisms between them are embeddings, that is to say, inclusions up to isomorphisms. Concerning isomorphism, we have:

Theorem 1.4.2. *Let $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda), \mathscr{M}' = (\mathcal{D}', \mathrm{Ap}', \lambda')$ be two models living in the main semantics, $\mathscr{C}, \mathscr{C}'$ be the corresponding λ-models, and $\Psi : \mathcal{D} \to \mathcal{D}'$ be a bijection. The following assertions are equivalent:*

(i) Ψ is an isomorphism between $\mathscr{C}$ and $\mathscr{C}'$,

(ii) for all $M \in \Lambda^o$, $\Psi(|M|^{\mathscr{M}}) = |M|^{\mathscr{M}'}$.

We will hence also speak in this case of an *isomorphism between the models $\mathscr{M}$ and $\mathscr{M}'$*, and of an *automorphism*, when $\mathscr{M} = \mathscr{M}'$ (and hence $\mathscr{C} = \mathscr{C}'$).

The next remark is clear from the definition of the equational theory of a model, and it is also consequence of Theorem 1.4.2.

Remark 1.4.3. *If $\mathscr{M}$ and $\mathscr{M}'$ are isomorphic models, then* $\mathrm{Th}(\mathscr{M}) = \mathrm{Th}(\mathscr{M}')$.

It has been noticed (see, e.g., [13]) that all the usual models living in the main semantics admit a uniform presentation as "webbed models". Roughly speaking, a webbed model is a model such that $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ is a subdomain of $(\mathscr{P}(D), \subseteq)$ for some set D. One of the major interests of working with classes of webbed models is that recursive equations on domains are replaced by simple set-theoretical (recursive) equations on their web.

1.4.2 Scott-continuous semantics

Scott-continuous semantics corresponds to the category whose objects are cpo's and morphisms are Scott-continuous functions. Given two cpo's $\mathcal{D}, \mathcal{D}'$ a function $f : \mathcal{D} \to \mathcal{D}'$ is Scott-continuous if it is monotonous and $f(\bigsqcup A) = \bigsqcup f^+(A)$ for all non-empty directed $A \subseteq \mathcal{D}$. We will denote by $[\mathcal{D} \to \mathcal{D}']$ the set of all Scott continuous functions from $\mathcal{D}$ into $\mathcal{D}'$ considered as a cpo by pointwise ordering. If $\mathcal{D}, \mathcal{D}'$ are Scott domains then $[\mathcal{D} \to \mathcal{D}']$ is a Scott domain.

For each function $f \in [\mathcal{D} \to \mathcal{D}']$, we define the *trace of f* as $tr(f) = \{(d, e) \in \mathcal{K}(\mathcal{D}) \times \mathcal{K}(\mathcal{D}') : e \sqsubseteq_{\mathcal{D}'} f(d)\}$. If $\mathcal{D}'$ is prime algebraic it is more interesting to work with $Tr(f) = \{(d,p) \in \mathcal{K}(\mathcal{D}) \times \mathcal{P}r(\mathcal{D}') : p \sqsubseteq_{\mathcal{D}'} f(d)\}$. Hence, if $\mathcal{D}, \mathcal{D}' = (\mathscr{P}(D), \subseteq)$, as it will be the case for graph models, we can use $Tr(f) = \{(a,\alpha) \in D^* \times D : \alpha \in f(a)\}$.

The classes of webbed models living in Scott-continuous semantics which are interesting for our purposes are, using the terminology of [13]: graph models, K-models [72], pcs-models [13] and filter models [38]. In this thesis we are mainly interested in graph models, first because they constitute the simplest (but very rich) class of continuous models, and second because all the other classes of webbed models can be seen as variations of this one. Hence, in the next subsection we will provide a more detailed description of graph models.

We just recall now some basic facts about the other classes of continuous webbed models we are interested in.

The class of K-models, isolated by Krivine [72], contains all graph models and also extensional models, such as, e.g., Scott's $\mathscr{D}_\infty$. The domain underlying a K-model is the complete lattice $(\mathfrak{S}(D), \subseteq)$, where $\mathfrak{S}(D)$ is the set of all the initial segments of some preordered set $(D, \preceq)$. Note that graph models can be seen as K-models having a trivial preorder on D, indeed in this case $\mathfrak{S}(D) = \mathscr{P}(D)$.

The class of pcs-models, introduced by Berline in [13], is the simplest class including all K-models and allowing us to work outside the framework of complete lattices. The domain underlying a pcs-model is $(\mathfrak{S}_{\mathrm{coh}}(D), \subseteq)$ where $\mathfrak{S}_{\mathrm{coh}}(D)$ denotes the set of all coherent initial segments of some preordered set with coherences $(D, \preceq, \frown)$. All binary prime algebraic domains can be described in such a way.

The class of filter models was defined by Coppo et al. in [38], but the first examples of filter models were given in [39, 9]. Here, the underlying domain is of the form $(\mathfrak{F}(D), \subseteq)$ where $\mathfrak{F}(D)$ is the set of all filters of a preordered set $(D, \preceq)$. Note that the usual definition of filter models (see, e.g., [37]) entails that they are semi-extensional, in the sense that they are asked to satisfy a condition equivalent to $\lambda \circ \mathrm{Ap} \leq \mathrm{Id}_{\mathcal{D}}$. Semi-extensionality makes easier the proof theoretic study of the models when viewed as "intersection type assignment systems" [44] but excludes, for instance, all graph models.

Definition of graph models

The class of graph models belongs to Scott-continuous semantics, it is the simplest class of models of the untyped λ-calculus; nevertheless it is very rich. All known classes of webbed models can be presented as variations of this class (see [13]), and, even more, as variations of the simplest graph model, which is Engeler's model $\mathscr{E}$ (Example 5.1.12(i)). Moreover $\mathscr{E}$ is, from far, the simplest of all non-syntactical models. Historically, the first graph model was Plotkin and Scott's $\mathscr{P}_\omega$ [85, 96], and it was followed soon by $\mathscr{E}$ [47, 89]. The word *graph* refers to the fact that the continuous functions are encoded in the model via (a sufficient fragment of) their graphs, namely their traces, as recalled below. For more details we refer to [13], and to [14] which is a recent survey of the known properties of this class.

Definition 1.4.4. *A* total pair $\mathcal{G}$ *is a pair* $(G, i_{\mathcal{G}})$ *where* G *is an infinite set and* $i_{\mathcal{G}} : G^* \times G \to G$ *is an injective total function.*

Definition 1.4.5. *The* graph model *generated by the total pair* $\mathcal{G}$ *is the reflexive cpo* $\mathscr{G} = ((\mathscr{P}(G), \subseteq), \lambda^{\mathcal{G}}, \mathrm{Ap}^{\mathcal{G}})$*, where* $\lambda^{\mathcal{G}} = i^+_{\mathcal{G}} \circ Tr$ *and* $\mathrm{Ap}^{\mathcal{G}}$ *is a left inverse of* $\lambda^{\mathcal{G}}$*. More precisely:*

(i) $\lambda^{\mathcal{G}}(f) = \{i_{\mathcal{G}}(a, \alpha) : a \in G^*$ *and* $\alpha \in f(a)\}$,

(ii) $\mathrm{Ap}^{\mathcal{G}}(X)(Y) = \{\alpha \in G : (\exists a \subseteq_{\mathrm{f}} Y)\ i_{\mathcal{G}}(a, \alpha) \in X\}$.

In particular, the function $i_{\mathcal{G}}$ encodes the trace of the Scott continuous function $f : \mathscr{P}(G) \to \mathscr{P}(G)$ by $\lambda^{\mathcal{G}}(f) \subseteq G$. The total pair $\mathcal{G} = (G, i_{\mathcal{G}})$ is called the "*web*" of the model. Hereafter, unless otherwise specified, we suppose that a graph model $\mathscr{G}$ has a web denoted by $\mathcal{G}$.

It is easy to check that, in the case of a graph model $\mathscr{G}$, the interpretation $|M|^{\mathscr{G}} : Env_{\mathscr{P}(G)} \to \mathscr{P}(G)$ of $M \in \Lambda$ becomes:

- $|x|^{\mathscr{G}}_\rho = \rho(x)$,
- $|MN|^{\mathscr{G}}_\rho = \{\alpha \in G : (\exists a \subseteq_{\mathrm{f}} |N|^{\mathscr{G}}_\rho)\ i_{\mathcal{G}}(a, \alpha) \in |M|^{\mathscr{G}}_\rho\}$,
- $|\lambda x.M|^{\mathscr{G}}_\rho = \{i_{\mathcal{G}}(a, \alpha) : a \in G^*$ and $\alpha \in |M|^{\mathscr{G}}_{\rho[x:=a]}\}$.

Example 1.4.6. *Given a graph model* $\mathscr{G}$*:*
$|\mathbf{I}|^{\mathscr{G}} \equiv |\lambda x.x|^{\mathscr{G}} = \{i_{\mathcal{G}}(a, \alpha) : a \in G^*$ *and* $\alpha \in a\}$,
$|\mathbf{T}|^{\mathscr{G}} \equiv |\lambda xy.x|^{\mathscr{G}} = \{i_{\mathcal{G}}(a, i_{\mathcal{G}}(b, \alpha)) : a, b \in G^*$ *and* $\alpha \in a\}$,
$|\mathbf{F}|^{\mathscr{G}} \equiv |\lambda xy.y|^{\mathscr{G}} = \{i_{\mathcal{G}}(a, i_{\mathcal{G}}(b, \alpha)) : a, b \in G^*$ *and* $\alpha \in b\}$.

Concerning $|\Omega|^{\mathscr{G}}$ we only use the following characterization (the details of the proof are, for example, worked out in [16, Lemma 4]).

Lemma 1.4.7. *If* $\mathscr{G}$ *is a graph model, then*

$$|\Omega|^{\mathscr{G}} \equiv |\delta\delta|^{\mathscr{G}} = \{\alpha : (\exists a \subseteq |\delta|^{\mathscr{G}})\ i_{\mathcal{G}}(a, \alpha) \in a\}.$$

In the following "*graph theory*" will abbreviate "λ-theory of a graph model".

Proposition 1.4.8. *For all graph models* $\mathscr{G}$*,* $\mathrm{Th}(\mathscr{G}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

Indeed, it was long ago noticed that no graph model could be extensional, and recently noticed in [29] that $|\Omega_3|^{\mathscr{G}} \subseteq |\mathbf{1}\Omega_3|^{\mathscr{G}}$, where $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$, holds in all graph models $\mathscr{G}$ (because $|\Omega_3|^{\mathscr{G}} \subseteq \mathrm{rg}(i_{\mathcal{G}})$). Hence, Selinger's result [99, Cor. 4] stating that in any partially ordered model whose theory is λ_β or $\lambda_{\beta\eta}$ the interpretations of closed λ-terms are discretely ordered, implies that the theory of a graph model cannot be $\lambda_\beta, \lambda_{\beta\eta}$.

Nevertheless, graph models represent a wealth of different λ-theories, as pointed out in the following proposition.

Proposition 1.4.9. *(Berline and Salibra [16]) The set of graph theories is* $2^{\aleph_0}$*-broad (and hence,* $2^{\aleph_0}$*-wide [66]).*

1.4.3 Stable and strongly stable semantics

The stable semantics and the strongly stable semantics are refinements of Scott-continuous semantics which were successively introduced respectively by Berry [17, 18] and Ehrhard [25], mainly for proving some properties of typed λ-calculi with a flavour of sequentiality [87, 17, 18, 21]. For this thesis it is enough to know the following. In this framework, the objects are particular Scott domains called *DI-domains* (resp. *DI-domains with coherences*) where, in particular, $u \sqcap v$ is defined for all pairs (u, v) of compatible elements. The morphisms are, respectively, the stable and strongly stable functions between such domains.

A function between DI-domains is *stable* if it is Scott continuous and furthermore commutes with "infs of compatible elements". A *strongly stable function* between DI-domains with coherence, is a stable function preserving coherence. The relevant order on the corresponding cpo's of functions, respectively $[\mathcal{D} \to_s \mathcal{D}]$ and $[\mathcal{D} \to_{ss} \mathcal{D}]$ is, in both cases, Berry's order $\leq_s$ which is defined as follows:

$$f \leq_s g \text{ if, and only if, } \forall x \forall y\ (x \sqsubseteq_{\mathcal{D}} y \Rightarrow f(x) = f(y) \sqcap g(x)).$$

The following basic properties of Berry's order are easy to check.

Remark 1.4.10.

(i) $f \leq_s g$ *implies that* f *is pointwise smaller than* g*,*

(ii) $f \leq_s g$ *and* g *constant imply* f *constant.*

As soon as we are working with stable functions, the following alternative notion of trace makes sense: $Tr_s(f)$ is defined in the same way as $Tr(f)$ in Section 1.4.2 but retains only the pairs (d, e) of compact elements satisfying: d is minimal such that $e \sqsubseteq_{\mathcal{D}} f(d)$; and similarly with the pairs (a, α) if $\mathcal{D}$ is furthermore a prime algebraic cpo. For example, if $\mathcal{D} = (\mathscr{P}(D), \subseteq)$, for some set D, then $Tr(\mathrm{Id}_{\mathcal{D}}) = \{(a, \alpha) : \alpha \in a \in D^*\}$ while $Tr_s(\mathrm{Id}_{\mathcal{D}}) = \{(\{\alpha\}, \alpha) : \alpha \in D\}$.

The classes of webbed models which have been considered inside these semantics are: Girard's G-models (reflexive coherences) for the stable semantics and Ehrhard's H-models (reflexive hypercoherences) for the strongly stable semantics.

Compared to the models studied so far, the interpretation of terms in G- and H- models is more economical, because the encoding of (strongly) stable functions is done via a more economical notion of trace. Another advantage of these classes is that we have extensional models without having to introduce a preorder. The price to pay is first that we have to check for the minimality condition, and second that the definition of the class of webs of H-models is much more complicate than for graph or even for G- models.

2

Working outside concrete categories

Haskell Curry may be surprised to hear that he has spent a lifetime doing fundamental work in category theory.
(Joachim Lambek, from [73])

In denotational semantics, ccc's without enough points arise naturally when morphisms are not functions, as e.g., sequential algorithms [21] or strategies in various categories of games [3, 61]. In this chapter we show that any categorical model of λ-calculus can be presented as a λ-model, even when the underlying category has not enough points. We also provide sufficient conditions for categorical models living in cpo-enriched ccc's to have $\mathcal{H}^*$ as equational theory. In the next chapter, we will build a rather simple categorical model having not enough points and satisfying these conditions.

2.1 Introduction

The first non-syntactical model of the untyped λ-calculus, namely $\mathscr{D}_\infty$, has been constructed by Scott [95] in 1969, but only at the end of the seventies researchers were able to provide general definitions of a model of λ-calculus (e.g., Barendregt [7, 8], Lambek [73], Berry [19, 20], Hindley and Longo [56], Meyer [80] and Scott [98]). We refer the reader to [8, 36, 19] for more details.

Barendregt, inspired by proof theoretical considerations such as ω-incompleteness [86], proposed as models of λ-calculus both the class of λ-algebras and that of λ-models. All the other notions of model coincide essentially with λ-models, except for categorical models which have been proved equivalent to λ-algebras.

More specifically, given a λ-model, it is always possible to define a ccc where it can be viewed as a reflexive object with enough points [5, Sec. 9.5]. On the other side, by applying a construction due to Koymans [71] and based on work of Scott, arbitrary reflexive objects in ccc's give rise to λ-algebras and to all of them. Moreover, using this method, it turns out that the λ-models are exactly those λ-algebras that come from reflexive objects with enough points.

The class of λ-algebras is not sound for λ-theories since, with the failure of the Meyer-Scott axiom, it can happen for two non β-convertible λ-terms M, N that (under the interpretation) $M = N$ but $\lambda x.M \neq \lambda x.N$. Nevertheless, λ-algebras can be viewed as desirable since they satisfy all the provable equations of λ-calculus (i.e., if $M =_{\lambda_\beta} N$, then $\mathcal{A} \models M = N$ holds for every λ-algebra $\mathcal{A}$) and since they constitute an equational class. Hence, Koymans had only as *aim* to provide λ-algebras. But this led to a common belief that only the reflexive objects having enough points can give rise to λ-models!

We will show in the first part of this chapter that this belief is not true, and prove that there exists a simple method for turning *any* reflexive object $\mathscr{U}$ of a ccc *into a λ-model*. Using this method, we can easily switch from the categorical to the algebraic interpretation of λ-terms and *vice versa*. We notice that the resulting λ-model is isomorphic to the λ-model obtained by freely adjoining the variables of λ-calculus as indeterminates to the λ-algebra associated with $\mathscr{U}$ by Koymans' construction.

It is well known that there exist several models of λ-calculus having as equational theory $\mathcal{H}^*$, the unique maximal consistent sensible λ-theory [8, 53, 45]. The most general result in this context is in Gouy's thesis [53]. Gouy introduces a notion of "regular ccc" and characterizes a class of models, all living in regular ccc's, which can be suitably stratified (using stratification is an original idea of Hyland [59]) yielding $\mathcal{H}^*$ as equational theory. Regular ccc's are concrete by definition. For instance the Scott-continuous, stable and strongly stable semantics are regular. In the second part of the chapter we generalize Gouy's result in order to cover also models living in non-concrete (but cpo-enriched) ccc's. This generalization is necessary to cover the categorical model we will build in the next chapter.

2.2 From ccc's to λ-models

We will prove that any reflexive object of an arbitrary ccc gives rise to a λ-model, when choosing appropriately the underlying combinatory algebra. Before going further, let us remark that our construction does not give anything new for the categories which do have enough points. Moreover, there is no absolute need of considering the combinatory algebra associated with a categorical model, in order to study the λ-theory thereof: it is often a matter of taste whether to use categorical or algebraic approaches. For example, in [45] and [65], the authors provide reflexive objects in categories of games (hence without enough points) and prove that the induced λ-theories are $\mathcal{H}^*$ and $\mathcal{B}_{\mathcal{T}}$, respectively. What we are proposing here is simply an algebraic counterpart of any categorical model, which satisfies weak extensionality.

We briefly recall below the classic construction due to Koymans [71] of the λ-algebra associated with a reflexive object and show how we can obtain a λ-model getting rid of the "enough points" hypothesis.

2.2.1 The crucial point of the construction

Given any reflexive object $\mathscr{U}$ of a ccc $\mathbf{C}$, Koymans' construction takes as combinatory algebra associated with $\mathscr{U}$ the set $\mathbf{C}(\mathbb{1}, U)$ equipped with the following application operator: $a \bullet b = ev \circ \langle \mathrm{Ap} \circ a, b\rangle$. This combinatory algebra[1] is, in general, only a λ-algebra; it is a λ-model if, and only if, $\mathscr{U}$ has enough points.

Hence, the choice of $\mathbf{C}(\mathbb{1}, U)$, although canonical, it is not appropriate if we want to provide a λ-model in all cases. We will prove in the next subsection that a suitable choice for the carrier set of the combinatory algebra associated with $\mathscr{U}$ is the set $\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$ of those morphisms in $\mathbf{C}(U^{\mathrm{Var}}, U)$ only depending on a finite number of "arguments".

Speaking of U^{Var} asks for countable products in $\mathbf{C}$, a hypothesis which usually holds in practice. Nevertheless, we can also get rid of this additional hypothesis; the price to pay is to take a quotient over $\cup_{I \subseteq_{\mathrm{f}} \mathrm{Var}} \mathbf{C}(U^I, U)$, instead of $\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$. This approach is sketched in Subsection 2.2.3.

2.2.2 From reflexive objects to λ-models

The aim is here to turn a (fixed) categorical model $\mathscr{U} = (U, \mathrm{Ap}, \lambda)$, living in a ccc $\mathbf{C}$ with countable products, into a λ-model $\mathscr{C}_{\mathscr{U}}$.

This construction is performed in two steps: (i) we define an environment λ-model $\mathscr{S}_{\mathscr{U}}$ associated with $\mathscr{U}$, (ii) we use the correspondence between environment λ-models and λ-models recalled in Subsection 1.3.4 to obtain $\mathscr{C}_{\mathscr{U}}$.

The associated applicative structure

The categorical interpretation of a λ-term can be viewed as a morphism in $\mathbf{C}(U^{\mathrm{Var}}, U)$ only depending on a finite number of variables. In order to capture this informal idea, we now focus our attention on the "finitary" morphisms in $\mathbf{C}(U^{\mathrm{Var}}, U)$.

Definition 2.2.1. *A morphism $f \in \mathbf{C}(U^{\mathrm{Var}}, U)$ is* finitary *if there exist a finite set J of variables, and a morphism $f_J \in \mathbf{C}(U^J, U)$ such that $f = f_J \circ \Pi^{\mathrm{Var}}_J$.*

Notation 2.2.2. *The set of all finitary morphisms will be denoted by $\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$.*

Of course, a finitary morphism can be decomposed in many different ways, and there exists a minimum set of variables on which it depends. Hence, for all $f \in \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$, we let:

- $\mathrm{Ad}(f) = \{(f_J, J) : J \subseteq_{\mathrm{f}} \mathrm{Var},\ f = f_J \circ \Pi^{\mathrm{Var}}_J\}$ be the set of *adequate pairs* of f,

- $\mathrm{dom}_{\mathrm{f}}(f) = \cap_{(f_J,J)\in \mathrm{Ad}(f)} J$ be the *finitary domain* of f.

[1] Indeed, it is possible to prove that there exist two morphisms in $\mathbf{C}(\mathbb{1}, U)$ which play the roles of $\mathbf{k}$ and $\mathbf{s}$.

The set $\mathbf{C}(U^{\mathrm{Var}}, U)$ can be naturally seen as an applicative structure whose application is defined by $a \bullet b = ev \circ \langle \mathrm{Ap} \circ a, b\rangle$. This operation preserves the property of being finitary, hence the set of finitary morphisms is closed under this operation. Indeed, given $f, g \in \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$, it is easy to verify that:

- $f \bullet g \in \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$,
- if $(f_J, J) \in \mathrm{Ad}(f)$ and $(g_I, I) \in \mathrm{Ad}(g)$, then $((f_J \circ \Pi_J^{J\cup I}) \bullet (g_I \circ \Pi_I^{J\cup I}), J \cup I) \in \mathrm{Ad}(f \bullet g)$,
- $\mathrm{dom}_{\mathrm{f}}(f \bullet g) = \mathrm{dom}_{\mathrm{f}}(f) \cup \mathrm{dom}_{\mathrm{f}}(g)$.

Therefore, we can choose the set $\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$ of finitary morphisms together with the operation $\bullet$ as applicative structure associated with $\mathscr{U}$.

Definition 2.2.3. *The applicative structure $\mathscr{A}_{\mathscr{U}} = (A_{\mathscr{U}}, \bullet)$ associated with the categorical model $\mathscr{U}$ is defined by:*

- $A_{\mathscr{U}} = \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$,
- $a \bullet b = ev \circ \langle \mathrm{Ap} \circ a, b\rangle$.

We are going to show that $\mathscr{A}_{\mathscr{U}}$ gives rise to an environment λ-model $\mathscr{S}_{\mathscr{U}}$, which is extensional precisely when $\mathscr{U}$ is an extensional categorical model (i.e., $\lambda \circ \mathrm{Ap} = \mathrm{Id}_U$). We first prove some basic properties of the categorical interpretation which are necessary to ensure that our definition of $\mathscr{S}_{\mathscr{U}}$ will be sound. Roughly speaking, we show that the value of $|M|_I$ only depends on the subset $\mathrm{FV}(M) \subseteq I$.

Some properties of the interpretation

As a matter of notation, given a set $I \subseteq \mathrm{Var}$ and an environment $\rho \in Env_{A_{\mathscr{U}}}$, we denote by ρ^I the morphism $\langle \rho(x)\rangle_{x\in I} \in \mathbf{C}(U^{\mathrm{Var}}, U^I)$.

Lemma 2.2.4. *Let M be a λ-term and I be adequate for M. For all $J \subseteq_{\mathrm{f}} \mathrm{Var}$ such that $I \subseteq J$ we have:*

(i) $\Pi_I^J \circ \rho^J = \rho^I$, *for every* $\rho \in Env_{A_{\mathscr{U}}}$,

(ii) $|M|_J = |M|_I \circ \Pi_I^J$.

Proof. (i) From the definition of Π_I^J given in Subsection 1.1.6 we have that $\Pi_I^J \circ \rho^J = \langle \pi_x^J\rangle_{x\in I} \circ \rho^J$. By applying the axiom (pair) this is equal to $\langle \pi_x^J \circ \rho^J\rangle_{x\in I}$ which is, by definition of π_x^J, exactly $\langle \rho(x)\rangle_{x\in I} = \rho^I$.
(ii) By induction over the structure of M.
If $M \equiv x$ then $|x|_J = \pi_x^J$ which is equal, by Remark 1.1.3(i), to $\pi_x^I \circ \Pi_I^J = |x|_I \circ \Pi_I^J$.
If $M \equiv NP$, then

$$\begin{array}{rlll} |NP|_J = & ev \circ \langle \mathrm{Ap} \circ |N|_J, |P|_J\rangle & \text{by def. of } |-|_J \\ = & ev \circ \langle \mathrm{Ap} \circ |N|_I \circ \Pi_I^J, |P|_I \circ \Pi_I^J\rangle & \text{by I.H.} \\ = & ev \circ \langle \mathrm{Ap} \circ |N|_I, |P|_I\rangle \circ \Pi_I^J & \text{by (pair)} \\ = & |NP|_I \circ \Pi_I^J & \text{by def. of } |-|_I \end{array}$$

If $M \equiv \lambda x.N$, then

$$\begin{array}{rcll} |\lambda x.N|_J & = & \lambda \circ \Lambda(|N|_{J\cup\{x\}}) & \text{by def. of } |-|_J \\ & = & \lambda \circ \Lambda(|N|_{I\cup\{x\}} \circ \Pi^{J\cup\{x\}}_{I\cup\{x\}}) & \text{by I.H.} \\ & = & \lambda \circ \Lambda(|N|_{I\cup\{x\}} \circ (\Pi^J_I \times \mathrm{Id})) & \text{by Rem. 1.1.3}(ii) \\ & = & \lambda \circ \Lambda(|N|_{I\cup\{x\}}) \circ \Pi^J_I & \text{by (Curry)} \\ & = & |\lambda x.N|_I \circ \Pi^J_I & \text{by def. of } |-|_I. \end{array}$$

□

Lemma 2.2.5. *Let $M \in \Lambda$ and $\rho \in Env_{A_{\mathscr{U}}}$. For all $J_1, J_2 \subseteq_{\mathrm{f}} \mathrm{Var}$ adequate for M, we have that $|M|_{J_1} \circ \rho^{J_1} = |M|_{J_2} \circ \rho^{J_2}$.*

Proof. Let us consider a set $I \subseteq_{\mathrm{f}} \mathrm{Var}$ such that $J_1 \subseteq I$ and $J_2 \subseteq I$.

$$\begin{array}{rcll} |M|_{J_1} \circ \rho^{J_1} & = & |M|_{J_1} \circ \Pi^I_{J_1} \circ \rho^I & \text{by Lemma 2.2.4}(i) \\ & = & |M|_I \circ \rho^I & \text{by Lemma 2.2.4}(ii) \\ & = & |M|_{J_2} \circ \Pi^I_{J_2} \circ \rho^I & \text{by Lemma 2.2.4}(ii) \\ & = & |M|_{J_2} \circ \rho^{J_2} & \text{by Lemma 2.2.4}(i) \end{array}$$

□

The associated environment λ-model

As a consequence of Lemma 2.2.5, the following definition is sound.

Definition 2.2.6. *Let us set $\mathscr{S}_{\mathscr{U}} = (\mathscr{A}_{\mathscr{U}}, [\![-]\!])$, where:*

- *$\mathscr{A}_{\mathscr{U}} = (A_{\mathscr{U}}, \bullet)$ is the applicative structure associated with $\mathscr{U}$ by Definition 2.2.3,*
- *$[\![-]\!] : \Lambda \times Env_{A_{\mathscr{U}}} \to A_{\mathscr{U}}$ is defined by $[\![M]\!]_\rho = |M|_I \circ \rho^I$ for some adequate I.*

We are going to show that the structure $\mathscr{S}_{\mathscr{U}}$ is an environment λ-model (Theorem 2.2.12). In order to prove this result we need to define both an *"updating" morphism* η_z and a *canonical injection* $\iota_{J,x}$, and to prove some technical lemmata. We start defining $\eta_z \in \mathbf{C}(U^{\mathrm{Var}} \times U, U^{\mathrm{Var}})$ whose intuitive behaviour is to replace the z-th component of U^{Var} by a new value which is obtained applying π_2.

Definition 2.2.7. *For all $z \in \mathrm{Var}$, we define componentwise a morphism $\eta_z \in \mathbf{C}(U^{\mathrm{Var}} \times U, U^{\mathrm{Var}})$ as follows:*

$$\eta^x_z = \begin{cases} \pi_2 & \text{if } x \equiv z, \\ \pi^{\mathrm{Var}}_x \circ \pi_1 & \text{otherwise,} \end{cases}$$

where $\pi_1 \in \mathbf{C}(U^{\mathrm{Var}} \times U, U^{\mathrm{Var}})$ and $\pi_2 \in \mathbf{C}(U^{\mathrm{Var}} \times U, U)$ are the projections.

Remark 2.2.8. *From the definition of η_z, the following equalities hold:*

(i) $\pi^{\mathrm{Var}}_x \circ \eta_x = \pi_2$,

(ii) $\pi^{\mathrm{Var}}_y \circ \eta_x = \pi^{\mathrm{Var}}_y \circ \pi_1$ *if* $x \not\equiv y$.

The following lemma is essential for showing that, under the interpretation in $\mathscr{S}_{\mathscr{U}}$, $M = N$ implies $\lambda x.M = \lambda x.N$.

Lemma 2.2.9. *Let* $f_1, \ldots, f_n \in \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$*. For all* $z \in \mathrm{Var}$*, if* $z \notin \cup_{k \leq n} \mathrm{dom}_{\mathrm{f}}(f_k)$*, then:*

$$(\langle f_1, \ldots, f_n \rangle \times \mathrm{Id}) = \langle \langle f_1, \ldots, f_n \rangle, \pi_z^{\mathrm{Var}} \rangle \circ \eta_z, \tag{2.1}$$

i.e., the following diagram commutes:

$$\begin{tikzcd}
U^{\mathrm{Var}} \arrow[r, "\langle \mathrm{Id}, \pi_z^{\mathrm{Var}} \rangle"] & U^{\mathrm{Var}} \times U \arrow[r, "\langle f_1, \ldots, f_n \rangle \times \mathrm{Id}"] & U^n \times U \\
U^{\mathrm{Var}} \times U \arrow[u, "\eta_z"] \arrow[ur, "\mathrm{Id}"'] & &
\end{tikzcd}$$

Proof. By (pair) and Remark 2.2.8(i) we obtain that:

$$\langle \langle f_1, \ldots, f_n \rangle, \pi_z^{\mathrm{Var}} \rangle \circ \eta_z = \langle \langle f_1, \ldots, f_n \rangle \circ \eta_z, \pi_2 \rangle.$$

Hence, it is sufficient to prove that $\langle f_1, \ldots, f_n \rangle \circ \eta_z = \langle f_1, \ldots, f_n \rangle \circ \pi_1$. Since every f_k is finitary and $z \notin \mathrm{dom}_{\mathrm{f}}(f_k)$ there exist $(f_1', J_1) \in \mathrm{Ad}(f_1), \ldots, (f_n', J_n) \in \mathrm{Ad}(f_n)$ such that $z \notin \cup_{k \leq n} J_k$. We conclude the proof as follows.

$$\begin{array}{lll}
& \langle f_1, \ldots, f_n \rangle \circ \eta_z = & \\
= & \langle f_1' \circ \Pi_{J_1}^{\mathrm{Var}}, \ldots, f_n' \circ \Pi_{J_n}^{\mathrm{Var}} \rangle \circ \eta_z & \text{since } f_k = f_k' \circ \Pi_{J_k}^{\mathrm{Var}} \\
= & \langle f_1' \circ \Pi_{J_1}^{\mathrm{Var}} \circ \eta_z, \ldots, f_n' \circ \Pi_{J_n}^{\mathrm{Var}} \circ \eta_z \rangle & \text{by (pair)} \\
= & \langle f_1' \circ \Pi_{J_1}^{\mathrm{Var}} \circ \pi_1, \ldots, f_n' \circ \Pi_{J_n}^{\mathrm{Var}} \circ \pi_1 \rangle & \text{by Rem. 2.2.8(ii) since } z \notin \cup_{k \leq n} J_k \\
= & \langle f_1 \circ \pi_1, \ldots, f_n \circ \pi_1 \rangle & \text{since } f_k = f_k' \circ \Pi_{J_k}^{\mathrm{Var}} \\
= & \langle f_1, \ldots, f_n \rangle \circ \pi_1 & \text{by (pair)} \qquad \Box
\end{array}$$

We define now a "canonical" injection $\iota_{J,x} \in \mathbf{C}(U^{J \cup \{x\}}, U^{\mathrm{Var}})$ in such a way that, for all $J \subseteq_{\mathrm{f}} \mathrm{Var}$ and $x \notin J$, we have:

$$\Pi_{J \cup \{x\}}^{\mathrm{Var}} \circ \iota_{J,x} = \mathrm{Id}_{U^{J \cup \{x\}}}. \tag{2.2}$$

Definition 2.2.10. *For all* $x \in \mathrm{Var}$ *and* $J \subseteq_{\mathrm{f}} \mathrm{Var}$ *we define componentwise a morphism* $\iota_{J,x} \in \mathbf{C}(U^{J \cup \{x\}}, U^{\mathrm{Var}})$ *as follows:*

$$\iota_{J,x}^z = \begin{cases} \pi_z^{J \cup \{x\}} & \text{if } z \in J \cup \{x\}, \\ \lambda \circ \Lambda(\mathrm{Id}_U) \circ !_{U^{J \cup \{x\}}} & \text{otherwise.} \end{cases}$$

Of course, it is easy to verify that Property (2.2) is guaranteed by the above definition. We claim that the choice of $\lambda \circ \Lambda(\mathrm{Id}_U) \circ !_{U^{J \cup \{x\}}}$ is canonical since $\lambda \circ \Lambda(\mathrm{Id}_U)$ is nothing else than the point of U corresponding to Id_U under the morphism λ. However, this fact does not play any role in the following proof.

Lemma 2.2.11. *Let $f \in \mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U)$ and $(f_J, J) \in \mathrm{Ad}(f)$. Then for all $x \notin J$, we have:*

$$f \times \mathrm{Id} = \langle f, \pi_x^{\mathrm{Var}} \rangle \circ \iota_{J,x} \circ (\Pi_J^{\mathrm{Var}} \times \mathrm{Id})$$

i.e., the following diagram commutes:

$$\begin{tikzcd}
U^{\mathrm{Var}} \times U \arrow[r, "\Pi_J^{\mathrm{Var}} \times \mathrm{Id}"] \arrow[drr, "f \times \mathrm{Id}"'] & U^J \times U \simeq U^{J \cup \{x\}} \arrow[r, "\iota_{J,x}"] & U^{\mathrm{Var}} \arrow[d, "\langle f, \pi_x^{\mathrm{Var}} \rangle"] \\
 & & U \times U
\end{tikzcd}$$

Proof. Since by hypothesis $f = f_J \circ \Pi_J^{\mathrm{Var}}$, this is equivalent to ask that the following diagram commutes, and this is obvious by (2.2) since $\langle \Pi_J^{\mathrm{Var}}, \pi_x^{\mathrm{Var}} \rangle = \Pi_{J \cup \{x\}}^{\mathrm{Var}}$.

$$\begin{tikzcd}
U^{\mathrm{Var}} \times U \arrow[r, "\Pi_J^{\mathrm{Var}} \times \mathrm{Id}"] & U^J \times U \simeq U^{J \cup \{x\}} \arrow[r, "\iota_{J,x}"] \arrow[dr, "f_J \times \mathrm{Id}"] & U^{\mathrm{Var}} \arrow[dr, "\langle \Pi_J^{\mathrm{Var}}, \pi_x^{\mathrm{Var}} \rangle"] & \\
 & & U \times U & U^{J \cup \{x\}} \arrow[l, "f_J \times \mathrm{Id}"]
\end{tikzcd}$$

$\square$

We are now able to prove the promised theorem stating that the structure $\mathscr{S}_{\mathscr{U}}$ associated with $\mathscr{U}$ by Definition 2.2.6 is actually an environment λ-model.

Theorem 2.2.12. *Let $\mathscr{U}$ be a categorical model living in a ccc $\mathbf{C}$ with countable product. Then:*

1) $\mathscr{S}_{\mathscr{U}}$ is an environment λ-model,

2) $\mathscr{S}_{\mathscr{U}}$ is extensional if, and only if, $\mathscr{U}$ is extensional.

Proof. 1) We prove that $\mathscr{S}_{\mathscr{U}}$ satisfies the conditions $(i) - (v)$ of the definition of environment λ-model recalled in Subsection 1.3.4. In each item we let $I \subseteq_{\mathrm{f}} \mathrm{Var}$ be any set adequate for the λ-term in the left hand sight of the equality.

(i) $[\![z]\!]_\rho = |z|_I \circ \rho^I = \pi_z^I \circ \rho^I = \rho(z)$.

(ii)
$$\begin{array}{rcll}
[\![PQ]\!]_\rho & = & |PQ|_I \circ \rho^I & \text{by def. of } [\![-]\!] \\
 & = & (|P|_I \bullet |Q|_I) \circ \rho^I & \text{by def. of } \bullet \\
 & = & ev \circ \langle \mathrm{Ap} \circ |P|_I, |Q|_I \rangle \circ \rho^I & \text{by (pair)} \\
 & = & ev \circ \langle \mathrm{Ap} \circ |P|_I \circ \rho^I, |Q|_I \circ \rho^I \rangle & \text{by def. of } \bullet \text{ and } [\![-]\!] \\
 & = & [\![P]\!]_\rho \bullet [\![Q]\!]_\rho &
\end{array}$$

(iii)
$$\begin{array}{rcll}
[\![\lambda x.P]\!]_\rho \bullet a & = & (|\lambda x.P|_I \circ \rho^I) \bullet a & \text{by def. of } [\![-]\!]_\rho \\
 & = & ev \circ \langle \mathrm{Ap} \circ (|\lambda x.P|_I \circ \rho^I), a \rangle & \text{by def. of } \bullet \\
 & = & ev \circ \langle \mathrm{Ap} \circ \boldsymbol{\lambda} \circ \Lambda(|P|_{I \cup \{x\}}) \circ \rho^I, a \rangle & \text{by def. of } |-|_I \\
 & = & ev \circ \langle \Lambda(|P|_{I \cup \{x\}}) \circ \rho^I, a \rangle & \text{since } \mathrm{Ap} \circ \boldsymbol{\lambda} = \mathrm{Id}_{[U \Rightarrow U]} \\
 & = & ev \circ \langle \Lambda(|P|_{I \cup \{x\}} \circ (\rho^I \times \mathrm{Id})), a \rangle & \text{by (Curry)} \\
 & = & |P|_{I \cup \{x\}} \circ (\rho^I \times \mathrm{Id}) \circ \langle \mathrm{Id}, a \rangle & \text{by (beta)} \\
 & = & |P|_{I \cup \{x\}} \circ \langle \rho^I, a \rangle & \text{by (pair)} \\
 & = & [\![P]\!]_{\rho[x:=a]} & \text{by def. of } [\![-]\!]_\rho
\end{array}$$

(iv) Obvious since, by Lemma 2.2.5, $[\![M]\!]_\rho = |M|_{\mathrm{FV}(M)} \circ \rho^{\mathrm{FV}(M)}$.

(v)
$$\begin{array}{rcll} [\![\lambda z.M]\!]_\rho &=& |\lambda z.M|_I \circ \rho^I & \text{by def. of } [\![-]\!] \\ &=& \lambda \circ \Lambda(|M|_{I\cup\{z\}}) \circ \rho^I & \text{by def. of } |-|_I \\ &=& \lambda \circ \Lambda(|M|_{I\cup\{z\}} \circ (\rho^I \times \mathrm{Id})) & \text{by (Curry).} \end{array}$$

Since $\rho(x)$ is finitary for all $x \in I$, we can suppose by α-conversion that $z \notin \cup_{x\in I}\mathrm{dom}_{\mathrm{f}}(\rho(x))$. Hence we can apply Lemma 2.2.9 and obtain:

$$\begin{array}{rcll} \lambda \circ \Lambda(|M|_{I\cup\{z\}} \circ (\rho^I \times \mathrm{Id})) &=& \lambda \circ \Lambda(|M|_{I\cup\{z\}} \circ \langle \rho^I, \pi_z^{\mathrm{Var}}\rangle \circ \eta_z) & \text{by Lemma 2.2.9} \\ &=& \lambda \circ \Lambda([\![M]\!]_{\rho[z:=\pi_z^{\mathrm{Var}}]} \circ \eta_z) & \text{by def. of } [\![-]\!] \end{array}$$

Since, by hypothesis, $[\![M]\!]_{\rho[z:=a]} = [\![N]\!]_{\rho[z:=a]}$ for every $a \in A_{\mathscr{U}}$ this also holds for $a = \pi_z$, hence $\lambda \circ \Lambda([\![M]\!]_{\rho[z:=\pi_z]} \circ \eta_z) = \lambda \circ \Lambda([\![N]\!]_{\rho[z:=\pi_z]} \circ \eta_z)$. It is, now, routine to check that $\lambda \circ \Lambda([\![N]\!]_{\rho[z:=\pi_z]} \circ \eta_z) = [\![\lambda z.N]\!]_\rho$.

2) ($\Rightarrow$) Let $x \in \mathrm{Var}$. Since π_x^{Var} is finitary we have that $\pi_x^{\mathrm{Var}} \in A_{\mathscr{U}}$. For all $a \in A_{\mathscr{U}}$ we have:

$$\begin{array}{rcll} (\lambda \circ \mathrm{Ap} \circ \pi_x^{\mathrm{Var}}) \bullet a &=& ev \circ \langle \mathrm{Ap} \circ \lambda \circ \mathrm{Ap} \circ \pi_x^{\mathrm{Var}}, a\rangle & \text{by def. of } \bullet \\ &=& ev \circ \langle \mathrm{Ap} \circ \pi_x^{\mathrm{Var}}, a\rangle & \text{by } \mathrm{Ap} \circ \lambda = \mathrm{Id}_{U\Rightarrow U} \\ &=& \pi_x^{\mathrm{Var}} \bullet a & \text{by def. of } \bullet \end{array}$$

If $\mathscr{S}_{\mathscr{U}}$ is extensional, this implies $\lambda \circ \mathrm{Ap} \circ \pi_x^{\mathrm{Var}} = \pi_x^{\mathrm{Var}}$. Since π_x^{Var} is an epimorphism, we get $\lambda \circ \mathrm{Ap} = \mathrm{Id}_U$.

($\Leftarrow$) Let $a, b \in A_{\mathscr{U}}$, then there exist $(a_J, J) \in \mathrm{Ad}(a)$ and $(b_I, I) \in \mathrm{Ad}(b)$ such that $I = J$. Let us set $\phi = \iota_{J,x} \circ (\Pi_J^{\mathrm{Var}} \times \mathrm{Id})$ where $x \notin J$ and $\iota_{J,x}$ is defined in Definition 2.2.10. Suppose that for all $c \in A_{\mathscr{U}}$ we have ($a \bullet c = b \bullet c$) then, in particular, $ev\circ\langle \mathrm{Ap} \circ a, \pi_x^{\mathrm{Var}}\rangle = ev\circ\langle \mathrm{Ap} \circ b, \pi_x^{\mathrm{Var}}\rangle$ and this implies that $\langle \mathrm{Ap} \circ a, \pi_x^{\mathrm{Var}}\rangle\circ \phi = \langle \mathrm{Ap} \circ b, \pi_x^{\mathrm{Var}}\rangle \circ \phi$. By applying Lemma 2.2.11, we get $\langle \mathrm{Ap} \circ a, \pi_x^{\mathrm{Var}}\rangle \circ \phi = (\mathrm{Ap} \circ a) \times \mathrm{Id}$ and $\langle \mathrm{Ap} \circ b, \pi_x^{\mathrm{Var}}\rangle \circ \phi = (\mathrm{Ap} \circ b) \times \mathrm{Id}$. Then $\mathrm{Ap} \circ a = \mathrm{Ap} \circ b$ which implies $\lambda \circ \mathrm{Ap} \circ a = \lambda \circ \mathrm{Ap} \circ b$. We conclude since $\lambda \circ \mathrm{Ap} = \mathrm{Id}_U$. □

Note that, by using a particular environment $\hat\rho$, it is possible to "recover" the categorical interpretation $|M|_I$ from the interpretation $[\![M]\!]_\rho$ in the environment λ-model. Let us fix the environment $\hat\rho(x) = \pi_x^{\mathrm{Var}}$ for all $x \in \mathrm{Var}$, then we have that:

$$[\![M]\!]_{\hat\rho} = |M|_I \circ \Pi_I^{\mathrm{Var}},$$

i.e., $[\![M]\!]_{\hat\rho}$ is the morphism $|M|_I$ "viewed" as an element of $\mathbf{C}(U^{\mathrm{Var}}, U)$.

The associated λ-model

The second step of our construction is very simple. Indeed, it is easy to check that the λ-model corresponding to the environment λ-model $\mathscr{S}_{\mathscr{U}} = ((\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U), \bullet), [\![-]\!])$ under the correspondence of Subsection 1.3.4 is exactly

$$\mathscr{C}_{\mathscr{U}} = (\mathbf{C}_{\mathrm{f}}(U^{\mathrm{Var}}, U), \bullet, [\![\mathbf{K}]\!], [\![\mathbf{S}]\!]).$$

Of course, the categorical model $\mathscr{U}$ and the λ-model $\mathscr{C}_{\mathscr{U}}$ have the same equational theory. Moreover, if $\mathscr{U}$ does have enough points, then the λ-model associated with $\mathscr{U}$ by Koymans' construction embeds canonically into $\mathscr{C}_{\mathscr{U}}$.

2.2.3 Working without countable products

The construction provided in the previous section works if the underlying category $\mathbf{C}$ has countable products. We remark, once again, that this hypothesis is not really restrictive since all the categories used in the literature in order to obtain models of λ-calculus satisfy this requirement. Nevertheless, the discussions on the relation between categorical and algebraic models of λ-calculus in [73, 74, 101] can help us to get rid of this additional hypothesis. We give here the basic ideas of this approach.

In [101], Selinger implicitly suggests that every λ-algebra $\mathscr{A}$ can be embedded into a λ-model $\mathscr{A}[\mathrm{Var}]$, which is obtained from $\mathscr{A}$ by freely adjoining the variables of λ-calculus as indeterminates (see also the discussion of C-monoids in [74]). More precisely he shows that, under the interpretation in $\mathscr{A}[x_1,\dots,x_n]$, $M = N$ implies $\lambda x.M = \lambda x.N$ as soon as M, N are λ-terms with free variables among $x_1,\dots,x_n$. Moreover, if $\mathscr{A}$ is the λ-algebra associated with a categorical model $\mathscr{U}$ by Koymans' construction, then for all $I \subseteq_{\mathrm{f}} \mathrm{Var}$ the free extension $\mathscr{A}[I]$ is isomorphic (in the category of combinatory algebras and homomorphisms between them) to $\mathbf{C}(U^I, U)$ endowed with the natural structure of combinatory algebra.

Since there exist canonical homomorphisms $\mathscr{A}[I] \mapsto \mathscr{A}[J]$ and $\mathbf{C}(U^I, U) \mapsto \mathbf{C}(U^J, U)$ which are one-to-one if $I \subseteq J \subseteq_{\mathrm{f}} \mathrm{Var}$, we can construct the inductive limit of both $\mathcal{P}_{\mathrm{f}}(\mathrm{Var})$-indexed diagrams. From one side we obtain a λ-model isomorphic to $\mathscr{A}[\mathrm{Var}]$ and from the other side we get $A' = \bigcup_{I \subseteq_{\mathrm{f}} \mathrm{Var}} \mathbf{C}(U^I, U)/_\sim$, where $\sim$ is the equivalence relation defined as follows: if $f \in \mathbf{C}(U^J, U)$ and $g \in \mathbf{C}(U^I, U)$, then

$$f \sim g \text{ if, and only if, } f \circ \Pi_J^{I\cup J} = g \circ \Pi_I^{I\cup J}.$$

The above isomorphism is obviously preserved at the limit; hence A', endowed with the natural application operator on the equivalence classes, is also a λ-model. This approach, although less simple and natural, also works in case the underlying category $\mathbf{C}$ does not have countable products. Finally, it is easy to check that if $\mathbf{C}$ does have countable products then the λ-model A' is isomorphic to $\mathscr{C}_{\mathscr{U}}$.

2.3 Well stratifiable categorical models

The λ-theory $\mathcal{H}^*$ was first introduced by Hyland [59] and Wadsworth [108], who proved (independently) that $\mathrm{Th}(\mathscr{D}_\infty) = \mathcal{H}^*$ (see also [8, Thm. 19.2.9]). This proof has been extended by Gouy in [53] with the aim of showing that also the stable analogue of $\mathscr{D}_\infty$ had $\mathcal{H}^*$ as equational theory. However, Gouy's result is more powerful and covers all "well stratifiable extensional $\perp$-models" living in "regular" Cartesian closed categories. The definition of regular ccc given in [53] is general enough for including the Scott-continuous, stable and strongly stable semantics. However, all regular ccc's have (possibly special) cpo's as objects and (possibly special) continuous functions as morphisms, hence only concrete categories can be regular. Concerning models without enough points, Di Gianantonio et al. provided in [45] a similar proof, but it works only for non-concrete *categories of games*. In Franco's phd thesis [49] it is claimed that the proof in [45] can be generalized to all well stratifiable extensional $\perp$-models, but the proof is omitted and the hypotheses which are listed there seem too weak to obtain [49, Thm. 5.23].

In this section we will generalize Gouy's proof to all (possibly non-concrete) cpo-enriched ccc's. All syntactic notions and results we will use were already present in the literature, whilst the semantic results are our own contribution.

2.3.1 Outline of the proof

The structure of our proof and the techniques we will use are quite classic. The idea is that we want to find a class of models (as large as possible) satisfying a (strong) Approximation Theorem. More precisely, we want to be able to characterize the interpretation of a λ-term M as the least upper bound of the interpretations of its approximants. These approximants are particular terms of an auxiliary calculus, due to Wadsworth and called *labelled* $\lambda\perp$*-calculus*, which is strongly normalizable and Church-Rosser.

Then, we define the "well stratifiable $\perp$-models", we show that they model also Wadsworth's calculus and satisfy the Approximation Theorem. As a consequence, we get that every well stratifiable $\perp$-model $\mathscr{U}$ satisfies $\mathrm{Th}(\mathscr{U}) \supseteq \mathcal{B}_{\mathcal{T}}$; in particular, $\mathscr{U}$ is sensible. Finally we prove, under the additional hypothesis that $\mathscr{U}$ is extensional, that $\mathrm{Th}(\mathscr{U}) = \mathcal{H}^*$ (using the characterization of $\mathcal{H}^*$ given in terms of Böhm trees; cf. Subsection 1.2.3).

2.3.2 A more uniform interpretation of λ-terms

The fact that λ-terms are interpreted in different homsets $\mathbf{C}(U^I, U)$ depending on the choice of $I \subseteq_{\mathrm{f}} \mathrm{Var}$, is tedious to treat when dealing with the equalities induced by a model. Fortunately, if the underlying category has countable products we are able to interpret all λ-terms in the homset $\mathbf{C}(U^{\mathrm{Var}}, U)$ just slightly modifying the definition of interpretation. Indeed, given $M \in \Lambda$ we can define $|M|_{\mathrm{Var}} \in \mathbf{C}(U^{\mathrm{Var}}, U)$ by structural induction on M, as follows:

- $|x|_{\mathrm{Var}} = \pi_x^{\mathrm{Var}}$,
- $|NP|_{\mathrm{Var}} = |N|_{\mathrm{Var}} \bullet |P|_{\mathrm{Var}}$,
- $|\lambda x.N|_{\mathrm{Var}} = \boldsymbol{\lambda} \circ \Lambda(|N|_{\mathrm{Var}} \circ \eta_x)$,

where $\eta_x \in \mathbf{C}(U^{\mathrm{Var}} \times U, U^{\mathrm{Var}})$ is the "updating" morphism of Definition 2.2.7. In the next proposition we show that it is possible to characterize the interpretation $[\![-]\!]$ of the λ-model $\mathscr{C}_{\mathscr{U}}$ associated with $\mathscr{U}$ in terms of $|-|_{\mathrm{Var}}$.

Proposition 2.3.1. *Let $\mathscr{C}_{\mathscr{U}}$ be the λ-model associated with $\mathscr{U}$. For all $M \in \Lambda$ and $\rho \in Env_{A_{\mathscr{U}}}$ we have $[\![M]\!]_\rho = |M|_{\mathrm{Var}} \circ \rho^{\mathrm{Var}}$.*

Proof. Once recalled that $[\![M]\!]_\rho = |M|_I \circ \rho^I$ for any adequate $I \subseteq_{\rm f} \mathrm{Var}$, we proceed by induction on M. The only non-trivial case is $M \equiv \lambda x.N$:

$$\begin{array}{rcll}
[\![\lambda x.N]\!]_\rho & = & |\lambda x.N|_I \circ \rho^I & \text{for some adequate } I \\
& = & \mathrm{Ap} \circ \Lambda(|N|_{I\cup\{x\}} \circ (\rho^I \times \mathrm{Id})) & \text{by def. of } |-|_I \text{ plus (Curry)} \\
& = & \mathrm{Ap} \circ \Lambda(|N|_{I\cup\{x\}} \circ \langle \rho^I, \pi_x^{\mathrm{Var}} \rangle \circ \eta_x) & \text{by Lemma 2.2.9} \\
& = & \mathrm{Ap} \circ \Lambda(|N|_{I\cup\{x\}} \circ (\rho[x := \pi_x^{\mathrm{Var}}])^{I\cup\{x\}} \circ \eta_x) & \text{by def. of } \rho[x := \pi_x^{\mathrm{Var}}] \\
& = & \mathrm{Ap} \circ \Lambda(|N|_{\mathrm{Var}} \circ (\rho[x := \pi_x^{\mathrm{Var}}])^{\mathrm{Var}} \circ \eta_x) & \text{by I.H.}
\end{array}$$

Since easy calculations provide $(\rho[x := \pi_x^{\mathrm{Var}}])^{\mathrm{Var}} \circ \eta_x = \eta_x \circ (\rho^{\mathrm{Var}} \times \mathrm{Id})$, we can conclude the proof as follows: $\mathrm{Ap} \circ \Lambda(|N|_{\mathrm{Var}} \circ (\rho[x := \pi_x^{\mathrm{Var}}])^{\mathrm{Var}} \circ \eta_x) = \mathrm{Ap} \circ \Lambda(|N|_{\mathrm{Var}} \circ \eta_x \circ (\rho^{\mathrm{Var}} \times \mathrm{Id})) = \mathrm{Ap} \circ \Lambda(|N|_{\mathrm{Var}} \circ \eta_x) \circ \rho^{\mathrm{Var}} = |\lambda x.N|_{\mathrm{Var}} \circ \rho^{\mathrm{Var}}$. □

Hence, for the sake of simplicity, we will work in ccc's having countable products. We remark once again that this is just a simplification: all the work done in this section could be adapted to cover also categorical models living in ccc's without countable products but the statements and the proofs would be significantly more technical.

2.3.3 Stratifiable models in cpo-enriched ccc's

The classic method for proving that the theory of a categorical model is $\mathcal{H}^*$ requires that the λ-terms are interpreted as elements of a cpo and that the morphisms involved in the definition of the interpretation are continuous functions. Thus, working *possibly outside* concrete categories, it becomes natural to consider categorical models living in cpo-enriched ccc's. We recall that, roughly speaking, a ccc $\mathbf{C}$ is cpo-enriched if every homset $\mathbf{C}(A, B)$ has a structure of cpo: $(\mathbf{C}(A, B), \sqsubseteq_{(A,B)}, \perp_{(A,B)})$. See Subsection 1.1.6 for more details.

From now on, and until the end of the chapter, we consider a fixed (non-trivial) categorical model $\mathscr{U} = (U, \mathrm{Ap}, \lambda)$ living in a cpo-enriched ccc $\mathbf{C}$ having countable products and we work with the interpretation function defined in the previous subsection.

As an easy consequence of Lemma 1.1.4 we get the following corollary.

Corollary 2.3.2. *The operations $\bullet$ and $\lambda \circ \Lambda(- \circ \eta_x)$ are continuous.*

To lighten the notation we will write $\sqsubseteq$ and $\perp$ respectively for $\sqsubseteq_{(U^{\mathrm{Var}},U)}$ and $\perp_{(U^{\mathrm{Var}},U)}$.

Definition 2.3.3. *The model $\mathscr{U}$ is a* $\perp$-model *if the following two conditions are satisfied:*

(i) $\perp \bullet a = \perp$ *for all* $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$,

(ii) $\lambda \circ \Lambda(\perp_{(U^{\mathrm{Var}} \times U, U)}) = \perp$.

Stratifications of models are done by using special morphisms, acting at the level of $\mathbf{C}(U,U)$ and called *projections*.

Definition 2.3.4. *Given an object U of a category $\mathbf{C}$, a morphism $p \in \mathbf{C}(U,U)$ is a* projection from U to U *if $p \sqsubseteq_{(U,U)} \mathrm{Id}_U$ and $p \circ p = p$.*

From now on, we also fix a family $(p_k)_{k\in\mathbb{N}}$ of projections from U to U such that $(p_k)_{k\in\mathbb{N}}$ is increasing with respect to $\sqsubseteq_{(U,U)}$ and $\sqcup_{k\in\mathbb{N}} p_k = \mathrm{Id}_U$.

Notation 2.3.5. *Given a morphism $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$ we write a_k for $p_k \circ a$.*

Since the p_k's are increasing, $\sqcup_{k\in\mathbb{N}} p_k = \mathrm{Id}_U$, and composition is continuous, we have for every morphism $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$:

$$a_k \sqsubseteq a, \tag{2.3}$$

$$a = \sqcup_{k\in\mathbb{N}} a_k. \tag{2.4}$$

Definition 2.3.6. *The model $\mathscr{U}$ is called:*

(i) stratified *(by $(p_k)_{k\in\mathbb{N}}$) if $a_{k+1} \bullet b = (a \bullet b_k)_k$;*

(ii) well stratified *(by $(p_k)_{k\in\mathbb{N}}$) if, moreover, $a_0 \bullet b = (a \bullet \perp)_0$.*

Of course, the fact that $\mathscr{U}$ is a (well) stratified model depends on the family $(p_k)_{k\in\mathbb{N}}$ we are considering. Hence, it is natural and convenient to introduce the notion of (well) stratifiable model.

Definition 2.3.7. *The model $\mathscr{U}$ is* stratifiable *(*well stratifiable*) if there exists a family $(p_k)_{k\in\mathbb{N}}$ making $\mathscr{U}$ stratified (well stratified).*

The aim of this section is in fact to prove that every extensional well stratifiable $\perp$-model has $\mathcal{H}^*$ as equational theory.

2.3.4 Interpreting the labelled $\lambda\perp$-terms in $\mathscr{U}$

We recall now the definition of the *labelled $\lambda\perp$-calculus* (see [108] or [8, Sec. 14.1]). As usual, we consider a set $C = \{c_k : k \in \mathbb{N}\}$ of constants called *labels*, together with a constant $\underline{\perp}$ to indicate lack of information. The set $\Lambda_{\underline{\perp}}^{lab}$ of *labelled $\lambda\perp$-terms* is inductively defined as follows:

- $\underline{\perp} \in \Lambda_{\underline{\perp}}^{lab}$;
- $x \in \Lambda_{\underline{\perp}}^{lab}$, for every variable x;
- if $M, N \in \Lambda_{\underline{\perp}}^{lab}$ then $MN \in \Lambda_{\underline{\perp}}^{lab}$;
- if $M \in \Lambda_{\underline{\perp}}^{lab}$ then $\lambda x.M \in \Lambda_{\underline{\perp}}^{lab}$, for every variable x;

- if $M \in \Lambda_{\perp}^{lab}$ then $c_k M \in \Lambda_{\perp}^{lab}$, for every label $c_k \in C$.

We will denote by $\Lambda_{\perp}$ the subset of $\Lambda_{\perp}^{lab}$ consisting of those terms that do not contain any label; note that $\Lambda \subsetneq \Lambda_{\perp} \subsetneq \Lambda_{\perp}^{lab}$.

We recall, from Subsection 2.3.3, that we work with a fixed categorical model $\mathscr{U}$ living in a cpo-enriched category $\mathbf{C}$ having countable products, as well as an increasing sequence $(p_k)_{k\in\mathbb{N}}$ of projections such that $\sqcup_{k\in\mathbb{N}} p_k = \mathrm{Id}_U$.

The labelled $\lambda\perp$-terms can be interpreted in $\mathscr{U}$, the intuitive meaning of $c_k M$ is the k-th projection applied to the meaning of M. Hence, we define the interpretation function as the unique extension of the interpretation function of λ-terms such that:

- $|\perp|_{\mathrm{Var}} = \perp$,
- $|c_k M|_{\mathrm{Var}} = p_k \circ |M|_{\mathrm{Var}} = (|M|_{\mathrm{Var}})_k$, for all $M \in \Lambda_{\perp}^{lab}$ and $k \in \mathbb{N}$.

Since the ccc $\mathbf{C}$ is cpo-enriched, all labelled $\lambda\perp$-terms are interpreted in the cpo $(\mathbf{C}(U^{\mathrm{Var}}, U), \sqsubseteq, \perp)$. Hence, we can transfer this ordering, and the corresponding equality, on $\Lambda_{\perp}^{lab}$ as follows.

Definition 2.3.8. *For all $M, N \in \Lambda_{\perp}^{lab}$ we set:*

- *$M \sqsubseteq_{\mathscr{U}} N$ if and only if $|M|_{\mathrm{Var}} \sqsubseteq |N|_{\mathrm{Var}}$,*
- *$M =_{\mathscr{U}} N$ if and only if $M \sqsubseteq_{\mathscr{U}} N$ and $N \sqsubseteq_{\mathscr{U}} M$.*

It is straightforward to check that both $\sqsubseteq_{\mathscr{U}}$ and $=_{\mathscr{U}}$ are contextual.

The notion of substitution can be extended to $\Lambda_{\perp}^{lab}$ by setting: $\perp[M/x] = \perp$ and $(c_k M)[N/x] = c_k(M[N/x])$ for all $M, N \in \Lambda_{\perp}^{lab}$. We now show that $\mathscr{U}$ is sound for the β-conversion extended to $\Lambda_{\perp}^{lab}$.

Lemma 2.3.9. *For all $M, N \in \Lambda_{\perp}^{lab}$ we have:*

$$(\lambda x.M)N =_{\mathscr{U}} M[N/x].$$

Proof. It is well known (cf. [8, Prop. 5.5.5]) that $(\lambda x.M)N =_{\mathscr{U}} M[N/x]$ still holds for λ-calculi extended with constants c, if $|c|_{\mathrm{Var}} = u \circ !_{U^{\mathrm{Var}}}$ for some $u \in \mathbf{C}(\mathbb{1}, U)$. Hence, this lemma holds since the interpretation defined above it is equal to the one obtained by setting: $|c_k|_{\mathrm{Var}} = \lambda \circ \Lambda(p_k) \circ !_{U^{\mathrm{Var}}}$ and $|\perp|_{\mathrm{Var}} = \perp_{(\mathbb{1},U)} \circ !_{U^{\mathrm{Var}}}$. □

2.3.5 Modelling the labelled $\lambda\perp$-calculus

We now introduce the reduction rules on labelled $\lambda\perp$-terms which generate the labelled $\lambda\perp$-calculus.

Definition 2.3.10.

- *The* ω-reduction *is defined by:*

  - $\underline{\perp} M \to_\omega \underline{\perp}$,
  - $\lambda x.\underline{\perp} \to_\omega \underline{\perp}$.

- *The* γ-reduction *is defined by:*

  - $c_0(\lambda x.M)N \to_\gamma c_0(M[\underline{\perp}/x])$,
  - $c_{k+1}(\lambda x.M)N \to_\gamma c_k(M[c_k N/x])$.

- *The* ϵ-reduction *is defined by:*

  - $c_k\underline{\perp} \to_\epsilon \underline{\perp}$,
  - $c_k(c_n M) \to_\epsilon c_{\min(k,m)} M$.

The calculus on $\Lambda_{\underline{\perp}}^{lab}$ generated by the ω-, γ-, ϵ-reductions is called *labelled* $\lambda\perp$-*calculus*. Note that the β-reduction is not considered here. The main properties of this calculus are summarized in the next theorem.

Theorem 2.3.11. *[8, Thm. 14.1.12 and 14.2.3] The labelled $\lambda\perp$-calculus is strongly normalizable and Church Rosser.*

We now show that the interpretation of a labelled $\lambda\perp$-term, in a well stratified $\perp$-model, is invariant along its ω-, ϵ-, γ-reduction paths.

Proposition 2.3.12. *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M, N \in \Lambda_{\underline{\perp}}^{lab}$:*

(i) $\underline{\perp} M =_{\mathscr{U}} \underline{\perp}$,

(ii) $\lambda x.\underline{\perp} =_{\mathscr{U}} \underline{\perp}$,

(iii) $c_k\underline{\perp} =_{\mathscr{U}} \underline{\perp}$,

(iv) $c_n(c_m M) =_{\mathscr{U}} c_{\min(n,m)} M$,

(v) $(c_0\lambda x.M)N =_{\mathscr{U}} c_0(M[\underline{\perp}/x])$,

(vi) $(c_{k+1}\lambda x.M)N =_{\mathscr{U}} c_k(M[c_k N/x])$.

Proof. *(i)* $|\underline{\perp} M|_{\mathrm{Var}} = |\underline{\perp}|_{\mathrm{Var}} \bullet |M|_{\mathrm{Var}} = \perp \bullet |M|_{\mathrm{Var}}$. Hence, the result follows from Definition 2.3.3(i).
(ii) $|\lambda x.\underline{\perp}|_{\mathrm{Var}} = \lambda \circ \Lambda(|\underline{\perp}|_{\mathrm{Var}} \circ \eta_x) = \lambda \circ \Lambda(\perp \circ \eta_x)$. Using the strictness condition (l-strict) this is equal to $\lambda \circ \Lambda(\perp_{(U^{\mathrm{Var}} \times U, U)})$, which is $\perp$ by Definition 2.3.3(ii). On the other side $|\underline{\perp}|_{\mathrm{Var}} = \perp$ always by definition.
(iii) $|c_k\underline{\perp}|_{\mathrm{Var}} = \perp_k$, hence by Inequality (2.3) and Equation (2.4) we obtain $\perp_k \sqsubseteq \sqcup_{k\in\mathbb{N}} \perp_k = \perp$. The other inequality is clear.
(iv) $|c_n(c_m M)|_{\mathrm{Var}} = p_n \circ p_m \circ |M|_{\mathrm{Var}}$. By Lemma 1.1.4, and since the sequence $(p_k)_{k\in\mathbb{N}}$ is increasing and every $p_k \sqsubseteq_{(U,U)} \mathrm{Id}_U$ we obtain $p_n \circ p_m = p_{\min(n,m)}$.

$$\begin{array}{llll}
(v) & |(c_0\lambda x.M)N|_{\mathrm{Var}} = & (|\lambda x.M|_{\mathrm{Var}})_0 \bullet |N|_{\mathrm{Var}} & \text{by def. of } |-|_{\mathrm{Var}} \\
 & = & (|\lambda x.M|_{\mathrm{Var}} \bullet \perp)_0 & \text{by Def. 2.3.6(ii)} \\
 & = & |c_0((\lambda x.M)\underline{\perp})|_{\mathrm{Var}} & \text{by def. of } |-|_{\mathrm{Var}} \\
 & = & |c_0(M[\underline{\perp}/x])|_{\mathrm{Var}} & \text{by Lemma 2.3.9.} \\
(vi) & |(c_{k+1}\lambda x.M)N|_{\mathrm{Var}} = & (|\lambda x.M|_{\mathrm{Var}})_{k+1} \bullet |N|_{\mathrm{Var}} & \text{by def. of } |-|_{\mathrm{Var}} \\
 & = & (|\lambda x.M|_{\mathrm{Var}} \bullet (|N|_{\mathrm{Var}})_k)_k & \text{by Def. 2.3.6(i)} \\
 & = & |c_k((\lambda x.M)(c_k N))|_{\mathrm{Var}} & \text{by def. of } |-|_{\mathrm{Var}} \\
 & = & |c_k(M[c_k N/x])|_{\mathrm{Var}} & \text{by Lemma 2.3.9.}
\end{array}$$
□

As a consequence of the above proposition we get that every well stratifiable $\perp$-model is also a model of the labelled $\lambda\perp$-calculus. In other words, the following corollary holds.

Corollary 2.3.13. *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M, N \in \Lambda_{\underline{\perp}}^{lab}$, $M =_{\omega\gamma\epsilon} N$ implies $M =_{\mathscr{U}} N$.*

Proof. The result follows from Proposition 2.3.12, since the relation $=_{\mathscr{U}}$ is contextual. □

2.3.6 Completely labelled $\lambda\perp$-terms

We are now interested in studying the properties of those labelled $\lambda\perp$-terms M which are completely labelled. Roughly speaking, this means that every subterm of M "has" a label.

Definition 2.3.14. *The set of* completely labelled $\lambda\perp$*-terms is defined by induction: $c_k\underline{\perp}$ is a completely labelled $\lambda\perp$-term, for every k; $c_k x$ is a completely labelled $\lambda\perp$-term, for every x and k; if $M, N \in \Lambda_{\underline{\perp}}^{lab}$ are completely labelled then also $c_k(MN)$ and $c_k(\lambda x.M)$ are completely labelled for every x and k.*

Note that every completely labelled $\lambda\perp$-term is β-normal, since every lambda abstraction is "blocked" by a c_k.

Definition 2.3.15. *A* complete labelling *L of a term $M \in \Lambda_{\underline{\perp}}$ is a map which assigns to each subterm of M a natural number.*

Notation 2.3.16. *Given a term $M \in \Lambda_{\underline{\perp}}$ and a complete labelling L of M, we denote by M^L the resulting completely labelled $\lambda\perp$-term. In other words, M^L is the term defined by induction on the subterms of M as follows:*

- $\underline{\perp}^L = c_{L(\underline{\perp})}\underline{\perp}$,
- $x^L = c_{L(x)}x$,
- $(NP)^L = c_{L(NP)}(N^L P^L)$,
- $(\lambda x.N)^L = c_{L(\lambda x.N)}(\lambda x.N^L)$.

It is easy to check that the set of all complete labellings of M is directed with respect to the following partial ordering: $L_1 \sqsubseteq_{lab} L_2$ if, and only if, for each subterm N of M we have $L_1(N) \leq L_2(N)$. By structural induction on the subterms of M one proves that $L_1 \sqsubseteq_{lab} L_2$ implies $M^{L_1} \sqsubseteq_{\mathscr{U}} M^{L_2}$. Therefore, the set of M^L such that L is a complete labelling of M, is also directed with respect to $\sqsubseteq_{\mathscr{U}}$.

Lemma 2.3.17. *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M \in \Lambda_{\underline{\perp}}$ we have:*

$$|M|_{\mathrm{Var}} = \sqcup_L |M^L|_{\mathrm{Var}}.$$

Proof. By straightforward induction on M, using $a = \sqcup_{k\in\mathbb{N}} a_k$ (Equation 2.4) and Corollary 2.3.2. □

2.3.7 The Approximation Theorem and applications

Approximation theorems are an important tool in the analysis of the λ-theories induced by the models of λ-calculus. In this section we provide an Approximation Theorem for the class of well stratified $\perp$-models: we show that the interpretation of a λ-term in a well stratified $\perp$-model $\mathscr{U}$ is the least upper bound of the interpretations of its direct approximants. From this it follows first that $\mathrm{Th}(\mathscr{U})$ is sensible, and second that $\mathcal{B}_{\mathcal{T}} \subseteq \mathrm{Th}(\mathscr{U})$.

Definition 2.3.18. *Let $M, N \in \Lambda_{\underline{\perp}}$, then:*

(i) *N is an approximant of M if there exist a context $C[\xi_1, \dots, \xi_k]$ over $\Lambda_{\underline{\perp}}$, with $k \geq 0$, and $M_1, \dots, M_k \in \Lambda_{\underline{\perp}}$ such that $N \equiv C[\underline{\perp}, \dots, \underline{\perp}]$ and $M \equiv C[M_1, \dots, M_k]$;*

(ii) *N is an* approximate normal form *(*app-nf*, for short) of M if, furthermore, it is $\beta\omega$-normal.*

Given $M \in \Lambda$, we define the set $\mathcal{A}(M)$ of *all direct approximants* of M as follows:

$$\mathcal{A}(M) = \{W \in \Lambda_{\underline{\perp}} : \exists N, (M \twoheadrightarrow_\beta N) \text{ and } W \text{ is an app-nf of } N\}.$$

Remark 2.3.19. *The set $\mathcal{A}(M)$ admits an alternative characterization:*

- *$\mathcal{A}(M) = \{\underline{\perp}\}$ if M is unsolvable.*
- *Otherwise, M has a principal hnf $\lambda x_1 \dots x_n.xM_1 \cdots M_k$ and $\mathcal{A}(M) = \{\underline{\perp}\} \cup \{\lambda x_1 \dots x_n.xW_1 \cdots W_k : W_i \in \mathcal{A}(M_i)\}$.*

The proof of the following lemma is straightforward once recalled that, if $N \in \mathcal{A}(M)$, then M results (up to β-conversion) from N by replacing some $\underline{\perp}$ in N by other terms.

Lemma 2.3.20. *If $\mathscr{U}$ is a well stratifiable $\perp$-model and $M \in \Lambda$, then for all $N \in \mathcal{A}(M)$ we have $N \sqsubseteq_{\mathscr{U}} M$.*

Notation 2.3.21. *Given $M \in \Lambda_{\perp}^{lab}$ we will denote by $\overline{M} \in \Lambda_{\perp}$ the term obtained from M by erasing all labels.*

Lemma 2.3.22. *For all $M \in \Lambda_{\perp}^{lab}$, we have that $M \sqsubseteq_{\mathscr{U}} \overline{M}$.*

Proof. By Inequality (2.3) we have $(|M|_{\mathrm{Var}})_k \sqsubseteq |M|_{\mathrm{Var}}$, and this implies $c_k M \sqsubseteq_{\mathscr{U}} M$. We conclude the proof since $\sqsubseteq_{\mathscr{U}}$ is contextual. □

The following syntactic property, which is borrowed from [53] (but it is also a consequence of the results present in [8, Sec. 14.3]), will be used for proving the Approximation Theorem.

Proposition 2.3.23. *[53, Prop. 1.9] Let $M \in \Lambda$ and L be a complete labelling of M. If $\mathrm{nf}(M^L)$ is the $\omega\gamma\epsilon$-normal form of M^L, then $\overline{\mathrm{nf}(M^L)} \in \mathcal{A}(M)$.*

We prove now the Approximation Theorem for well stratified $\perp$-models.

Theorem 2.3.24. *(Approximation Theorem) If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M \in \Lambda$:*

$$|M|_{\mathrm{Var}} = \bigsqcup \mathcal{A}(M),$$

where $\bigsqcup \mathcal{A}(M) = \bigsqcup \{|W|_{\mathrm{Var}} : W \in \mathcal{A}(M)\}$.

Proof. Let L be a complete labelling for M. We know from Theorem 2.3.11 that the labelled $\lambda\perp$-calculus is Church Rosser and strongly normalizable, hence there exists a unique $\omega\epsilon\gamma$-normal form of M^L. We denote this normal form by $\mathrm{nf}(M^L)$. Since $M^L \twoheadrightarrow_{\epsilon\gamma\omega} \mathrm{nf}(M^L)$, and $\mathscr{U}$ is a model of the labelled $\lambda\perp$-calculus (Corollary 2.3.13), we have $M^L =_{\mathscr{U}} \mathrm{nf}(M^L)$. Moreover, Proposition 2.3.23 implies that $\overline{\mathrm{nf}(M^L)} \in \mathcal{A}(M)$ and hence $\mathrm{nf}(M^L) \sqsubseteq_{\mathscr{U}} \overline{\mathrm{nf}(M^L)}$ by Lemma 2.3.22. This implies that $|\mathrm{nf}(M^L)|_{\mathrm{Var}} \sqsubseteq \bigsqcup \mathcal{A}(M)$. Since L is an arbitrary complete labelling for M, we have:

$$\begin{array}{rcll} |M|_{\mathrm{Var}} & = & \sqcup_L |M^L|_{\mathrm{Var}} & \text{(by Lemma 2.3.17)} \\ & = & \sqcup_L |\mathrm{nf}(M^L)|_{\mathrm{Var}} & \\ & \sqsubseteq & \bigsqcup \mathcal{A}(M). & \end{array}$$

The opposite inequality is clear. □

Corollary 2.3.25. *$M \in \Lambda$ is unsolvable $\iff$ $M =_{\mathscr{U}} \perp$.*

Proof. ($\Rightarrow$) If M is unsolvable, then Remark 2.3.19 implies that $\mathcal{A}(M) = \{\perp\}$. Hence, $M =_{\mathscr{U}} \perp$ by Theorem 2.3.24.
($\Leftarrow$) If M is solvable, then by [8, Thm. 8.3.14] there exist $N_1, \ldots, N_k \in \Lambda$, with $k \geq 0$, such that $MN_1 \cdots N_k =_{\mathscr{U}} \mathbf{I}$. Since $\mathscr{U}$ is a $\perp$-model, $M =_{\mathscr{U}} \perp$ would imply $\mathbf{I} =_{\mathscr{U}} \perp$ (by Definition 2.3.3(i)) and $\mathscr{U}$ would be trivial. Contradiction. □

Corollary 2.3.26. *If $\mathscr{U}$ is a well stratifiable $\perp$-model, then $\mathrm{Th}(\mathscr{U})$ is sensible.*

We show now that the notion of Böhm tree can be also generalized to terms in $\Lambda_{\perp}$.

Definition 2.3.27. *For all $M \in \Lambda_{\perp}$ we write* $\mathrm{BT}(M)$ *for the Böhm tree of the λ-term obtained by substituting Ω for all occurrences of $\perp$ in M.* Vice versa, *for all $M \in \Lambda$ we denote by $M^{[k]} \in \Lambda_{\perp}$ the (unique) $\beta\omega$-normal form such that* $\mathrm{BT}(M^{[k]}) = \mathrm{BT}^k(M)$ *(where* $\mathrm{BT}^k(M)$ *is the Böhm tree of M pruned at level k).*

It is straightforward to check that, for every λ-term M, $M^{[k]} \in \mathcal{A}(M)$. *Vice versa*, the following proposition is a consequence of the Approximation Theorem.

Proposition 2.3.28. *If $\mathscr{U}$ is a well stratifiable $\perp$-model, then for all $M \in \Lambda$:*

$$|M|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}} |M^{[k]}|_{\mathrm{Var}}.$$

Proof. For all $W \in \mathcal{A}(M)$, there exists a $k \in \mathbb{N}$ such that all the nodes in $\mathrm{BT}(W)$ have depth less than k. Thus $W \sqsubseteq_{BT} M^{[k]}$ and $W \sqsubseteq_{\mathscr{U}} M^{[k]}$ by Theorem 2.3.24. Hence, the result follows. □

Corollary 2.3.29. *If $N \sqsubseteq_{BT} M$ then $N \sqsubseteq_{\mathscr{U}} M$.*

Proof. If $N \sqsubseteq_{BT} M$ then for all $k \in \mathbb{N}$ we have $N^{[k]} \sqsubseteq_{BT} M$. By Lemma 2.3.20 $N^{[k]} \sqsubseteq_{\mathscr{U}} M$. Thus $|N|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}} |N^{[k]}|_{\mathrm{Var}} \sqsubseteq |M|_{\mathrm{Var}}$ by Proposition 2.3.28. □

As a direct consequence we get the following result.

Theorem 2.3.30. *If $\mathscr{U}$ is a well stratifiable $\perp$-model, then $\mathcal{B}_{\mathcal{T}} \subseteq \mathrm{Th}(\mathscr{U})$.*

2.3.8 A class of models of $\mathcal{H}^*$

We recall that the λ-theory $\mathcal{H}^*$ can be characterized in terms of Böhm trees as follows:

$$M =_{\mathcal{H}^*} N \iff M \simeq_\eta N \qquad \text{cf. Subsection 1.2.3.} \tag{2.5}$$

The definition of $\simeq_\eta$ has been recalled in Subsection 1.2.2, together with those of $\sqsubseteq_{BT}$, $\sqsubseteq_{\eta,\infty}$, $\precsim_\eta$. However, for proving that $\mathrm{Th}(\mathscr{U}) = \mathcal{H}^*$, the following alternative characterization of $\sqsubseteq_{\eta,\infty}$ will be useful.

Theorem 2.3.31. *[8, Lemma 10.2.26] The following conditions are equivalent:*

- *$M \sqsubseteq_{\eta,\infty} N$,*
- *for all $k \in \mathbb{N}$ there exists $P_k \in \Lambda$ such that $P_k \twoheadrightarrow_\eta M$, and $P_k^{[k]} = N^{[k]}$.*

We need now some technical lemmata.

Lemma 2.3.32. *If $\mathscr{U}$ is an extensional well stratified $\perp$-model then, for all $M \in \Lambda_{\perp}$ and $x \in \mathrm{Var}$, $x \sqsubseteq_{\eta,\infty} M$ implies $c_n x \sqsubseteq_{\mathscr{U}} M$ for all $n \in \mathbb{N}$.*

Proof. From [8, Def. 10.2.10], we can assume that $M \equiv \lambda y_1 \dots y_m.xM_1 \cdots M_m$ with $y_i \sqsubseteq_{\eta,\infty} M_i$. The proof is done by induction on n.
If $n = 0$, then:

$$\begin{array}{rlll} c_0 x & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_0 x y_1 \cdots y_m & \text{since } \mathscr{U} \text{ is extensional,} \\ & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_0 (x\bot) y_2 \cdots y_m & \text{since } \mathscr{U} \text{ is well stratified (Def. 2.3.6(ii)),} \\ & \vdots & \vdots & \vdots \\ & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_0 (x\bot \cdots \bot) & \text{since } \mathscr{U} \text{ is well stratified (Def. 2.3.6(ii)),} \\ & \sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.x\bot \cdots \bot & \text{by Lemma 2.3.22,} \\ & \sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.xM_1 \cdots M_m & \text{by } \bot \sqsubseteq_{\mathscr{U}} M_i. \end{array}$$

If $n > 0$, then

$$\begin{array}{rlll} c_n x & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_n x y_1 \cdots y_m & \text{since } \mathscr{U} \text{ is extensional,} \\ & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_{n-1}(x(c_{n-1}y_1)) y_2 \dots y_m & \text{since } \mathscr{U} \text{ is stratified (Def. 2.3.6(i)),} \\ & \vdots & \vdots & \vdots \\ & =_{\mathscr{U}} & \lambda y_1 \dots y_m.c_{n-m}(x(c_{n-1}y_1) \cdots (c_{n-m}y_m)) & \text{since } \mathscr{U} \text{ is stratified (Def. 2.3.6(i)).} \end{array}$$

Recalling that $y_i \sqsubseteq_{\eta,\infty} M_i$, we have:

$$\begin{array}{rll} & \lambda y_1 \dots y_m.c_{n-m}(x(c_{n-1}y_1) \cdots (c_{n-m}y_m)) & \\ \sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.c_{n-m}(xM_1 \cdots M_m) & \text{since } c_{n-i}y_i \sqsubseteq_{\mathscr{U}} M_i \text{ by I.H.,} \\ \sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.xM_1 \cdots M_m & \text{by Lemma 2.3.22.} \end{array}$$

□

Lemma 2.3.33. *Let $\mathscr{U}$ be an extensional well stratified $\bot$-model and $M, N, W \in \Lambda_\bot$. If W is a $\beta\omega$-normal form such that $W \sqsubseteq_{BT} M$ and $M \sqsubseteq_{\eta,\infty} N$, then $W \sqsubseteq_{\mathscr{U}} N$.*

Proof. The proof is done by induction on the structure of W.
If $W \equiv \bot$, then it is trivial.
If $W \equiv x$ then also $M \equiv x$ and the result follows from Lemma 2.3.32 since $|x|_{\mathrm{Var}} = \sqcup_{n\in\mathbb{N}}(|x|_{\mathrm{Var}})_n$.
If $W \equiv \lambda x_1 \dots x_m.yW_1 \cdots W_r$, then $M =_{\lambda\beta} \lambda x_1 \dots x_m.yM_1 \cdots M_r$ and every W_i is a $\beta\omega$-normal form such that $W_i \sqsubseteq_{BT} M_i$ (for $i \le r$). By $M \sqsubseteq_{\eta,\infty} N$, we can assume that $N =_{\lambda_{\beta\eta}} \lambda x_1 \dots x_{m+s}.yN_1 \cdots N_{r+s}$, with $x_{m+k} \sqsubseteq_{\eta,\infty} N_{r+k}$ (for $1 \le k \le s$) and $M_i \sqsubseteq_{\eta,\infty} N_i$ (for $i \le r$). From $x_{m+k} \sqsubseteq_{\eta,\infty} N_{r+k}$ we obtain, using the previous lemma, that $x_{m+k} \sqsubseteq_{\mathscr{U}} N_{r+k}$. Moreover, since $W_i \sqsubseteq_{BT} M_i \sqsubseteq_{\eta,\infty} N_i$, the induction hypothesis implies $W_i \sqsubseteq_{\mathscr{U}} N_i$. Hence, we can conclude that $W \sqsubseteq_{\mathscr{U}} N$. □

Lemma 2.3.34. *If $\mathscr{U}$ is an extensional well stratifiable $\bot$-model then for all $M, N \in \Lambda$ we have:*

(i) $M \sqsubseteq_{\eta,\infty} N$ *implies* $M =_{\mathscr{U}} N$,

(ii) $M \precsim_\eta N$ *implies* $M \sqsubseteq_{\mathscr{U}} N$.

Proof. (i) Let us suppose that $M \sqsubseteq_{\eta,\infty} N$. Since all $W \in \mathcal{A}(M)$ are $\beta\omega$-normal forms such that $W \sqsubseteq_{BT} M$, the Approximation Theorem and Lemma 2.3.33 imply that $M \sqsubseteq_{\mathscr{U}} N$. We prove now that also $N \sqsubseteq_{\mathscr{U}} M$ holds. By the characterization of $\sqsubseteq_{\eta,\infty}$ given in Theorem 2.3.31 we know that for all $k \in \mathbb{N}$ there exists a λ-term P_k such that $P_k \twoheadrightarrow_\eta M$ and $P_k^{[k]} = N^{[k]}$. Since every $P_k^{[k]} \in \mathcal{A}(P_k)$, we have $P_k^{[k]} \sqsubseteq_{\mathscr{U}} P_k$; also, from the extensionality of $\mathscr{U}$, $P_k =_{\mathscr{U}} M$. Thus, by Proposition 2.3.28, we have $|N|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}}|N^{[k]}|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}}|P_k^{[k]}|_{\mathrm{Var}} \sqsubseteq |M|_{\mathrm{Var}}$. This implies that $N \sqsubseteq_{\mathscr{U}} M$.

(ii) Suppose now that $M \precsim_\eta N$. By definition, there exist two λ-terms M' and N' such that $M \sqsubseteq_{\eta,\infty} M' \sqsubseteq_{BT} N' \sqsupseteq_{\eta,\infty} N$. We conclude as follows: $M =_{\mathscr{U}} M'$ by (i), $M' \sqsubseteq_{\mathscr{U}} N'$ by Theorem 2.3.30, and $N' =_{\mathscr{U}} N$, again by (i). □

Theorem 2.3.35. *If $\mathscr{U}$ is a well stratifiable extensional $\perp$-model living in a cpo-enriched ccc (having countable products), then* $\mathrm{Th}(\mathscr{U}) = \mathcal{H}^*$.

Proof. By Lemma 2.3.34(ii) we have that $M \simeq_\eta N$ implies $M =_{\mathscr{U}} N$. Then, by the characterization (2.5), it follows that $\mathrm{Th}(\mathscr{U}) \supseteq \mathcal{H}^*$. We conclude since $\mathcal{H}^*$ is the maximal sensible consistent λ-theory. □

2.4 Conclusions

For historical reasons, most of the work on models of λ-calculus, and its extensions, has been carried out in subcategories of **CPO**. *A posteriori*, we can propose two motivations: (i) because of the seminal work of Scott, the Scott continuity of morphisms has been seen as *the* natural way of allowing the *existence* of reflexive objects; (ii) the classic result relating algebraic and categorical models of λ-calculus asks for reflexive objects *with enough points*.

In this chapter we have shown that *any* categorical model can be presented as a λ-model, even when the underlying category does not have enough points. We have defined a notion of well stratified $\perp$-model in general cpo-enriched ccc's and showed that every well stratified $\perp$-model living in such a category has $\mathcal{H}^*$ as equational theory.

It remains to be proved that, working outside concrete categories, we can get new interesting classes of models. A first step in this direction is done in the next chapter, where we build an extensional model in a category of sets and relations which has not enough points.

3

A relational model of λ-calculus

It is an ongoing natural question, from a category theoretic perspective, to see the various models of the untyped λ-calculus as reflexive objects in natural cartesian closed categories. [...] Our analysis of the Engeler model is that it naturally arises from a category without enough points. (Martin Hyland et al., from [60])

In this section we build a simple example of a categorical model $\mathscr{D}$ living in a ccc without enough points: the Kleisli-category of the comonad $\mathcal{M}_{\mathrm{f}}(-)$ of "finite multisets" over the category **Rel** of sets and relations. In [60], Hyland et al. define in this ccc a relational version, based on $\mathcal{M}_{\mathrm{f}}(-)$, of graph models and provide, as a paradigmatic example, the analogue of Engeler's model $\mathscr{E}$. Our model $\mathscr{D}$ is extensional and its construction is similar to that of Scott's $\mathscr{D}_\infty$ when built as a K-model, but simpler, and we will also prove that $\mathrm{Th}(\mathscr{D}) = \mathrm{Th}(\mathscr{D}_\infty) = \mathcal{H}^*$; hence, $\mathscr{D}$ can be viewed as a relational version of $\mathscr{D}_\infty$.

Finally, we will present some algebraic properties of the λ-model $\mathscr{C}_{\mathscr{D}}$ obtained from the categorical model $\mathscr{D}$ by applying the construction described in the previous chapter, which make it suitable for modelling non-deterministic extensions of the untyped λ-calculus.

3.1 Introduction

Having described, in the previous chapter, a general construction for extracting a λ-model from a reflexive object (possibly without enough points) of a ccc, we build here a simple example to which this construction can be applied.

In denotational semantics, ccc's without enough points may arise naturally when morphisms are not functions, like for instance sequential algorithms [21] or strategies in various categories of games [3, 61], and carry more "intensional" information than usual. The original motivation for these constructions was to obtain a semantic characterization of sequentiality, in the simply typed case.

A framework simpler than game semantics where reflexive objects cannot have enough points is the following. Given the category **Rel** of sets and relations, consider the comonad $\mathcal{M}_{\mathrm{f}}(-)$ of "finite multisets". **MRel**, the Kleisli category of $\mathcal{M}_{\mathrm{f}}(-)$, is a ccc which has been studied in particular as a semantic framework for linear logic [52, 4, 26]. Here, we study **MRel** as a *relational semantics* of λ-calculus.

An even simpler framework, based on **Rel**, would be provided by taking the functor "finite sets" instead of the comonad "finite multisets". The point is that the former is not a comonad. Nevertheless, a ccc may eventually be obtained in this case too, via a "quasi Kleisli" construction [60]. Interestingly, from the perspective of the work done in this chapter, these Kleisli categories over **Rel** are advocated in [60] as the "natural" categories in which standard models of the λ-calculus like Engeler's model $\mathscr{E}$, and graph models in general, *should* be considered.

In Section 3.3 we define a relational version, in **MRel**, of another classical model: Scott's $\mathscr{D}_\infty$. The construction of this reflexive object $\mathscr{D}$ is performed by an iterated completion process similar to the one for obtaining graph models from partial pairs (which will be recalled in Subsection 5.1.4). However, $\mathscr{D}$ happens to be trivially extensional. In fact, we will prove that $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^*$: this follows directly from the last result of Section 2.3 once observed that $\mathscr{D}$ is a well stratifiable $\perp$-model and **MRel** a cpo-enriched ccc.

Finally, in Section 3.4 we show that the λ-model $\mathscr{C}_{\mathscr{D}}$ associated with $\mathscr{D}$ by the construction described above has a rich algebraic structure. In particular, we define two operations of sum and product which are left distributive with respect to application and give to $\mathscr{C}_{\mathscr{D}}$ a structure of commutative semiring. This opens the way to the interpretation of conjunctive-disjunctive λ-calculi (see, e.g., [43]) in this relational framework.

3.2 A ccc of sets and relations

It is quite well known [52, 4, 60, 26] that, by endowing the monoidal closed category **Rel** of sets and relations with a suitable comonad, one gets a ccc via the co-Kleisli construction. In this section we present the ccc **MRel** obtained by using the comonad $\mathcal{M}_{\mathrm{f}}(-)$, without explicitly going through the monoidal structure of **Rel**. As we will see below, the categorical product in **MRel** is the disjoint union, hence we prefer to slightly modify the classic notation to avoid confusion.

Notation 3.2.1. *We denote the categorical product in* **MRel** *by* & *instead of* $\times$. *The symbol* $\times$ *is kept to denote the usual set-theoretical Cartesian product.*

Let us define directly the category **MRel** as follows:

- The objects of **MRel** are all the sets.
- Given two sets S and T, a morphism from S to T is a relation from $\mathcal{M}_{\mathrm{f}}(S)$ to T, in other words, $\mathbf{MRel}(S,T) = \mathscr{P}(\mathcal{M}_{\mathrm{f}}(S) \times T)$.
- The identity morphism of S is the relation:

$$\mathrm{Id}_S = \{([a], a) : a \in S\} \in \mathbf{MRel}(S,S)\,.$$

- Given two morphisms $s \in \mathbf{MRel}(S,T)$ and $t \in \mathbf{MRel}(T,U)$, we define:
$$t \circ s = \{(m,c) : \ \exists k \geq 0,\ \exists (m_1,b_1),\ldots,(m_k,b_k) \in s \text{ such that } m = m_1 \uplus \ldots \uplus m_k \text{ and } ([b_1,\ldots,b_k],c) \in t\}.$$

It is easy to check that this composition law is associative, and that the identity morphisms defined above are neutral for this composition.

Theorem 3.2.2. *The category* **MRel** *is Cartesian closed and has countable products.*

Proof. The terminal object $\mathbb{1}$ is the empty set $\emptyset$, and the unique element of $\mathbf{MRel}(S,\emptyset)$ is the empty relation.

Given two sets S_1 and S_2, their categorical product $S_1 \,\&\, S_2$ in **MRel** is their disjoint union:
$$S_1 \& S_2 = (\{1\} \times S_1) \cup (\{2\} \times S_2)$$
and the projections π_1, π_2 are given by:
$$\pi_i = \{([(i,a)],a) : a \in S_i\} \in \mathbf{MRel}(S_1 \& S_2, S_i), \text{ for } i = 1,2.$$
It is easy to check that this is actually the categorical product of S_1 and S_2 in **MRel**; given $s \in \mathbf{MRel}(U,S_1)$ and $t \in \mathbf{MRel}(U,S_2)$, the corresponding morphism $\langle s,t\rangle \in \mathbf{MRel}(U, S_1 \& S_2)$ is given by:
$$\langle s,t\rangle = \{(m,(1,a)) : (m,a) \in s\} \cup \{(m,(2,b)) : (m,b) \in t\}\,.$$
This definition extends to arbitrary I-indexed families $(S_i)_{i\in I}$ of sets in the obvious way:
$$\&_{i\in I} S_i = \cup_{i\in I}(\{i\} \times S_i),$$
$$\pi_i = \{([(i,a)],a) : a \in S_i\} \in \mathbf{MRel}(\&_{i\in I} S_i, S_i), \text{ for } i \in I.$$
In particular, **MRel** has countable products.

Notice now that there exists a canonical bijection between $\mathcal{M}_{\mathrm{f}}(S_1) \times \mathcal{M}_{\mathrm{f}}(S_2)$ and $\mathcal{M}_{\mathrm{f}}(S_1 \,\&\, S_2)$ which maps the pair $([a_1,\ldots,a_p],[b_1,\ldots,b_q])$ to the multiset $[(1,a_1),\ldots,(1,a_p),(2,b_1),\ldots,(2,b_q)]$. We will confuse this bijection with an equality, hence we will still denote by (m_1,m_2) the corresponding element of $\mathcal{M}_{\mathrm{f}}(S_1 \& S_2)$.

Given two objects S and T, the exponential object $[S \Rightarrow T]$ is $\mathcal{M}_{\mathrm{f}}(S) \times T$ and the evaluation morphism is given by:
$$ev_{ST} = \{(([(m,b)],m),b) : m \in \mathcal{M}_{\mathrm{f}}(S) \text{ and } b \in T\} \in \mathbf{MRel}([S \Rightarrow T] \& S, T)\,.$$
Again, it is easy to check that in this way we defined an exponentiation. Indeed, given any set U and any morphism $s \in \mathbf{MRel}(U \,\&\, S, T)$, there is exactly one morphism $\Lambda(s) \in \mathbf{MRel}(U,[S \Rightarrow T])$ such that:
$$ev_{ST} \circ (\Lambda(s) \times \mathrm{Id}_S) = s.$$
where $\Lambda(s) = \{(p,(m,b)) : ((p,m),b) \in s\}$. $\square$

Here, the points of an object S, i.e. the elements of $\mathbf{MRel}(\mathbb{1}, S)$, are the relations between $\mathcal{M}_{\mathrm{f}}(\emptyset)$ and S, and hence, up to isomorphism, the subsets of S.

In the next section we will present an extensional model of λ-calculus which lives in **MRel**, although **MRel** which is "strongly" non-extensional in the sense expressed by the following theorem. The existence of such a model contradicts a common belief.

Theorem 3.2.3. *No object $U \neq \mathbb{1}$ of* **MRel** *has enough points.*

Proof. We can always find $t_1, t_2 \in \mathbf{MRel}(U, U)$ such that $t_1 \neq t_2$ and, for all $s \in \mathbf{MRel}(\mathbb{1}, U)$, $t_1 \circ s = t_2 \circ s$. Recall that, by definition of composition, $t_1 \circ s = \{([], b) : \exists a_1, \dots, a_n \in U \;\; ([], a_i) \in s \quad ([a_1, \dots, a_n], b) \in t_1\} \in \mathbf{MRel}(\mathbb{1}, U)$, and similarly for $t_2 \circ s$. Hence it is sufficient to choose $t_1 = \{(m_1, b)\}$ and $t_2 = \{(m_2, b)\}$ such that m_1, m_2 are different multisets with the same support. $\square$

Corollary 3.2.4. **MRel** *has not enough points*

3.3 An extensional relational model of λ-calculus

In this section we build a reflexive object $\mathscr{D}$ in **MRel**, which is extensional by construction.

3.3.1 Constructing an extensional reflexive object.

We build a family of sets $(D_n)_{n \in \mathbb{N}}$ as follows[1]:

- $D_0 = \emptyset$,
- $D_{n+1} = \mathcal{M}_{\mathrm{f}}(D_n)^{(\omega)}$.

Since the operation $S \mapsto \mathcal{M}_{\mathrm{f}}(S)^{(\omega)}$ is monotonic on sets, and since $D_0 \subseteq D_1$, we have $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$. Finally, we set $D = \cup_{n \in \mathbb{N}} D_n$.

So we have $D_0 = \emptyset$ and $D_1 = \{*\} = \{([], [], \dots)\}$. The elements of D_2 are quasi-finite sequences of multisets over a singleton, i.e., quasi-finite sequences of natural numbers. More generally, an element of D can be represented as a finite tree which alternates two kinds of layers:

- ordered nodes (the quasi-finite sequences), where immediate subtrees are indexed by distinct natural numbers,
- unordered nodes where subtrees are organised in a *non-empty* multiset.

[1] We could more generally start from a set A of "atoms" and take: $D_0 = \emptyset$, $D_{n+1} = \mathcal{M}_{\mathrm{f}}(D_n)^{(\omega)} \times A$. Our model $\mathscr{D}$ corresponds to take as A any singleton. We did not check yet whether $\mathscr{D}$ would remain extensional if $\mathrm{card}(A) \geq 2$.

In order to define an isomorphism in **MRel** between D and $[D \Rightarrow D] = \mathcal{M}_{\mathrm{f}}(D) \times D$ it is enough to remark that every element $\sigma \in D$ is canonically associated with the pair $(\sigma_0, (\sigma_1, \sigma_2, \ldots))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_{\mathrm{f}}(D)$, we write $m :: \sigma$ for the element $\tau \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_{\mathrm{f}}(D) \times D$ and D, and hence an isomorphism in **MRel** as follows:

Proposition 3.3.1. *The triple* $\mathscr{D} = (D, \mathrm{Ap}, \lambda)$ *where:*

- $\lambda = \{([(m, \sigma)], m :: \sigma) : m \in \mathcal{M}_{\mathrm{f}}(D), \sigma \in D\} \in \mathbf{MRel}([D \Rightarrow D], D)$,
- $\mathrm{Ap} = \{([m :: \sigma], (m, \sigma)) : m \in \mathcal{M}_{\mathrm{f}}(D), \sigma \in D\} \in \mathbf{MRel}(D, [D \Rightarrow D])$,

is an extensional categorical model of λ*-calculus.*

Proof. It is trivial that $\lambda \circ \mathrm{Ap} = \mathrm{Id}_D$ and $\mathrm{Ap} \circ \lambda = \mathrm{Id}_{[D \Rightarrow D]}$. □

Since $\mathscr{D}$ is such that $D \cong [D \Rightarrow D]$ and **MRel** has countable products, the construction given in Subsection 2.2.2 provides first an applicative structure $\mathscr{A}_{\mathscr{D}} = (\mathbf{MRel}_{\mathrm{f}}(D^{\mathrm{Var}}, D), \bullet)$, and second an associated (environment) λ-model $\mathscr{S}_{\mathscr{D}} = (\mathscr{A}_{\mathscr{D}}, [\![-]\!])$ which is extensional by Theorem 2.2.12(2).

3.3.2 Interpreting untyped λ-calculus in $\mathscr{D}$

In Subsection 1.3.1, we have recalled how we can interpret a λ-term in any reflexive object of a ccc. We provide the result of the corresponding computation, when it is performed in $\mathscr{D}$.

Given a λ-term M and a sequence[2] $\vec{x}$ of length n containing all the free variables of M, the interpretation $|M|_{\vec{x}}$ is an element of $\mathbf{MRel}(D^n, D)$, where $D^n = D \,\&\, \ldots \,\&\, D$, i.e., $|M|_{\vec{x}} \subseteq \mathcal{M}_{\mathrm{f}}(D)^n \times D$. $|M|_{\vec{x}}$ is defined by structural induction on M.

- $|x_i|_{\vec{x}} = \{(([], \ldots, [], [\sigma], [], \ldots, []), \sigma) : \sigma \in D\}$, where the only non-empty multiset occurs in the i-th position.
- $$\begin{array}{ll} |NP|_{\vec{x}} = \{((m_1, \ldots, m_n), \sigma) : \exists k \in \mathbb{N} & \\ \quad \exists (m_1^j, \ldots, m_n^j) \in \mathcal{M}_{\mathrm{f}}(D)^n & \text{for } j = 0, \ldots, k \\ \quad \exists \sigma_1, \ldots, \sigma_k \in D & \text{such that} \\ \quad m_i = m_i^0 \uplus \ldots \uplus m_i^k & \text{for } i = 1, \ldots, n \\ \quad ((m_1^0, \ldots, m_n^0), [\sigma_1, \ldots, \sigma_k] :: \sigma) \in |N|_{\vec{x}} & \\ \quad ((m_1^j, \ldots, m_n^j), \sigma_j) \in |P|_{\vec{x}} & \text{for } j = 1, \ldots, k\} \end{array}$$
- $|\lambda z.P|_{\vec{x}} = \{((m_1, \ldots, m_n), m :: \sigma) : ((m_1, \ldots, m_n, m), \sigma) \in |P|_{\vec{x},z}\}$, where we assume that z does not occur in $\vec{x}$.

Note that if $M \in \Lambda^o$ then $|M| \subseteq D$. If M is moreover solvable and $M =_{\lambda_\beta} \lambda x_1 \ldots x_n.x_i M_1 \cdots M_k$ $(n, k \geq 0)$, then $|M| \neq \emptyset$. Indeed, it is easy to check that $[] :: \cdots :: [] :: [*] :: * \in |M|$ (where $[*]$ is precedeed by $i - 1$ occurrences of $[]$).

[2] It is convenient, here, to use sequences instead of sets of variables. This allows us to simplify the presentation of the interpretation.

3.3.3 The equational theory of $\mathscr{D}$ is $\mathcal{H}^*$

The first question naturally arising when a new model of λ-calculus is introduced concerns its equational theory. Not surprisingly, it turns out that the λ-theory induced by $\mathscr{D}$ is $\mathcal{H}^*$. From Theorem 2.3.35 it is enough to check that **MRel** is a cpo-enriched ccc and that $\mathscr{D}$ is a well stratifiable $\perp$-model.

Theorem 3.3.2. *The ccc* **MRel** *is cpo-enriched.*

Proof. It is clear that, for all sets S, T, the homset $(\mathbf{MRel}(S,T), \subseteq, \emptyset)$ is a cpo, that composition is continuous, and pairing and currying are monotonic. Finally, it is easy to check that the strictness conditions hold. □

Theorem 3.3.3. *$\mathscr{D}$ is a well stratifiable $\perp$-model.*

Proof. By definition of Ap and λ it is straigthforward to check that $\emptyset \bullet a = \emptyset$, for all $a \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$, and that $\lambda \circ \Lambda(\emptyset) = \emptyset$, hence $\mathscr{D}$ is a $\perp$-model. Let now $p_n = \{([\sigma], \sigma) : \sigma \in D_n\}$, where $(D_n)_{n\in\mathbb{N}}$ is as in Subsection 3.3.1. Since $(D_n)_{n\in\mathbb{N}}$ is increasing also $(p_n)_{n\in\mathbb{N}}$ is, and furthermore $\sqcup_{n\in\mathbb{N}} p_n = \{([\sigma], \sigma) : \sigma \in D\} = \mathrm{Id}_D$. Then, easy calculations show that $\mathscr{D}$ enjoys conditions (i) and (ii) of Definition 2.3.6. □

Corollary 3.3.4. $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^*$.

Proof. By Theorem 2.3.35. □

3.4 Modelling non-determinism via $\mathscr{D}$

In this section we show that the λ-model $\mathscr{C}_{\mathscr{D}}$ is suitable for modelling non-determinism. A variety of non-deterministic and parallel operators have been introduced by several authors with different aims [1, 2, 24, 55, 82]. Here, we focus our attention on the non-deterministic extensions of λ-calculus $\Lambda_{(+)}$ and $\Lambda_{(+,\|)}$, respectively introduced by de'Liguoro and Piperno in [42] and Dezani et al. in [43].

Recall that $\mathscr{C}_{\mathscr{D}} = (A_{\mathscr{D}}, \bullet, [\![\mathbf{K}]\!], [\![\mathbf{S}]\!])$, where $A_{\mathscr{D}} = \mathbf{MRel}_{\mathrm{f}}(D^{\mathrm{Var}}, D)$. For modelling non-determinism, we are going to define two operations of sum and product, respectively denoted by $\oplus$ and $\odot$, on $A_{\mathscr{D}}$. In order to show easily that these operations are well defined, we provide a characterization of the finitary morphisms in $\mathbf{MRel}(D^{\mathrm{Var}}, D)$.

Proposition 3.4.1. *$f \in \mathbf{MRel}_{\mathrm{f}}(D^{\mathrm{Var}}, D)$ whenever there exists $J \subseteq_{\mathrm{f}} \mathrm{Var}$ such that for all $((m_{x_1}, \ldots, m_{x_n}, \ldots), \sigma) \in f$ we have $m_{x_i} = []$ for every $x_i \notin J$.*

Proof. Straightforward. □

3.4.1 Non-deterministic choice

We define a first binary operation on $A_{\mathscr{D}}$, denoted by $\oplus$, which can be thought of as non-deterministic choice.

Definition 3.4.2. *For all $a, b \in \mathbf{MRel}_{\mathrm{f}}(D^{\mathrm{Var}}, D)$ we set $a \oplus b = a \cup b$.*

From Proposition 3.4.1, it is clear that $\mathbf{MRel}_{\mathrm{f}}(D^{\mathrm{Var}}, D)$ is closed under $\oplus$. De'Liguoro and Piperno proposed, as underlying structure of the models of the non-deterministic λ-calculus $\Lambda_{(+)}$, the notion of *semilinear applicative structure.*

Definition 3.4.3. *(de'Liguoro and Piperno [42]) A* semilinear applicative structure *is a pair $((A, \cdot), +)$ such that:*

(i) $(A, \cdot)$ *is an applicative structure.*

(ii) $+ : A^2 \to A$ *is an idempotent, commutative and associative operation.*

(iii) $\forall x, y, z \in A \ \ (x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

Straightforwardly, the operation $\oplus$ makes the applicative structure $(A_{\mathscr{D}}, \bullet)$ associated with $\mathscr{D}$ semilinear.

Proposition 3.4.4. $((A_{\mathscr{D}}, \bullet), \oplus)$ *is a semilinear applicative structure.*

The interpretation mapping of an environment λ-model can now be extended to the non-deterministic λ-calculus $\Lambda_{(+)}$ of [42], by stipulating that $[\![M \oplus N]\!]_\rho = [\![M]\!]_\rho \oplus [\![N]\!]_\rho$. Hence, we get that $(\mathscr{A}_{\mathscr{D}}, \oplus, [\![-]\!])$ is an extensional environment λ-model of $\Lambda_{(+)}$ in the sense of [42].

3.4.2 Parallel composition

We define another binary operation on $A_{\mathscr{D}}$, denoted by $\odot$, which can be thought of as parallel composition.

Definition 3.4.5.

- *Given $\sigma, \tau \in D$, we set $\sigma \odot \tau = (\sigma_1 \uplus \tau_1, \ldots, \sigma_n \uplus \tau_n, \ldots)$.*
- *Given $a, b \in A_{\mathscr{D}}$, we set $a \odot b = \{(m_1 \uplus m_2, \sigma \odot \tau) : (m_1, \sigma) \in a, (m_2, \tau) \in b\}$.*

Once again, from Proposition 3.4.1 it follows that $A_{\mathscr{D}}$ is closed under $\odot$. Note that $A_{\mathscr{D}}$, equipped with $\odot$, is *not* a semilinear applicative structure, simply because the operator $\odot$ is not idempotent. Nevertheless, right distributivity of $\bullet$ with respect to $\odot$ holds, as expressed in the following proposition whose proof is straightforward.

Proposition 3.4.6. *For all $a, b, c \in A_{\mathscr{D}}$, $(a \odot b) \bullet c = (a \bullet c) \odot (b \bullet c)$.*

$(A_{\mathscr{D}}, \oplus, 0)$ and $(A_{\mathscr{D}}, \odot, 1)$ are commutative monoids, where $0 = \emptyset$ and $1 = \{([], *)\}$. Moreover, 0 annihilates $\odot$, and multiplication distributes over addition. Summing up, we have:

Proposition 3.4.7.

(i) $(A_{\mathscr{D}}, \oplus, \odot, 0, 1)$ *is a commutative semiring.*

(ii) $\bullet$ *is right distributive over* $\oplus$ *and* $\odot$*.*

(iii) $\oplus$ *is idempotent.*

In order to interpret conjunctive-disjunctive λ-calculi, endowed with both "non-deterministic choice" and "parallel composition", a notion of λ-*lattice* have been introduced in [43]. It is interesting to notice that our structure $(A_{\mathscr{D}}, \subseteq, \bullet, \oplus, \odot)$ does not give rise to a real λ-lattice, essentially because $\odot$ is not idempotent. Roughly speaking, this means that in the model $\mathscr{C}_{\mathscr{D}}$ of the conjunctive-disjunctive λ-calculus $[\![M||M]\!] \neq [\![M]\!]$. In other words, *this model is "resource sensible"*.

3.5 Conclusions and further works

We have proved that **MRel** is a cpo-enriched ccc having countable products. We have built an extensional reflexive object $\mathscr{D}$ of **MRel**, and applied the construction described in the previous chapter to obtain a λ-model $\mathscr{C}_{\mathscr{D}}$. We have noticed that, by construction, $\mathscr{D}$ is a well-stratified $\perp$-model. Hence, $\mathrm{Th}(\mathscr{D}) = \mathrm{Th}(\mathscr{D}_\infty) = \mathcal{H}^*$ by Theorem 2.3.35. Therefore, $\mathscr{D}$ may be considered as a relational version of Scott's $\mathscr{D}_\infty$.

Finally, we have shown that $\mathscr{C}_{\mathscr{D}}$ has a quite rich algebraic structure, which make it suitable for modelling non-determinism. We aim to investigate in the future full abstraction results for must/may semantics in $\mathscr{C}_{\mathscr{D}}$.

4

The indecomposable semantics

Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world.
(Alfred North Whitehead)

Stone representation theorem, which is one of the milestones of universal algebra, states that every Boolean algebra is isomorphic to a Boolean product of direct indecomposable Boolean algebras (i.e., of algebras which cannot be decomposed as the Cartesian product of two other non-trivial algebras). We recover the usual formulation of Stone representation theorem since there exists a unique directly indecomposable Boolean algebra, namely the Boolean algebra of truth values.

Using *central elements*, which play here the role of idempotent elements in rings, we will show that combinatory algebras satisfy a similar theorem: every combinatory algebra can be decomposed as a *weak* Boolean product of directly indecomposable algebras which constitute, then, the "building blocks" in the variety of combinatory algebras. The notion of directly indecomposable combinatory algebra appears to be so relevant that we find it even interesting to speak of the "*indecomposable semantics*" to denote the class of models of λ-calculus which are directly indecomposable as combinatory algebras. This semantics is very general since, as we will see, it encompasses the Scott-continuous, the stable and the strongly stable semantics, as well as the term models of all semi-sensible λ-theories.

We will also show that "being an indecomposable combinatory algebra" can be expressed by a universal formula, and this property will be then fruitfully applied for proving that the indecomposable semantics, although so general, is (largely) incomplete. More precisely, we will prove that it omits a set of λ-theories which is $2^{\aleph_0}$-broad and also contains countably many $2^{\aleph_0}$-high intervals.

This gives in particular a *new* and uniform proof of the large incompleteness of the three main semantics.

4.1 Introduction

The λ-calculus is not a genuine equational theory since the variable-binding properties of lambda abstraction prevent variables in λ-calculus from playing the role of real algebraic variables. Consequently the general methods that have been developed in universal algebra are not directly applicable.

Algebraic reformulations of λ-calculus. There have been several attempts to reformulate λ-calculus as a purely algebraic theory.

The earliest, and best known, algebraic models are the *λ-models* that we presented in Subsection 1.3.3. They are special combinatory algebras which provide a first-order, but not equational, characterization of the models of λ-calculus.

Much more recently, Pigozzi and Salibra proposed the *Lambda Abstraction Algebras* as an alternative first-order description of the models of λ-calculus [84, 90]. Lambda Abstraction Algebras form an equational class and allow to keep the lambda-notation and, hence, all the functional intuitions.

Negative algebraic results. Combinatory algebras are considered algebraically pathological since they are never commutative, associative, finite or recursive [8, Prop. 5.1.15]. Moreover, Salibra and Lusin [76] showed that only trivial lattice identities are satisfied by all congruence lattices of combinatory algebras. Thus, it is not possible to apply the results developed in universal algebra (see [31, 79]) in the last thirty years which connect the lattice identities satisfied by all the congruence lattices of algebras belonging to a variety with Mal'cev conditions.

A positive result: a representation theorem for combinatory algebras. One of the milestones of modern algebra is the Stone representation theorem for Boolean algebras. This result was first generalized by Pierce to commutative rings with unit and next by Comer to the class of algebras with Boolean factor congruences (see [35, 63, 83]). By applying a theorem due to Vaggione [106], we show that Comer's generalization holds for combinatory algebras: any combinatory algebra is isomorphic to a weak Boolean product of directly indecomposable combinatory algebras (i.e., algebras which cannot be decomposed as the Cartesian product of two other non-trivial algebras). This can be expressed as follows in terms of sheaves: every combinatory algebra is isomorphic to the algebra of global sections of a sheaf of indecomposable combinatory algebras over a Boolean space.

The proof of the representation theorem for combinatory algebras is based on the fact that every combinatory algebra has *central elements*, i.e., elements which induce a direct decomposition of the algebra as the Cartesian product of two other combinatory algebras, just like idempotent elements in rings or complemented elements in bounded distributive lattices.

We show that the central elements of a combinatory algebra constitute a Boolean algebra, whose Boolean operations can moreover be internally defined by suitable combinators. This result suggests a connection between propositional classic logic and combinatory logic; what is the real meaning of this connection remains to be investigated.

The indecomposable semantics. We investigate the class of all models of λ-calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short). We show that the indecomposable semantics includes the Scott-continuous, the stable and the strongly stable semantics, as well as the term models of all semi-sensible λ-theories.

However, we also prove that the indecomposable semantics is incomplete, and that this incompleteness is, also, as large as possible: (i) there exists a continuum of pairwise incompatible λ-theories which are omitted by the indecomposable semantics (ii) for every recursively enumerable λ-theory $\mathcal{T}$ there is a continuum of λ-theories including $\mathcal{T}$, and forming an interval, which are omitted by the indecomposable semantics. This gives a *new* and uniform proof of the large incompleteness of each of the main semantics.

In one of the last results of this chapter we show that the set of λ-theories representable in each of the classic semantics of λ-calculus is not closed under finite intersection, in particular it is not a sublattice of $\lambda\mathscr{T}$.

An historical excursus on the previous incompleteness results. It was already known that the main semantics are *equationally incomplete*: they do not even match the most natural operational semantics of λ-calculus. The problem of the equational completeness was negatively solved by Honsell and Ronchi Della Rocca [58] for the Scott-continuous semantics, by Bastonero and Gouy [11, 53] for the stable semantics and by Bastonero [10] for H-models. In [91, 92] Salibra proved that the "monotonous semantics" (i.e., the class of all λ-models involving monotonicity with respect to some partial order and having a bottom element) is incomplete, thus giving a first uniform proof of incompleteness encompassing the three main semantics.

Outline of the chapter. This chapter is organized as follows: In Section 4.2 we review the basic definitions of universal algebra which are involved in the rest of the chapter. In particular, we recall the formal definitions of a Boolean product. The Stone representation theorem for combinatory algebras is presented in Section 4.3. Section 4.4 is devoted to the equational incompleteness of the indecomposable semantics.

4.2 Algebras and Boolean products

In this section we briefly recall the concepts of universal algebra which will be useful in the sequel.

4.2.1 Algebras

Definition 4.2.1. *A* congruence ϑ *on an algebra* $\mathscr{A}$ *is an equivalence relation which is compatible with the basic operations of* $\mathscr{A}$.

We will denote by $\mathrm{Con}(\mathscr{A})$ the set of congruences of $\mathscr{A}$. Since a congruence ϑ on $\mathscr{A}$ can be viewed as a subset of $A \times A$, $(\mathrm{Con}(\mathscr{A}), \subseteq)$ is a complete lattice (meet is set-theoretical intersection).

The lattice $(\mathrm{Con}(\mathscr{A}), \subseteq)$ contains a top and a bottom element:

$$\nabla^{\mathscr{A}} = A \times A$$
$$\Delta^{\mathscr{A}} = \{(a,a) : a \in A\}.$$

A congruence ϑ on $\mathscr{A}$ is called *trivial* if it is equal to $\nabla^{\mathscr{A}}$ or $\Delta^{\mathscr{A}}$.

Notation 4.2.2. *Given* $a, b \in A$ *we write* $\vartheta(a,b)$ *for the least congruence relating* a *and* b.

Given two congruences σ and τ on the algebra $\mathscr{A}$, we can form their *relative product*:

$$\tau \circ \sigma = \{(a,c) : \exists b \in A \ a\sigma b\tau c\}.$$

It is easy to check that $\tau \circ \sigma$ is still a compatible relation on $\mathscr{A}$, but not necessarily a congruence (transitivity or symmetry could fail).

Definition 4.2.3. *An algebra* $\mathscr{A}$ *is* simple *when its only congruences are* $\Delta^{\mathscr{A}}$ *and* $\nabla^{\mathscr{A}}$.

Given two algebras $\mathscr{A}, \mathscr{B}$, we denote by $\mathscr{A} \times \mathscr{B}$ their *(direct) product* and we let $\mathscr{A} \cong \mathscr{B}$ mean that they are isomorphic. Recall that the *product congruence* of $\vartheta_1 \in \mathrm{Con}(\mathscr{A})$ and $\vartheta_2 \in \mathrm{Con}(\mathscr{B})$ is the congruence $\vartheta_1 \times \vartheta_2$ on $\mathscr{A} \times \mathscr{B}$ defined by: $(b,c)\ \vartheta_1 \times \vartheta_2\ (b',c')$ if, and only if, $b\ \vartheta_1\ b'$ and $c\ \vartheta_2\ c'$.

An algebra is *trivial* if its underlying set is a singleton.

Definition 4.2.4. *An algebra* $\mathscr{A}$ *is* directly decomposable *if there exist two non-trivial algebras* $\mathscr{B}, \mathscr{C}$ *such that* $\mathscr{A} \cong \mathscr{B} \times \mathscr{C}$.

Definition 4.2.5. *An algebra* $\mathscr{A}$ *is a* subdirect product *of the algebras* $(\mathscr{B}_i)_{i \in I}$, *written* $\mathscr{A} \leq \Pi_{i \in I} \mathscr{B}_i$, *if there exists an embedding* f *of* $\mathscr{A}$ *into the direct product* $\Pi_{i \in I} \mathscr{B}_i$ *such that the projection* $\pi_i \circ f : \mathscr{A} \to \mathscr{B}_i$ *is onto for every* $i \in I$.

Definition 4.2.6. *A non-empty class* $\mathbb{K}$ *of algebras of the same similarity type is:*

(i) a variety *if it is closed under subalgebras, homomorphic images and direct products,*

(ii) an equational class *if it is axiomatizable by a set of equations.*

Birkhoff proved in [23] (see also [79, Thm. 4.131]) that conditions (i) and (ii) in the above definition are equivalent.

4.2.2 Factor congruences

Definition 4.2.7. *A congruence ϑ on an algebra $\mathscr{A}$ is a* factor congruence *if there exists another congruence $\overline{\vartheta}$ such that $\vartheta \cap \overline{\vartheta} = \Delta^{\mathscr{A}}$ and $\vartheta \circ \overline{\vartheta} = \nabla^{\mathscr{A}}$. In this case we call $(\vartheta, \overline{\vartheta})$ a* pair of complementary factor congruences.

It is easy to check that the homomorphism $f : \mathscr{A} \to \mathscr{A}/\vartheta \times \mathscr{A}/\overline{\vartheta}$ defined by $f(x) = (x/\vartheta, x/\overline{\vartheta})$ is:

- injective if and only if $\vartheta \cap \overline{\vartheta} = \Delta^{\mathscr{A}}$,
- onto if and only if $\vartheta \circ \overline{\vartheta} = \nabla^{\mathscr{A}}$ if and only if $\overline{\vartheta} \circ \vartheta = \nabla^{\mathscr{A}}$.

Hence, $(\vartheta, \overline{\vartheta})$ is a pair of complementary factor congruences of $\mathscr{A}$ if, and only if, $\mathscr{A} \cong \mathscr{B} \times \mathscr{C}$, where $\mathscr{B} \cong \mathscr{A}/\vartheta$ and $\mathscr{C} \cong \mathscr{A}/\overline{\vartheta}$.

So, the existence of factor congruences is just another way of saying "this algebra is a direct product of simpler algebras".

The set of factor congruences of $\mathscr{A}$ is not, in general, a sublattice of $\mathrm{Con}(\mathscr{A})$. $\Delta^{\mathscr{A}}$ and $\nabla^{\mathscr{A}}$ are the *trivial* factor congruences, corresponding to $\mathscr{A} \cong 1 \times \mathscr{A}$; of course, 1 is isomorphic to $\mathscr{A}/\nabla^{\mathscr{A}}$ and $\mathscr{A}$ is isomorphic to $\mathscr{A}/\Delta^{\mathscr{A}}$.

Lemma 4.2.8. *An algebra $\mathscr{A}$ is* directly indecomposable *when $\mathscr{A}$ admits only the two trivial factor congruences ($\Delta^{\mathscr{A}}$ and $\nabla^{\mathscr{A}}$).*

Clearly, every simple algebra is directly indecomposable, while there are algebras which are directly indecomposable but not simple: they have congruences, which however do not split the algebra up neatly as a Cartesian product.

4.2.3 Decomposition operators

Factor congruences can be characterized in terms of certain algebra homomorphisms called *decomposition operators* as follows (see [79, Def. 4.32] for more details).

Definition 4.2.9. *A* decomposition operation *for an algebra $\mathscr{A}$ is a function $f : A \times A \to A$ such that*

- *$f(x, x) = x$;*
- *$f(f(x, y), z) = f(x, z) = f(x, f(y, z))$;*
- *f is an algebra homomorphism from $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$.*

There exists a bijective correspondence between pairs of complementary factor congruences and decomposition operations, and thus, between decomposition operations and factorizations like $\mathscr{A} \cong \mathscr{B} \times \mathscr{C}$.

Proposition 4.2.10. *Given a decomposition operator f the binary relations ϑ and $\overline{\vartheta}$ defined by:*

$$x\ \vartheta\ y \quad \textit{if, and only if,} \quad f(x,y) = y,$$
$$x\ \overline{\vartheta}\ y \quad \textit{if, and only if,} \quad f(x,y) = x,$$

form a pair of complementary factor congruences. Conversely, given a pair $(\vartheta, \overline{\vartheta})$ of complementary factor congruences, the map f defined by:

$$f(x,y) = u \quad \textit{if, and only if,} \quad x\ \vartheta\ u\ \overline{\vartheta}\ y, \tag{4.1}$$

is a decomposition operation.

Proof. The proof is easy (see [79, Thm. 4.33] for more details). Note that the definition of f is sound because of the following remark. □

Remark 4.2.11. *If $(\vartheta, \overline{\vartheta})$ is a pair of complementary factor congruences, then for all x and y there is just one element u such that $x\ \vartheta\ u\ \overline{\vartheta}\ y$.*

4.2.4 Boolean and factorable congruences

Definition 4.2.12. *An algebra has* Boolean factor congruences *if its factor congruences form a Boolean sublattice of the congruence lattice.*

Definition 4.2.13. *A variety $\mathbb{C}$ of algebras has* factorable congruences *if for every $\mathscr{A}, \mathscr{B} \in \mathbb{C}$ we have* $\mathrm{Con}(\mathscr{A} \times \mathscr{B}) \cong \mathrm{Con}(\mathscr{A}) \times \mathrm{Con}(\mathscr{B})$.

Lemma 4.2.14. *(Bigelow and Burris [22, Cor. 1.4]) If a variety $\mathbb{C}$ has factorable congruences, then every $\mathscr{A} \in \mathbb{C}$ has Boolean factor congruences.*

Most known examples of varieties in which all algebras have Boolean factor congruences are those with factorable congruences. This is the case of combinatory algebras as we will show in the next lemma.

Recall that the combinators $\mathbf{t}, \mathbf{f}$ have been defined in Subsection 1.3.2.

Lemma 4.2.15. *For all combinatory algebras $\mathscr{A}, \mathscr{B}$ we have that* $\mathrm{Con}(\mathscr{A} \times \mathscr{B}) \cong \mathrm{Con}(\mathscr{A}) \times \mathrm{Con}(\mathscr{B})$.

Proof. Let $\mathscr{A}, \mathscr{B}$ be combinatory algebras; it is clear that, up to isomorphism, $\mathrm{Con}(\mathscr{A}) \times \mathrm{Con}(\mathscr{B}) \subseteq \mathrm{Con}(\mathscr{A} \times \mathscr{B})$. Conversely, let $\vartheta \in \mathrm{Con}(\mathscr{A} \times \mathscr{B})$. The "projections" ϑ_1, ϑ_2 of ϑ are the binary relations on $\mathscr{A}$ and $\mathscr{B}$, respectively, defined as follows:

$$a_1 \vartheta_1 a_2 \iff \exists b_1, b_2 \in B \text{ such that } (a_1, b_1)\ \vartheta\ (a_2, b_2),$$
$$b_1 \vartheta_2 b_2 \iff \exists a_1, a_2 \in A \text{ such that } (a_1, b_1)\ \vartheta\ (a_2, b_2).$$

It is obvious that $\vartheta \subseteq \vartheta_1 \times \vartheta_2$. We now prove the opposite inclusion. Suppose that $(a_1, b_1)\ \vartheta_1 \times \vartheta_2\ (a_2, b_2)$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then, by definition of $\vartheta_1 \times \vartheta_2$, we have that $a_1 \vartheta_1 a_2$ and $b_1 \vartheta_2 b_2$. Hence, there exist $a_3, a_4 \in A$, $b_3, b_4 \in B$

such that $(a_1, b_3)\ \vartheta\ (a_2, b_4)$ and $(a_3, b_1)\ \vartheta\ (a_4, b_2)$. Since $(\mathbf{t}, \mathbf{f})\ \vartheta\ (\mathbf{t}, \mathbf{f})$ and ϑ is a compatible relation, we get:

$$(a_1, b_1) = (\mathbf{t}a_1a_3, \mathbf{f}b_3b_1)\ \vartheta\ (\mathbf{t}a_2a_4, \mathbf{f}b_4b_2) = (a_2, b_2).$$

Thus we get $\vartheta = \vartheta_1 \times \vartheta_2$. It is easy to check that ϑ_1, ϑ_2 are reflexive, symmetric and compatible with application. We now show that ϑ_1 is also transitive. Let $a_1\vartheta_1a_2\vartheta_1a_3$, then there exist b_1, b_2, b_3, b_4 such that $(a_1, b_1)\ \vartheta\ (a_2, b_2)$ and $(a_2, b_3)\ \vartheta\ (a_3, b_4)$; from the symmetry of ϑ we have also $(a_3, b_4)\ \vartheta\ (a_2, b_3)$. Since $(\mathbf{t}, \mathbf{f})\ \vartheta\ (\mathbf{t}, \mathbf{f})$ and ϑ is a compatible relation, we get:

$$(a_1, b_4) = (\mathbf{t}a_1a_3, \mathbf{f}b_1b_4)\ \vartheta\ (\mathbf{t}a_2a_2, \mathbf{f}b_2b_3) = (a_2, b_3).$$

Finally, from $(a_1, b_4)\ \vartheta\ (a_2, b_3)$ and $(a_2, b_3)\ \vartheta\ (a_3, b_4)$ we get $(a_1, b_4)\ \vartheta\ (a_3, b_4)$ and, hence, $a_1\vartheta_1a_3$; thus $\vartheta_1 \in \mathrm{Con}(\mathscr{A})$. An analogous reasoning gives $\vartheta_2 \in \mathrm{Con}(\mathscr{B})$. From this it is easy to conclude that $\mathrm{Con}(\mathscr{A} \times \mathscr{B}) \cong \mathrm{Con}(\mathscr{A}) \times \mathrm{Con}(\mathscr{B})$. □

Corollary 4.2.16. *All combinatory algebras have Boolean factor congruences.*

4.2.5 Boolean product

The Boolean product construction allows us to transfer numerous fascinating properties of Boolean algebras into other varieties of algebras (see [31, Ch. IV]). Actually, this construction has been presented for several years as "the algebra of global sections of sheaves of algebras over Boolean spaces" (see [35, 63]); however, these notions were unnecessarily complex and we prefer to adopt here the following equivalent presentation (see [32]). We recall that a Boolean space is a compact, Hausdorff and totally disconnected topological space.

Definition 4.2.17. *A* weak Boolean product *of a family $(\mathscr{A}_i)_{i\in I}$ of algebras is a subdirect product $\mathscr{A} \leq \Pi_{i\in I}\mathscr{A}_i$, where I can be endowed with a Boolean space topology such that:*

(i) the set $\{i \in I : a_i = b_i\}$ is open for all $a, b \in A$, and

(ii) if $a, b \in A$ and N is a clopen subset of I, then the element c, defined by $c_i = a_i$ for every $i \in N$ and $c_i = b_i$ for every $i \in I - N$, belongs to A.

Definition 4.2.18. *A* Boolean product *is a weak Boolean product such that the set $\{i \in I : a_i = b_i\}$ is clopen for all $a, b \in A$.*

In the next section we will see that every combinatory algebra admits a **weak** Boolean product representation, whilst there exist combinatory algebras which cannot be decomposed as a Boolean product of algebras.

4.3 The Stone representation theorem for combinatory algebras

In this section we show that combinatory algebras satisfy a theorem which is similar to the Stone representation theorem for Boolean algebras.

4.3.1 The classical Stone and Pierce theorem

The Stone representation theorem for Boolean rings (the observation that Boolean algebras could be regarded as rings is due to Stone) admits a generalization, due to Pierce, to commutative rings with unit (see [83] and [63, Ch. V]). To help the reader to get familiar with the argument, we outline now Pierce's construction.

Let $\mathscr{A} = (A, +, \cdot, 0, 1)$ be a commutative ring with unit, and let $\mathrm{IE}(\mathscr{A}) = \{a \in A : a \cdot a = a\}$ be the set of its *idempotent elements*. One defines a structure of Boolean algebra on $\mathrm{IE}(\mathscr{A})$ as follows. For all $a, b \in \mathrm{IE}(\mathscr{A})$:

- $a \wedge b = a \cdot b$;
- $a^- = 1 - a$.

Then it is possible to show that for every $a \in \mathrm{IE}(\mathscr{A})$, $a \neq 0, 1$ induces a pair $(\vartheta(a, 1), \vartheta(a, 0))$ of non-trivial complementary factor congruences. In other words, the ring $\mathscr{A}$ can be decomposed in a non-trivial way as $\mathscr{A} \cong \mathscr{A}/\vartheta(a, 1) \times \mathscr{A}/\vartheta(a, 0)$. If $\mathrm{IE}(\mathscr{A}) = \{0, 1\}$, then A is directly indecomposable. Then Pierce's theorem for commutative rings with unit can be stated as follows:

"Every commutative ring with unit is isomorphic to a Boolean product of directly indecomposable rings."

If $\mathscr{A}$ is a Boolean ring, we get the Stone representation theorem for Boolean algebras, because the ring of truth values is the unique directly indecomposable Boolean ring.

The remaining part of this section is devoted to provide the statement and the proof of the representation theorem for combinatory algebras.

4.3.2 The Boolean algebra of central elements

Combinatory logic and λ-calculus internalize many important things (computability, for example). "To be directly decomposable" is another internalizable property of these formalisms, as it will be shown in this subsection.

The combinators **t** and **f**, representing the booleans *true* and *false*, correspond to the constants 0 and 1 in a commutative ring with unit. More generally, the central elements of a combinatory algebra, as defined below, correspond to the idempotent elements of a ring. The notion of central element was introduced by Vaggione in universal algebra [105]. Here we give a new characterization which works for combinatory algebras.

Definition 4.3.1. *Let $\mathscr{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ be a combinatory algebra. We say that an element $e \in C$ is* central *when it satisfies the following equations, where x, y, z, t range over C:*

(i) $exx = x$.

(ii) $e(exy)z = exz = ex(eyz)$.

(iii) $e(xy)(zt) = exz(eyt)$.

(iv) $e = e\mathbf{tf}$.

Such a characterization works more generally for all varieties of algebras for which there are two constants $0, 1$ and a term $u(x, y, z)$ such that $u(x, y, 1) = x$ and $u(x, y, 0) = y$ (for combinatory algebras one takes $1 \equiv \mathbf{t}$, $0 \equiv \mathbf{f}$ and $u(x, y, z) \equiv zyx$). Note that this is also the case for Lemma 4.2.15 above and Proposition 4.3.3 below.

Notation 4.3.2. $\mathrm{CE}(\mathscr{C})$ *denotes the set of central elements of $\mathscr{C}$.*

Every combinatory algebra admits at least two central elements: $\mathbf{t}$ and $\mathbf{f}$. Now we show that central elements, similarly as idempotent elements in a ring, decompose a combinatory algebra $\mathscr{C}$ as a Cartesian product: if $e \in \mathrm{CE}(\mathscr{C})$, then $\mathscr{C} \cong \mathscr{C}/\vartheta(e, \mathbf{t}) \times \mathscr{C}/\vartheta(e, \mathbf{f})$. This will be shown in the next proposition via decomposition operators.

Proposition 4.3.3. *There is a (natural) bijective correspondence between central elements and decomposition operators (resp. pairs of complementary factor congruences).*

Proof. Given a central element e we obtain a decomposition operator by taking $f_e(x, y) = exy$. It is a simple exercise to show that axioms (i)-(iii) of a central element make f_e a decomposition operator.

Conversely, given a decomposition operator f, we show that the element $f(\mathbf{t}, \mathbf{f})$ is central. From Remark 4.2.11 we have that $f(\mathbf{t}, \mathbf{f})$ is the unique element u satisfying $\mathbf{t}\ \vartheta\ u\ \overline{\vartheta}\ \mathbf{f}$, where $(\vartheta, \overline{\vartheta})$ is the pair of complementary factor congruences associated with the decomposition operator f. Since ϑ and $\overline{\vartheta}$ are compatible equivalence relations, it follows that for all x, y:

$$\mathbf{t}xy\ \vartheta\ f(\mathbf{t}, \mathbf{f})xy\ \overline{\vartheta}\ \mathbf{f}xy,$$

which implies $x\ \vartheta\ f(\mathbf{t}, \mathbf{f})xy\ \overline{\vartheta}\ y$. Since, by definition, $f(x, y)$ is the unique element satisfying $x\ \vartheta\ f(x, y)\ \overline{\vartheta}\ y$, we obtain:

$$f(x, y) = f(\mathbf{t}, \mathbf{f})xy. \tag{4.2}$$

Finally, the identities defining f as decomposition operator make $f(\mathbf{t}, \mathbf{f})$ a central element.

We now check that these correspondences form the two sides of a bijection. If e is central, then the central element $f_e(\mathbf{t}, \mathbf{f})$ is equal to e, because $f_e(\mathbf{t}, \mathbf{f}) = e\mathbf{tf} = e$ by Definition 4.3.1(iv). If f is a decomposition operator, then by (4.2) we have that $f_{f(\mathbf{t},\mathbf{f})}(x, y) = f(\mathbf{t}, \mathbf{f})xy = f(x, y)$ for all x, y. □

Summing up, by Proposition 4.3.3 and Proposition 4.2.10 (together with [79, Thm. 4.33]) we get the following theorem.

Theorem 4.3.4. *Given a combinatory algebra $\mathscr{C}$, the following conditions are equivalent:*

(i) $\mathscr{C}$ *is directly indecomposable,*

(ii) $\mathscr{C}$ *has no non-trivial pair of complementary factor congruences,*

(iii) $\mathrm{CE}(\mathscr{C}) = \{\mathbf{t}, \mathbf{f}\}$.

For every central element e, we denote respectively by f_e and by $(\vartheta_e, \overline{\vartheta}_e)$ the decomposition operator and the pair of complementary factor congruences determined by e.

Corollary 4.3.5. *If e is central, then we have:*

(i) $x\ \vartheta_e\ exy\ \overline{\vartheta}_e\ y$.

(ii) $x\ \vartheta_e\ y$ *if and only if* $exy = y$, *and* $x\ \overline{\vartheta}_e\ y$ *if and only if* $exy = x$.

(iii) $\vartheta_e = \vartheta(e, \mathbf{t})$ *and* $\overline{\vartheta}_e = \vartheta(e, \mathbf{f})$.

We now show that the partial ordering between central elements, defined by:

$$d \leq e \text{ if, and only if, } \vartheta_d \subseteq \vartheta_e \tag{4.3}$$

is a Boolean ordering and the meet operation and the complementation are internally representable by the combinators $\lambda^* xy.x\mathbf{t}y$ and $\lambda^* x.x\mathbf{ft}$ respectively (recall that λ^* is defined in Subsection 1.3.2); it is clear that the combinators $\mathbf{t}$ and $\mathbf{f}$ are respectively the bottom element and the top element of this ordering.

Theorem 4.3.6. *Let $\mathscr{C}$ be a combinatory algebra. Then the algebra $\mathscr{E}(\mathscr{C}) = (\mathrm{CE}(\mathscr{C}), \wedge, ^-)$ of central elements of $\mathscr{C}$, defined by*

$$e \wedge d = e\mathbf{t}d; \quad e^- = e\mathbf{ft},$$

is a Boolean algebra, which is isomorphic to the Boolean algebra of factor congruences.

Proof. By Corollary 4.2.16 $\mathscr{C}$ has Boolean factor congruences. It follows that the partial ordering on central elements, defined in (4.3), is a Boolean ordering. There only remains to show that, for all central elements d, e, the elements $e^- = e\mathbf{ft}$ and $e \wedge d = e\mathbf{t}d$ are central and are respectively associated with the pairs $(\overline{\vartheta}_e, \vartheta_e)$ and $(\vartheta_e \cap \vartheta_d, \overline{\vartheta}_e \vee \overline{\vartheta}_d)$ of complementary factor congruences (recall that $(\vartheta_e, \overline{\vartheta}_e)$ is the pair of complementary factor congruences associated with the central element e).

We check the details for $e\mathbf{ft}$. By Corollary 4.3.5(i) we have that $e\mathbf{ft}$ is the unique element u such that $\mathbf{t}\ \overline{\vartheta}_e\ u\ \vartheta_e\ \mathbf{f}$. By (4.1) in Subsection 4.2.2 this means that $e\mathbf{ft} = g(\mathbf{t}, \mathbf{f})$ for the decomposition operator g associated with the pair $(\overline{\vartheta}_e, \vartheta_e)$

of complementary factor congruences. We have the conclusion that $e\mathbf{ft}$ is central associated with the pair $(\overline{\vartheta}_e, \vartheta_e)$ as in the proof of Proposition 4.3.3.

We now consider $e \wedge d = e\mathbf{t}d$. First of all, we show that $e\mathbf{t}d = d\mathbf{t}e$. By Corollary 4.3.5(i) we have that $\mathbf{t}\ \vartheta_e\ e\mathbf{t}d\ \overline{\vartheta}_e\ d$, while $\mathbf{t}\ \vartheta_e\ d\mathbf{t}e\ \overline{\vartheta}_e\ d$ can be obtained as follows:

$$\begin{array}{rcll} \mathbf{t} & = & d\mathbf{tt} & \text{by Definition 4.3.1(i)}, \\ d\mathbf{tt} & \vartheta_e & d\mathbf{t}e & \text{by } \mathbf{t}\vartheta_e e, \\ d\mathbf{t}e & \overline{\vartheta}_e & d\mathbf{tf} & \text{by } e\overline{\vartheta}_e\mathbf{f}, \\ d\mathbf{tf} & = & d & \text{by Definition 4.3.1(iv)}. \end{array}$$

Since by Remark 4.2.11 there is a unique element u such that $\mathbf{t}\ \vartheta_e\ u\ \overline{\vartheta}_e\ d$, then we have the conclusion $d\mathbf{t}e = e\mathbf{t}d$. We now show that $e\mathbf{t}d$ is the central element associated with the factor congruence $\vartheta_e \cap \vartheta_d$, i.e.,

$$\mathbf{t}\ (\vartheta_e \cap \vartheta_d)\ e\mathbf{t}d\ (\overline{\vartheta}_e \vee \overline{\vartheta}_d)\ \mathbf{f}.$$

From $d\mathbf{t}e = e\mathbf{t}d$ we easily get that $\mathbf{t}\ \vartheta_e\ e\mathbf{t}d$ and $\mathbf{t}\ \vartheta_d\ e\mathbf{t}d$, that is, $\mathbf{t}\ (\vartheta_e \cap \vartheta_d)\ e\mathbf{t}d$. Finally, by Corollary 4.3.5, we have: $e\mathbf{t}d\ \overline{\vartheta}_e\ d = d\mathbf{tf}\ \overline{\vartheta}_d\ \mathbf{f}$, i.e., $e\mathbf{t}d\ (\overline{\vartheta}_e \vee \overline{\vartheta}_d)\ \mathbf{f}$. □

We now provide the promised representation theorem. If I is a maximal ideal of the Boolean algebra CE($\mathscr{A}$), then ϑ_I denotes the congruence on $\mathscr{A}$ defined by:

$$x\ (\vartheta_I)\ y \text{ if, and only if, } x\ \vartheta_e\ y \text{ for some } e \in I.$$

By a *Pierce variety* (see [106] for the general definition) we mean here a variety of algebras for which there are two constants $0, 1$ and a term $u(x, y, z, v)$ such that the following identities hold: $u(x, y, 0, 1) = x$ and $u(x, y, 1, 0) = y$.

Obviously, the variety of combinatory algebras is a Pierce Variety: it is sufficient to take $1 \equiv \mathbf{t}$, $0 \equiv \mathbf{f}$ and $u(x, y, z, v) \equiv zyx$.

Theorem 4.3.7. (Representation Theorem for combinatory algebras) *Let $\mathscr{C} = (C, \cdot, \mathbf{k}, \mathbf{s})$ be a combinatory algebra and X be the Boolean space of maximal ideals of the Boolean algebra $\mathscr{E}(\mathscr{C})$ of central elements. Then, for all $I \in X$ the quotient algebra $\mathscr{C}/\vartheta_I$ is directly indecomposable and the map*

$$f : C \to \Pi_{I \in X}(C/\vartheta_I),$$

defined by

$$f(x) = (x/\vartheta_I : I \in X),$$

gives a **weak** *Boolean product representation of $\mathscr{C}$.*

Proof. By Corollary 4.2.16 the factor congruences of $\mathscr{C}$ constitute a Boolean sublattice of Con($\mathscr{C}$). Then by [35] f gives a weak Boolean product representation of $\mathscr{C}$. The quotient algebras $\mathscr{C}/\vartheta_I$ are directly indecomposable by [106, Thm. 8], because combinatory algebras form a Pierce variety. □

Note that, in general, it is not possible to obtain a (non-weak) Boolean product representation of a combinatory algebra. This follows from Lemma 4.2.15 and two results due to Vaggione [105] and Plotkin-Simpson [100, Lemma 3.14]. Vaggione has shown that, if a variety has factorable congruences and every member of the variety can be represented as a Boolean product of directly indecomposable algebras, then the variety is a discriminator variety (see [31] for the terminology). Discriminator varieties satisfy very strong algebraic properties, in particular they are congruence permutable (i.e., in each algebra the join of two congruences is just their composition). Plotkin and Simpson have shown that this last property is inconsistent with combinatory logic, hence by Lemma 4.2.15 and Vaggione's theorem not all combinatory algebras have a Boolean product representation.

4.4 The indecomposable semantics

The representation theorem of combinatory algebras can be roughly summarized as follows: the directly indecomposable combinatory algebras are the "building blocks" in the variety of combinatory algebras. Then it is natural to investigate the class of models of λ-calculus, which are directly indecomposable as combinatory algebras (*indecomposable semantics*, for short).

In this section we show that the indecomposable semantics includes: the Scott-continuous, stable and strongly stable semantics, as well as the term models of all semi-sensible λ-theories. In spite of this richness, in the last results of this chapter we show that the indecomposable semantics is incomplete, and that this incompleteness is as large as possible: the set of λ-theories which are omitted by the indecomposable semantics is $2^{\aleph_0}$-broad, and moreover contains countably many $2^{\aleph_0}$-high intervals.

Finally, we will show that the set of λ-theories induced by each of the main semantics is not closed under finite intersection, and hence it is not a sublattice of $\lambda\mathscr{T}$.

4.4.1 Internalizing "indecomposable"

We have shown that any factor congruence could be internally represented by a central element, and in particular that a combinatory algebra is directly indecomposable if, and only if, $\mathrm{CE}(\mathscr{C}) = \{\mathbf{t}, \mathbf{f}\}$. We now deduce from this that "being a directly indecomposable combinatory algebra" can be expressed by a universal formula.

Let us define the following combinatory terms:

- $\mathbf{z} \equiv \lambda^* e.[\lambda^* x.exx, \lambda^* xyz.e(exy)z, \lambda^* xyz.exz, \lambda^* xyzu.e(xy)(zu), e\mathbf{tf}]$;
- $\mathbf{u} \equiv \lambda^* e.[\lambda^* x.x, \lambda^* xyz.exz, \lambda^* xyz.ex(eyz), \lambda^* xyzu.exz(eyu), e]$.

Proposition 4.4.1. *The class $\mathbb{CA}_{DI}$ of the directly indecomposable combinatory algebras is a universal class (i.e., it can be axiomatized by universal sentences).*

Proof. By Definition 4.3.1 we have that e is central if, and only if, the equation $\mathbf{z}e = \mathbf{u}e$ holds. Hence, the class $\mathbb{CA}_{DI}$ can be axiomatized by the following universal formula ϕ:

$$\phi \equiv \forall e((\mathbf{z}e = \mathbf{u}e \Rightarrow e = \mathbf{t} \vee e = \mathbf{f}) \wedge \neg(\mathbf{t} = \mathbf{f})).$$

□

Corollary 4.4.2. *The class $\mathbb{CA}_{DI}$ of the directly indecomposable combinatory algebras is closed under subalgebras and ultraproducts.*

4.4.2 Incompleteness of the indecomposable semantics

The closure of the class of directly indecomposable combinatory algebras under subalgebras is the key trick in the proof of the algebraic incompleteness theorem. Recall that $\mathscr{M}_{\mathcal{T}}$ denotes the term model of a λ-theory $\mathcal{T}$.

Lemma 4.4.3. *Given a λ-theory $\mathcal{T}$ the following conditions are equivalent:*

(i) *$\mathscr{M}_{\mathcal{T}}$ has a non-trivial central element.*

(ii) *$\mathscr{M}_{\mathcal{T}}$ is directly decomposable.*

(iii) *All λ-models $\mathscr{C}$ such that $\mathrm{Th}(\mathscr{C}) = \mathcal{T}$ are directly decomposable.*

(iii') *The indecomposable semantics omits $\mathcal{T}$.*

Proof. ($i \iff ii$): By Proposition 4.3.3.

($ii \iff iii$): follows from Corollary 4.4.2, since $\mathrm{Th}(\mathscr{C}) = \mathcal{T}$ if, and only if, $\mathscr{M}_{\mathcal{T}}$ is isomorphic to a subalgebra of $\mathscr{C}$.

($iii \iff iii$'): By Definition 1.3.1. □

In every λ-model the interpretations of the combinators $\mathbf{t}$ and $\mathbf{f}$ coincide with those of the λ-terms $\boldsymbol{T} \equiv \lambda xy.x$ and $\boldsymbol{F} \equiv \lambda xy.y$. It follows that $\boldsymbol{T}$ and $\boldsymbol{F}$ can cover the role of trivial central elements in every λ-model.

Lemma 4.4.4. *Let $Q \in \Lambda^o$ and $\mathcal{T}_{\mathbf{T}}, \mathcal{T}_{\mathbf{F}}$ be two consistent λ-theories such that $\mathcal{T}_{\mathbf{T}} \vdash Q = \mathbf{T}$ and $\mathcal{T}_{\mathbf{F}} \vdash Q = \mathbf{F}$. Then, for $\mathcal{T} = \mathcal{T}_{\mathbf{T}} \cap \mathcal{T}_{\mathbf{F}}$, $[Q]_{\mathcal{T}}$ is a non-trivial central element of $\mathscr{M}_{\mathcal{T}}$.*

Proof. Since $[\mathbf{T}]_{\mathcal{T}_{\mathbf{T}}}$ and $[\mathbf{F}]_{\mathcal{T}_{\mathbf{F}}}$ are central elements in $\mathscr{M}_{\mathcal{T}_{\mathbf{T}}}$ and $\mathscr{M}_{\mathcal{T}_{\mathbf{F}}}$ respectively, the λ-theory $\mathcal{T}$ contains all equations $(i) - (iv)$ of Definition 4.3.1 for $e = Q$, making $[Q]_{\mathcal{T}}$ a central element of $\mathscr{M}_{\mathcal{T}}$. Moreover, $[Q]_{\mathcal{T}}$ is non-trivial since $\mathcal{T} \nvdash Q = \mathbf{T}$ and $\mathcal{T} \nvdash Q = \mathbf{F}$. □

Theorem 4.4.5. *The indecomposable semantics is incomplete.*

Proof. By Lemma 4.4.3 it is sufficient to produce a λ-theory $\mathcal{T}$ such that $\mathscr{M}_{\mathcal{T}}$ has a non-trivial central element. Since Ω is an easy term, there exist two consistent λ-theories $\mathcal{T}_{\mathbf{T}}, \mathcal{T}_{\mathbf{F}}$ such that $\mathcal{T}_{\mathbf{T}} \vdash \Omega = \boldsymbol{T}$ and $\mathcal{T}_{\mathbf{F}} \vdash \Omega = \boldsymbol{F}$. Obviously, the λ-theory $\mathcal{T} = \mathcal{T}_{\mathbf{T}} \cap \mathcal{T}_{\mathbf{F}}$ is consistent. Then, we conclude by Lemma 4.4.4 that $[\Omega]_{\mathcal{T}}$ is a non-trivial central element of $\mathscr{M}_{\mathcal{T}}$. □

In the following theorem we show that, although the indecomposable semantics is incomplete, it is large enough to represent all semi-sensible λ-theories.

Lemma 4.4.6. *Let $\mathcal{T}$ be a consistent λ-theory and e be a non-trivial central element of $\mathscr{M}_{\mathcal{T}}$. Then, every λ-term U belonging to the equivalence class e is unsolvable.*

Proof. By Corollary 4.3.5 the congruences $\vartheta_e = \vartheta(e, [\boldsymbol{T}]_{\mathcal{T}})$ and $\overline{\vartheta}_e = \vartheta(e, [\boldsymbol{F}]_{\mathcal{T}})$ on $\mathscr{M}_{\mathcal{T}}$ are non-trivial. Then, for every λ-term $U \in e$, the λ-theories $\mathcal{T}_{\mathbf{F}}$ and $\mathcal{T}_{\mathbf{T}}$, generated respectively by $\mathcal{T} \cup \{\boldsymbol{F} = U\}$ and $\mathcal{T} \cup \{\boldsymbol{T} = U\}$, are consistent. Assume, by the way of contradiction, that U is solvable. Then, U should be simultaneously equivalent (as hnf) to $\boldsymbol{F}$ and to $\boldsymbol{T}$. Contradiction. ☐

Theorem 4.4.7. *The indecomposable semantics represents all semi-sensible λ-theories.*

Proof. Let $\mathcal{T}$ be a semi-sensible λ-theory. Assume, by the way of contradiction, that $\mathscr{M}_{\mathcal{T}}$ has a non-trivial central element e (cf. Lemma 4.4.3). Then, $\mathscr{M}_{\mathcal{T}}$ satisfies $exx = x$ (see Definition 4.3.1). Let $U \in \Lambda^o$ such that $U \in e$, then $\mathcal{T} \vdash Uxx = x$. By Lemma 4.4.6, U is unsolvable. Thus, the unsolvable λ-term Uxx is provably equal in $\mathcal{T}$ to the solvable λ-term x, which contradicts the semi-sensibility of $\mathcal{T}$. ☐

4.4.3 Continuous, stable and strongly stable semantics

In the next proposition we show that all λ-models living in the main semantics are simple algebras. We recall that an algebra is *simple* when it has just the two trivial congruences, and is hence directly indecomposable.

Proposition 4.4.8.

(i) All λ-models living in the Scott-continuous semantics are simple combinatory algebras.

(ii) All λ-models living in the stable or strongly stable semantics are simple combinatory algebras.

Proof. Let us consider a λ-model $\mathscr{C} = (\mathcal{D}, \cdot, \mathbf{k}, \mathbf{s})$.

(i) Suppose that $\mathscr{C}$ is a continuous λ-model. It is easy to check that, for all $b, c \in \mathcal{D}$, the function $g_{b,c}$ defined by

$$g_{b,c}(x) = \begin{cases} c & \text{if } x \not\sqsubseteq_{\mathcal{D}} b, \\ \bot & \text{otherwise,} \end{cases}$$

is Scott continuous. Let ϑ be a congruence on $\mathscr{C}$ and suppose that there exist a, d such that $a \;\vartheta\; d$ with $a \neq d$. We have $a \not\sqsubseteq_{\mathcal{D}} d$ or $d \not\sqsubseteq_{\mathcal{D}} a$. Suppose, without loss of generality, that we are in the first case. Since the continuous function $g_{d,c}$ is representable in the model (for all c), we have: $\bot = g_{d,c}(a) \;\vartheta\; g_{d,c}(d) = c$, hence $c\vartheta\bot$.

By the arbitrariness of c we get that ϑ is trivial, so that $\mathscr{C}$ is simple. Note that $g_{d,c}$ is neither stable nor strongly stable hence it cannot be used for proving item (*ii*).

(*ii*) Suppose that $\mathscr{C}$ is a (strongly) stable λ-model. Consider two elements $a, b \in \mathcal{D}$ such that $a \neq b$. We have $a \not\sqsubseteq_{\mathcal{D}} b$ or $b \not\sqsubseteq_{\mathcal{D}} a$. Suppose, without loss of generality, that we are in the first case. Then there is a compact element d of $\mathscr{C}$ such that $d \sqsubseteq_{\mathcal{D}} a$ and $d \not\sqsubseteq_{\mathcal{D}} b$. The step function $f_{d,c}$ defined by :

$$f_{d,c}(x) = \begin{cases} c & \text{if } d \sqsubseteq_{\mathcal{D}} x, \\ \perp & \text{otherwise,} \end{cases}$$

is stable (strongly stable) for every element c. This function $f_{d,c}$ can be used to show that every congruence on $\mathscr{C}$ is trivial as in the proof of item (*i*). □

As a consequence of Proposition 4.4.8, we get, in a uniform way, the incompleteness for the main semantics of λ-calculus. We will see later on that this incompleteness is very large.

Corollary 4.4.9. *The Scott-continuous, the stable and the strongly stable semantics are incomplete.*

Proof. By Proposition 4.4.8 and Theorem 4.4.5. □

Recall that, given a class $\mathbb{C}$ of λ-models, $\lambda\mathbb{C}$ denotes the set of λ-theories which are represented in $\mathbb{C}$. In the remaining part of this subsection we show that, for each of the classic semantics of λ-calculus, the set $\lambda\mathbb{C}$ is not closed under finite intersection, so that it is not a sublattice of the lattice $\lambda\mathscr{T}$ of λ-theories.

Theorem 4.4.10. *Let $\mathbb{C}$ be a class of directly indecomposable models of λ-calculus. If there are two consistent λ-theories $\mathcal{T}_{\mathbf{T}}, \mathcal{T}_{\mathbf{F}} \in \lambda\mathbb{C}$ such that*

$$\mathcal{T}_{\mathbf{T}} \vdash \Omega = \boldsymbol{T}; \quad \mathcal{T}_{\mathbf{F}} \vdash \Omega = \boldsymbol{F},$$

then $\lambda\mathbb{C}$ is not closed under finite intersection, so it is not a sublattice of $\lambda\mathscr{T}$.

Proof. Let $\mathcal{T} = \mathcal{T}_{\mathbf{T}} \cap \mathcal{T}_{\mathbf{F}}$. By Lemma 4.4.4, $[\Omega]_{\mathcal{T}}$ is a non-trivial central element of $\mathscr{M}_{\mathcal{T}}$. It follows that $\mathcal{T} \notin \lambda\mathbb{C}$. □

Corollary 4.4.11. *Let $\mathbb{C}$ be one of the following semantics: graph semantics, G-semantics, H-semantics, filter semantics, Scott-continuous semantics, stable semantics, strongly stable semantics. Then $\lambda\mathbb{C}$ is not a sublattice of $\lambda\mathscr{T}$.*

Proof. Semantic proofs that Ω is an easy term were given (see [13]) for the graph, G- and H- semantics (and hence still hold for all the larger classes). Then the conclusion follows from Theorem 4.4.10, and Proposition 4.4.8. □

4.4.4 Concerning the number of decomposable and indecomposable λ-models

From the work done in the previous subsection, it is moreover easy to conclude that there is a wealth of directly indecomposable λ-models inducing different λ-theories.

Theorem 4.4.12. *Let $\mathbb{IND}$ be the indecomposable semantics, then $\lambda\mathbb{IND}$ is $2^{\aleph_0}$-high and $2^{\aleph_0}$-broad (hence, $2^{\aleph_0}$-wide).*

Proof. We know from Theorem 4.4.7 that $\lambda\mathbb{IND}$ contains the interval $[\lambda_\beta, \mathcal{H}^*]$, which is $2^{\aleph_0}$-high by Theorem 1.2.2(ii). Moreover, Proposition 4.4.8(i) implies that $\lambda\mathbb{IND}$ also contains the set of all graph theories which is, by Proposition 1.4.9, $2^{\aleph_0}$-broad. □

Now, we show that also the incompleteness of the indecomposable semantics is as large as possible.

First of all we need some results about λ-theories. The proof of the following lemma is similar to that of [8, Prop. 17.1.9], where the case $k = 1$ (due to Visser) is shown, and it is omitted.

Lemma 4.4.13. *Suppose $\mathcal{T}$ is an r.e. λ-theory and fix arbitrary λ-terms M_i, N_i for $1 \le i \le k$ such that $\mathcal{T} \not\vdash M_i = N_i$ for all $i \le k$. Then there is a $\mathcal{T}$-easy term $M \in \Lambda^o$ such that*

$$\mathcal{T} \cup \{M = P\} \not\vdash M_i = N_i, \text{ for all } i \le k \text{ and all closed terms } P.$$

Then the following theorems are corollaries of the algebraic incompleteness theorem.

Theorem 4.4.14. *Let $\mathcal{T}$ be an r.e. λ-theory. Then, the interval $[\mathcal{T}[= \{\mathcal{S} : \mathcal{T} \subseteq \mathcal{S}\}$ contains a subinterval $[\mathcal{S}_1, \mathcal{S}_2] = \{\mathcal{S} : \mathcal{S}_1 \subseteq \mathcal{S} \subseteq \mathcal{S}_2\}$ satisfying the following conditions:*

- *$\mathcal{S}_1$ and $\mathcal{S}_2$ are distinct r.e. λ-theories,*
- *every $\mathcal{S} \in [\mathcal{S}_1, \mathcal{S}_2]$ is omitted by the indecomposable semantics,*
- $\mathrm{card}([\mathcal{S}_1, \mathcal{S}_2]) = 2^{\aleph_0}$.

Proof. Since $\mathcal{T}$ is r.e. we know that there exists a $\mathcal{T}$-easy λ-term Q. In particular $\mathcal{T} \not\vdash Q = \boldsymbol{T}$ and $\mathcal{T} \not\vdash Q = \boldsymbol{F}$.

Let $\mathcal{S}_1 = \mathcal{T}_{\mathbf{T}} \cap \mathcal{T}_{\mathbf{F}}$, where $\mathcal{T}_{\mathbf{T}}, \mathcal{T}_{\mathbf{F}}$ are the consistent λ-theories generated respectively by $\mathcal{T} \cup \{Q = \boldsymbol{T}\}$ and $\mathcal{T} \cup \{Q = \boldsymbol{F}\}$. Obviously, the λ-theory $\mathcal{S}_1$ is consistent, r.e. and contains $\mathcal{T}$. By Lemma 4.4.4, $[Q]_{\mathcal{S}_1}$ is a non-trivial central element of $\mathscr{M}_{\mathcal{S}_1}$.

We apply Lemma 4.4.13 to the λ-theory $\mathcal{S}_1$ and to the equations $Q = \boldsymbol{T}$ and $Q = \boldsymbol{F}$. We get an $\mathcal{S}_1$-easy term $R \in \Lambda^o$ such that $\mathcal{S}_1 \cup \{R = P\} \not\vdash Q = \boldsymbol{T}$ and

$\mathcal{S}_1 \cup \{R = P\} \not\vdash Q = \boldsymbol{F}$, for all λ-terms $P \in \Lambda^o$. Let $\mathcal{S}_2 = \mathcal{S}_1 \cup \{R = \mathbf{I}\}$. Since R is $\mathcal{S}_1$-easy, we have that $\mathcal{S}_2$ is a proper extension of $\mathcal{S}_1$.

The term model $\mathscr{M}_{\mathcal{S}_2}$ of $\mathcal{S}_2$ is a homomorphic image of the term model $\mathscr{M}_{\mathcal{S}_1}$ of $\mathcal{S}_1$, then every equation satisfied by $\mathscr{M}_{\mathcal{S}_1}$ is also satisfied by $\mathscr{M}_{\mathcal{S}_2}$. In particular, the equations expressing that Q is a central element. Finally, $[Q]_{\mathcal{S}_2}$ is non-trivial as a central element because $\mathcal{S}_2 \not\vdash Q = \boldsymbol{T}$ and $\mathcal{S}_2 \not\vdash Q = \boldsymbol{F}$.

Hence, for every λ-theory $\mathcal{S}$ such that $\mathcal{S}_1 \subseteq \mathcal{S} \subseteq \mathcal{S}_2$ the equivalence class of Q is non-trivial central element of the term model of $\mathcal{S}$.

We get the conclusion of the theorem because $\mathrm{card}([\mathcal{S}_1, \mathcal{S}_2]) = 2^{\aleph_0}$ by Theorem 1.2.2(i'). $\square$

Remark 4.4.15. *From Lemma 4.4.3 it follows that all the λ-models $\mathscr{C}$ such that $\mathrm{Th}(\mathscr{C})$ belongs to the interval $[\mathcal{S}_1, \mathcal{S}_2]$ above, are directly decomposable.*

Theorem 4.4.16. *Let $\mathbb{DEC}$ be the class of all directly decomposable λ-models. Then $\lambda\mathbb{DEC}$ is:*

(i) $2^{\aleph_0}$*-broad.*

(ii) $2^{\aleph_0}$*-high, and even contains countably many "pairwise incompatible"* $2^{\aleph_0}$*-high intervals.*

Proof. (i) Let $U_n \equiv \Omega(\lambda x_1 \dots x_n.\mathbf{I})$ and $\underline{k} \equiv \lambda xy.x^k(y)$ be the k-th Church's numeral. Given a permutation σ of the set of Church's numerals, we write $\mathcal{T}_\sigma, \mathcal{S}_\sigma$ for the λ-theories respectively generated by:

$$E_\sigma^{\mathbf{T}} = \{U_0 = \mathbf{T}\} \cup \{U_n = \sigma(\underline{n-1}) : n \geq 1\},$$
$$E_\sigma^{\mathbf{F}} = \{U_0 = \mathbf{F}\} \cup \{U_n = \sigma(\underline{n-1}) : n \geq 1\}.$$

Obviously, no equation in $E_\sigma^{\mathbf{T}}$ (resp. $E_\sigma^{\mathbf{F}}$) is consequence of the other ones. From [16, Thm. 22] we get that $\mathcal{T}_\sigma, \mathcal{S}_\sigma$ are consistent and hence, by Lemma 4.4.4, we have that the equivalence class of U_0 is a non-trivial central element of $\mathscr{M}_{\mathcal{T}_\sigma \cap \mathcal{S}_\sigma}$. Thus, $\mathcal{T}_\sigma \cap \mathcal{S}_\sigma \in \lambda\mathbb{DEC}$ by Lemma 4.4.3.

If σ_1, σ_2 are two distinct permutations of the set of Church's numerals, then $\mathcal{T}_{\sigma_1} \cap \mathcal{S}_{\sigma_1}$ and $\mathcal{T}_{\sigma_2} \cap \mathcal{S}_{\sigma_2}$ are incompatible, because it is inconsistent to equate $\underline{n} = \underline{m}$ for every $n \neq m$.

Hence, (i) follows since there exist $2^{\aleph_0}$ permutations σ of the set of Church's numerals which give rise to pairwise incompatible λ-theories $\mathcal{T}_\sigma \cap \mathcal{S}_\sigma \in \lambda\mathbb{DEC}$.

(ii) Let σ be a permutation of the Church's numerals and $\mathcal{T}_\sigma, \mathcal{S}_\sigma$ be as in the proof of (i). Suppose that σ is computable, then both $\mathcal{T}_\sigma$ and $\mathcal{S}_\sigma$ are r.e. λ-theories, hence also $\mathcal{T}_\sigma \cap \mathcal{S}_\sigma \in \lambda\mathbb{DEC}$ is r.e. Thus, by Theorem 4.4.14 the interval $[\mathcal{T}_\sigma \cap \mathcal{S}_\sigma[$ contains an interval of $2^{\aleph_0}$ λ-theories belonging to $\lambda\mathbb{DEC}$.

The theorem follows since there exist countably many computable permutations σ. $\square$

Corollary 4.4.17. *The indecomposable semantics omits a set of λ-theories which is $2^{\aleph_0}$-broad, $2^{\aleph_0}$-high, and even contains countably many "pairwise incompatible" $2^{\aleph_0}$-high intervals.*

Corollary 4.4.18. *The Scott-continuous, the stable and the strongly stable semantics omit a set of λ-theories which is $2^{\aleph_0}$-broad, $2^{\aleph_0}$-high, and even contains countably many "pairwise incompatible" $2^{\aleph_0}$-high intervals.*

4.5 Conclusions and further works

We generalized the Stone representation theorem to combinatory algebras showing that every combinatory algebra can be decomposed as a weak product of directly indecomposable combinatory algebras. We showed that the semantics of λ-calculus given in terms of directly indecomposable λ-models, although huge enough to include all the main semantics, is hugely incomplete. This gives a strong, uniform and elegant proof of the incompleteness of the continuous, stable and strongly-stable semantics.

A related question is whether there exists a partially ordered λ-model which admits a non-trivial decomposition. Of course, in this case, there is no reason why the decomposition operators introduced in this chapter should decompose the λ-model respecting the associated ordering. Hence, it would be interesting to find new kinds of decompositions which take into account also the partial order. On the other hand, the result of incompleteness in [92], stating that any semantics of λ-calculus given in terms of partial orderings with a bottom element is incomplete, removed the belief that partial orderings were intrinsic to λ-models. It is an open problem to find new Cartesian closed categories, where the partial orderings play no role and where the reflexive objects are directly indecomposable as combinatory algebras.

Moreover, it would be interesting to find other relations on λ-theories which are more informative than set-theoretical inclusion. A recent proposal, due to Hyland[1] is the following. Recall that the Karubi envelope of a λ-model $\mathscr{C}$ [8, Def. 5.5.11], denoted here by $\mathbf{Ret}(\mathscr{C})$, is the category having as objects the elements $a \in C$ satisfying $a \circ a = a$ and as arrows the elements $f \in C$ such that $a \circ f \circ b = f$. Scott has shown in [97] that $\mathbf{Ret}(\mathscr{C})$ is a Cartesian closed category in which $\mathbf{I}$ is a reflexive object and Koymans [71] proved that $\mathrm{Th}(\mathbf{I}) = \mathrm{Th}(\mathscr{C})$. Hyland's suggestion is to define a preorder on λ-theories as follows:

$$\mathcal{S} \preceq \mathcal{T} \text{ if there exists a reflexive object } \mathscr{R} \text{ in } \mathbf{Ret}(\mathscr{M}_{\mathcal{T}}) \text{ such that } \mathrm{Th}(\mathscr{R}) = \mathcal{S}.$$

We will investigate in the future whether this relation can be fruitfully used to relate models and theories of λ-calculus.

[1] 2007, J.M.E. Hyland. Personal communication.

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Towards a model theory for graph models

Given a class of models in a given semantics, is there a minimal theory represented in it? [...] Can the size of the web affect the equational theory of a model?
(Chantal Berline, from [13])

Even thought the models of λ-calculus have been object of study of many researchers since 1969, we still lack general results of model theory for them. However, for the classes of webbed models, it is possible to infer properties of the models by analyzing the structure of their web. Here, we focus our attention on the simplest kind of webbed models: the graph models.

In this chapter we recall a simple completion process due to Longo [75], which allows to build (the web of) a graph model starting from a "partial web", called *partial pair*. Moreover, we provide some notions and results which are useful for studying the framework of partial pairs, and we show that they can be fruitfully applied for proving that: (*i*) there exists a minimum order graph theory (for equational graph theories this was proved in [29, 30]); (*ii*) every equational/order graph theory is the theory of a graph model having a countable web.

This last result proves that graph models enjoy a kind of (downwards) Löwenheim-Skolem theorem, and it answers positively Question 3 in [13, Sec. 6.3] for the class of graph models.

5.1 Partial pairs

The definition of graph models (and hence of total pairs) has been recalled in Subsection 1.4.2. We need now to develop the wider framework of partial pairs. We will see that from each partial pair we can "freely generate" a graph model (see Definition 5.1.8, below).

5.1.1 Definition and ordering of partial pairs

Definition 5.1.1. *A* partial pair $\mathcal{A}$ *is a pair* $(A, j_\mathcal{A})$ *where* A *is an arbitrary set and* $j_\mathcal{A} : A^* \times A \to A$ *is a partial (possibly total) injection.*

A partial pair $\mathcal{A}$ is *finite* if A is finite, and it is total if $j_\mathcal{A}$ is total. The simplest examples of partial pairs are $(A, \emptyset)$, where $\emptyset$ denotes the empty function, and $(\emptyset, \emptyset)$ which is called the *trivial* or the *empty* partial pair. In the sequel the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ will always denote partial pairs.

Definition 5.1.2. *We say that* $\mathcal{A}$ *is a* subpair *of* $\mathcal{B}$*, and we write* $\mathcal{A} \sqsubseteq \mathcal{B}$*, if:*

- $A \subseteq B$ *and*
- $j_\mathcal{A}(a, \alpha) = j_\mathcal{B}(a, \alpha)$ *for all* $(a, \alpha) \in \mathrm{dom}(j_\mathcal{A})$.

The set of subpairs of $\mathcal{A}$ *will be denoted by* $\mathrm{Sub}(\mathcal{A})$.

It is easy to check that, for all partial pairs $\mathcal{A}$, $(\mathrm{Sub}(\mathcal{A}), \sqsubseteq)$ is a DI-domain. Moreover, given a family $(\mathcal{A}_k)_{k\in K}$ of partial pairs belonging to $\mathrm{Sub}(\mathcal{A})$ we have that $\sqcup_{k\in K}\mathcal{A}_k = (\cup_{k\in K} A_k, \cup_{k\in K} j_{\mathcal{A}_k})$.

5.1.2 Interpretation with respect to a partial pair

Definition 5.1.3. *An* $\mathcal{A}$-environment *is a function* $\rho : \mathrm{Var} \to \mathscr{P}(A)$.

To lighten the notation we will denote by $Env_\mathcal{A}$, instead of $Env_{\mathscr{P}(A)}$, the set of all $\mathcal{A}$-environments.

The definition of the *partial interpretation* $|M|^\mathcal{A}_\rho$ of a λ-term M with respect to a partial pair $\mathcal{A}$, generalizes in the obvious way the one given in Subsection 1.4.2 for graph models. For all $\rho \in Env_\mathcal{A}$ we let:

- $|x|^\mathcal{A}_\rho = \rho(x)$,
- $|PQ|^\mathcal{A}_\rho = \{\alpha : (\exists a \subseteq_\mathrm{f} |Q|^\mathcal{A}_\rho)[(a, \alpha) \in \mathrm{dom}(j_\mathcal{A}) \text{ and } j_\mathcal{A}(a, \alpha) \in |P|^\mathcal{A}_\rho]\}$,
- $|\lambda x.N|^\mathcal{A}_\rho = \{j_\mathcal{A}(a, \alpha) : (a, \alpha) \in \mathrm{dom}(j_\mathcal{A}) \text{ and } \alpha \in |N|^\mathcal{A}_{\rho[x:=a]}\}$.

Note that, if $\mathscr{G}$ is a graph model with web $\mathcal{G}$, then $|M|^\mathscr{G}_\rho = |M|^\mathcal{G}_\rho$ for every λ-term M and $\mathcal{G}$-environment ρ. Remark also that, if $\mathcal{A}$ is not total, β-equivalent λ-terms do not necessarily have the same interpretation.

As a matter of notation, given $\rho \in Env_\mathcal{A}, \sigma \in Env_\mathcal{B}$ and a set C, we write:

- $\sigma = \rho \cap C$ if $\sigma(x) = \rho(x) \cap C$ for every variable x, and
- $\rho \subseteq \sigma$ if $\rho(x) \subseteq \sigma(x)$ for every variable x.

We provide now two lemmata on the partial interpretation of λ-terms with respect to partial pairs. The idea behind Lemma 5.1.5 below is that, for a fixed $M \in \Lambda$, the function from $\mathrm{Sub}(\mathcal{A}) \times Env_{\mathcal{A}}$ to $\mathscr{P}(A)$ mapping $(\mathcal{B}, \rho) \mapsto |M|^{\mathcal{B}}_{\rho \cap B}$ is continuous.

Lemma 5.1.4. *If $\mathcal{A} \sqsubseteq \mathcal{B}$, then $|M|^{\mathcal{A}}_{\rho} \subseteq |M|^{\mathcal{B}}_{\sigma}$ for all $\rho \in Env_{\mathcal{A}}$ and $\sigma \in Env_{\mathcal{B}}$ such that $\rho \subseteq \sigma$.*

Proof. By straigthforward induction on the structure of M. $\square$

Lemma 5.1.5. *Let $M \in \Lambda$, $\mathcal{B}$ be a partial pair and $\rho \in Env_{\mathcal{B}}$. Suppose $\alpha \in |M|^{\mathcal{B}}_{\rho}$ then there exists a finite pair $\mathcal{A} \sqsubseteq \mathcal{B}$ such that $\alpha \in |M|^{\mathcal{A}}_{\rho \cap A}$.*

Proof. The proof is by induction on M.

If $M \equiv x$, then $\alpha \in \rho(x)$, so that we define $\mathcal{A} = (\{\alpha\}, \emptyset)$.

If $M \equiv PQ$, then there is a finite set $a = \{\alpha_1, \dots, \alpha_n\}$, for some $n \geq 0$, such that $(a, \alpha) \in \mathrm{dom}(j_{\mathcal{B}})$, $j_{\mathcal{B}}(a, \alpha) \in |P|^{\mathcal{B}}_{\rho}$ and $a \subseteq |Q|^{\mathcal{B}}_{\rho}$. By the induction hypothesis there exist finite subpairs $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ of $\mathcal{B}$ such that $j_{\mathcal{B}}(a, \alpha) \in |P|^{\mathcal{A}_{n+1}}_{\rho \cap A_{n+1}}$ and $\alpha_k \in |Q|^{\mathcal{A}_k}_{\rho \cap A_k}$ for $k = 1, \dots, n$. We define $\mathcal{A} \sqsubseteq \mathcal{B}$ as $\sqcup_{k=0,\dots,n+1} \mathcal{A}_k$ where $\mathcal{A}_0 = (a \cup \{\alpha\}, j_{\mathcal{B}}\restriction_{\{(a,\alpha)\}})$. From Lemma 5.1.4 it follows the conclusion.

If $M \equiv \lambda x.N$, then $\alpha = j_{\mathcal{B}}(b, \beta)$ for some b and β such that $(b, \beta) \in \mathrm{dom}(j_{\mathcal{B}})$ and $\beta \in |N|^{\mathcal{B}}_{\rho[x:=b]}$. By the induction hypothesis there exists a finite pair $\mathcal{C} \sqsubseteq \mathcal{B}$ such that $\beta \in |N|^{\mathcal{C}}_{\rho[x:=b] \cap C}$. We define $\mathcal{A} \sqsubseteq \mathcal{B}$ as $\mathcal{C} \sqcup (b \cup \{\alpha, \beta\}, j_{\mathcal{B}}\restriction_{\{(b,\beta)\}})$. Then we have that $\mathcal{C} \sqsubseteq \mathcal{A}$ and $\rho[x := b] \cap C \subseteq \rho[x := b] \cap A$. From $\beta \in |N|^{\mathcal{C}}_{\rho[x:=b] \cap C}$ and from Lemma 5.1.4 it follows that $\beta \in |N|^{\mathcal{A}}_{\rho[x:=b] \cap A} = |N|^{\mathcal{A}}_{(\rho \cap A)[x:=b]}$. Then we conclude that $\alpha = j_{\mathcal{A}}(b, \beta) \in |\lambda x.N|^{\mathcal{A}}_{\rho \cap A}$. $\square$

5.1.3 Morphisms between partial pairs

Definition 5.1.6. *A total function $\theta : A \to B$ is a* morphism *from $\mathcal{A}$ to $\mathcal{B}$ if, for all $(a, \alpha) \in A^* \times A$, we have:*

$$(a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}}) \Longrightarrow [(\theta^+(a), \theta(\alpha)) \in \mathrm{dom}(j_{\mathcal{B}}) \text{ and } \theta(j_{\mathcal{A}}(a, \alpha)) = j_{\mathcal{B}}(\theta^+(a), \theta(\alpha))]$$

and it is an endomorphism *if, moreover, $\mathcal{A} = \mathcal{B}$.*

Note that an *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a bijection $\theta : A \to B$ such that both θ and θ^{-1} are morphisms; if, moreover, $\mathcal{A} = \mathcal{B}$ then θ is an *automorphism*.

This notion of morphism is in some way a generalization of the notion of subpair given in Subsection 5.1.1. Indeed, it is easy to verify that we have $\mathcal{A} \sqsubseteq \mathcal{B}$ exactly when the inclusion mapping $\iota : A \to B$ is a morphism.

As a matter of notation:

- $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ denotes the set of morphisms from $\mathcal{A}$ to $\mathcal{B}$,

- $\mathrm{Iso}(\mathcal{A},\mathcal{B})$ denotes the set of isomorphisms between $\mathcal{A}$ and $\mathcal{B}$,
- $\mathrm{Aut}(\mathcal{A})$ denotes the group of automorphisms of $\mathcal{A}$,

Moreover, $\theta : \mathcal{A} \to \mathcal{B}$ will be an alternative notation for $\theta \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$.

Lemma 5.1.7. *Let* $\phi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ *and* $\rho \in Env_{\mathcal{A}}$. *Then:*
(i) $\phi^+(|M|^{\mathcal{A}}_{\rho}) \subseteq |M|^{\mathcal{B}}_{\phi^+\circ\rho}$
(ii) $\phi^+(|M|^{\mathcal{A}}_{\rho}) = |M|^{\mathcal{B}}_{\phi^+\circ\rho}$ *if* $\phi \in \mathrm{Iso}(\mathcal{A},\mathcal{B})$.

Proof. (i) By straightforward induction on M one proves that, for all $\alpha \in \phi^+(|M|^{\mathcal{A}}_{\rho})$, we have $\phi(\alpha) \in |M|^{\mathcal{B}}_{\phi^+\circ\rho}$.

(ii) By (i) it is enough to prove that $|M|^{\mathcal{B}}_{\phi^+\circ\rho} \subseteq \phi^+(|M|^{\mathcal{A}}_{\rho})$. Let $\psi = \phi^{-1}$; then $\phi^+ \circ \psi^+ = \mathrm{Id}$. Thus $|M|^{\mathcal{B}}_{\phi^+\circ\rho} = \phi^+(\psi^+(|M|^{\mathcal{B}}_{\phi^+\circ\rho})) \subseteq \phi^+(|M|^{\mathcal{A}}_{\psi^+\circ\phi^+\circ\rho}) = \phi^+(|M|^{\mathcal{A}}_{\rho})$, the inclusion follows by (i). $\square$

As a consequence of Lemma 5.1.7(ii) we have that if $\phi \in \mathrm{Iso}(\mathcal{A},\mathcal{B})$, with $\mathcal{A},\mathcal{B}$ total, then ϕ^+ is an isomorphism of λ-models.

On the contrary, if ϕ is only a morphism of pairs, then ϕ^+ cannot be a morphism of combinatory algebras. Indeed, it is easy to check that $\phi^+(|\mathbf{K}|^{\mathcal{A}}) \subsetneq |\mathbf{K}|^{\mathcal{B}}$ if ϕ is not surjective and $\phi^+(|MN|^{\mathcal{A}}) \subsetneq \phi^+(|M|^{\mathcal{A}}) \cdot \phi^+(|N|^{\mathcal{A}})$ if ϕ is not injective.

5.1.4 Graph-completions of partial pairs

There are two known processes for building a graph model satisfying some additional requirements. Both consist in completing a partial pair $\mathcal{A}$ into a total pair. The *free completion*[1], which is due to Longo [75] and mimics the construction of Engeler's model $\mathscr{E}$, is a constructive way for building, canonically and as freely as possible, a total pair $\overline{\mathcal{A}}$ from a partial pair $\mathcal{A}$. This construction opens the possibility to induce some properties of the graph model with web $\overline{\mathcal{A}}$ from $\mathcal{A}$.

The other completion process, called *forcing completion* or simply "forcing", originates in [6]. Baeten and Boerboom built out of a partial pair $(G, \emptyset)$, for all $M \in \Lambda^o$, a graph model $\mathscr{G}$ with web (G, i^M_G) such that $|\Omega|^{\mathscr{G}} = |M|^{\mathscr{G}}$, thus proving semantically that Ω is easy. This technique is, in general, non constructive. Forcing was generalized in [16], where it is shown, in particular, that we can go far beyond Λ^o. In this thesis forcing will only have an auxiliary role allowing us to produce examples; hence "completion" will mean "free completion" unless otherwise stated.

Let us recall now the formal definition of free completion.

[1] This is the terminology used, e.g., in [13, 14]. Free completion is also termed *canonical completion* [30] and *Engeler completion* [28, 29].

Definition 5.1.8. *Let* $\mathcal{A} = (A, j_{\mathcal{A}})$ *be a partial pair. The* free completion *of* $\mathcal{A}$ *is the total pair* $\overline{\mathcal{A}} = (\overline{A}, i_{\bar{A}})$*, where* $\overline{A} = \cup_{n\in\mathbb{N}} A_n$*, with* $A_0 = A$*,* $A_{n+1} = A \cup ((A_n^* \times A_n) - \mathrm{dom}(j_{\mathcal{A}}))$ *and* $i_{\bar{A}}$ *is defined by:*

$$i_{\bar{A}}(a, \alpha) = \begin{cases} j_{\mathcal{A}}(a, \alpha) & \text{if } (a, \alpha) \in \mathrm{dom}(j_{\mathcal{A}}), \\ (a, \alpha) & \text{otherwise.} \end{cases}$$

An element of A *has rank* 0*, whilst an element* $\alpha \in \overline{A} - A$ *has rank* n *if* $\alpha \in A_n - A_{n-1}$.

Actually this completion construction requires that $((A^* \times A) - \mathrm{dom}(j_{\mathcal{A}})) \cap \mathrm{rg}(j_{\mathcal{A}}) = \emptyset$, otherwise $i_{\bar{A}}$ would not be injective, hence we will always suppose that no element of A is a pair. This is not restrictive because partial pairs can be considered up to isomorphism.

Notation 5.1.9. *In the sequel,* $\mathscr{G}_{\mathcal{A}}$ *will denote the graph model whose web is* $\overline{\mathcal{A}}$*, and it will be said* freely generated *by* $\mathcal{A}$.

Then, we note that the graph model $\mathscr{G}_{(\emptyset,\emptyset)}$ freely generated by the trivial pair is trivial. Hence, we will always suppose in the sequel that the partial pairs we are considering are non trivial.

Of course, not all graph models have a web which can be obtained as a free completion of a (proper) partial pair. In particular there is a recent result, which is recalled below, stating that the free completion process only generates semi-sensible graph models.

Theorem 5.1.10. *(Bucciarelli and Salibra [30, Thm. 29])*
If $\mathcal{A}$ *is a partial pair which is not total then* $\mathscr{G}_{\mathcal{A}}$ *is semi-sensible.*

Remark 5.1.11. *Let* $\mathcal{A}, \mathcal{B}$ *be two partial pairs. If* $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \overline{\mathcal{A}}$ *then* $\overline{\mathcal{A}} = \overline{\mathcal{B}}$ *and hence* $\mathscr{G}_{\mathcal{A}} = \mathscr{G}_{\mathcal{B}}$.

Example 5.1.12. *By definition:*

- *(i) the* Engeler's model $\mathscr{E}$ *is freely generated by* $\mathcal{A} = (A, \emptyset)$*, where* A *is a non-empty set. Thus, in fact, we have a family of graph models* $\mathscr{E}_A$*;*
- *(ii) the* graph-Scott models *are freely generated by* $\mathcal{A} = (A, j_{\mathcal{A}})$*, where* $j_{\mathcal{A}}(\emptyset, \alpha) = \alpha$ *for all* $\alpha \in A$*;*
- *(iii) the* graph-Park models *are freely generated by* $\mathcal{A} = (A, j_{\mathcal{A}})$*, where* $j_{\mathcal{A}}(\{\alpha\}, \alpha) = \alpha$ *for all* $\alpha \in A$*;*
- *(iv) the* mixed-Scott-Park graph models *are freely generated by* $\mathcal{A} = (A, j_{\mathcal{A}})$ *where* $j_{\mathcal{A}}(\emptyset, \alpha) = \alpha$ *for all* $\alpha \in Q$*,* $j_{\mathcal{A}}(\{\beta\}, \beta) = \beta$ *for all* $\beta \in R$ *and* Q, R *form a non-trivial partition of* A.

Remark 5.1.13. *(Longo [75]) The model* $\mathscr{P}_\omega$ *is isomorphic to the graph-Scott model of a pair* $\mathcal{A} = (\{0\}, j_{\mathcal{A}})$ *where* $j_{\mathcal{A}}(\emptyset, 0) = 0$.

Theorem 5.1.14.

(i) *(Kerth [66, 69]) There exist* $2^{\aleph_0}$ *graph models of the form* $\mathscr{G}_{\mathcal{A}}$*, with distinct theories, among which* $\aleph_0$ *are freely generated by finite pairs.*

(ii) *(Kerth [68] plus David [41]) The same is true for sensible graph models.*

Lemma 5.1.15.

(i) *For all* $\theta \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ *there is a unique* $\bar{\theta} \in \mathrm{Hom}(\overline{\mathcal{A}}, \overline{\mathcal{B}})$ *such that* $\bar{\theta}\restriction_A = \theta$.

(ii) *If* $\theta \in \mathrm{Iso}(\mathcal{A}, \mathcal{B})$*, then* $\bar{\theta} \in \mathrm{Iso}(\overline{\mathcal{A}}, \overline{\mathcal{B}})$.

Proof. The definition of $\bar{\theta}$ and the verification of the first point are by straightforward induction on the rank of the elements of $\overline{A}$. It is also easy to check that if θ is an isomorphism then $\bar{\theta}^{-1}$ is the inverse of $\bar{\theta}$. □

A morphism $\theta : \mathcal{A} \to \mathcal{B}$ does not induce, in general, a morphism of models except when θ is an isomorphism. In other words, the next corollary holds.

Corollary 5.1.16. *Let* $\theta \in \mathrm{Iso}(\mathcal{A}, \mathcal{B})$*, then:*

(i) $\bar{\theta}^{+} \in \mathrm{Iso}(\mathscr{G}_{\mathcal{A}}, \mathscr{G}_{\mathcal{B}})$,

(ii) $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}}) = \mathrm{Th}_{\sqsubseteq}(\mathscr{G}_{\mathcal{B}})$.

(iii) $\mathrm{Th}(\mathscr{G}_{\mathcal{A}}) = \mathrm{Th}(\mathscr{G}_{\mathcal{B}})$.

Proof. (i) By Lemma 5.1.7 and Lemma 5.1.15.
(ii) By (i) and Remark 1.4.3.
(iii) From (ii). □

Proposition 5.1.17. *Let* $\mathscr{G}$ *be a graph model with web* $\mathcal{G}$*, and suppose* $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$ *for some* $M, N \in \Lambda^o$*. Then there exists a finite* $\mathcal{A} \sqsubseteq \mathcal{G}$ *such that: for all pairs* $\mathcal{C} \sqsupseteq \mathcal{A}$*, if there is a morphism* $\theta : \mathcal{C} \to \mathcal{G}$ *such that* $\theta(\alpha) = \alpha$*, then* $\alpha \in |M|^{\mathcal{C}} - |N|^{\mathcal{C}}$.

Proof. By Lemma 5.1.5 there is a finite pair $\mathcal{A}$ such that $\alpha \in |M|^{\mathcal{A}}$. By Lemma 5.1.4 we have $\alpha \in |M|^{\mathcal{C}}$. Now, if $\alpha \in |N|^{\mathcal{C}}$ then, by Lemma 5.1.7, $\alpha = \theta(\alpha) \in |N|^{\mathscr{G}}$, which is a contradiction. □

Corollary 5.1.18. *Let* $\mathscr{G}$ *be a graph model, and suppose* $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$ *for some* $M, N \in \Lambda^o$*. Then there exists a finite* $\mathcal{A} \sqsubseteq \mathcal{G}$ *such that, for all pairs* $\mathcal{B}$ *satisfying* $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{G}$*, we have:*

(i) $\alpha \in |M|^{\mathcal{B}} - |N|^{\mathcal{B}}$ *and*

(ii) $\alpha \in |M|^{\mathscr{G}_{\mathcal{B}}} - |N|^{\mathscr{G}_{\mathcal{B}}}$.

5.1.5 Retracts

Definition 5.1.19. *Given two partial pairs $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$* is a retract of *$\mathcal{B}$, and we write $\mathcal{A} \lhd \mathcal{B}$, if there are morphims $e \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ and $\pi \in \mathrm{Hom}(\mathcal{B}, \mathcal{A})$ such that $\pi \circ e = \mathrm{Id}_A$. In this case we will also write $(e, \pi) : \mathcal{A} \lhd \mathcal{B}$.*

Notation 5.1.20. *Given two graph models $\mathscr{G}, \mathscr{G}'$ we write $\mathscr{G} \lhd \mathscr{G}'$ if $\mathcal{G}$ is a retract of $\mathcal{G}'$.*

From Lemma 5.1.15(i), and the fact that the identity $\mathrm{Id}_{\overline{A}}$ is the only endomorphism of $\overline{\mathcal{A}}$ whose restriction to A is Id_A, we get the following lemma.

Lemma 5.1.21. *Let $\mathcal{A}, \mathcal{B}$ be two partial pairs, then $\mathcal{A} \lhd \mathcal{B}$ implies $\overline{\mathcal{A}} \lhd \overline{\mathcal{B}}$.*

Proposition 5.1.22. *If $\mathscr{G} \lhd \mathscr{G}'$ then $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}') \subseteq \mathrm{Th}_{\sqsubseteq}(\mathscr{G})$ and $\mathrm{Th}(\mathscr{G}') \subseteq \mathrm{Th}(\mathscr{G})$.*

Proof. Let $(e, \pi) : \mathcal{G} \lhd \mathcal{G}'$. It is enough to prove that for all $M, N \in \Lambda^o$, if $\alpha \in |M|^{\mathscr{G}} - |N|^{\mathscr{G}}$ then $\theta(\alpha) \in |M|^{\mathscr{G}'} - |N|^{\mathscr{G}'}$. Now, by applying Lemma 5.1.7 twice, $\theta(\alpha) \in |M|^{\mathscr{G}'}$ and $\theta(\alpha) \in |N|^{\mathscr{G}'}$ would imply $\alpha = \theta'(\theta(\alpha)) \in |N|^{\mathscr{G}}$. □

Example 5.1.23. *Given the Engeler's model $\mathscr{E}_A$ freely generated by $\mathcal{A} = (A, \emptyset)$ and the graph-Scott model $\mathscr{P}_A$ freely generated by $\mathcal{A}' = (A, j_A)$, we have:*

(a) $\mathcal{A} \sqsubseteq \mathcal{A}'$ *but* not $\mathcal{A} \sqsubseteq \mathcal{A}' \sqsubseteq \overline{\mathcal{A}}$,

(b) $(\emptyset, \alpha) \in \overline{A} - \overline{A'}$,

(c) $\mathrm{Th}(\mathscr{E}_A) = \mathrm{Th}(\mathscr{P}_A) = \mathcal{B}_{\mathcal{T}}$ *[75]*,

(d) $\mathrm{Th}_{\sqsubseteq}(\mathscr{E}_A) \subsetneq \mathrm{Th}_{\sqsubseteq}(\mathscr{P}_A)$ *[75, Prop. 2.8]*,

(e) $\mathbf{I} \sqsubseteq \mathbf{1} \in \mathrm{Th}_{\sqsubseteq}(\mathscr{P}_A) - \mathrm{Th}_{\sqsubseteq}(\mathscr{E}_A)$ *(easy)*,

(f) $\alpha \in |\lambda x.\mathbf{I}|^{\mathscr{P}_A} - |\lambda x.\mathbf{I}|^{\mathscr{E}_A}$ *and* $(\emptyset, \alpha) \in |\lambda x.\mathbf{I}|^{\mathscr{E}_A} - |\lambda x.\mathbf{I}|^{\mathscr{P}_A}$.

5.2 The minimum order and equational graph theories

In [29, 30], Bucciarelli and Salibra defined a notion of "weak product" for graph models. In this paper we prefer to call this construction *gluing* since it does not enjoy the categorical properties of a weak product.

Definition 5.2.1. *The* gluing *$\Diamond_{k \in K} \mathscr{G}_k$ of a family $(\mathscr{G}_k)_{k \in K}$ of graph models with pairwise disjoint webs is the graph model freely generated by the partial pair $\sqcup_{k \in K} \mathcal{G}_k$; its web is denoted by $\Diamond_{k \in K} \mathcal{G}_k$ instead of $\overline{\sqcup_{k \in K} \mathcal{G}_k}$.*

Note that the gluing is commutative and associative up to isomorphism of graph models. More generally, for any family $(\mathscr{G}_k)_{k\in K}$ of graph models, $\Diamond_{k\in K}\mathscr{G}_k$ will denote any gluing of isomorphic copies of the $\mathscr{G}_k$'s with pairwise disjoint webs.

Lemma 5.2.2. *Let $(\mathscr{G}_k)_{k\in K}$, be a family of graph models such that $\mathscr{G}_k = \mathscr{G}_{\mathcal{A}_k}$ for some family $(\mathcal{A}_k)_{k\in K}$ of pairwise disjoint partial pairs. Then $\Diamond_{k\in K}\mathscr{G}_k = \mathscr{G}_{\mathcal{A}}$, where $\mathcal{A} = \overline{\sqcup_{k\in K}\mathcal{A}_k}$.*

Proof. By Remark 5.1.11 since, clearly, $\sqcup_{k\in K}\mathcal{A}_k \sqsubseteq \sqcup_{k\in K}\overline{\mathcal{A}_k} \sqsubseteq \overline{\sqcup_{k\in K}\mathcal{A}_k}$, and $\sqcup_{k\in K}\overline{\mathcal{A}_k} = \sqcup_{k\in K}\mathcal{G}_k$ by definition. ☐

Proposition 5.2.3. *(Bucciarelli and Salibra [28, Prop. 2])*
Let $(\mathscr{G}_k)_{k\in K}$ be a family of graph models and $\mathscr{G} = \Diamond_{k\in K}\mathscr{G}_k$, then:

(i) $|M|^{\mathscr{G}_k} = |M|^{\mathscr{G}} \cap G_k$ *for any* $M \in \Lambda^o$, *hence:*

(ii) $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}) \subseteq \mathrm{Th}_{\sqsubseteq}(\mathscr{G}_k)$,

(iii) $\mathrm{Th}(\mathscr{G}) \subseteq \mathrm{Th}(\mathscr{G}_k)$.

The existence of a minimum equational graph theory has been shown by Bucciarelli and Salibra in [29, 30]. By slightly modifying their proof, we are able to prove that there exists also a graph model whose *order* theory (and hence equational theory) is the minimum one.

Theorem 5.2.4. *There exists a graph model whose order/equational theory is the minimum order/equational graph theory.*

Proof. Let $(\mathcal{A}_k)_{k\in\mathbb{N}}$ be a family of pairwise disjoint finite partial pairs such that all other finite pairs are isomorphic to at least one $\mathcal{A}_k$. We now take $\mathscr{G} = \Diamond_{k\in K}\mathscr{G}_k$, where $\mathscr{G}_k = \mathscr{G}_{\mathcal{A}_k}$. By Lemma 5.2.2, $\mathscr{G} = \mathscr{G}_{\mathcal{A}}$ where $\mathcal{A} = \sqcup_{k\in\mathbb{N}}\mathcal{A}_k$.

We now prove that the order theory, and hence also the equational theory, of $\mathscr{G}$ is the minimum one. Let e be an inequality which fails in some graph model. By Corollary 5.1.18(ii) e fails in some $\mathscr{G}_{\mathcal{B}}$ where $\mathcal{B}$ is some finite pair, hence it fails in some $\mathscr{G}_k$. Thus, by Proposition 5.2.3(ii), e fails in $\mathscr{G}$. ☐

Recall that the minimum equational graph theory cannot be λ_β or $\lambda_{\beta\eta}$ by Proposition 1.4.8.

5.3 A Löwenheim-Skolem theorem for graph models

In this section we prove a kind of downwards Löwenheim-Skolem theorem for graph models: every equational/order graph theory is the theory of a graph model having a countable web. This result positively answers Question 3 in [13, Sec. 6.3] for the class of graph models. Note that applying the classical Löwenheim-Skolem theorem to a graph model $\mathscr{G}$, viewed as a combinatory algebra $\mathscr{C}$, would only give a countable elementary substructure $\mathscr{C}'$ of $\mathscr{C}$. Such a $\mathscr{C}'$ does not correspond to any graph model since there exists no countable graph model.

The class of total subpairs of a total pair $\mathcal{G}$ is closed under (finite or infinite) intersections and increasing unions.

Definition 5.3.1. *If $\mathcal{A} \sqsubseteq \mathcal{G}$ is a partial pair, then the* total subpair of $\mathcal{G}$ generated by $\mathcal{A}$ *is defined as the intersection of all the total pairs $\mathcal{G}'$ such that $\mathcal{A} \sqsubseteq \mathcal{G}' \sqsubseteq \mathcal{G}$.*

Theorem 5.3.2. (Löwenheim-Skolem Theorem for graph models)
For all graph models $\mathscr{G}$ there exists a graph model $\mathscr{G}'$ with a countable web $\mathcal{G}' \sqsubseteq \mathcal{G}$ such that $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}') = \mathrm{Th}_{\sqsubseteq}(\mathscr{G})$, and hence such that $\mathrm{Th}(\mathscr{G}') = \mathrm{Th}(\mathscr{G})$.

Proof. We will define an increasing sequence of countable subpairs $\mathcal{A}_n$ of $\mathcal{G}$, and take for $\mathcal{G}'$ the total subpair of $\mathcal{G}$ generated by $\mathcal{A} = \sqcup_{n\in\mathbb{N}}\mathcal{A}_n$.

We start defining $\mathcal{A}_0$. Let I be the countable set of inequalities between closed λ-terms which fail in $\mathscr{G}$. Let $e \in I$. By Corollary 5.1.18(i) there exists a finite partial pair $\mathcal{A}_e \sqsubseteq \mathcal{G}$ such that e fails in every partial pair $\mathcal{B}$ satisfying $\mathcal{A}_e \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{G}$. Then we define $\mathcal{A}_0 = \sqcup_{e\in I}\mathcal{A}_e \sqsubseteq \mathcal{G}$. Assume now that $\mathcal{A}_n$ has been defined, and we define $\mathcal{A}_{n+1}$ as follows. Let $\mathscr{G}'_n$ be the graph model whose web $\mathcal{G}'_n$ is the total subpair of $\mathcal{G}$ generated by $\mathcal{A}_n$. For each inequality $e = M \sqsubseteq N$ which holds in $\mathscr{G}$ and fails in $\mathscr{G}'_n$, we consider the set $L_e = \{\alpha \in G'_n : \alpha \in |M|^{\mathscr{G}'_n} - |N|^{\mathscr{G}'_n}\}$. Let $\alpha \in L_e$. Since $\mathcal{G}'_n \sqsubseteq \mathcal{G}$ and $\alpha \in |M|^{\mathscr{G}'_n}$, then by Lemma 5.1.4 we have that $\alpha \in |M|^{\mathscr{G}}$. By $|M|^{\mathscr{G}} \subseteq |N|^{\mathscr{G}}$ we also obtain $\alpha \in |N|^{\mathscr{G}}$. By Lemma 5.1.5 there exists a partial pair $\mathcal{C}_{\alpha,e} \sqsubseteq \mathcal{G}$ such that $\alpha \in |N|^{\mathcal{C}_{\alpha,e}}$. We define $\mathcal{A}_{n+1}$ as the union of the partial pair $\mathcal{A}_n$ and the partial pairs $\mathcal{C}_{\alpha,e}$ for every $\alpha \in L_e$.

As announced, we take for $\mathcal{G}'$ the total subpair of $\mathcal{G}$ generated by $\mathcal{A} = \sqcup_{n\in\mathbb{N}}\mathcal{A}_n$. By construction we have, for every inequality e which fails in $\mathscr{G}$: $\mathcal{A}_e \sqsubseteq \mathcal{G}'_n \sqsubseteq \mathcal{G}' \sqsubseteq \mathcal{G}$. Now, $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}') \subseteq \mathrm{Th}_{\sqsubseteq}(\mathscr{G})$ follows from Corollary 5.1.18(i) and from the choice of $\mathcal{A}_e$.

Suppose now, by contradiction, that there exists an inequality $M \sqsubseteq N$ which fails in $\mathscr{G}'$ but not in $\mathscr{G}$. Then there is an $\alpha \in |M|^{\mathscr{G}'} - |N|^{\mathscr{G}'}$. By Corollary 5.1.18(i) there is a finite partial pair $\mathcal{B} \sqsubseteq \mathcal{G}'$ satisfying the following condition: for every partial pair $\mathcal{C}$ such that $\mathcal{B} \sqsubseteq \mathcal{C} \sqsubseteq \mathcal{G}'$, we have $\alpha \in |M|^{\mathcal{C}} - |N|^{\mathcal{C}}$. Since $\mathcal{B}$ is finite, we have that $\mathcal{B} \sqsubseteq \mathcal{G}'_n$ for some n. This implies that $\alpha \in |M|^{\mathscr{G}'_n} - |N|^{\mathscr{G}'_n}$. By construction of $\mathscr{G}'_{n+1}$ we have that $\alpha \in |N|^{\mathscr{G}'_{n+1}}$; this implies $\alpha \in |N|^{\mathscr{G}'}$. Contradiction. □

5.4 Conclusions

In this chapter we have developed several mathematical tools for studying the framework of partial pairs. These tools have been fruitfully used here for proving the existence of a minimum order graph theory, and a Löwenheim-Skolem theorem for graph models, and they will have other interesting consequences in the next chapter.

A concluding remark is that an alternative proof of Theorem 5.3.2 could be given by using the Löwenheim-Skolem theorem for weak monadic second order structures. Moreover, using this approach, it is possible to provide a tool generic enough for treating simultaneously all the several classes of webbed models studied in the literature changing, when necessary, weak monadic second order logic for another logic.

We will keep this development for a later work: providing all the details here would add several pages, by forcing us to introduce the various adequate logics and enter more deeply in the definitions and particularities of the stable and strongly stable semantics and of the diverse classes.

6

Effective models of λ-calculus

Is there a continuously complete CPO model of the λ-calculus whose theory is precisely λ_β or $\lambda_{\beta\eta}$? I asked myself this question in 1983. In 1984, on different occasions I asked it to Dana Scott and Gordon Plotkin. Both told me that they had already thought about it.
(Furio Honsell, from [57])

A longstanding open problem is whether there exists a non-syntactical continuous model of the untyped λ-calculus whose equational theory is exactly the least (least extensional) λ-theory λ_β ($\lambda_{\beta\eta}$). In this chapter we investigate the more general question of whether the equational/order theory of a model of the untyped λ-calculus living in one of the main semantics can be recursively enumerable. We introduce a notion of effective model of λ-calculus, which covers in particular all the models individually introduced in the literature. We prove that the order theory of an effective model is never r.e.; from this it follows that its equational theory cannot be λ_β or $\lambda_{\beta\eta}$. Then, we show that no effective model living in the stable or strongly stable semantics has an r.e. equational theory. Concerning Scott-continuous semantics, we investigate the class of graph models and prove that no order graph theory can be r.e., and that there exists an effective graph model whose equational/order theory is the minimum one.

6.1 Introduction

6.1.1 Description of the problem

The initial problem. The question of the existence of a non-syntactical continuous model of λ_β or $\lambda_{\beta\eta}$ was proposed by Honsell and Ronchi Della Rocca in 1984 (see [58]). This problem, which is still open, generated a wealth of interesting research and results (surveyed first in [13] and later on in [14]). Here, we briefly recall what is relevant for the present work.

The first results. In 1995, Di Gianantonio, Honsell and Plotkin succeeded to build an extensional model of $\lambda_{\beta\eta}$ living in a weakly continuous semantics [46].

However, the construction of this model (as an inverse limit) starts from the term model of $\lambda_{\beta\eta}$, and hence involves the syntax of λ-calculus. Furthermore the problem of whether there exists a model of λ_β or $\lambda_{\beta\eta}$ *living in one of the main semantics* remains completely open. Nevertheless, the authors showed in the same paper that the set of extensional λ-theories induced by Scott-continuous models has a least element. At the same time Selinger proved that if an ordered model has theory λ_β or $\lambda_{\beta\eta}$ then the order is discrete on the interpretations of λ-terms [102].

First extension: the minimality problem. In view of [46], it became natural to ask whether, given a (uniform) class of models of λ-calculus, there was a minimum λ-theory represented in it; a question which was raised in [13]. Bucciarelli and Salibra showed [29, 30] that the answer is also positive for the class of graph models, and that the least graph theory was different from λ_β. At the moment the problem remains open for the other classes.

Each class of models represents and omits many λ-theories. Ten years ago, it was proved that, in each of the known (uniform) classes $\mathbb{C}$ of models living in one of the main semantics, it is possible to build $2^{\aleph_0}$ models inducing pairwise distinct λ-theories [69, 70]. More recently, it has been proved in [92] that there are $2^{\aleph_0}$ λ-theories which are omitted by all the above mentioned $\mathbb{C}$'s, among which $\aleph_0$ are finitely axiomatizable.

From these results, and since there are only $\aleph_0$ recursively enumerable λ-theories, it follows that each class $\mathbb{C}$ represents $2^{\aleph_0}$ non r.e. λ-theories and omits $\aleph_0$ r.e. λ-theories. Note also that there are only very few λ-theories of non-syntactical models which are known to admit an alternative description (e.g., via syntactical considerations), and that all happen to coincide either with the λ-theory $\mathcal{B}_\mathcal{T}$ of Böhm trees [8], or some variations of it, and hence are non r.e. Thus we find natural to ask the following question.

Can a non-syntactical model have an r.e. theory? This problem was first raised in [14], where it is conjectured that no graph model can have an r.e. theory. But we expect that this could indeed be true for all models living in the Scott-continuous semantics, and its refinements (but of course not in its weakenings, because of [46]). Here we extend officially this conjecture.

Conjecture 1. *No λ-calculus model living in Scott-continuous semantics or one of its refinements has an r.e. equational theory.*

6.1.2 Methodology

1) *Look also at order theories.* Since all the models we are interested in are partially ordered, and since, in this case, the equational theory $\mathrm{Th}(\mathscr{M})$ is easily expressible from its order theory $\mathrm{Th}_\sqsubseteq(\mathscr{M})$ (in particular if $\mathrm{Th}_\sqsubseteq(\mathscr{M})$ is r.e. then also $\mathrm{Th}(\mathscr{M})$ is

r.e.) we will also address the analogue problem for order theories.

2) *Look at models with built-in effectivity properties.* There are two reasons for this. First, it seems reasonable to think that, if effective models do not even succeed to have an r.e. theory, then it is unlikely that the other ones may succeed. Second, because effective (webbed) models, in our sense, are omni-present in the continuous, stable and strongly stable semantics; by this we mean that they are omnipresent in the literature! Starting from the known notion of an effective domain, we introduce an appropriate notion of an *effective model of λ-calculus* and we study the main properties of these models[1]. Note that, in the absolute, effective models happen to be rare, since each (uniform) class $\mathbb{C}$ of models represents $2^{\aleph_0}$ λ-theories, but contains only $\aleph_0$ non-isomorphic effective models! However, and this is a third *a posteriori* reason to work with them, it happens that they can be used to prove properties of non effective models (Theorem 3 below is the first example we know of such a result).

3) A previous result obtained for *typed λ-calculus* also justifies the above methodology. Indeed, it was proved in [12] that there exists a (webbed) model of Girard's system F, living in the Scott-continuous semantics, whose λ-theory is $\lambda_{\beta\eta}$, and whose construction does not involve the syntax of λ-calculus. Furthermore, this model can easily be checked to be "effective" in the same spirit as in the present work (see [12, App. C] for a sketchy presentation of the model). Note that this model has no analogue in the stable semantics.

4) *Look at the class of graph models.* Studying graph models illustrates the spirit of the tools we aim at developing, while keeping technicalities at the lowest possible level.

5) *Mention when the results extend* to some other class(es) of webbed models, and when they do not (sometimes we do not know). All these classes indeed appear to be (more or less) sophisticated variations of graph models. We will not work out the details, since this would lead us too far, and would be teadious, with no special added interest. We rather aim at searching for generic tools, when possible, in further work.

6.1.3 Main results of the chapter

Main results I: Effective models. The central technical device here is Visser's result [107] stating that the complements of β-closed r.e. sets of λ-terms enjoy the finite intersection property (Theorem 6.2.4). We will be able to prove the following.

Theorem 1. *Let $\mathscr{M}$ be an* effective *model of λ-calculus. Then:*

(i) $\mathrm{Th}_{\sqsubseteq}(\mathscr{M})$ *is not r.e.*

[1] As far as we know, only Giannini and Longo [50] have introduced a notion of an effective model; moreover their definition is *ad hoc* for two particular models (Scott's $\mathscr{P}_\omega$ and Plotkin's T_ω) and their results depend on the fact that these models have a very special (and well known) common theory.

(ii) $\mathrm{Th}(\mathscr{M}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

(iii) *If* $\perp$ *is* λ*-definable then* $\mathrm{Th}(\mathscr{M})$ *is not r.e., more generally:*

(iv) *If there is a* λ*-term* M *such that in* $\mathscr{M}$ *there are only finitely many* λ*-definable elements below the interpretation of* M *then* $\mathrm{Th}(\mathscr{M})$ *is not r.e.*

Concerning the existence of a non-syntactical effective model with an r.e. equational theory, we are able to give a definite answer for all (effective) stable and strongly stable models:

Theorem 2. *No effective model living in the stable or in the strongly stable semantics has an r.e. equational theory.*

Concerning Scott-continuous semantics, the problem looks much more difficult and we concentrate on the class of graph models.

Main results II. Graph models.

Theorem 3. *If* $\mathscr{G}$ *is a graph model then* $\mathrm{Th}_{\sqsubseteq}(\mathscr{G})$ *is not r.e.*

We emphasize that Theorem 3, which happens to be a consequence of Theorem 5 below, plus the work on effective models, concerns all the graph models and not only the effective ones. Concerning the equational theories of graph models we only give below, as Theorem 4, the more flashy example of the results we will prove in Subsection 6.4.3. The stronger versions are however natural, and needed for covering all the traditional models (for example the Engeler's model is covered by Theorem 4 only if it is generated from a finite set of atoms, while it is well known that its theory is $\mathcal{B}_\mathcal{T}$, independently of the number of its atoms).

Theorem 4. *If* $\mathscr{G}$ *is a graph model which is freely generated by a finite pair, then* $\mathrm{Th}(\mathscr{G})$ *is not r.e.*

Theorem 5. *There exists an* effective *graph model whose equational/order theory is minimal among all theories of graph models.*

A few more specific conjectures for Scott-continuous semantics. Thus, concerning effective models, Conjecture 1 is solved for the two refinements of Scott-continuous semantics which are mainly considered in the literature, but for Scott-continuous semantics it remains open, as well as its two following instances (from the weaker to the stronger conjecture).

Conjecture 2. *The minimal equational graph theory is non r.e.*

Conjecture 3. *All effective graph models have non r.e. theories.*

Conjecture 4. *All effective models living in the Scott-continuous semantics have non r.e. theories*

6.2 Recursion in λ-calculus

We now recall the main properties of recursion theory concerning λ-calculus that will be applied in the following sections.

From now on, we fix an effective bijective numeration $\nu_\Lambda : \mathbb{N} \to \Lambda$ of the set of λ-terms.

6.2.1 Co-r.e. sets of λ-terms

Definition 6.2.1. *A set $O \subseteq \Lambda$ is* r.e. *(*co-r.e.*) if it is r.e. (co-r.e.) with respect to ν_Λ; it is* trivial *if either $O = \emptyset$ or $O = \Lambda$.*

If $\mathcal{T}$ is a λ-theory, then an r.e. (co-r.e.) set of λ-terms closed under $=_\mathcal{T}$ will be called a *$\mathcal{T}$-r.e.* (*$\mathcal{T}$-co-r.e.*) set. When $\mathcal{T} = \lambda_\beta$ we simply speak of β-r.e. (β-co-r.e.) sets. The following theorem is due to Scott.

Theorem 6.2.2. *(Scott [8, Thm. 6.6.2]) A set of λ-terms which is both β-r.e. and β-co-r.e. is trivial.*

Definition 6.2.3. *A family $(X_i)_{i\in I}$ of sets has the* FIP *(finite intersection property) if $X_{i_1} \cap \cdots \cap X_{i_n} \neq \emptyset$ for all $i_1, \dots, i_n \in I$.*

In [107] Visser has shown that the topology on Λ generated by the β-co-r.e. sets of λ-terms is hyperconnected (see also [8, Ch. 17]), i.e., the intersection of two non-empty open sets is non-empty. In other words, the following theorem holds.

Theorem 6.2.4. *(Visser [107]) The family of all non-empty β-co-r.e. subsets of Λ has the FIP.*

As shown in the next lemma, the set $\mathcal{U}$ of all unsolvable λ-terms constitutes an interesting example of β-co-r.e. set.

Lemma 6.2.5.

(i) $\mathcal{U}$ is β-co-r.e.; moreover, given a λ-theory $\mathcal{T}$:

(ii) $\mathcal{U}$ is $\mathcal{T}$-co-r.e. if, and only if, $\mathcal{T}$ is semi-sensible.

Proof. (i) Indeed, $\mathcal{U}$ is co-r.e. and β-closed.
(ii) By definition of semi-sensibility, $\mathcal{U}$ is closed under $=_\mathcal{T}$ exactly when $\mathcal{T}$ is semi-sensible. $\square$

From this lemma and from Theorem 6.2.4 we easily get the following remark.

Remark 6.2.6. *Every non-empty β-co-r.e. set O of λ-terms contains a non-empty β-co-r.e. set $\mathcal{V}$ of unsolvable λ-terms. Indeed, it is sufficient to choose $\mathcal{V} = O \cap \mathcal{U}$.*

Lemma 6.2.7. *Let $\mathcal{T}$ be an r.e. λ-theory and O be a non-empty set of λ-terms. If O is β-co-r.e., then either $O/\mathcal{T}$ is infinite or $\mathcal{T}$ is inconsistent.*

Proof. Let V be the $\mathcal{T}$-closure of O, and $O' = \Lambda - V$. If $\mathcal{T}$ is r.e. and $O/\mathcal{T}$ is finite, then V is r.e. and hence O' is β-co-r.e. Since $O' \cap O = \emptyset$, O' must be empty by Theorem 6.2.4. Hence $V = \Lambda$, and $\Lambda/\mathcal{T}$ is finite. Hence $\mathcal{T}$ is inconsistent. $\square$

6.2.2 Effective domains

The notion of computability recalled in Subsection 1.1.3 cannot be directly applied to domains. A trivial reason is that many domains of interest are uncountable. Now, ω-algebraic domains are conceived as the *completion* of a countable set of concrete elements (the compact elements) and computations on an element in the completion are determined by the way the computations act on its approximations (the compact elements below it). The theory of computability on domains reflects this idea in the sense that a domain is effective when an effective numeration of its compact elements is provided.

All the material developed in this subsection can be found in [103, Ch. 10]; its adaptation to DI-domains and DI-domains with coherences can be found in [54].

Definition of effective domains

Definition 6.2.8. *A triple $\mathcal{D} = (\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ is called an* effective domain *if $(\mathcal{D}, \sqsubseteq_{\mathcal{D}})$ is a Scott domain and $d : \mathbb{N} \to \mathcal{K}(\mathcal{D})$ is a numeration of $\mathcal{K}(\mathcal{D})$ such that:*

(i) the relation "d_m and d_n have an upper bound" is decidable in (m, n),

(ii) the relation "$d_n = d_m \sqcup d_k$" is decidable in (m, n, k).

Note that it is equivalent to replace (ii) by (ii)': "the join operator restricted to pairs of compact elements is total recursive and the equality relation is decidable on compact elements". Note also that $d_m \sqsubseteq_{\mathcal{D}} d_n$ is decidable in (m, n) as it is equivalent to $d_n = d_m \sqcup d_n$.

As usual, when there is no ambiguity, we denote by $\mathcal{D}$ the effective domain $(\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$.

Notation 6.2.9. *For all $v \in \mathcal{D}$ we set $\widehat{v} = \{n : d_n \sqsubseteq_{\mathcal{D}} v\}$.*

Definition 6.2.10. *An element v of an effective domain $\mathcal{D}$ is said* r.e. *(*decidable*) if the set $\widehat{v}$ is r.e. (decidable).*

In the literature, r.e. elements of effective domains are called "computable", "recursive" or "effective" elements, while our decidable elements were apparently not addressed. We choose the alternative terminology of "r.e. elements" for the following two reasons: (1) it is more coherent with the usual terminology for elements of $\mathscr{P}(\mathbb{N})$ (see Example 6.2.13); (2) it makes clear the difference between r.e. elements and decidable elements of $\mathcal{D}$.

Notation 6.2.11. *$\mathcal{D}^{r.e.}$ ($\mathcal{D}^{dec}$) denotes the set of the r.e. (decidable) elements of the effective domain $\mathcal{D}$.*

Note that $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{D}^{dec} \subseteq \mathcal{D}^{r.e.}$ and that, in general, $\mathcal{D}^{r.e.}$ and $\mathcal{D}^{dec}$ are not cpo's.

Example 6.2.12. *Given an effective numeration of a countable set D, the flat domain $D_\perp$ is effective and all its elements are decidable since they are compact. In particular $\Lambda_\perp$ is effective for the following numeration:*

$$\nu_{\Lambda_\perp}(n) = \begin{cases} \perp & \text{if } n = 0, \\ \nu_\Lambda(n-1) & \text{if } n > 0. \end{cases}$$

where ν_Λ has been defined at the beginning of Section 6.2.

Example 6.2.13. *The key example of an effective domain is $(\mathscr{P}(\mathbb{N}), \subseteq, d)$ where d is some standard bijective numeration $d : \mathbb{N} \to \mathbb{N}^*$ of the finite subsets of $\mathbb{N}$. Here the r.e. (decidable) elements are the r.e. (decidable) sets.*

Characterizations of r.e. continuous functions

Given two effective domains $\mathcal{D}$ and $\mathcal{D}'$ it is essentially straightforward to obtain, in a canonical way, a numeration $\nu_{[\mathcal{D}\to\mathcal{D}']}$ of the compact elements of $[\mathcal{D} \to \mathcal{D}']$ (the details can be found in [103, Ch. 10, Thm. 3.6]).

This numeration gives $[\mathcal{D} \to \mathcal{D}']$ a structure of effective domain. Thus, we already know when a function $f : \mathcal{D} \to \mathcal{D}'$ is an r.e. element of $[\mathcal{D} \to \mathcal{D}']$ (i.e., when $f \in [\mathcal{D} \to \mathcal{D}']^{r.e.}$). However, we also have the intuition that a continuous function *should* be r.e. if its values on r.e. elements can be effectively approximated. The next proposition gives two other characterizations of r.e. continuous functions which capture this idea.

Proposition 6.2.14. *Let $(\mathcal{D}, \sqsubseteq_\mathcal{D}, d)$ and $(\mathcal{D}', \sqsubseteq_{\mathcal{D}'}, d')$ be two effective domains. For all functions $f : \mathcal{D} \to \mathcal{D}'$, the following conditions are equivalent:*

(a) $f \in [\mathcal{D} \to \mathcal{D}']^{r.e.}$,

(b) the relation $d'_m \sqsubseteq_{\mathcal{D}'} f(d_n)$ is r.e. in (m, n),

(c) $\{(m, n) : (d_m, d'_n) \in Tr(f)\}$ *is r.e.*

and the same holds when "decidable" replaces "r.e.".

The proof is easy; of course (c) is just a reformulation of (b). We refer to [103, Ch. 10, Prop. 3.7] for more details (at least in the r.e. case).

Adequate numerations of $\mathcal{D}^{r.e.}$

We are now interested in defining a numeration of $\mathcal{D}^{r.e.}$ amenable to the effective numeration of $\mathcal{K}(\mathcal{D})$. The natural surjection $\zeta' : \mathbb{N} \to \mathcal{D}^{r.e.}$ defined by $\zeta'(n) = v$ if and only if $\mathcal{W}_n = \widehat{v}$ is not a numeration in general, since it can be partial. This partiality would create technical difficulties. However, using standard techniques of recursion theory, it is not difficult to get in a uniform way a *total* numeration $\zeta^{\mathcal{D}}$ of $\mathcal{D}^{r.e.}$ [103, Ch. 10, Thm. 4.4]. Moreover, in the sequel, we will need some further constraints on $\zeta^{\mathcal{D}}$, whose satisfiability is guaranteed by the following proposition.

Proposition 6.2.15. *For every effective domain $\mathcal{D}$, there exists a total numeration $\zeta^{\mathcal{D}} : \mathbb{N} \to \mathcal{D}^{r.e.}$ such that:*

(i) $d_n \sqsubseteq_{\mathcal{D}} \zeta^{\mathcal{D}}_m$ is r.e. in (m, n),

(ii) the inclusion mapping $\iota : \mathcal{K}(\mathcal{D}) \to \mathcal{D}^{r.e.}$ is computable with respect to $d, \zeta^{\mathcal{D}}$.

A numeration $\zeta^{\mathcal{D}}$ of $\mathcal{D}^{r.e.}$ is called *adequate* if it fulfills the conditions (i) and (ii) of Proposition 6.2.15.

Example 6.2.16. *The usual map $n \mapsto \mathcal{W}_n$ is an adequate numeration of the r.e. elements of the effective domain $(\mathscr{P}(\mathbb{N}), \subseteq, d)$. From Proposition 6.2.14 it follows that a Scott continuous function $f : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ is r.e. if, and only if, its trace $Tr(f) = \{(a, n) \in \mathbb{N}^* \times \mathbb{N} : n \in f(a)\}$ is an r.e. set.*

Lemma 6.2.17. *[103, Ch. 10, Cor. 4.12] For all adequate numerations $\zeta^{\mathcal{D}}, \zeta'^{\mathcal{D}}$ there is a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\zeta^{\mathcal{D}} = \zeta'^{\mathcal{D}} \circ \varphi$.*

Hereafter we will always suppose that $\zeta^{\mathcal{D}}$ is an adequate numeration of $\mathcal{D}^{r.e.}$. Adequate numerations allow us to provide another characterization of r.e. continuous functions which highlights the connection with the classical notion of computability. Note that, as a consequence of Lemma 6.2.17, the following theorem is independent of the choice of $\zeta^{\mathcal{D}}, \zeta^{\mathcal{D}'}$.

Theorem 6.2.18. *Let $\mathcal{D}, \mathcal{D}'$ be effective domains, then a continuous function $f : \mathcal{D} \to \mathcal{D}'$ is r.e. in its effective domain if, and only if, its restriction $f\lceil : \mathcal{D}^{r.e.} \to \mathcal{D}'^{r.e.}$ is computable with respect to $\zeta^{\mathcal{D}}, \zeta^{\mathcal{D}'}$.*

Proof. Relatively easy (the details are worked out in [103, Ch. 10, Prop. 4.14]). $\square$

In particular the previous theorem states that r.e. continuous functions preserve the r.e. elements.

The category of effective domains and continuous functions

Definition 6.2.19. ED *is the category with effective domains as objects, and continuous functions as morphisms.*

ED is a full subcategory of the category of Scott domains; it is Cartesian closed since, if $\mathcal{D}, \mathcal{D}'$ are effective domains, also $\mathcal{D}\times\mathcal{D}'$ and $[\mathcal{D} \to \mathcal{D}']$ are effective domains. The reader can easily check these facts by himself or find the proofs in [48].

Remark 6.2.20. *It is clear that the composition of r.e. functions is an r.e. function, moreover it is straightforward to check that the maps Λ and ev defined in Subsection 1.1.6, and the composition operator $C(f, g) = g \circ f$, are r.e. at all types. Hence, by Theorem 6.2.18, their restrictions to r.e. elements are computable.*

For the stable and strongly stable semantics, we take respectively: **EDID**, the category having effective DI-domains as objects and stable functions as morphisms; $\mathbf{EDID^{coh}}$, the category having effective DI-domains with coherences as objects and strongly stable functions as morphisms.

Before introducing the notions of effective and weakly effective models we recall some properties of effective domains.

6.2.3 Completely co-r.e. sets of r.e. elements

Our aim is to infer properties of weakly effective models using standard techniques of recursion theory. For this purpose, given an effective domain $\mathcal{D}$ and an adequate numeration $\zeta^{\mathcal{D}} : \mathbb{N} \to \mathcal{D}^{r.e.}$, we study the properties of the completely co-r.e. subsets of $\mathcal{D}^{r.e.}$. The work done here could also be easily adapted to DI-domains and DI-domains with coherences.

Definition 6.2.21. *A subset $A \subseteq \mathcal{D}^{r.e.}$ is called* completely r.e. *if A is r.e. with respect to $\zeta^{\mathcal{D}}$; it is called* trivial *if $A = \emptyset$ or $A = \mathcal{D}^{r.e.}$. In a similar way we define* completely co-r.e. sets *and* completely decidable sets.

This terminology (i.e., the use of the adjective "completely") is coherent with the terminology classically used in recursion theory (see, e.g., [81]). We will see in Corollary 6.2.24(iii) below that there exist no non-trivial completely decidable sets.

Remark 6.2.22. *The set $\{\perp_{\mathcal{D}}\}$ is a non-trivial completely co-r.e. subset of $\mathcal{D}^{r.e.}$ (therefore it is not completely decidable).*

We will use the following extension of a well known result of classical recursion theory due to Rice, Myhill and Shepherdson [81, Thm. 10.5.2] to prove that the FIP (finite intersection property) still holds in this framework.

Theorem 6.2.23. *[103, Thm. 5.2] Let $\mathcal{D}$ be an effective domain and let $A \subseteq \mathcal{D}^{r.e.}$, then A is completely r.e. if, and only if, there is an r.e. set $E \subseteq \mathbb{N}$ such that:*

$$A = \{v \in \mathcal{D}^{r.e.} : \exists n \in E \ (\zeta_n^{\mathcal{D}} \in \mathcal{K}(\mathcal{D}) \ \textit{and} \ \zeta_n^{\mathcal{D}} \sqsubseteq_{\mathcal{D}} v)\}.$$

As direct consequences of this theorem we obtain that the following interesting closure properties hold.

Corollary 6.2.24. *With respect to the partial order $\sqsubseteq_{\mathcal{D}}$:*

(i) completely r.e. sets are upward closed (in $\mathcal{D}^{r.e.}$),

(ii) completely co-r.e. sets are downward closed (in $\mathcal{D}^{r.e.}$),

(iii) completely decidable sets are trivial.

Theorem 6.2.25. *The family of all non-empty completely co-r.e. subsets of $\mathcal{D}^{r.e.}$ has the FIP.*

Proof. It follows from Corollary 6.2.24(ii) that $\perp_{\mathcal{D}}$ belongs to every non-empty completely co-r.e. subset of $\mathcal{D}^{r.e.}$. □

6.2.4 Weakly effective models

In this section we will consider models living in Scott-continuous semantics; but analogous notions can be defined for the stable and the strongly stable semantics. The following definition of weakly effective models is completely natural in this context; however, in order to obtain stronger results, we will need a slightly more powerful notion. That is the reason why we only speak of "weak effectivity" here.

Definition 6.2.26. *A continuous model $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$ is* weakly effective *if:*

(i) $\mathscr{M}$ is a reflexive object in the category **ED**,

(ii) $\mathrm{Ap} \in [\mathcal{D} \to [\mathcal{D} \to \mathcal{D}]]$ and $\lambda \in [[\mathcal{D} \to \mathcal{D}] \to \mathcal{D}]$ are r.e.

For the stable or strongly stable semantics we take **EDID** and $\mathbf{EDID^{coh}}$ instead of **ED**.

Remark 6.2.27. *Let $\mathbf{ED^{r.e.}}$ be the subcategory of* **ED** *with the same objects as* **ED** *(and the same exponential objects) but r.e. continuous functions as morphisms. Using Remark 6.2.20 it is easy to check that $\mathbf{ED^{r.e.}}$ inherits the structure of ccc from* **ED**. *The weakly effective models of Definition 6.2.26 above are exactly the reflexive objects of $\mathbf{ED^{r.e.}}$. We prefer to use the category* **ED** *first because we think that it is more coherent with the definition of the exponential objects to take all continuous functions as morphisms, and second to put in major evidence the only effectiveness conditions which are required.*

We recall here a consequence of Theorem 6.2.18 that will be often used later on.

Remark 6.2.28. *Let $\mathscr{M}$ be a weakly effective model, then:*

(i) If $u, v \in \mathcal{D}^{r.e.}$ then $uv \in \mathcal{D}^{r.e.}$,

(ii) If $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$, then $\lambda(f) \in \mathcal{D}^{r.e.}$.

In the rest of this section it is understood that we are speaking of a fixed continuous model $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$, where $\mathcal{D} = (\mathcal{D}, \sqsubseteq_{\mathcal{D}}, d)$ is an effective domain and that $\mathcal{T} = \mathrm{Th}(\mathscr{M})$. Furthermore, we fix an effective bijective numeration ν_{Var} from $\mathbb{N}$ to the set Var of variables of λ-calculus. This gives $Env_{\mathcal{D}}$ a structure of effective domain.

Proposition 6.2.29. *If $\mathscr{M}$ is weakly effective, then $(\mathcal{D}^{r.e.}, \cdot, |\mathbf{K}|, |\mathbf{S}|)$ is a combinatory subalgebra of $(\mathcal{D}, \cdot, |\mathbf{K}|, |\mathbf{S}|)$.*

Proof. It follows from Remark 6.2.28 that $|\mathbf{K}| \in \mathcal{D}^{r.e.}$ and that $\mathcal{D}^{r.e.}$ is closed under application. The fact that $|\mathbf{S}| \in \mathcal{D}^{r.e.}$ is a direct consequence of the next result. $\square$

Theorem 6.2.30. *If $\mathscr{M}$ is weakly effective, then $|M| \in \mathcal{D}^{r.e.}$ for all $M \in \Lambda^o$.*

Proof. From the next proposition and by Theorem 6.2.18 it follows that $|M|_\rho \in \mathcal{D}^{r.e.}$ for all $\rho \in Env_{\mathcal{D}}^{r.e.}$. Hence it is sufficient to remark that $|M| = |M|_{\rho_\perp}$ since M is closed, and that $\rho_\perp \in \mathcal{K}(Env_{\mathcal{D}}) \subseteq Env_{\mathcal{D}}^{r.e.}$. $\square$

Proposition 6.2.31. *If $\mathscr{M}$ is weakly effective then, for all $M \in \Lambda_\perp$, the function $|M| : Env_{\mathcal{D}} \to \mathcal{D}$ is r.e.*

Proof. If $M \equiv \perp$ then $|\perp|$ is the constant function mapping ρ to $\perp_{\mathcal{D}}$ which is obviously r.e. Otherwise, M is a λ-term and we conclude the proof by structural induction.

If $M \equiv x$ then $|M|$ is the map $\rho \mapsto \rho(x)$, i.e., the evaluation of the environment ρ on the variable x. It is easy to check that this function is r.e.

If $M \equiv NP$ then $|M| = ev \circ \langle \mathrm{Ap} \circ |N|, |P| \rangle$. By the induction hypothesis and Remark 6.2.20, $|M|$ is a composition of r.e. functions, hence it is r.e.

If $M \equiv \lambda x.N$ then $|M| = \lambda \circ C \circ \langle \Lambda(f_x), k \rangle$, where f_x is the function $(\rho, d) \mapsto \rho[x := d]$, C is the composition operator and k is the constant function mapping ρ to $|N|$. We note that f_x is r.e. because its restriction f'_x to $Env_{\mathcal{D}}^{r.e.} \times \mathcal{D}^{r.e.}$ is computable. Indeed $f'_x(\rho, d)$ differs from the r.e. environment ρ only on x where it takes as value the r.e. element d. Then this case follows again from the induction hypothesis and Remark 6.2.20. $\square$

Theorem 6.2.32. *If $\mathscr{M}$ is weakly effective, then the function $f : \Lambda_\perp \times Env_{\mathcal{D}} \to \mathcal{D}$ defined by $f(M, \rho) = |M|_\rho$ is r.e.*

Proof. Since the function $|M|$ is r.e. for every $M \in \Lambda_\perp$, we have that $|M|_\rho \in \mathcal{D}^{r.e.}$ for all r.e. environments ρ. Moreover, whenever $M \in \Lambda_\perp$ and $\rho \in Env_{\mathcal{D}}^{r.e.}$, the proof of Proposition 6.2.31 gives an effective algorithm to compute in a uniform way the code of $|M|_\rho$ starting from the codes of M and ρ. $\square$

Corollary 6.2.33. *If $\mathscr{M}$ is weakly effective and $\rho \in Env_{\mathcal{D}}^{r.e.}$, then the function $| - |_\rho : \Lambda_\perp \to \mathcal{D}$ is r.e. and its restriction to Λ is computable with respect to $\nu_\Lambda, \zeta^{\mathcal{D}}$.*

Corollary 6.2.34. *If $\mathscr{M}$ is weakly effective and $V \subseteq \mathcal{D}^{r.e.}$ is completely co-r.e. then $\{M \in \Lambda^o : |M| \in V\}$ is β-co-r.e.*

Proof. Let $\rho \in Env_{\mathcal{D}}^{r.e.}$. By Corollary 6.2.33 there exists a recursive map φ_ρ tracking the interpretation function $M \mapsto |M|_\rho$ (for $M \in \Lambda$) with respect to $\nu_\Lambda, \zeta^{\mathcal{D}}$. Since the set $E = \{n : \zeta_n^{\mathcal{D}} \in V\}$ is co-r.e. it follows from Remark 1.1.1 that $\varphi_\rho^-(E)$, which is equal to $\{\nu_\Lambda^{-1}(M) : |M|_\rho \in V\}$, is also co-r.e. We get the conclusion because Λ^o is a decidable subset of Λ. $\square$

Notation 6.2.35. *Given a partially ordered model $\mathscr{M}$ we set:*

(i) $M^- = \{N \in \Lambda^o : |N| \sqsubseteq_{\mathcal{D}} |M|\}$, *for all* $M \in \Lambda^o(\mathcal{D})$.

(ii) $\bot_\omega^- = \cup_{n\in\mathbb{N}} \bot_n^-$ *where* $\bot_n \equiv \lambda x_1 \dots x_n.\bot_{\mathcal{D}}$.

We will write $(M)^-_{\mathscr{M}}$ instead of M^-, when the model $\mathscr{M}$ is not clear from the context.

Remark 6.2.36. *If $\mathscr{M}$ is a partially ordered model and $\mathcal{T} = \mathrm{Th}(\mathscr{M})$, then it is easy to check that:*

(i) M^- *is a union of* $\mathcal{T}$*-classes and* $M \in M^-$ *for all* $M \in \Lambda^o(\mathcal{D})$,

(ii) $\bot_{\mathcal{D}}^- = \bot_0^- = \{N \in \Lambda^o : |N| = \bot_{\mathcal{D}}\}$,

(iii) either $\bot_{\mathcal{D}}^- = \emptyset$ *or* $\bot_{\mathcal{D}}^-$ *consists of a single* $\mathcal{T}$*-class.*

Lemma 6.2.37. *For all partially ordered models $\mathscr{M}$, and $M, N \in \Lambda^o$ we have:*

(i) $\bot_{\mathcal{D}}^- \cup [M]_{\mathcal{T}} \subseteq M^-$,

(ii) if M, N *are non equivalent hnfs then* $M^- \cap N^- \subseteq \mathcal{U}$.

Proof. (i) Trivial.
(ii) Given $Q \in \Lambda^o$ we let Q^{ins} be the set of terms P which are inseparable from Q. Then item (ii) is immediate, once noted that no solvable λ-term belongs to $M^{ins} \cap N^{ins}$, since an hnf in the intersection should be simultaneously equivalent to M and N. $\square$

Note that under the hypothesis of Lemma 6.2.37 it can be true that $M^- \cap N^- = \emptyset$ for all M, N which are not $\beta\eta$-equivalent as shows the model of Di Gianantonio et al. [46].

It is interesting to note the following related result, which holds only for graph models and which, as the preceeding, does not need any hypothesis of effectivity.

Lemma 6.2.38. *For all graph models $\mathscr{G}$, if $N \in \mathbf{I}^-$ then either $N =_{\lambda_\beta} \mathbf{I}$ or N is unsolvable.*

Proof. Suppose that $N \in \mathbf{I}^-$ and N is solvable. Without loss of generality N is an hnf equivalent to $\mathbf{I}$, hence of the form $N \equiv \lambda x.\lambda\vec{z}.x\vec{N}$, with $\vec{z}$ and $\vec{N}$ of the same length $k \geq 0$. If $k \geq 1$ it is easy to check that

$$\gamma = i_{\mathcal{G}}(\{i_{\mathcal{G}}(\emptyset^k, \alpha)\}, i_{\mathcal{G}}(\{\alpha\}, i_{\mathcal{G}}(\emptyset^{k-1}, \alpha))) \in |N|^{\mathscr{G}} - |\mathbf{I}|^{\mathscr{G}},$$

where $i_{\mathcal{G}}(\emptyset^n, \alpha)$ is a shorthand for $i_{\mathcal{G}}(\emptyset, i_{\mathcal{G}}(\emptyset \ldots, i_{\mathcal{G}}(\emptyset, \alpha) \ldots))$. $\square$

Note that this lemma is false for the other classes of webbed models living in the Scott-continuous, stable or strongly stable semantics which have been introduced in the literature, since they all contain extensional models. Looking at the proof we can observe that the reasons why it does not work differ according to the semantics: $\gamma \in |\mathbf{I}|$ in the case of K-models, and $\gamma \notin |N|$ in the stable and strongly stable case (because the injective function i of the web is defined via Tr_s).

Proposition 6.2.39. *If $\mathscr{M}$ is weakly effective and $\mathcal{T} = \mathrm{Th}(\mathscr{M})$, then:*

(i) $\perp_{\mathcal{D}}^- \subseteq \mathcal{U}$,

(ii) $\perp_{\mathcal{D}}^-$ *is $\mathcal{T}$-co-r.e.*

Proof. (i) is true by Lemma 6.2.37(ii) since $\perp_{\mathcal{D}}^- \subseteq M^- \cap N^-$ for all M, N.
(ii) follows from Remark 6.2.22 and Corollary 6.2.34. $\square$

Any sensible model satisfies $\mathcal{U} = [\Omega]_{\mathcal{T}} \subseteq \Omega^-$. Thus, in all sensible models which interpret Ω by $\perp_{\mathcal{D}}$ we have $\perp_{\mathcal{D}}^- = \Omega^- = \mathcal{U}$ (this is the case for example of all sensible graph models). On the other hand it is easy to build models satisfying $\Omega^- = \Lambda^o$: for example, finding a graph model $\mathscr{G}$ with carrier set G such that $|\Omega|^{\mathscr{G}} = G$, is an exercise, which also appears as the simplest application of the generalized forcing developed in [16]. Finally (usual) forcing also allows us to build, for all $M \in \Lambda^o$, a graph model satisfying $|\Omega| = |M|$ and hence $\Omega^- = M^-$, and this is still true for $M \in \Lambda^o(\mathcal{D})$, and beyond, using generalized forcing.

Proposition 6.2.40. *If $\mathscr{M}$ is weakly effective and $\mathcal{T} = \mathrm{Th}(\mathscr{M})$ is r.e., then* $\perp_\omega^- = \emptyset = \perp_{\mathcal{D}}^-$.

Proof. Since $\perp_{\mathcal{D}}^-$ consists of zero or one $\mathcal{T}$-class, it follows from Lemma 6.2.7 and Proposition 6.2.39(i) that $\perp_{\mathcal{D}}^- = \emptyset$. Now it follows, by easy induction on n, that $\perp_n^- = \emptyset$ for all n since, if $N \in \perp_{k+1}^-$, then $N\mathbf{I} \in \perp_k^-$. $\square$

Notation 6.2.41. *For all $E \subseteq \mathbb{N}$ we set $\Lambda_E^o = \{N \in \Lambda^o : \widehat{|N|} \subseteq E\}$, where* $\widehat{|N|} = \{n : d_n \sqsubseteq_{\mathcal{D}} |N|\}$.

Note that Λ_E^o is a union of $\mathcal{T}$-classes, which depends on $\mathscr{M}$ (and not only on $\mathcal{T}$). Furthermore, for all $M \in \Lambda^o$, $\Lambda^o_{\widehat{|M|}} = M^-$.

Theorem 6.2.42. *Let $\mathscr{M}$ be weakly effective, $\mathcal{T} = \mathrm{Th}(\mathscr{M})$ and $E \subseteq \mathbb{N}$.*

(i) If E is co-r.e. then Λ^o_E is $\mathcal{T}$-co-r.e.,

(ii) If E is decidable then either $\Lambda^o_E = \emptyset$ or $\Lambda^o_{E^c} = \emptyset$.

Proof. (i) We first note that $E' = \{n : (\exists m \notin E)\ d_m \sqsubseteq_{\mathcal{D}} \zeta_n^{\mathcal{D}}\}$ is r.e. Hence $\{\zeta_n : n \notin E'\}$ is completely co-r.e. We conclude by Corollary 6.2.34.
(ii) follows from the FIP, since $\Lambda^o_E \cap \Lambda^o_{E^c} = \emptyset$. □

Theorem 6.2.43. *Let $\mathscr{M}$ be weakly effective, $\mathcal{T} = \mathrm{Th}(\mathscr{M})$ and $M_1, \ldots, M_n \in \Lambda^o$. If $|M_i| \in \mathcal{D}^{dec}$ for all $1 \leq i \leq n$, then $M_1^- \cap \cdots \cap M_n^-$ is a non-empty $\mathcal{T}$-co-r.e. set.*

Proof. Since, for all $1 \leq i \leq n$, the set $\widehat{|M_i|}$ is decidable and $M_i^- = \Lambda^o_{\widehat{|M_i|}}$, then every M_i^- is non-empty and $\mathcal{T}$-co-r.e. by Theorem 6.2.42(i). Hence the theorem follows from the FIP. □

Theorem 6.2.44. *Let $\mathscr{M}$ be weakly effective and $\mathcal{T} = \mathrm{Th}(\mathscr{M})$. If there exists $M \in \Lambda^o$ such that $|M| \in \mathcal{D}^{dec}$ and $M^- - [M]_{\mathcal{T}}$ is finite modulo $\mathcal{T}$, then $\mathcal{T}$ is not r.e.*

Proof. Since $|M| \in \mathcal{D}^{r.e.}$ we have, by Theorem 6.2.43, that M^- is $\mathcal{T}$-co-r.e. If $\mathcal{T}$ is r.e. then $M^-/\mathcal{T}$ is infinite by Lemma 6.2.7. □

6.2.5 Effective models

As proved in Proposition 6.2.31 weakly effective models interpret λ-terms by r.e. elements. The notion of effective model introduced below has the further key advantage that normal terms are interpreted by decidable elements and this leads to interesting consequences. As we will see later on, all the models introduced individually in the literature and living in one of the main semantics are effective. Furthermore, in the case of webbed models, easy sufficient conditions can be given at the level of the web in order to guarantee the effectiveness of the model.

Definition 6.2.45. *A weakly effective model $\mathscr{M} = (\mathcal{D}, \mathrm{Ap}, \lambda)$ is called* effective *if it satisfies the following two conditions:*

(i) if $d \in \mathcal{K}(D)$ and $e_1, \ldots, e_k \in \mathcal{D}^{dec}$, then $de_1 \cdots e_k \in \mathcal{D}^{dec}$,

(ii) if $f \in [\mathcal{D} \to \mathcal{D}]^{r.e.}$ and $f(d) \in \mathcal{D}^{dec}$ for all $d \in \mathcal{K}(\mathcal{D})$, then $\lambda(f) \in \mathcal{D}^{dec}$.

It is easy to check that condition (i) of Definition 6.2.45 is always verified in every graph model and in every extensional model. We will see in Section 6.3 and 6.4 that many models are effective.

Theorem 6.2.46. *If $\mathscr{M}$ is effective, then for all normal λ-terms $M \in \Lambda^o$ we have $|M| \in \mathcal{D}^{dec}$.*

Proof. Since the interpretation of a closed λ-term is independent of the context, it is enough to show that $|M|_\rho \in \mathcal{D}^{dec}$ for all normal $M \in \Lambda$ and for all $\rho \in \mathcal{K}(Env_{\mathcal{D}})$. This proof is done by induction over the complexity of M.
If $M \equiv x$ then $|M|_\rho = \rho(x)$ is a compact element, hence it is decidable.
Suppose $M \equiv yN_1 \cdots N_k$ with N_i normal for all $1 \leq i \leq k$. By definition $|M|_\rho$ is equal to $|y|_\rho \cdot |N_1|_\rho \cdots |N_k|_\rho$. Hence this case follows from Definition 6.2.45(i), the fact that $\rho(y)$ is compact and the induction hypothesis.
If $M \equiv \lambda x.N$ then $|M|_\rho = \lambda(d \mapsto |N|_{\rho[x:=d]})$. Note that, since $\rho \in \mathcal{K}(Env_{\mathcal{D}})$, also $\rho[x := d]$ is compact for all $d \in \mathcal{K}(\mathcal{D})$. Hence the result follows from the induction hypothesis and Definition 6.2.45(ii). □

We are now able to prove that the order theory of an effective model is never r.e. From this it follows that no effective model can have λ_β or $\lambda_{\beta\eta}$ as equational theory.

Corollary 6.2.47. *If $\mathscr{M}$ is effective, then* $\mathrm{Th}_\sqsubseteq(\mathscr{M})$ *is not r.e.*

Proof. Let $M \in \Lambda^o$ be normal. If $\mathrm{Th}_\sqsubseteq(\mathscr{M})$ were r.e., then we could enumerate the set M^-. However, by Theorem 6.2.46 and Theorem 6.2.43, this set is co-r.e. and it is non-empty because clearly $M \in M^-$. Hence M^- would be a non-empty decidable set of λ-terms closed under β-conversion, i.e., $M^- = \Lambda^o$. Since the model is non-trivial and M is arbitrary this lead us to a contradiction. □

Corollary 6.2.48. *If $\mathscr{M}$ is effective and* $\mathrm{Th}(\mathscr{M})$ *is r.e. then $\sqsubseteq_{\mathcal{D}}$ induces a non-trivial partial order on the interpretations of closed λ-terms.*

Corollary 6.2.49. *If $\mathscr{M}$ is effective then* $\mathrm{Th}(\mathscr{M}) \neq \lambda_\beta, \lambda_{\beta\eta}$.

Proof. By Selinger's result stating that in any partially ordered model whose theory is λ_β or $\lambda_{\beta\eta}$ the interpretations of closed λ-terms are discretely ordered [99, Cor. 4]. □

Recall that in the case of graph models we know a much stronger result, since we already know that *for all graph models $\mathscr{G}$* we have $\mathrm{Th}(\mathscr{G}) \neq \lambda_\beta$, $\lambda_{\beta\eta}$.

6.3 Effective stable and strongly stable models

There are also many effective models in the stable and strongly stable semantics. Indeed, the material developed in Subsections 6.4.2 and 6.4.3 below for graph models could be adapted for G-models, even if it is more delicate to complete partial pairs in this case (we refer the reader to [67, 70] for the details of this construction). A free completion process could also be developed for H-models. This result has been worked out for particular models [53, 10], but works in greater generality[2], even though working in the strongly stable semantics certainly adds technical difficulties.

[2] R. Kerth and O. Bastonero, *personal communication.*

Lemma 6.3.1. *If $\mathscr{M}$ belongs to the stable or to the strongly stable semantics, then:*

$$\mathbf{F}^- \cap \mathbf{T}^- \subseteq \{N : \mathbf{1}N \in \perp_2^-\}.$$

Proof. Suppose $N \in \mathbf{F}^- \cap \mathbf{T}^-$. Let $f, g, h \in [\mathcal{D} \to_s \mathcal{D}]$ (resp. $[\mathcal{D} \to_{ss} \mathcal{D}]$) be $f = \mathrm{Ap}(|\mathbf{T}|)$, $g = \mathrm{Ap}(|\mathbf{F}|)$ and $h = \mathrm{Ap}(|N|)$. By monotonicity of Ap we have $h \leq_s f, g$. Now, g is the constant function taking value $|\mathbf{I}|$, and $f(\perp_\mathcal{D}) = |\lambda y.\perp_\mathcal{D}|$. The first assertion forces h to be a constant function (Remark 1.4.10) and the fact that h is pointwise smaller than f forces $\lambda(h) = |\lambda x.\lambda y.\perp_\mathcal{D}|$. Therefore $\lambda(h) \in \perp_2^-$. It is now enough to notice that, in all models $\mathscr{M}$ we have $\lambda(\mathrm{Ap}(u)) = |\mathbf{1}u|$ for all $u \in \mathcal{D}$ and, in particular, $\lambda(h) = |\mathbf{1}N|$. Hence $\mathbf{1}N \in \perp_2^-$. □

Theorem 6.3.2. *If $\mathscr{M}$ is effective and belongs to the stable or to the strongly stable semantics then $\mathcal{T} = \mathrm{Th}(\mathscr{M})$ is not r.e.*

Proof. Since $\mathscr{M}$ is effective and $\mathbf{F}, \mathbf{T}$ are closed and normal we have that $\mathbf{F}^- \cap \mathbf{T}^-$ is non-empty by Theorem 6.2.43. Now, if $\mathcal{T}$ is r.e. then every $\perp_n^-$ is empty by Proposition 6.2.40. In particular this implies, by Lemma 6.3.1, that $\mathbf{F}^- \cap \mathbf{T}^- = \emptyset$ leading us to a contradiction. □

It is easy to check that Lemma 6.3.1 is false for the Scott-continuous semantics. We can even give a counter-example in the class of graph models. Indeed we know from [16] that there exists a graph model $\mathscr{G}$, which is built by forcing, where Ω acts like intersection. In other words, in $\mathscr{G}$ we have $|\Omega|^\mathscr{G} = |\mathbf{T}|^\mathscr{G} \cap |\mathbf{F}|^\mathscr{G}$ and $\Omega \in (\mathbf{T}^- \cap \mathbf{F}^-) - \perp_\mathcal{D}^-$.

6.4 Effective graph models

The aim of this section is to show that effective models are omni-present in the Scott-continuous semantics. In Subsection 6.4.1 we will introduce a notion of weakly effective (effective) partial pairs, and in Subsection 6.4.2 we will prove that they generate weakly effective (effective) models. An analogue of the work done in these two subsections could clearly be developed for each of the other classes of webbed models, e.g., K-models, pcs-models, filter models (for the Scott-continuous semantics), G-models and H-models (respectively, for the stable and strongly stable semantics). Note that all the models which have been introduced individually in the literature, to begin with $\mathscr{P}_\omega$, $\mathscr{E}$ (graph models) and Scott's $\mathscr{D}_\infty$ (K-model) can easily be proved to be effective models in our sense.

6.4.1 Weakly effective and effective pairs

Definition 6.4.1. *A partial pair $\mathcal{A}$ is* weakly effective *if it is isomorphic to some pair (E, ℓ) where E is a decidable subset of $\mathbb{N}$ and ℓ is a partial recursive function with decidable domain. It is* effective *if, moreover,* $\mathrm{rg}(\ell)$ *is decidable.*

Lemma 6.4.2. *A total pair $\mathcal{G}$ is weakly effective if, and only if, it is isomorphic to a total pair $(\mathbb{N}, \ell)$ where ℓ is total recursive and it is effective if, moreover, we can choose ℓ with a decidable range.*

Proof. Straightforward. □

Hence hereafter we can suppose, without loss of generality, that all effective and weakly effective pairs have as underlying set a subset of $\mathbb{N}$.

Example 6.4.3. *$\mathscr{P}_\omega$ in its original definition (see, e.g., [8]), since ℓ is defined by $\ell(a, n) = \#\langle a, n\rangle$ (with the notation of Subsection 1.1.3); note that ℓ is also surjective here.*

Proposition 6.4.4. *If $\mathcal{G}$ is a weakly effective (effective) total pair then $\mathscr{G}$ is a weakly effective (effective) model.*

Proof. By Lemma 6.4.2, it is enough to prove it for weakly effective pairs of the form $(\mathbb{N}, \ell)$. Then it is easy to check, using Definition 1.4.5, that $\mathrm{Ap}^{\mathcal{G}}, \lambda^{\mathcal{G}}$ are r.e. and that condition (i) of Definition 6.2.45 (effective models) is always satisfied. It is also straightforward to check that condition (ii) holds when $\mathrm{rg}(\ell)$ is decidable. □

Next, we show that the free completion process preserves the effectivity of the partial pairs.

6.4.2 Free completions of (weakly) effective pairs

Theorem 6.4.5. *If $\mathcal{A}$ is weakly effective (effective) then $\overline{\mathcal{A}}$ is weakly effective (effective).*

Proof. Suppose $\mathcal{A} = (A, j_{\mathcal{A}})$ is a weakly effective partial pair. Without loss of generality we can suppose $A = \{2^k : k < \mathrm{card}(A)\}$. For all $n \in \mathbb{N}$, we will denote by j_n the restriction $i_{\bar{\mathcal{A}}}\lceil_{A_n^* \times A_n}$ where A_n has been introduced in Definition 5.1.8. We now build $\theta : \overline{A} \to \mathbb{N}$ as an increasing union of functions $\theta_n : A_n \to \mathbb{N}$ which are defined by induction on n. At each step we set $E_n = \mathrm{rg}(\theta_n)$ and define $\ell_n : E_n^* \times E_n \to E_n$ such that θ_n is an isomorphism between (A_n, j_n) and (E_n, ℓ_n). We will take $E = \cup_{n\in\mathbb{N}} E_n$ and $\ell = \cup_{n\in\mathbb{N}} \ell_n$.
Case $n = 0$. We take for θ_0 the identity on A, then $E_0 = A$ and we take $\ell_0 = j_{\mathcal{A}}$. By hypothesis E_0 is decidable and $j_{\mathcal{A}}$ has a decidable domain and, if $\mathcal{A}$ is moreover effective, also a decidable range.
Case $n + 1$. We define

$$\theta_{n+1}(x) = \begin{cases} \theta_n(x) & \text{if } x \in A_n, \\ p_{n+1}^{\#\langle a,\alpha\rangle} & \text{if } x = (a, \alpha) \in (A_{n+1} - A_n). \end{cases}$$

where p_{n+1} denotes the $(n + 1)$-th prime number and $\#\langle -, -\rangle$ is the encoding defined in Section 1.1.3. Since A and $\mathrm{dom}(j_{\mathcal{A}})$ are decidable by hypothesis and A_n is decidable by the induction hypothesis then also $A_{n+1} = A \cup ((A_n^* \times A_n) - \mathrm{dom}(j_{\mathcal{A}}))$ is decidable. θ_{n+1} is injective, by construction and the induction hypothesis. Moreover θ_{n+1} is computable and $E_{n+1} = \mathrm{rg}(\theta_{n+1})$ is decidable since A_n and $A_{n+1} - A_n$ are decidable and θ_n and $\#\langle -, - \rangle$ are computable with decidable range.

We define $\ell_{n+1} : E_{n+1}^* \times E_{n+1} \to E_{n+1}$ as follows:

$$\ell_{n+1}(a, \alpha) = \begin{cases} \ell_n(a, \alpha) & \text{if } q(a, \alpha) \in E_n, \\ q(a, \alpha) & \text{if } q(a, \alpha) \in E_{n+1} - E_n, \end{cases}$$

where $q = \theta_{n+1} \circ (\theta_{n+1}^-, \theta_{n+1}^{-1})$. The map ℓ_{n+1} is partial recursive since ℓ_n is partial recursive by the induction hypothesis, E_n and $E_{n+1} - E_n$ are decidable and $\theta_{n+1}, \theta^+, \theta^{-1}$ are computable.

It is clear that for all $(a, \alpha) \in A_{n+1}^* \times A_{n+1}$ we have

$$\theta_{n+1}(j_{n+1}(a, \alpha)) \simeq \ell_{n+1}(\theta_{n+1}^+(a), \theta_{n+1}(\alpha)),$$

where the symbol $\simeq$ denotes Kleene's equality[3]. Hence θ_{n+1} is an isomophism between (A_{n+1}, j_{n+1}) and (E_{n+1}, ℓ_{n+1}). Note that, if ℓ_n has a decidable range, also ℓ_{n+1} has a decidable range.

Then $\theta = \cup_{n \in \mathbb{N}} \theta_n$ is an isomorphism between $(\overline{A}, i_{\bar{A}})$ and (E, ℓ) where $E = \cup_{n \in \mathbb{N}} \mathrm{rg}(\theta_n)$ and $\ell = \cup_{n \in \mathbb{N}} \ell_n$. It is now routine to check that θ is computable, $E = \mathrm{rg}(\theta)$ is decidable, $\ell : E^* \times E \to E$ is partial recursive, $\mathrm{dom}(\ell)$ is decidable and, in the case of effectivity, that $\mathrm{rg}(\ell)$ is decidable. $\square$

Corollary 6.4.6. *If $\mathcal{A}$ is weakly effective (effective) then $\mathscr{G}_{\mathcal{A}}$ is a weakly effective (effective) graph model.*

Proof. By Proposition 6.4.4 and Theorem 6.4.5. $\square$

Remark 6.4.7. *The above corollary implies, in particular, that $\mathscr{G}_{\mathcal{A}}$ is effective for all finite $\mathcal{A}$.*

Recall that M^- has been introduced in Notation 6.2.35 for all $M \in \Lambda^o(\mathcal{D})$ and that the reflexive object associated with a graph model $\mathscr{G}$ is $\mathcal{D} = \mathscr{P}(G)$.

Lemma 6.4.8. *If $\mathcal{A}$ is weakly effective then A^- is $\mathcal{T}$-co-r.e., where $\mathcal{T} = \mathrm{Th}(\mathscr{G}_{\mathcal{A}})$.*

Proof. $A^- = \Lambda_E^o$ for $E = \{n : d_n \subseteq A\}$. If A is decidable then E is decidable, hence A^- is co-r.e. by Theorem 6.2.42, moreover it is obviously $\mathcal{T}$-closed. $\square$

All the results of this Section would hold for G- and H-models (even though the corresponding partial pairs and free completion process are somewhat more complex than for graph models).

[3] In other words, $f(x) \simeq g(y)$ abbreviates "$f(x)$ is undefined if and only if $g(y)$ is undefined and, if they are both defined, $f(x) = g(y)$".

6.4.3 Are there r.e. graph theories?

In this section we will prove, in particular, that Conjecture 1 holds for all graph models freely generated by finite partial pairs.

Lemma 6.4.9. *If $\mathcal{A}$ is a partial pair, then $|\Omega|^{\mathscr{G}_\mathcal{A}} \subseteq A$, hence $A^- \neq \emptyset$ for the model $\mathscr{G}_\mathcal{A}$.*

Proof. It is well known, and provable in a few lines, that $\alpha \in |\Omega|^{\mathscr{G}_\mathcal{A}}$ implies that $i_{\bar{\mathcal{A}}}(a, \alpha) \in a$ for some $a \in \overline{A}^*$ (the details are, for example, worked out in [16]). Immediate considerations on the rank show that this is possible only if $(a, \alpha) \in \mathrm{dom}(j_\mathcal{A})$, which forces $\alpha \in A$. $\square$

Corollary 6.4.10. *If $\mathcal{A}$ is a partial pair and $|U|^{\mathscr{G}_\mathcal{A}} \subseteq |\Omega|^{\mathscr{G}_\mathcal{A}}$, for some $U \in \Lambda^o$, then U is unsolvable.*

Proof. Solvable λ-terms have an interpretation which contains elements of any rank, while $|\Omega|^{\mathscr{G}_\mathcal{A}}$ contains only elements of rank 0. $\square$

In the next theorem, which constitutes one of the main results of this section, we provide sufficient conditions for graph models generated by weakly effective partial pairs to have a non r.e. equational theory.

Theorem 6.4.11. *Let $\mathcal{A}$ be a weakly effective partial pair. If there exists $E \subseteq A$ such that E is co-r.e., $E^- \neq \emptyset$ and $E/\mathrm{Aut}(\mathcal{A})$ is finite, then $\mathcal{T} = \mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Proof. We first show that if $\mathrm{card}(E/\mathrm{Aut}(\mathcal{A})) = k$, for some $k \in \mathbb{N}$, then $\mathrm{card}(E^-/\mathcal{T}) \leq 2^k$.

Assume $M \in E^-$ and $\alpha \in |M|^{\mathscr{G}_\mathcal{A}} \subseteq E$ then $O(\alpha)$ is included in $|M|^{\mathscr{G}_\mathcal{A}}$ where $O(\alpha)$ is the orbit of α in A modulo $\mathrm{Aut}(\mathcal{A})$. Indeed if $\theta \in \mathrm{Aut}(\mathcal{A})$ then $\theta(\alpha) = \bar{\theta}(\alpha) \in \bar{\theta}^+(|M|^{\mathscr{G}_\mathcal{A}}) = |M|^{\mathscr{G}_\mathcal{A}}$ since $\bar{\theta}^+ \in \mathrm{Aut}(\mathscr{G}_\mathcal{A})$ (Lemma 5.1.15, Theorem 1.4.2(ii)). By hypothesis the number of orbits is k; hence the number of all possible interpretations $|M|^{\mathscr{G}_\mathcal{A}} \subseteq E$ cannot overcome 2^k, hence E^- is a finite union of $\mathcal{T}$-classes.

Since E^- is co-r.e. by Theorem 6.2.42 and $E^- \neq \emptyset, \Lambda^o$, it cannot be decidable; hence $\mathcal{T}$ cannot be r.e. $\square$

From Theorem 6.4.11 and Lemma 6.4.9 we get the following results.

Corollary 6.4.12. *If $\mathcal{A}$ is finite, then $\mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Corollary 6.4.13. *If $\mathcal{A}$ is weakly effective and $A/\mathrm{Aut}(\mathcal{A})$ is finite, then $\mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Corollary 6.4.14. *If $\mathcal{A}$ is weakly effective and there is a co-r.e. set E such that $|\Omega|^{\mathscr{G}_\mathcal{A}} \subseteq E \subseteq A$ and $E/\mathrm{Aut}(\mathcal{A})$ is finite, then $\mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Corollary 6.4.15. *If $\mathcal{A}$ is weakly effective, $|\Omega|^{\mathscr{G}_\mathcal{A}}$ is decidable and $|\Omega|^{\mathscr{G}_\mathcal{A}}/\mathrm{Aut}(\mathcal{A})$ is finite, then $\mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Corollary 6.4.16. *If $\mathcal{A}$ is effective, $|\Omega|^{\mathscr{G}_\mathcal{A}}$ is decidable and $|\Omega|^{\mathscr{G}_\mathcal{A}} \cap |N_1|^{\mathscr{G}_\mathcal{A}} \cap \ldots \cap |N_k|^{\mathscr{G}_\mathcal{A}}/\mathrm{Aut}(\mathcal{A})$ is finite (possibly empty) for some normal terms $N_1, \ldots, N_k \in \Lambda^o$, with $k \in \mathbb{N}$, then $\mathrm{Th}(\mathscr{G}_\mathcal{A})$ is not r.e.*

Let us now give applications of the various corollaries.

Example 6.4.17. *Corollary 6.4.13 applies to all the usual graph (or webbed) models, indeed:*

(i) The Engeler's model $\mathscr{E}$ is freely generated by $\mathcal{A} = (A, \emptyset)$, thus all the elements of A play exactly the same role and any permutation of A is an automorphism of $\mathcal{A}$; hence the pair has only one orbit whatever the cardinality of A is. Of course if $\mathcal{A}$ is finite, then Corollary 6.4.12 applies.

(ii) Idem for the graph-Scott models (including $\mathscr{P}_\omega$) and the graph-Park models introduced in Example 5.1.12. Similarly, the graph model freely generated by $(\{\alpha, \beta\}, j)$ where $j(\{\alpha\}, \beta) = \beta$ and $j(\{\beta\}, \alpha) = \alpha$ only has one orbit.

(iii) Consider now the mixed-Scott-Park graph models defined in Example 5.1.12(iv). Then, only the permutations of A which leave Q and R invariant will be automorphisms of $(A, j_\mathcal{A})$, and we will have two orbits.

Example 6.4.18. *Corollary 6.4.15 (and hence Corollary 6.4.14) applies to the following effective pair $\mathcal{A}$.*

$A = \{\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \ldots, \beta_n, \ldots\}$ *and $j_\mathcal{A}$ defined by:*
$j_\mathcal{A}(\{\beta_n\}, \beta_n) = \beta_n,$ *for every $n \in \mathbb{N}$,*
$j_\mathcal{A}(\{\alpha_1\}, \alpha_2) = \alpha_2,$
$j_\mathcal{A}(\{\alpha_1, \alpha_2\}, \alpha_3) = \alpha_3,$
$\ldots$
$j_\mathcal{A}(\{\alpha_1, \ldots, \alpha_{n+1}\}, \alpha_{n+2}) = \alpha_{n+2}.$

Here we have that $|\Omega|^{\mathscr{G}_\mathcal{A}} = \{\beta_n : n \in \mathbb{N}\}$ is decidable and that $|\Omega|^{\mathscr{G}_\mathcal{A}}/\mathrm{Aut}(\mathcal{A})$ has cardinality 1, since every permutation of the β_n's extends into an automorphism of $\mathcal{A}$. Note that the orbits of A are: $|\Omega|^{\mathscr{G}_\mathcal{A}}$ and all the singletons $\{\alpha_n\}$; in particular $A/\mathrm{Aut}(\mathcal{A})$ is infinite.

Example 6.4.19. *Corollary 6.4.15 applies to the following pair (against the appearance it is an effective pair). Consider the set $A = \{\beta_1, \ldots, \beta_n, \ldots\}$ and the function $j_\mathcal{A}$ defined as follows: $j_\mathcal{A}(\{\beta_n\}, \beta_n) = \beta_n$ if, and only if, n belongs to a non co-r.e. set $E \subseteq \mathbb{N}$.*

Then $|\Omega|^{\mathscr{G}_\mathcal{A}} = \{\beta_n : n \in E\}$ consists of only one orbit but is not co-r.e. However (starting for example from any bijection between E and the set of even numbers) it is easy to find an isomorphism of pairs such that $j_\mathcal{A}$ is partial recursive with decidable range, and hence $|\Omega|^{\mathscr{G}_\mathcal{A}}$ becomes decidable.

Example 6.4.20. *Corollary 6.4.16 (and no other corollary) applies to the following effective pair $\mathcal{A}$.*

$$\begin{array}{l} A = \{\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \ldots, \beta_n, \ldots, \alpha'_1, \ldots, \alpha'_n, \ldots, \beta'_1, \ldots, \beta'_n, \ldots\}, \\ \text{and } j_{\mathcal{A}} \text{ is defined by}: \\ j_{\mathcal{A}}(\{\alpha'_n\}, \beta_n) = \alpha'_n, \\ j_{\mathcal{A}}(\{\alpha_1, \ldots, \alpha_n\}, \beta_n) = \beta'_n, \end{array}$$

for all $n \in \mathbb{N}$. In this case $|\Omega|^{\mathscr{G}_{\mathcal{A}}} = \{\beta_n : n \in N\}$ is decidable and $|\Omega|^{\mathscr{G}_{\mathcal{A}}} \cap |\mathbf{I}|^{\mathscr{G}_{\mathcal{A}}} = \emptyset$ (note that $|\Omega|^{\mathscr{G}_{\mathcal{A}}}/\mathrm{Aut}(\mathcal{A})$ is infinite).

Example 6.4.21. *(Example of an effective graph model outside the scope of Theorem 6.4.11) Take the total pair $\mathcal{G} = (\mathbb{N}, \ell)$ where ℓ is defined as follows:*

$$\ell(\#_*^{-1}(n), m) = \begin{cases} 2k & \text{if } \#_*^{-1}(n) = \{2k\} \text{ for some } k \in \mathbb{N}, \\ \ell(\#_*^{-1}(n), m) = 3^n 5^m & \text{otherwise.} \end{cases}$$

Recall that the effective encoding $\#_* : \mathbb{N}^* \to \mathbb{N}$ has been introduced in Section 1.1.3. It is easy to check that $\mathcal{G}$ is effective and that $|\Omega|^{\mathscr{G}} = \mathbb{N}$. Then $\Omega^- = \Lambda^o$, hence $|\Omega|^{\mathscr{G}}/\mathrm{Aut}(\mathcal{G})$ is infinite.

Another example of an effective graph model to which Theorem 6.4.11 is not applicable will be provided by Theorem 6.4.22.

6.4.4 What about the other classes of webbed models?

Theorem 6.4.11 and all its corollaries will hold for all webbed models. But for K-models it could be the case that A^- (and hence E^-) is empty. To give an idea of the power of Theorem 6.4.11, note that all the webbed models that have been introduced individually in the literature are generated by partial webs W which are weakly effective and such that W is finite with respect to $\mathrm{Aut}(W)$. Of course, the notion of automorphism of (partial) webs and automorphism of these webs should be defined in a suitable way for each class of models.

Concerning Lemma 6.4.9, it holds not only for graph models but also for G- and H-models. However, still concerning the Scott-continuous semantics, it is unclear to us whether it holds for the wider class of K-models, and *a fortiori* to that of filter models. The problem is the following: in the case of K-models the web is a tuple $(D, \preceq, i)$ where $\preceq$ is a preorder on D and $i : D^* \times D \to D$ is an injection such that $\preceq$ and i are compatible in a certain sense. The elements of the associated reflexive domain are the downward closed subsets of D, thus, we should already ask for $|\Omega| \subseteq A{\downarrow}$, where $A{\downarrow}$ is the downwards closure of A. But the real problem is that the control we have on $|\Omega|$ in K-models is much looser than in graph models. The only thing we know (from Ying Jiang's thesis [62]) is the following. If $\alpha \in |\Omega|$ then there are two sequences $\alpha_n \in D$ and $a_n \in D^*$ such that $\alpha = \alpha_0 \preceq \alpha_1 \preceq \ldots \alpha_n \preceq \ldots$, $|\delta| \supseteq a_0{\downarrow} \supseteq a_1{\downarrow} \supseteq \ldots \supseteq a_n{\downarrow} \supseteq \ldots$ and $\beta_n = i(a_{n+1}, \alpha_{n+1}) \in a_n$ for all n. This forces

β_n to be an increasing sequence, included in $\cap_{n\in\mathbb{N}}(a_n\downarrow)$. Moreover, if the model is extensional, we have that $\alpha_n = \alpha$ for all $n \in \mathbb{N}$.

The proof of Theorem 6.4.11 holds for all webbed models, but of course to get the conclusions one has to check whether the three hypotheses hold. For example, for K-models, it could be the case that $E^- = \emptyset$.

6.4.5 An effective graph model having the minimum graph theory, and applications

In this section we show another main theorem of the chapter: the minimum order graph theory is the theory of an effective graph model. This result has the following interesting consequences: (i) no order graph theories can be r.e.; (ii) for any closed normal term M, there exists a non-empty β-co-r.e. set $\mathcal{V}$ of unsolvable terms whose interpretations are below that of M in all graph models.

Theorem 6.4.22. *There exists an effective graph model whose order/equational theory is the minimum order/equational graph theory.*

Proof. It is not difficult to define an effective numeration $\mathcal{N}$ of all the finite partial pairs whose carrier set is a subset of $\mathbb{N}$. We now make the carrier sets N_k, for $k \in \mathbb{N}$, pairwise disjoint. Let p_k be the k-th prime number. Then we define another finite partial pair $\mathcal{A}_k$ as follows: $A_k = \{p_k^{n+1} : n \in N_k\}$ and $j_{\mathcal{A}_k}(\{p_k^{\alpha_1+1}, \dots, p_k^{\alpha_n+1}\}, p_k^{\alpha+1}) = p_k^{j_{\mathcal{N}_k}(\{\alpha_1,\dots,\alpha_n\},\alpha)+1}$ for all $(\{\alpha_1, \dots, \alpha_n\}, \alpha) \in \mathrm{dom}(j_{\mathcal{N}_k})$. In this way we get an effective bijective numeration of all the finite partial pairs $\mathcal{A}_k$.
Let us take $\mathcal{A} = \sqcup_{k\in\mathbb{N}}\mathcal{A}_k$. It is an easy matter to prove that A is a decidable subset of $\mathbb{N}$ and that $j_{\mathcal{A}}$ is a computable map with decidable domain and range. It follows from Theorem 6.4.5 that $\mathscr{G}_{\mathcal{A}}$ is an effective graph model.

Finally, with the same reasoning done in the proof of Theorem 5.2.4, we can conclude that $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}})$ ($\mathrm{Th}(\mathscr{G}_{\mathcal{A}})$) is the minimum order (equational) graph theory. □

Let $\mathcal{T}^{min}$ and $\mathcal{T}^{min}_{\sqsubseteq}$ be, respectively, the minimum equational graph theory and the minimum order graph theory. We will denote by $\mathscr{G}_{min}$ *any* effective graph model whose order/equational graph theory is the minimum one. As shown in the next proposition, $\mathscr{G}_{min}$ is far from being unique, also if considered up to isomorphism.

Proposition 6.4.23.

(i) $\mathcal{T}^{min}$ is an intersection of a countable set of non-r.e. equational graph theories.

(ii) $\mathcal{T}^{min}$ (and $\mathcal{T}^{min}_{\sqsubseteq}$) is the theory of countably many non-isomorphic effective graph models.

Proof. (i) By the proof of Theorem 5.2.4 $\mathcal{T}_{\sqsubseteq}^{min} = \cap \mathrm{Th}_{\sqsubseteq}(\mathscr{G}_{\mathcal{A}_k})$ where $\mathcal{A}_k$ ranges over all finite pairs. By Corollary 6.4.12 these theories are not r.e.
(ii) Since, in the proof of Theorem 6.4.22, there exist countably many choices for the effective numeration $\mathcal{N}$ which give rise to non-isomorphic graph models $\mathscr{G}_{min}$. Indeed, given an effective bijective numeration $\mathcal{N}'$ of all finite pairs, we can take, for every recursive sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers, an effective numeration which repeats n_k-times the pair $\mathcal{N}'_k$. □

The following two results are consequences of Theorem 6.4.22. We remark that here we are using the effectiveness of $\mathscr{G}_{min}$ for proving properties of all graph models.

Theorem 6.4.24. *For all graph models $\mathscr{G}$, $\mathrm{Th}_{\sqsubseteq}(\mathscr{G})$ is not r.e.*

Proof. Let M be any closed normal λ-term. Since $\mathscr{G}_{min}$ is effective, Theorem 6.2.46 implies that $|M|^{\mathscr{G}_{min}}$ is decidable, hence $(M)^-_{\mathscr{G}_{min}} = \{N \in \Lambda^o : |N|^{\mathscr{G}_{min}} \subseteq |M|^{\mathscr{G}_{min}}\}$ is β-co-r.e. by Theorem 6.2.43.

Suppose, now, that $\mathscr{G}$ is a graph model such that $\mathrm{Th}_{\sqsubseteq}(\mathscr{G})$ is r.e. Then $(M)^-_{\mathscr{G}} = \{N \in \Lambda^o : |N|^{\mathscr{G}} \subseteq |M|^{\mathscr{G}}\}$ is a β-r.e. set which contains the co-r.e. set $(M)^-_{\mathscr{G}_{min}}$. Thus, by the FIP we get $(M)^-_{\mathscr{G}} = \Lambda^o$.

By the arbitrariness of M, it follows that $(\mathbf{T})^-_{\mathscr{G}} = (\mathbf{F})^-_{\mathscr{G}}$. Since $\mathbf{F} \in (\mathbf{T})^-_{\mathscr{G}}$, and *vice versa*, we get $|\mathbf{F}|^{\mathscr{G}} = |\mathbf{T}|^{\mathscr{G}}$, contradiction. □

Proposition 6.4.25. *For all normal $M_1, \dots, M_n \in \Lambda^o$ there exists a non-empty β-co-r.e. set $\mathcal{V}$ of closed unsolvable terms such that:*

$$\textit{for all graph models } \mathscr{G}\textit{:}\ (\forall U \in \mathcal{V})|U|^{\mathscr{G}} \subseteq |M_1|^{\mathscr{G}} \cap \dots \cap |M_n|^{\mathscr{G}}.$$

Proof. Since $\mathscr{G}_{min}$ is effective and $M_1, \dots, M_n$ are closed normal λ-terms Theorem 6.2.46 implies that every $|M_i|^{\mathscr{G}_{min}}$ is decidable. Thus, from Theorem 6.2.43, it follows that $|M_1|^{\mathscr{G}_{min}} \cap \dots \cap |M_n|^{\mathscr{G}_{min}}$ is a non-empty β-co-r.e. set. Therefore, by Remark 6.2.6 there exists a β-co-r.e. set $\mathcal{V}$ of unsolvable λ-terms such that for every $U \in \mathcal{V}$ we have $|U|^{\mathscr{G}_{min}} \subseteq |M_i|^{\mathscr{G}_{min}}$ for all $1 \le i \le n$. Then the conclusion follows since $\mathrm{Th}_{\sqsubseteq}(\mathscr{G}_{min})$ is the minimum order graph theory. □

We do not know any example of unsolvable satisfying the above condition, or even of an unsolvable U such that, for all graph model $\mathscr{G}$, we have $|U|^{\mathscr{G}} \subseteq |\mathbf{I}|^{\mathscr{G}}$.

6.5 Conclusions

We have investigated the question of whether the equational/order theory of a model of λ-calculus living in one of the main semantics can be recursively enumerable. This is a generalization of the longstanding open problem due to Honsell and Ronchi Della Rocca of whether there exists a Scott-continuous model of λ-calculus having λ_β or $\lambda_{\beta\eta}$ as equational theory.

For this purpose, we have introduced a notion of effective model of λ-calculus, which covers in particular all the models individually introduced in the literature. We have proved that the order theory of an effective model is never r.e. and the corresponding equational theory cannot be λ_β, or $\lambda_{\beta\eta}$. Moreover, we have shown that no effective model living in the stable or strongly stable semantics has an r.e. equational theory. Concerning the Scott-continuous semantics, we have focused our attention on the class of graph models. We have proved that no order graph theory can be r.e., that many effective graph models do not have an r.e. equational theory, and that there exists an effective graph model whose equational/order theory is the minimum one.

Conclusions

Although Alonzo Church introduced the untyped λ-calculus in the thirties, the study of its models and theories is, today, a research field which is still full of life. The wealth of results which have been discovered in the last years allow us to understand much better the known semantics of λ-calculus and the structure of the lattice of λ-theories, but they also generated a lot of new interesting open questions.

In this thesis we have mainly focused our attention on the models of λ-calculus living in the main semantics, but we have also studied two new kinds of semantics: the *relational* and the *indecomposable semantics*.

Since the models living in the relational semantics have not enough points, and the "enough points condition" is advocated in the literature as necessary to obtain a λ-model, we have found it natural to reinvestigate, first, the relationship between the categorical and algebraic definitions of model of λ-calculus. In Chapter 2 we have given a new construction which allows us to present *any* categorical model as a λ-model, and hence proved that there is a *unique* definition of model of λ-calculus. Moreover, we have provided sufficient conditions for categorical models living in arbitrary cpo-enriched Cartesian closed categories to have $\mathcal{H}^*$ as equational theory.

In Chapter 3 we have built a categorical model $\mathscr{D}$ living in the relational semantics, and we have proved that its equational theory is $\mathcal{H}^*$ since it fulfills the conditions described in Chapter 2. Then, we have applied to $\mathscr{D}$ our construction and shown that the associated λ-model satisfies suitable algebraic properties for modelling a λ-calculus extended with both non-deterministic choice and parallel composition.

Concerning the indecomposable semantics we have proved that it encompasses the main semantics, as well as the term models of all semi-sensible λ-theories and that, however, it is still largely incomplete. This gives a new and shorter common proof of the (large) incompleteness of the Scott-continuous, stable, and strongly stable semantics.

In Chapter 5 we have developed some mathematical tools for studying the framework of partial webs of graph models. These tools have been fruitfully used to prove, for example, that there exists a minimum order/equational graph theory and that graph models enjoy a kind of Löwenheim-Skolem theorem.

Finally, in Chapter 6, we have investigated the problem of whether the equational/order theory of a non-syntactical model of λ-calculus living in one of the main semantics can be r.e. For this reason we have introduced an appropriate notion of effective model of λ-calculus, which covers in particular all the models individually introduced in the literature. We have proved that the order theory of an effective model is never r.e., and hence that its equational theory cannot be λ_β or $\lambda_{\beta\eta}$. Then, we have shown that no effective model living in the stable or in the strongly stable semantics has an r.e. equational theory. Concerning Scott-continuous semantics, we have investigated the class of graph models and proved that no order graph theory can be r.e., that many effective graph models do not have an r.e. equational theory and that there exists an effective graph model whose equational/order theory is the minimum one.

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