The Bang Calculus and the Two Girard's Translations

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We study the two Girard's translations of intuitionistic implication into linear logic by exploiting the bang calculus, a paradigmatic functional language with an explicit box-operator that allows both the call-by-name and call-by-value λ-calculi to be encoded in. We investigate how the bang calculus subsumes both call-by-name and call-by-value λ-calculi from a syntactic and a semantic viewpoint.

1 Introduction

The λ-calculus is a simple framework formalizing many features of functional programming languages. For instance, the λ-calculus can be endowed with two different evaluation mechanisms, call-by-name (CbN) and call-by-value (CbV), which have quite different properties. A CbN discipline re-evaluates an argument each time it is used. On the contrary, a CbV discipline evaluates an argument just once and recalls its value each time it is used. CbN and CbV λ-calculi are usually defined by means of operational rules giving rise to two different rewriting systems on the same set of λ-terms: in CbN there is no restriction on firing a β-redex, whereas in CbV a β-redex can be fired only when the argument is a value, *i.e.* a variable or an abstraction. The standard categorical setting for describing denotational models of the λ-calculus, cartesian closed categories, provides models which are adequate for CbN, but typically not for CbV. For CbV, the introduction of an additional computational monad (in the sense of Moggi [22, 23]) is necessary. While CbN λ-calculus [4] has a rich and refined semantic and syntactic theory featuring advanced concepts such as separability, solvability, Böhm trees, classification of λ-theories, full-abstraction, *etc.*, this is not the case for CbV λ-calculus [25], in the sense that concerning the CbV counterpart of these theoretical notions there are only partial and not satisfactory results (or they do not exist at all!).

Quoting from [17], "the existence of two separate paradigms is troubling" for at least two reasons:

- it makes each language appear arbitrary (whereas a unified language might be more canonical);
- each time we create a new style of semantics, *e.g.* Scott semantics, operational semantics, game semantics, continuation semantics, *etc.*, we always need to do it twice — once for each paradigm.

Girard's Linear Logic (LL, [15]) provides a unifying setting where this discrepancy could be solved since both CbN and CbV λ-calculi can be faithfully translated, via two different translations, into LL proof-nets. Following [17], we can claim that, via these translations, LL proof-nets "subsume" the CbN and CbV paradigms, in the sense that both operational and denotational semantics for those paradigms can be seen as arising, via these translations, from similar semantics for LL.

Indeed, LL can be understood as a refinement of intuitionistic logic (and hence λ-calculus) in which resource management is made explicit thanks to the introduction of a new pair of dual connectives: the exponentials "!" and "?". In proof-nets, the standard syntax for LL proofs, *boxes* (introducing the

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modality "!") mark the sub-proofs available at will: during cut-elimination, such boxes can be erased (by weakening rules), can be duplicated (by contraction rules), can be opened (by dereliction rules) or can enter other boxes. The categorical counterpart of this refinement is well known: it is the notion of a cartesian $*$-autonomous[1] category, equipped with a comonad endowed with a strong monoidal structure. Every instance of such a kind of structure yields a denotational model of LL.

In his seminal article [15], Girard proposes a standard translation of intuitionistic logic (and hence simply typed λ-calculus) in multiplicative-exponential LL proof-nets whose semantic counterpart is well known: the Kleisli category of the exponential comonad "!" is cartesian closed thanks to the strong monoidal structure of "!". This translation $(\cdot)^{\mathsf{N}}$ maps the intuitionistic implication $A \Rightarrow B$ to the LL formula $!A^{\mathsf{N}} \multimap B^{\mathsf{N}}$. In [15] Girard proposes also another translation $(\cdot)^{\mathsf{V}}$ that he calls "boring": it maps the the intuitionistic implication $A \Rightarrow B$ to the LL formula $!(A^{\mathsf{V}} \multimap B^{\mathsf{V}})$ (or equivalently $!A^{\mathsf{V}} \multimap !B^{\mathsf{V}}$). Since the untyped λ-calculus can be seen as simply typed with only one ground type o satisfying the recursive identity $o = o \Rightarrow o$, the two Girard's translations $(\cdot)^{\mathsf{N}}$ and $(\cdot)^{\mathsf{V}}$ decompose this identity into $o = !o \multimap o$ and $o = !(o \multimap o)$ (or equivalently, $o = !o \multimap !o$), respectively. At the λ-term level, these two translations differ only by the way they use logical exponential rules (*i.e.* box and dereliction), whereas they use multiplicative and structural (*i.e.* contraction and weakening) ingredients in the same way. Because of this difference, the translation $(\cdot)^{\mathsf{N}}$ encodes the CbN λ-calculus into LL proof-nets (in the sense that CbN evaluation $\to_\beta$ is simulated by cut-elimination via $(\cdot)^{\mathsf{N}}$), while $(\cdot)^{\mathsf{V}}$ encodes the CbV λ-calculus into LL proof-nets (CbV evaluation $\to_{\beta^v}$ is simulated by cut-elimination via $(\cdot)^{\mathsf{V}}$). Indeed, since in CbN λ-calculus there is no restriction on firing a β-redex (its argument can be freely copied or erased), the translation $(\cdot)^{\mathsf{N}}$ puts the argument of every application into a box (see [7, 27, 16]); on the other hand, the translation $(\cdot)^{\mathsf{V}}$ puts only values into boxes (see [2]) since in CbV λ-calculus values are the only duplicable and discardable λ-terms. Thus, as deeply studied in [20], the two Girard's logical translations explain the two different evaluation mechanisms, bringing them into the scope of the Curry-Howard isomorphism.

The syntax of multiplicative-exponential LL proof-nets is extremely expressive and powerful, but it is too general and sophisticated for the computational purpose of representing purely functional programs. For instance, simulation of β-reduction on LL proof-nets passes through intermediate states/proof-nets that cannot be expressed as λ-terms, since LL proof-nets have many spurious cuts with axioms that have no counterpart on λ-terms. More in general, LL proof-nets are manipulated in their graphical form, and while this is a handy formalism for intuitions, it is far from practical for formal reasoning.

From the analysis of Girard's translations it seems worthwhile to extend the syntax of the λ-calculus to internalize the insights coming from LL *in a λ-like syntax*. The idea is to enrich the λ-calculus with explicit *boxes* marking the "values" of the calculus, *i.e.* the terms that can be freely duplicated and discarded: such a *linear* λ-calculus subsumes both CbN and CbV λ-calculi, via suitable translations. This, of course, has been done quite early in the history of LL by defining various linear λ-calculi, such as [19, 1, 5, 28, 20]. All these calculi require a clear distinction between linear and non-linear variables, structural rules being freely (and implicitly) available for the latter and forbidden for the former. This distinction complicates the formalism and is actually useless as far as we are interested in subsuming λ-calculi.

Inspired by Ehrhard [13], in [14] it has been introduced an intermediate formalism enjoying at the same time the conceptual simplicity of λ-calculus (without any distinction between linear and non-linear variables) and the operational expressiveness of LL proof-nets: the *bang calculus*. It is a variant of the λ-calculus which is "linear" in the sense that the exponential rules of LL (box and dereliction) are part of the syntax, so as to subsume CbN and CbV λ-calculi via two translations $(\cdot)^{\mathsf{n}}$ and $(\cdot)^{\mathsf{v}}$, respectively,

[1] Actually the full symmetry of such a category is not really essential as far as the λ-calculus is concerned, it is however quite natural from the LL viewpoint: LL restores the classical involutivity of negation in a constructive setting.

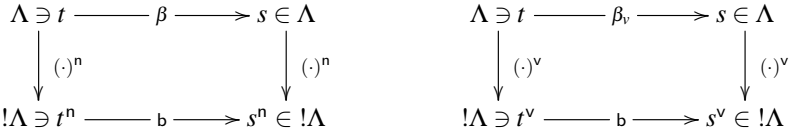
from the set Λ of λ-terms to the set $!\Lambda$ of terms of the bang calculus (see §3). These two translations are deeply related to Girard's encodings $(\cdot)^{\mathsf{N}}$ and $(\cdot)^{\mathsf{V}}$ of CbN and CbV λ-calculi into LL proof-nets. Indeed, Girard's translations $(\cdot)^{\mathsf{N}}$ and $(\cdot)^{\mathsf{V}}$ decompose in such a way that the following diagrams commute:

where $(\cdot)^\circ$ is a natural translation of the bang calculus into multiplicative-exponential LL proof-nets. Thus, the bang calculus *internalizes* the two Girard's translations in a λ-like calculus instead of LL proof-nets. It subsumes both CbN and CbV λ-calculi in the *same* rewriting system and denotational model, so that it may be a general setting to compare CbN and CbV. The bang calculus can be seen as a metalanguage where the choice of CbN or CbV evaluation depends on the way the term is built up. If we consider the syntax of the λ-calculus as a programming language, issues like CbN versus CbV evaluations affect the way the λ-calculus is translated in this metalanguage, but does not affect the metalanguage itself.

It turns out that this bang calculus was already known in the literature: it is an untyped version of the implicative fragment of Paul Levy's Call-By-Push-Value calculus [17, 18]. Interestingly, his work was not motivated by an investigation of the two Girard's translations. This link is not casual, since it holds even when the bang calculus is extended to a PCF-like system, as shown by Ehrhard [13].

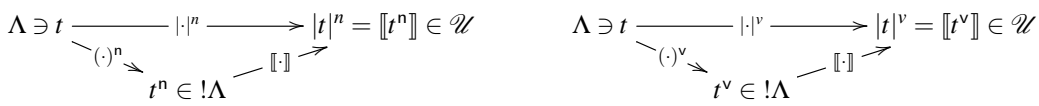
The aim of our paper is to further investigate the way the bang calculus subsumes CbN and CbV λ-calculi, refining and extending some results already obtained in [14].

1. From a syntactic viewpoint, we show in §3 that the bang calculus subsumes in the same rewriting system both CbN and CbV λ-calculi, in the sense that the translations $(\cdot)^{\mathsf{n}}$ and $(\cdot)^{\mathsf{v}}$ from the λ-calculus to the bang calculus are sound and complete with respect to β-reduction and β_v-reduction, respectively (in [14] only soundness was proven, and in a less elegant way). In other words, the diagrams



   commute in the two ways: starting from the β-reduction step $\to_\beta$ for the CbN λ-calculus (on the left) or the β_v-reduction step $\to_{\beta^v}$ for the CbV λ-calculus (on the right), and starting from the b-reduction step $\to_{\mathsf{b}}$ of the bang calculus.[2]

2. From a semantic viewpoint, we show in §4 that the *every* LL-based model $\mathscr{U}$ of the bang calculus (as categorically defined in [14]) provides a model for both CbN and CbV λ-calculi (in [14] this was done only for the special case of relational semantics). Moreover, given a λ-term t, we investigate the relation between its interpretations $|t|^n$ in CbN (resp. $|t|^v$ in CbV) and the interpretation $[\![\cdot]\!]$ of its translation t^{n} (resp. t^{v}) into the bang calculus. We prove that the diagram below on the left (for CbN) commutes, whereas we give a counterexample (in the relational semantics) to the commutation of the diagram below on the right (for CbV). We conjecture that there still exists a relationship in CbV between $|t|^v$ and $[\![t^{\mathsf{v}}]\!]$, but it should be more sophisticated than in CbN.



In order to achieve these results in a clearer and simpler way, we have slightly modified (see §2) the syntax and operational semantics of the bang calculus with respect to its original formulation in [14].

[2] Actually, for the CbV λ-calculus the diagram is slightly more complex, as we will see in §3, but the essence does not change.

$$
\begin{array}{rll}
\textit{Terms}: & T,S,R ::= x \mid \lambda x T \mid \langle T\rangle S \mid \mathrm{der}\, T \mid T^! & (\text{set: } !\Lambda)\\
\textit{Contexts}: & \mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x \mathtt{C} \mid \langle \mathtt{C}\rangle T \mid \langle T\rangle \mathtt{C} \mid \mathrm{der}\, \mathtt{C} \mid \mathtt{C}^! & (\text{set: } !\Lambda_{\mathtt{C}})\\
\textit{Ground contexts}: & \mathtt{G} ::= (\!|\cdot|\!) \mid \lambda x \mathtt{G} \mid \langle \mathtt{G}\rangle T \mid \langle T\rangle \mathtt{G} \mid \mathrm{der}\, \mathtt{G} & (\text{set: } !\Lambda_{\mathtt{G}})\\
\textit{Root-steps}: & \langle \lambda x T\rangle S^! \mapsto_\ell T\{S/x\} \qquad \mathrm{der}(T^!) \mapsto_{\mathsf{d}} T \qquad \mapsto_{\mathsf{b}} := \mapsto_\ell \cup \mapsto_{\mathsf{d}} &\\
\mathsf{r}\textit{-reduction}: & T \to_{\mathsf{r}} S \Leftrightarrow \exists \mathtt{C} \in !\Lambda_{\mathtt{C}}, \exists T',S' \in !\Lambda : T = \mathtt{C}(\!|T'|\!), S = \mathtt{C}(\!|S'|\!), T' \mapsto_{\mathsf{r}} S' &\\
\mathsf{r}_{\mathsf{g}}\textit{-reduction}: & T \to_{\mathsf{r}_{\mathsf{g}}} S \Leftrightarrow \exists \mathtt{G} \in !\Lambda_{\mathtt{G}}, \exists T',S' \in !\Lambda : T = \mathtt{G}(\!|T'|\!), S = \mathtt{G}(\!|S'|\!), T' \mapsto_{\mathsf{r}} S' &
\end{array}
$$

Figure 1: The bang calculus: its syntax and its reduction rules, where $\mathsf{r} \in \{\ell, \mathsf{d}, \mathsf{b}\}$.

Preliminaries and notations. Let $\to_{\mathsf{r}}$ and $\to_{\mathsf{r}'}$ be binary relations on a set X.

The composition of $\to_{\mathsf{r}}$ and $\to_{\mathsf{r}'}$ is denoted by $\to_{\mathsf{r}}\to_{\mathsf{r}'}$ or $\to_{\mathsf{r}} \cdot \to_{\mathsf{r}'}$. The transpose of $\to_{\mathsf{r}}$ is denoted by ${}_{\mathsf{r}}\!\leftarrow$. The reflexive-transitive (resp. reflexive) closure of $\to_{\mathsf{r}}$ is denoted by $\to_{\mathsf{r}}^*$ (resp. $\to_{\mathsf{r}}^=$). The r-*equivalence* $\simeq_{\mathsf{r}}$ is the reflexive-transitive and symmetric closure of $\to_{\mathsf{r}}$.

Let $t \in X$: t is r-*normal* if there is no term s such that $t \to_{\mathsf{r}} s$; t is r-*normalizable* if there is a r-normal term s such that $t \to_{\mathsf{r}}^* s$, and we then say that s is a r-*normal form of* t.

The relation $\to_{\mathsf{r}}$ is *confluent* if ${}_{\mathsf{r}}^*\!\leftarrow \cdot \to_{\mathsf{r}}^* \subseteq \to_{\mathsf{r}}^* \cdot {}_{\mathsf{r}}^*\!\leftarrow$; it is *quasi-storngly confluent* if ${}_{\mathsf{r}}\!\leftarrow \cdot \to_{\mathsf{r}} \subseteq (\to_{\mathsf{r}} \cdot {}_{\mathsf{r}}\!\leftarrow) \cup =$. From confluence it follows that: $t \simeq_{\mathsf{r}} s$ iff $t \to_{\mathsf{r}}^* r\ {}_{\mathsf{r}}^*\!\leftarrow s$ for some term r; and any r-normalizable term has a *unique* r-normal form. Clearly, quasi-strong confluence implies confluence.

2 Syntax and reduction rules of the bang calculus

The syntax and operational semantics of the *bang calculus* are defined in Fig. 1.

Terms are built up from a countably infinite set $\mathscr{V}ar$ of *variables* (denoted by $x,y,z,\ldots$). Terms of the form $T^!$ (resp. $\lambda x T$; $\langle T\rangle S$; $\mathrm{der}\, T$) are called or *boxes* (resp. *abstractions*; *(linear) applications*; *derelictions*). The set of boxes is denoted by $!\Lambda_!$. The set of free variables of a term T, denoted by $\mathsf{fv}(T)$, is defined as expected, λ being the only binding constructor. All terms are considered up to α-conversion. Given $T,S \in !\Lambda$ and a variable x, $T\{S/x\}$ denotes the term obtained by the *capture-avoiding substitution* of S (and not $S^!$) for each free occurrence of x in T: so, $T^!\{S/x\} = (T\{S/x\})^! \in !\Lambda_!$.

Contexts $\mathtt{C}$ and *ground contexts* $\mathtt{G}$ (both with exactly one hole $(\!|\cdot|\!)$) are defined in Fig. 1. All ground contexts are contexts but the converse fails: $(\!|\cdot|\!)^!$ is a non-ground context. We write $\mathtt{C}(\!|T|\!)$ for the term obtained by the capture-allowing substitution of the term T for the hole $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

Reductions in the bang calculus are defined in Fig. 1 as follows: given a *root-step* rule $\mapsto_{\mathsf{r}} \subseteq !\Lambda \times !\Lambda$, we define the r-*reduction* $\to_{\mathsf{r}}$ (resp. r_{g}-*reduction* $\to_{\mathsf{r}_{\mathsf{g}}}$) as the closure of $\mapsto_{\mathsf{r}}$ under contexts (resp. ground contexts). Note that $\to_{\mathsf{r}_{\mathsf{g}}} \subsetneq \to_{\mathsf{r}}$ as $!\Lambda_{\mathtt{G}} \subsetneq !\Lambda_{\mathtt{C}}$: the only difference between $\to_{\mathsf{r}}$ and $\to_{\mathsf{r}_{\mathsf{g}}}$ is that the latter does not reduce under $!$ (but both reduce under λ). The root-steps used in the bang calculus are $\mapsto_\ell$ and $\mapsto_{\mathsf{d}}$ and $\mapsto_{\mathsf{b}} := \mapsto_\ell \cup \mapsto_{\mathsf{d}}$. From the definitions in Fig. 1 it follows that $\to_{\mathsf{b}} = \to_\ell \cup \to_{\mathsf{d}}$ and $\to_{\mathsf{b}_{\mathsf{g}}} = \to_{\ell_{\mathsf{g}}} \cup \to_{\mathsf{d}_{\mathsf{g}}}$.

Intuitively, the basic idea behind the root-steps $\mapsto_\ell$ and $\mapsto_{\mathsf{d}}$ is that the box-constructor $!$ marks the only terms that can be erased and duplicated. When the argument of a constructor der is a box $T^!$, the root-step $\mapsto_{\mathsf{d}}$ opens the box, *i.e.* accesses its content T, destroying its status of availability at will (but T, in turn, might be a box). The root-step $\mapsto_\ell$ says that a β-like redex $\langle \lambda x T\rangle S$ can be fired only when its argument is a box, *i.e.* $S = R^!$: if it so, the content R of the box S replaces any free occurrence of x in T.

Example 1. Let $\Delta := \lambda x \langle x\rangle x^!$ and $\Delta' := \lambda x \langle \mathrm{der}(x^!)\rangle x^!$. Then, $\Delta' \to_{\mathsf{d}_{\mathsf{g}}} \Delta$ and $\langle \Delta\rangle \Delta^! \to_{\ell_{\mathsf{g}}} \langle \Delta\rangle \Delta^! \to_{\ell_{\mathsf{g}}} \ldots$ and $\langle \mathrm{der}(\Delta'^!)\rangle \Delta'^! \to_{\mathsf{d}_{\mathsf{g}}} \langle \Delta'\rangle \Delta'^! \to_{\ell_{\mathsf{g}}} \langle \mathrm{der}(\Delta'^!)\rangle \Delta'^! \to_{\mathsf{d}_{\mathsf{g}}} \ldots$ Note that $(\langle \Delta\rangle \Delta^!)^!$ is b_{g}-normal but not b-normalizable.

The *bang* (resp. *ground bang*) *calculus* is the set $!\Lambda$ endowed with the reduction $\to_{\mathsf{b}}$ (resp. $\to_{\mathsf{b}_{\mathsf{g}}}$).

$$\frac{}{x \Rightarrow_\ell x}\ \mathsf{var} \qquad \frac{T \Rightarrow_\ell S}{\lambda x T \Rightarrow_\ell \lambda x S}\ \lambda \qquad \frac{T \Rightarrow_\ell S}{T^! \Rightarrow_\ell S^!}\ ! \qquad \frac{T \Rightarrow_\ell S}{\mathrm{der}\, T \Rightarrow_\ell \mathrm{der}\, S}\ \mathrm{der} \qquad \frac{T \Rightarrow_\ell S \quad R \Rightarrow_\ell Q}{\langle T\rangle R \Rightarrow_\ell \langle S\rangle Q}\ @ \qquad \frac{T \Rightarrow_\ell S \quad R \Rightarrow_\ell Q}{\langle \lambda x T\rangle R^! \Rightarrow_\ell S\{Q/x\}}\ \ell$$

Figure 2: Parallel ℓ-reduction.

Quasi-strong confluence of $\mathsf{b_g}$-reduction and confluence of b-reduction. To prove the confluence of $\to_{\mathsf{b}}$ (Prop. 4.2), first we show that $\to_\ell$ is confluent (Lemma 3.4). The latter is proved by a standard adaptation of Tait–Martin-Löf technique — as improved by Takahashi [29] — based on parallel reduction. For this purpose, we introduce *parallel ℓ-reduction*, denoted by $\Rightarrow_\ell$, a binary relation on $!\Lambda$ defined by the rules in Fig. 2. Intuitively, $\Rightarrow_\ell$ reduces simultaneously a number of ℓ-redexes existing in a term. It is immediate to check that $\Rightarrow_\ell$ is reflexive and $\to_\ell \subseteq \Rightarrow_\ell \subseteq \to_\ell^*$, whence $\Rightarrow_\ell^* = \to_\ell^*$.

For any term T, we denote by T^* the term obtained by reducing *all* ℓ-redexes in T simultaneously. Formally, T^* is defined by induction on $T \in !\Lambda$ as follows:

$$x^* := x \qquad (\lambda x T)^* := \lambda x T^* \qquad (T^!)^* := (T^*)^! \qquad (\mathrm{der}\, T)^* := \mathrm{der}(T^*)$$
$$(\langle T\rangle S)^* := \langle T^*\rangle S^* \text{ if } T \neq \lambda x R \text{ or } S \notin !\Lambda_! \qquad (\langle \lambda x T\rangle S^!)^* := T^*\{S^*/x\}$$

Lemma 2 (Development)**.** *Let $T, S \in !\Lambda$. If $T \Rightarrow_\ell S$, then $S \Rightarrow_\ell T^*$.* *Proof p. 17*

Lemma 2 is the key ingredient to prove the confluence of $\to_\ell$ (Lemma 3.4 below).

The next lemma lists a series of good rewriting properties of ℓ-, ℓ_{g}-, d- and $\mathsf{d_g}$-reductions that will be used to prove quasi-strong confluence of $\to_{\mathsf{b_g}}$ and confluence of $\to_{\mathsf{b}}$ (Prop. 4 below).

Lemma 3 (Basic properties of reductions)**.** *Proof p. 18*

1. *$\to_{\ell_{\mathsf{g}}}$ is quasi-strongly confluent: ${}_{\ell_{\mathsf{g}}}\!\leftarrow \cdot \to_{\ell_{\mathsf{g}}} \subseteq (\to_{\ell_{\mathsf{g}}} \cdot {}_{\ell_{\mathsf{g}}}\!\leftarrow) \cup =$.*
2. *$\to_{\mathsf{d_g}}$ and $\to_{\mathsf{d}}$ are quasi-strongly confluent (separately).*
3. *$\to_{\mathsf{d_g}}$ and $\to_{\ell_{\mathsf{g}}}$ strongly commute: ${}_{\mathsf{d_g}}\!\leftarrow \cdot \to_{\ell_{\mathsf{g}}} \subseteq \to_{\ell_{\mathsf{g}}} \cdot {}_{\mathsf{d_g}}\!\leftarrow$.*
   $\to_{\mathsf{d}}$ quasi-strongly commutes over $\to_\ell$: ${}_{\mathsf{d}}\!\leftarrow \cdot \to_\ell \subseteq \to_\ell \cdot {}^*_{\mathsf{d}}\!\leftarrow$.
   $\to_{\mathsf{d}}$ and $\to_\ell$ commute: ${}^*_{\mathsf{d}}\!\leftarrow \cdot \to_\ell^* \subseteq \to_\ell^* \cdot {}^*_{\mathsf{d}}\!\leftarrow$.
4. *$\to_\ell$ is confluent: ${}^*_\ell\!\leftarrow \cdot \to_\ell^* \subseteq \to_\ell^* \cdot {}^*_\ell\!\leftarrow$.*

Proposition 4 (Quasi-strong confluence of $\to_{\mathsf{b_g}}$ and confluence of $\to_{\mathsf{b}}$)**.** *Proof p. 20*

1. *The reduction $\to_{\mathsf{b_g}}$ is quasi-strongly confluent, i.e. ${}_{\mathsf{b_g}}\!\leftarrow \cdot \to_{\mathsf{b_g}} \subseteq (\to_{\mathsf{b_g}} \cdot {}_{\mathsf{b_g}}\!\leftarrow) \cup =$.*
2. *The reduction $\to_{\mathsf{b}}$ is confluent: ${}^*_{\mathsf{b}}\!\leftarrow \cdot \to_{\mathsf{b}}^* \subseteq \to_{\mathsf{b}}^* \cdot {}^*_{\mathsf{b}}\!\leftarrow$.*

3 The bang calculus with respect to CbN and CbV λ-calculi, syntactically

One of the interests of the bang calculus is that it is a general framework where both *call-by-name* (CbN *i.e.* ordinary, [4]) and Plotkin's *call-by-value* (CbV, [25]) λ-calculi can be embedded.[3] Syntax and reduction rules of CbN and CbV λ-calculi are in Fig. 3: β reduction $\to_\beta$ (resp. β^v-reduction $\to_{\beta^v}$) is the reduction for the CbN (resp. CbV) λ-calculus. CbN and CbV λ-calculi share the same term syntax (the set Λ of λ-terms of CbV and CbN λ-calculi can be seen as a subset of $!\Lambda$), whereas $\to_{\beta^v}$ is just the restriction of $\to_\beta$ allowing to fire a β-redex $(\lambda x T)S$ only when S is a λ-value, *i.e.* variable or abstraction. Ground β-(resp. β^v-)reduction $\to_{\beta_{\mathsf{g}}}$ (resp. $\to_{\beta_{\mathsf{g}}^v}$) is an interesting restriction of β-(resp. β^v-)reduction:

[3] Here, with CbN or CbV λ-calculus we refer to the whole calculus and its general reduction rules, not only to CbN or CbV (deterministic) evaluation strategy in the λ-calculus.

λ-*terms*:	$t,s,r ::= v \mid ts$	(set: Λ)
λ-*values*:	$v ::= x \mid \lambda x\, t$	(set: Λ_v)
λ-*contexts*:	$\mathtt{C} ::= (\!\lvert\cdot\rvert\!) \mid \lambda x \mathtt{C} \mid \mathtt{C}t \mid t\mathtt{C}$	(set: $\Lambda_{\mathtt{C}}$)
CbN ground λ-*contexts*:	$\mathtt{N} ::= (\!\lvert\cdot\rvert\!) \mid \lambda x \mathtt{N} \mid \mathtt{N}t$	(set: $\Lambda_{\mathtt{N}}$)
CbV ground λ-*contexts*:	$\mathtt{V} ::= (\!\lvert\cdot\rvert\!) \mid \mathtt{V}t \mid t\mathtt{V}$	(set: $\Lambda_{\mathtt{V}}$)
Root-steps:	$(\lambda x t)S \mapsto_\beta t\{S/x\}$ (CbN) $\quad (\lambda x t)v \mapsto_{\beta^v} t\{v/x\}$ (CbV)	
r-*reduction*:	$t \to_{\mathsf{r}} s \Leftrightarrow \exists \mathtt{C} \in \Lambda_{\mathtt{C}}, \exists t', s' \in \Lambda : t = \mathtt{C}(\!\lvert t' \rvert\!), s = \mathtt{C}(\!\lvert s' \rvert\!), t' \mapsto_{\mathsf{r}} s'$	
β_{g}-*reduction*:	$t \to_{\beta_{\mathsf{g}}} s \Leftrightarrow \exists \mathtt{N} \in \Lambda_{\mathtt{N}}, \exists t', s' \in \Lambda : t = \mathtt{N}(\!\lvert t' \rvert\!), s = \mathtt{N}(\!\lvert s' \rvert\!), t' \mapsto_{\beta} s'$	
β^v_{g}-*reduction*:	$t \to_{\beta^v_{\mathsf{g}}} s \Leftrightarrow \exists \mathtt{V} \in \Lambda_{\mathtt{V}}, \exists t', s' \in \Lambda : t = \mathtt{V}(\!\lvert t' \rvert\!), s = \mathtt{V}(\!\lvert s' \rvert\!), t' \mapsto_{\beta^v} s'$	

Figure 3: The CbN and CbV λ-calculi: their syntax and reduction rules, where $\mathsf{r} \in \{\beta, \beta^v\}$.

- $\to_{\beta_{\mathsf{g}}}$ contains head β-reduction and weak head β-reduction, two well-known evaluation strategies for CbN λ-calculus (both reduce the β-redex in head position, the latter does not reduce under λ's);
- $\to_{\beta^v_{\mathsf{g}}}$ contains (weak) head β^v-reduction (aka left reduction in [25, p. 136]), the well-known evaluation strategy for CbV λ-calculus firing the leftmost-outermost β^v-redex (if any) not under λ's.

CbN and CbV translations into the bang calculus. *CbN* and *CbV translations* are two functions $(\cdot)^{\mathsf{n}} \colon \Lambda \to !\Lambda$ and $(\cdot)^{\mathsf{v}} \colon \Lambda \to !\Lambda$, respectively, translating λ-terms into terms of the bang calculus:

$$x^{\mathsf{n}} := x \qquad (\lambda x\, t)^{\mathsf{n}} := \lambda x\, t^{\mathsf{n}} \qquad (ts)^{\mathsf{n}} := \langle t^{\mathsf{n}} \rangle s^{\mathsf{n}!}$$
$$x^{\mathsf{v}} := x^{!} \qquad (\lambda x\, t)^{\mathsf{v}} := (\lambda x\, t^{\mathsf{v}})^{!} \qquad (ts)^{\mathsf{v}} := \langle \mathrm{der}\, t^{\mathsf{v}} \rangle s^{\mathsf{v}}.$$

Example 5. Let $\Omega := (\lambda xxx)\lambda xxx$, the typical diverging λ-term for CbN and CbV λ-calculi: one has $\Omega^{\mathsf{n}} = \langle \Delta \rangle \Delta^{!}$ and $\Omega^{\mathsf{v}} = \langle \mathrm{der}(\Delta'^{!}) \rangle \Delta'^{!}$, which are not b_{g}- nor b-normalizable (Δ and Δ' are defined in Ex. 1).

For any λ-term t, t^{n} and t^{v} are just different decorations of t by means of the monadic operators ! and der (the latter does not occur in t^{n}). Note that the translation $(\cdot)^{\mathsf{n}}$ puts the argument of any application into a box: in CbN λ-calculus any λ-term is duplicable or discardable. On the other hand, only λ-values (*i.e.* abstractions and variables) are translated by $(\cdot)^{\mathsf{v}}$ into boxes, as they are the only λ-terms duplicable or discardable in CbV λ-calculus.

Lemma 6 (Substitution)**.** *Let t, s be λ-terms and x be a variable.*

1. *One has that $t^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = (t\{s/x\})^{\mathsf{n}}$.*
2. *If s is such that $s^{\mathsf{v}} = R^{!}$ for some $R \in !\Lambda$, then $t^{\mathsf{v}}\{R/x\} = (t\{s/x\})^{\mathsf{v}}$.*

Proof. The proofs of both points are by induction on the λ-term t.

1. - *Variable:* If t is a variable then there are two subcases. If $t := x$ then $t^{\mathsf{n}} = x$, so $t^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = s^{\mathsf{n}} = (t\{s/x\})^{\mathsf{n}}$. Otherwise $t := y \neq x$ and then $t^{\mathsf{n}} = y$, hence $t^{\mathsf{n}}\{r/x\} = y = (t\{s/x\})^{\mathsf{n}}$.
   - *Abstraction:* If $t := \lambda y\, r$ then $t^{\mathsf{n}} = \lambda y\, r^{\mathsf{n}}$ and we can suppose without loss of generality that $y \notin \mathsf{fv}(s) \cup \{x\}$. By *i.h.*, $r^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = (r\{s/x\})^{\mathsf{n}}$ and hence $t^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = \lambda y (r^{\mathsf{n}}\{s^{\mathsf{n}}/x\}) = \lambda y (r\{s/x\})^{\mathsf{n}} = (t\{s/x\})^{\mathsf{n}}$.

- *Application:* If $t \coloneqq RQ$, then $t^{\mathsf{n}} = \langle r^{\mathsf{n}} \rangle q^{\mathsf{n}!}$. By *i.h.*, $r^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = (r\{s/x\})^{\mathsf{n}}$ and $q^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = (q\{s/x\})^{\mathsf{n}}$. So, $t^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = \langle r^{\mathsf{n}}\{s^{\mathsf{n}}/x\}\rangle (q^{\mathsf{n}}\{s^{\mathsf{n}}/x\})^! = \langle (r\{s/x\})^{\mathsf{n}}\rangle ((q\{s/x\})^{\mathsf{n}})^! = (t\{s/x\})^{\mathsf{n}}$.

2. - *Variable:* If t is a variable then there are two subcases. If $t \coloneqq x$ then $t^{\mathsf{v}} = x^!$, so $t^{\mathsf{v}}\{R/x\} = R^! = s^{\mathsf{v}} = (t\{s/x\})^{\mathsf{v}}$. Otherwise $t \coloneqq y \neq x$ and then $t^{\mathsf{v}} = y^!$, hence $t^{\mathsf{v}}\{R/x\} = y^! = (t\{s/x\})^{\mathsf{v}}$.
   - *Application:* If $t \coloneqq pq$ then $t^{\mathsf{v}} = \langle \mathrm{der}\, p^{\mathsf{v}} \rangle q^{\mathsf{v}}$. By *i.h.*, $p^{\mathsf{v}}\{R/x\} = (p\{s/x\})^{\mathsf{v}}$ and $q^{\mathsf{v}}\{R/x\} = (q\{s/x\})^{\mathsf{v}}$. So, $t^{\mathsf{v}}\{R/x\} = \langle \mathrm{der}(p^{\mathsf{v}}\{R/x\})\rangle q^{\mathsf{v}}\{R/x\} = \langle \mathrm{der}(p\{s/x\}^{\mathsf{v}})\rangle (q\{s/x\})^{\mathsf{v}} = (t\{s/x\})^{\mathsf{v}}$.
   - *Abstraction:* If $t \coloneqq \lambda y q$ then $t^{\mathsf{v}} = (\lambda y q^{\mathsf{v}})^!$ and we can suppose without loss of generality that $y \notin \mathrm{fv}(s) \cup \{x\}$. By *i.h.*, $q^{\mathsf{v}}\{R/x\} = (q\{s/x\})^{\mathsf{v}}$, and thus $t^{\mathsf{v}}\{R/x\} = (\lambda y (q^{\mathsf{v}}\{R/x\}))^! = (\lambda y (q\{s/x\})^{\mathsf{v}})^! = (t\{s/x\})^{\mathsf{v}}$. □

Note that the hypothesis about s in Lemma 6.2 is fulfilled if and only if s is a λ-value.

Remark 7 (CbV translation is ℓ-normal). It is immediate to prove by induction on $t \in \Lambda$ that t^{v} is ℓ-normal, so if $t^{\mathsf{v}} \to_{\mathsf{d}} S_0 \to_{\ell} S$ then the only ℓ-redex in S_0 has been created by the step $t^{\mathsf{v}} \to_{\mathsf{d}} S_0$ and is absent in t^{v}.

Simulating CbN and CbV reductions into the bang calculus. We can now show that the CbN translation $(\cdot)^{\mathsf{n}}$ (resp. CbV translation $(\cdot)^{\mathsf{v}}$) from the CbN (resp. CbV) λ-calculus into the bang calculus is *sound* and *complete*: said differently, the target of the CbN (resp. CbV) translation into the bang calculus is a *conservative* extension of the CbN (resp. CbV) λ-calculus.

Theorem 8 (Simulation of CbN and CbV λ-calculi)**.** *Let t be a λ-term.*

1. Conservative extension of CbN λ-calculus:
   Soundness: *If $t \to_{\beta} t'$ then $t^{\mathsf{n}} \to_{\ell} t'^{\mathsf{n}}$ (and $t^{\mathsf{n}} \to_{\mathsf{b}} t'^{\mathsf{n}}$);*
   Completeness: *Conversely, if $t^{\mathsf{n}} \to_{\mathsf{b}} S$ then $t^{\mathsf{n}} \to_{\ell} S = t'^{\mathsf{n}}$ and $t \to_{\beta} t'$ for some λ-term t'.*
2. Conservative extension of ground CbN λ-calculus:
   Soundness: *If $t \to_{\beta_{\mathsf{g}}} t'$ then $t^{\mathsf{n}} \to_{\ell_{\mathsf{g}}} t'^{\mathsf{n}}$ (and $t^{\mathsf{n}} \to_{\mathsf{b}_{\mathsf{g}}} t'^{\mathsf{n}}$);*
   Completeness: *Conversely, if $t^{\mathsf{n}} \to_{\mathsf{b}_{\mathsf{g}}} S$ then $t^{\mathsf{n}} \to_{\ell_{\mathsf{g}}} S = t'^{\mathsf{n}}$ and $t \to_{\beta_{\mathsf{g}}} t'$ for some λ-term t'.*
3. Conservative extension of CbV λ-calculus:
   Soundness: *If $t \to_{\beta^v} t'$ then $t^{\mathsf{v}} \to_{\mathsf{d}} \to_{\ell} t'^{\mathsf{v}}$ (and hence $t^{\mathsf{v}} \to_{\mathsf{b}} \to_{\mathsf{b}} t'^{\mathsf{v}}$);*
   Completeness: *Conversely, if $t^{\mathsf{v}} \to_{\mathsf{d}} \to_{\ell} S$ then $S = t'^{\mathsf{v}}$ and $t \to_{\beta^v} t'$ for some λ-term t'.*
4. Conservative extension of ground CbV λ-calculus:
   Soundness: *If $t \to_{\beta_{\mathsf{g}}^v} t'$ then $t^{\mathsf{v}} \to_{\mathsf{d}_{\mathsf{g}}} \to_{\ell_{\mathsf{g}}} t'^{\mathsf{v}}$ (and hence $t^{\mathsf{v}} \to_{\mathsf{b}_{\mathsf{g}}} \to_{\mathsf{b}_{\mathsf{g}}} t'^{\mathsf{v}}$);*
   Completeness: *Conversely, if $t^{\mathsf{v}} \to_{\mathsf{d}_{\mathsf{g}}} \to_{\ell_{\mathsf{g}}} S$ then $S = t'^{\mathsf{v}}$ and $t \to_{\beta_{\mathsf{g}}^v} t'$ for some λ-term t'.*

Proof. 1. *Soundness:* We prove by induction on the λ-term t that if $t \to_{\beta} t'$ then $t^{\mathsf{n}} \to_{\ell} t'^{\mathsf{n}}$. By definition of $t \to_{\beta} t'$, there are the following cases:

- *Root-step, i.e.* $t \coloneqq (\lambda x r)s \mapsto_{\beta} R\{s/x\} =: t'$: by Lemma 6.1, $t^{\mathsf{n}} = \langle \lambda x r^{\mathsf{n}} \rangle s^{\mathsf{n}!} \mapsto_{\ell} r^{\mathsf{n}}\{s^{\mathsf{n}}/x\} = t'^{\mathsf{n}}$.
- *Abstraction, i.e.* $t \coloneqq \lambda x r \to_{\beta} \lambda x r' =: t'$ with $r \to_{\beta} r'$: by *i.h.*, $r^{\mathsf{n}} \to_{\ell} r'^{\mathsf{n}}$, thus $t^{\mathsf{n}} = \lambda x r^{\mathsf{n}} \to_{\ell} \lambda x r'^{\mathsf{n}} = t'^{\mathsf{n}}$.
- *Application left, i.e.* $t \coloneqq rs \to_{\beta} r's =: t'$ with $r \to_{\beta} r'$: by *i.h.* $r^{\mathsf{n}} \to_{\ell} r'^{\mathsf{n}}$, so $t^{\mathsf{n}} = \langle r^{\mathsf{n}} \rangle s^{\mathsf{n}!} \to_{\ell} \langle r'^{\mathsf{n}} \rangle s^{\mathsf{n}!} = t'^{\mathsf{n}}$.
- *Application right, i.e.* $t \coloneqq sr \to_{\beta} sr' =: t'$ with $r \to_{\beta} r'$: analogous to the previous case.

Completeness: First, observe that $t^{\mathsf{n}} \to_{\mathsf{b}} s$ entails $t^{\mathsf{n}} \to_{\ell} s$ since der does not occur in t^{n}, hence t^{n} is d-normal. We prove by induction on the λ-term t that if $t^{\mathsf{n}} \to_{\ell} S$ then $S = t'^{\mathsf{n}}$ and $t \to_{\beta} t'$ for some λ-term t'. According to the definition of $t^{\mathsf{n}} \to_{\ell} S$, there are the following cases:

- *Root-step*, *i.e.* $t^{\mathsf{n}} := \langle \lambda x r^{\mathsf{n}} \rangle q^{\mathsf{n}!} \mapsto_\ell r^{\mathsf{n}}\{q^{\mathsf{n}}/x\} =: S$: by Lemma 6.1 $S = (r\{q/x\})^{\mathsf{n}}$, so $t = (\lambda x r)q \mapsto_\beta r\{q/x\} =: t'$ where $t'^{\mathsf{v}} = S$.
- *Abstraction*, *i.e.* $t^{\mathsf{n}} := \lambda x r^{\mathsf{n}} \to_\ell \lambda x S' =: S$ with $r^{\mathsf{n}} \to_\ell S'$: by *i.h.*, there is a λ-term r' such that $r'^{\mathsf{n}} = S'$ and $r \to_\beta r'$, thus $t = \lambda x r \to_\beta \lambda x r' =: t'$ where $t'^{\mathsf{n}} = \lambda x r'^{\mathsf{n}} = S$.
- *Application left*, *i.e.* $t^{\mathsf{n}} := \langle r^{\mathsf{n}} \rangle q^{\mathsf{n}!} \to_\ell \langle S' \rangle q^{\mathsf{n}!} =: S$ with $r^{\mathsf{n}} \to_\ell S'$: analogously to above.
- *Application right*, *i.e.* $t^{\mathsf{n}} := \langle q^{\mathsf{n}} \rangle r^{\mathsf{n}!} \to_\ell \langle q^{\mathsf{n}} \rangle S'^{!} =: S$ with $r^{\mathsf{n}} \to_\ell S'$: by *i.h.*, there is a λ-term r' such that $r'^{\mathsf{n}} = S'$ and $r \to_\beta r'$, so $t = qr \to_\beta qr' =: t'$ with $t'^{\mathsf{n}} = \langle q^{\mathsf{n}} \rangle r'^{\mathsf{n}!} = S$.

2. Since $\mapsto_\beta$ is simulated by $\mapsto_\ell$ and vice-versa (see the root-cases above), Thm. 8.2 is proved analogously to the proof of Thm. 8.1 ($\to_{\ell_{\mathsf{g}}}$ replaces $\to_\ell$, and $\to_{\beta_{\mathsf{g}}}$ replaces $\to_\beta$), with the difference that, by definition, $\to_{\beta_{\mathsf{g}}}$ and $\to_{\ell_{\mathsf{g}}}$ do not give rise to the case *Application right*.

3. *Soundness:* We prove by induction on the λ-term t that if $t \to_{\beta^v} t'$ then $t^{\mathsf{v}} \to_{\mathsf{d}} \to_\ell t'^{\mathsf{v}}$. According to the definition of $T \to_{\beta^v} T'$, there are the following cases:

   - *Root-step*, *i.e.* $t := (\lambda x r)v \mapsto_{\beta^v} r\{v/x\} =: t'$ where v is a λ-value, *i.e.* a variable or an abstraction: then $v^{\mathsf{v}} = S^!$ for some $S \in !\Lambda$, hence $t^{\mathsf{v}} = \langle \mathrm{der}\,(\lambda x r^{\mathsf{v}})^! \rangle v^{\mathsf{v}} \to_{\mathsf{d}_{\mathsf{g}}} \langle \lambda x r^{\mathsf{v}} \rangle v^{\mathsf{v}} \mapsto_\ell r^{\mathsf{v}}\{S/x\} = t'^{\mathsf{v}}$ by Lemma 6.2 (recall that $\to_{\mathsf{d}_{\mathsf{g}}} \subseteq \to_{\mathsf{d}}$).
   - *Abstraction*, *i.e.* $t := \lambda x r \to_{\beta^v} \lambda x r' =: t'$ with $r \to_{\beta^v} r'$: by *i.h.*, $r^{\mathsf{v}} \to_{\mathsf{d}} \to_\ell r'^{\mathsf{v}}$; therefore $t^{\mathsf{v}} = (\lambda x r^{\mathsf{v}})^! \to_{\mathsf{d}} \to_\ell (\lambda x r'^{\mathsf{v}})^! = t'^{\mathsf{v}}$.
   - *Application left*, *i.e.* $t := rs \to_{\beta^v} r's =: t'$ with $r \to_{\beta^v} r'$: by *i.h.* $r^{\mathsf{v}} \to_{\mathsf{d}} \to_\ell r'^{\mathsf{v}}$; therefore $t^{\mathsf{v}} = \langle \mathrm{der}\, r^{\mathsf{v}} \rangle s^{\mathsf{v}} \to_{\mathsf{d}} \to_\ell \langle \mathrm{der}\, r'^{\mathsf{v}} \rangle s^{\mathsf{v}} = T'^{\mathsf{v}}$.
   - *Application right*, *i.e.* $t := sr \to_{\beta^v} sr' =: t'$ with $r \to_{\beta^v} r'$: analogous to the previous case.

   Completeness: We prove by induction on $S_0 \in !\Lambda$ that if $t^{\mathsf{v}} \to_{\mathsf{d}} S_0 \to_\ell S$ then $S = t'^{\mathsf{v}}$ and $t \to_{\beta^v} t'$ for some λ-term t'. According to the definition of $S_0 \to_\ell S$, there are the following cases:

   - *Root-step*, *i.e.* $S_0 := \langle \lambda x R \rangle Q^! \mapsto_\ell R\{Q/x\} =: S$. According to Rmk. 7, necessarily $t^{\mathsf{v}} = \langle \mathrm{der}\,(\lambda x R)^! \rangle Q^!$ and hence $t = (\lambda x r_0)v$ for some λ-term r_0 and some λ-value v such that $r_0^{\mathsf{v}} = R$ and $v^{\mathsf{v}} = Q^!$. Notice that $t^{\mathsf{v}} \to_{\mathsf{d}_{\mathsf{g}}} S_0$. Let $t' := r_0\{v/x\}$: then, $t \mapsto_{\beta^v} t'$ and $t'^{\mathsf{v}} = r_0^{\mathsf{v}}\{Q/x\} = S$ according to Lemma 6.2.
   - *Abstraction*, *i.e.* $S_0 := \lambda x R_0 \to_\ell \lambda x R' =: S$ with $R_0 \to_\ell R'$. This case is impossible because, according to Rmk. 7, necessarily $t^{\mathsf{v}} = \lambda x R$ for some ℓ-normal $R \in !\Lambda$ such that $R \to_{\mathsf{d}} R_0$, but there is no λ-term t such that t^{v} is an abstraction.
   - *Dereliction*, *i.e.* $S_0 := \mathrm{der}\, R_0 \to_\ell \mathrm{der}\, R' =: S$ with $R_0 \to_\ell R'$. This case is impossible because, according to Rmk. 7, necessarily $t^{\mathsf{v}} = \mathrm{der}\, R$ for some ℓ-normal $R \in !\Lambda$ such that $R \to_{\mathsf{d}} R_0$, but there is no λ-term t such that t^{v} is a dereliction.
   - *Application left*, *i.e.* $S_0 := \langle R_0 \rangle R_1 \to_\ell \langle R' \rangle R_1 =: S$ with $R_0 \to_\ell R'$. By Rmk. 7, $t^{\mathsf{v}} = \langle \mathrm{der}\, P \rangle R_1$ for some ℓ-normal $P \in !\Lambda$ such that $\mathrm{der}\, P \to_{\mathsf{d}} R_0$, and thus $R_0 = \mathrm{der}\, P_0$ where $P \to_{\mathsf{d}} P_0$ (indeed $P = R_0^!$ is impossible because P is ℓ-normal), and hence $R' = \mathrm{der}\, P'$ with $P_0 \to_\ell P'$. So, $t = qq_1$ for some λ-terms q and q_1 such that $q^{\mathsf{v}} = P$ and $q_1^{\mathsf{v}} = R_1$ with $q^{\mathsf{v}} \to_{\mathsf{d}} P_0 \to_\ell P'$. By *i.h.*, $P' = q'^{\mathsf{v}}$ and $q \to_{\beta^v} q'$ for some λ-term q'. Let $t' := q'q_1$: then, $t = qq_1 \to_{\beta^v} t'$ and $t'^{\mathsf{v}} = \langle \mathrm{der}\, q'^{\mathsf{v}} \rangle q_1^{\mathsf{v}} = S$.
   - *Application right*, *i.e.* $S_0 := \langle R_1 \rangle R_0 \to_\ell \langle R_1 \rangle R' =: S$ with $R_0 \to_\ell R'$. By Rmk. 7, necessarily $t^{\mathsf{v}} = \langle \mathrm{der}\, P_1 \rangle R$ for some ℓ-normal $R, P_1 \in !\Lambda$ such that $\mathrm{der}\, P_1 = R_1$ and $R \to_{\mathsf{d}} R_0$. So, $t = q_1 Q$ for some λ-terms q_1 and q such that $q_1^{\mathsf{v}} = P_1$ and $q^{\mathsf{v}} = R$ with $q^{\mathsf{v}} \to_{\mathsf{d}} R_0 \to_\ell R'$. By *i.h.*, $R' = q'^{\mathsf{v}}$ and $q \to_{\beta^v} q'$ for some λ-term q'. Let $t' := q_1 q'$: then, $t = q_1 q \to_{\beta^v} t'$ and $t'^{\mathsf{v}} = \langle \mathrm{der}\, q_1^{\mathsf{v}} \rangle q'^{\mathsf{v}} = S$.

$$\begin{array}{lllll} \textit{target of CbN translation into } !\Lambda: & T,S ::= x \mid \langle T\rangle S^{!} \mid \lambda x T & \text{(set: } !\Lambda^{\mathsf{n}}\text{)} & & \\ \textit{target of CbV translation into } !\Lambda: & M,N ::= U^{!} \mid \langle \mathrm{der} M\rangle N \mid \langle U\rangle M & \text{(set: } !\Lambda^{\mathsf{v}}\text{)} & U ::= x \mid \lambda x M & \text{(set: } !\Lambda^{\mathsf{v}}_{v}\text{)}. \end{array}$$

Figure 4: Targets of CbN and CbV translations into the bang calculus.

- *Box*, *i.e.* $S_0 := R_0^{!} \to_{\ell} R'^{!} =: S$ with $R_0 \to_{\ell} R'$. According to Rmk. 7, necessarily $t^{\mathsf{v}} = R^{!}$ for some ℓ-normal $R \in !\Lambda$ such that $R \to_{\mathsf{d}} R_0$. So, $t = \lambda x q$ (since t^{v} is a box and x^{v} is d-normal) for some λ-term q such that $R = \lambda x q^{\mathsf{v}}$, and hence there are $P_0, P' \in !\Lambda$ such that $R_0 = \lambda x P_0$ and $R' = \lambda x P'$ with $q^{\mathsf{v}} \to_{\mathsf{d}} P_0 \to_{\ell} P'$. By *i.h.*, $P' = q'^{\mathsf{v}}$ and $q \to_{\beta^v} q'$ for some λ-term q'. Let $t' := \lambda x q'$: then, $t = \lambda x q \to_{\beta^v} t'$ and $t'^{\mathsf{v}} = (\lambda x q'^{\mathsf{v}})^{!} = S$.

4. Since $\mapsto_{\beta^v}$ is simulated by $\to_{\mathsf{d}_{\mathsf{g}}}\mapsto_{\ell}$ and vice-versa (see the root-cases above), Thm. 8.4 is proved analogously to the proof of Thm. 8.3 (replace $\to_{\ell}$ with $\to_{\ell_{\mathsf{g}}}$, and $\to_{\mathsf{d}}$ with $\to_{\mathsf{d}_{\mathsf{g}}}$, as well as $\to_{\beta^v}$ with $\to_{\beta^v_{\mathsf{g}}}$), with the difference that $\to_{\beta^v_{\mathsf{g}}}$ does not give rise to the case *Abstraction* (in the soundness proof) and *Box* (in the completeness proof). □

So, the bang calculus can simulate β- and β^v-reductions via $(\cdot)^{\mathsf{n}}$ and $(\cdot)^{\mathsf{v}}$ and, conversely, ℓ-reductions in the targets of $(\cdot)^{\mathsf{n}}$ and $(\cdot)^{\mathsf{v}}$ correspond to β- and β^v-reductions. Also, these simulations are:

- *modular*, in the sense that *ground* β-reduction (including head β-reduction and weak head β-reduction) is simulated by *ground* ℓ-reduction, and vice-versa (Thm. 8.2); *ground* β^v-reduction (including head β^v-reduction) is simulated by *ground* d- and ℓ-reductions, and vice-versa (Thm. 8.4);
- *quantitative sensitive*, meaning that *one step* of (ground) β-reduction corresponds exactly, via $(\cdot)^{\mathsf{n}}$, to *one step* of (ground) ℓ-reduction, and vice-versa; *one step* of (ground) β^v-reduction corresponds exactly, via $(\cdot)^{\mathsf{v}}$, to *one step* of (ground) ℓ-reduction, and vice-versa.

According to the definition of CbN translation $(\cdot)^{\mathsf{n}}$, the target of $(\cdot)^{\mathsf{n}}$ into the bang calculus can be characterized syntactically: it is the subset $!\Lambda^{\mathsf{n}}$ of $!\Lambda$ defined in Fig. 4. This means that $t^{\mathsf{n}} \in !\Lambda^{\mathsf{n}}$ for any $t \in \Lambda$, and conversely, for any $T \in !\Lambda^{\mathsf{n}}$, $T^{\mathsf{n}} = t$ for some $t \in \Lambda$. Note that in $!\Lambda^{\mathsf{n}}$ the constructor der does not occur and hence reductions $\to_{\ell}$ and $\to_{\ell_{\mathsf{g}}}$ coincide with $\to_{\mathsf{b}}$ and $\to_{\mathsf{b}_{\mathsf{g}}}$, respectively, in $!\Lambda^{\mathsf{n}}$.

So, Thm. 8.1-2 says that $!\Lambda^{\mathsf{n}}$ endowed with the reduction $\to_{\ell}$ (resp. $\to_{\ell_{\mathsf{g}}}$) — which coincides with $\to_{\mathsf{b}}$ (resp. $\to_{\mathsf{b}_{\mathsf{g}}}$) in $!\Lambda^{\mathsf{n}}$ — is *isomorphic* to CbN (resp. ground CbN) λ-calculus. In particular:

Corollary 9 (Preservations with respect to CbN λ-calculus)**.** *Let $t, s \in \Lambda$.*

1. CbN equational theory: *$t \simeq_{\beta} s$ iff $t^{\mathsf{n}} \simeq_{\ell} s^{\mathsf{n}}$ iff $t^{\mathsf{n}} \simeq_{\mathsf{b}} s^{\mathsf{n}}$.*
2. CbN normal forms: *t is (ground) β-normal iff t^{n} is (ground) ℓ-normal iff t^{n} is (ground)* b*-normal.*

Proof. 1. If $t \simeq_{\beta} s$ then $t \to_{\beta}^{*} r {}^{*}_{\beta}\!\leftarrow s$ for some $r \in \Lambda$, as $\to_{\beta}$ is confluent. By Thm. 8.1 (soundness), $t^{\mathsf{n}} \to_{\ell}^{*} r^{\mathsf{n}} {}^{*}_{\ell}\!\leftarrow s^{\mathsf{n}}$; so, $t^{\mathsf{n}} \simeq_{\ell} s^{\mathsf{n}}$ and $t^{\mathsf{n}} \simeq_{\mathsf{b}} S^{\mathsf{n}}$ since $\to_{\ell} \subseteq \to_{\mathsf{b}}$.

Conversely, if $t^{\mathsf{n}} \simeq_{\mathsf{b}} s^{\mathsf{n}}$ then $t^{\mathsf{n}} \to_{\mathsf{b}}^{*} R {}^{*}_{\mathsf{b}}\!\leftarrow s^{\mathsf{n}}$ for some $R \in !\Lambda$, since $\to_{\mathsf{b}}$ is confluent (Prop. 4.2). By Thm. 8.1 (completeness), $t \to_{\beta}^{*} q {}^{*}_{\beta}\!\leftarrow s$ for some λ-term q such that $q^{\mathsf{n}} = R$, and therefore $t \simeq_{\beta} s$.

2. Immediate consequence of Thm. 8.1-2. □

The correspondence between CbV λ-calculus and bang calculus is slightly more delicate: CbV translation $(\cdot)^{\mathsf{v}}$ gives a sound and complete embedding of $\to_{\beta^v}$ into $\to_{\mathsf{d}}\to_{\ell}$ (and similarly for their ground variants), but it is not complete with respect to generic $\to_{\mathsf{b}}$. Indeed, Ex. 1 and Ex. 5 have shown that

$(\lambda xxx)^{\mathsf{v}} = \Delta'^{!} \to_{\mathsf{d}} \Delta^{!}$, where $\Delta^{!}$ is b-normal and there is no λ-term t such that $t^{\mathsf{v}} = \Delta^{!}$. Note that λxxx is β^{v}-normal but $(\lambda xxx)^{\mathsf{v}} = \Delta'^{!}$ is not b-normal: in CbV the analogous of Cor. 9.2 does not hold for $(\cdot)^{\mathsf{v}}$.

Actually, an analogous of Cor. 9.1 for CbV holds: CbV translation preserves β^{v}-equivalence in a sound and complete way with respect to b-equivalence (see Cor. 12 below). The proof requires a fine analysis of CbV translation $(\cdot)^{\mathsf{v}}$. First, we define two subsets $!\Lambda^{\mathsf{v}}$ and $!\Lambda^{\mathsf{v}}_{v}$ of $!\Lambda$, see Fig. 4.

Remark 10 (Image of CbV translation). If $t \in \Lambda$ then $t^{\mathsf{v}} \in !\Lambda^{\mathsf{v}}$ (proof by induction on $t \in \Lambda$). In particular, for any $v \in \Lambda_{v}$, $v^{\mathsf{v}} = U^{!}$ for some $U \in !\Lambda^{\mathsf{v}}_{v}$. Note that $\Delta^{!} \in !\Lambda^{\mathsf{v}}$ but there is no λ-term t such that $t^{\mathsf{v}} = \Delta^{!}$.

We have just shown that $(\cdot)^{\mathsf{v}}$ is not surjective in $!\Lambda^{\mathsf{v}}$. Anyway, it can be shown (Lemma 11.3) that $!\Lambda^{\mathsf{v}}$ is the set of terms in $!\Lambda$ reachable by b-reduction from CbV translations of λ-terms (*i.e.* for any $t \in \Lambda$, if $t^{\mathsf{v}} \to_{\mathsf{b}}^{*} S$ then $S \in !\Lambda^{\mathsf{v}}$) and b-equivalence in $!\Lambda^{\mathsf{v}}$ preserves β_{v}-equivalence (Cor. 12). To prove that, we define a forgetful translation $(\cdot)^{\dagger} : !\Lambda^{\mathsf{v}} \cup !\Lambda^{\mathsf{v}}_{v} \to \Lambda$ transforming terms $M \in !\Lambda^{\mathsf{v}}$ and $U \in !\Lambda^{\mathsf{v}}_{v}$ into λ-terms:

$$(U^{!})^{\dagger} := U^{\dagger} \quad (\langle \mathrm{der} M \rangle N)^{\dagger} := M^{\dagger} N^{\dagger} \quad (\langle U \rangle M)^{\dagger} := U^{\dagger} M^{\dagger}; \qquad x^{\dagger} := x \quad (\lambda x M)^{\dagger} := \lambda x M^{\dagger}.$$

Proof p. 20 **Lemma 11** (Properties of the forgetful translation $(\cdot)^{\dagger}$)**.**

1. $(\cdot)^{\dagger}$ is a right-inverse of $(\cdot)^{\mathsf{v}}$: *For every* $t \in \Lambda$*, one has* $t^{\mathsf{v}\dagger} = t$.
2. Substitution: $M\{U/x\} \in !\Lambda^{\mathsf{v}}$ *with* $(M\{U/x\})^{\dagger} = M^{\dagger}\{U^{\dagger}/x\}$*, for any* $M \in !\Lambda^{\mathsf{v}}$ *and* $U \in !\Lambda^{\mathsf{v}}_{v}$.
3. b-reduction vs. β^{v}-reduction: *For any* $M \in !\Lambda^{\mathsf{v}}$ *and* $T \in !\Lambda$*, if* $M \to_{\mathsf{b}} T$ *then* $T \in !\Lambda^{\mathsf{v}}$ *and* $M^{\dagger} \to_{\beta^{v}}^{=} T^{\dagger}$.

Rmk. 10 and Lemma 11.3 mean that $!\Lambda^{\mathsf{v}}$ is the set of terms in $!\Lambda$ reachable by b-reduction from CbV translations of λ-terms (*i.e.* for any $t \in \Lambda$, if $t^{\mathsf{v}} \to_{\mathsf{b}}^{*} S$ then $S \in !\Lambda^{\mathsf{v}}$). We can conclude:

Corollary 12 (Preservation of CbV equational theory)**.** *Let* $t, s \in \Lambda$*. One has* $t \simeq_{\beta^{v}} s$ *iff* $t^{\mathsf{v}} \simeq_{\mathsf{b}} s^{\mathsf{v}}$.

Proof. If $t \simeq_{\beta^{v}} s$ then $t \to_{\beta^{v}}^{*} r \,{}^{*}_{\beta^{v}}\!\leftarrow s$ for some $r \in \Lambda$, as $\to_{\beta^{v}}$ is confluent; by Thm. 8.3, $t^{\mathsf{v}} \to_{\mathsf{b}}^{*} r^{\mathsf{v}} \,{}^{*}_{\mathsf{b}}\!\leftarrow s^{\mathsf{v}}$, since $\to_{\ell} \subseteq \to_{\mathsf{b}}$ and $\to_{\mathsf{d}} \subseteq \to_{\mathsf{b}}$; therefore, $t^{\mathsf{v}} \simeq_{\mathsf{b}} s^{\mathsf{v}}$. Conversely, if $t^{\mathsf{v}} \simeq_{\mathsf{b}} s^{\mathsf{v}}$ then $t^{\mathsf{v}} \to_{\mathsf{b}}^{*} R \,{}^{*}_{\mathsf{b}}\!\leftarrow s^{\mathsf{v}}$ for some $R \in !\Lambda$, since $\to_{\mathsf{b}}$ is confluent (Prop. 4.2); by Rmk. 10, $t^{\mathsf{v}}, s^{\mathsf{v}} \in !\Lambda^{\mathsf{v}}$; so, $R \in !\Lambda^{\mathsf{v}}$ and $T^{\mathsf{v}\dagger} \to_{\beta^{v}}^{*} R^{\dagger} \,{}^{*}_{\beta^{v}}\!\leftarrow s^{\mathsf{v}\dagger}$ by Lemma 11.3, hence $t = t^{\mathsf{v}\dagger} \simeq_{\beta^{v}} s^{\mathsf{v}\dagger} = s$ by Lemma 11.1. □

So, Cor. 12 says that CbV translation $(\cdot)^{\mathsf{v}}$ — even if it is a sound but not complete embedding of β^{v}-reduction into b-reduction — is a sound and complete embedding of β^{v}-equivalence into b-equivalence.

A final remark on the good rewriting properties of the bang calculus: the embeddings of CbN and CbV λ-calculi into the bang calculus are *finer* than the ones into the linear calculus λ_{lin} introduced in [20]. For instance, in λ_{lin} it is impossible to define a fragment isomorphic to CbN λ-calculus; also, the CbV translation of the λ-calculus into λ_{lin} is sound but not complete, and equates more than β^{v}-equivalence. Moreover, the bang calculus can be modularly extended with other reduction rules and/or syntactic constructs so that our CbV translation embeds the extensions of CbV λ-calculus studied in [3] into the corresponding extended version of the bang calculus, with results analogous to those presented here.

4 The bang calculus with respect to CbN and CbV λ-calculi, semantically

The denotational models of the bang calculus we are interested in this paper are those induced by a denotational model of LL. We recall the basic definitions and notations, see [21, 13, 14] for more details.

Linear logic based denotational semantics of bang calculus. A denotational model of LL is given by:

- A $*$-autonomous category, namely a symmetric monoidal closed category $(\mathscr{L},\otimes,1,\lambda,\rho,\alpha,\sigma)$ with a dualizing object $\bot$. We use $X\multimap Y$ for the linear exponential object, $\mathsf{ev}\in\mathscr{L}((X\multimap Y)\otimes X,Y)$ for the evaluation morphism and cur for the linear curryfication map $\mathscr{L}(Z\otimes X,Y)\to\mathscr{L}(Z,X\multimap Y)$. We use $X^\bot$ for the object $X\multimap\bot$ of $\mathscr{L}$ (the *linear negation* of X). This operation $(\_)^\bot$ is a functor $\mathscr{L}^{\mathsf{op}}\to\mathscr{L}$. The category $\mathscr{L}$ is cartesian with terminal object $\top$, product &, projections pr_i. By $*$-autonomy, $\mathscr{L}$ is cocartesian with initial object 0, coproduct $\oplus$ and injections in_i.
- A functor $!\_\colon\mathscr{L}\to\mathscr{L}$ which is:
  - a comonad with counit $\mathsf{der}_X\in\mathscr{L}(!X,X)$ (*dereliction*) and comultiplication $\mathsf{dig}_X\in\mathscr{L}(!X,!!X)$ (*digging*), and
  - a strong symmetric monoidal functor—with Seely isos $\mathsf{m}^0\in\mathscr{L}(1,!\top)$ and $\mathsf{m}^2_{X,Y}\in\mathscr{L}(!X\otimes!Y,!(X\,\&\,Y))$—from the symmetric monoidal category $(\mathscr{L},\&,\top)$ to the symmetric monoidal category $(\mathscr{L},\otimes,1)$, satisfying an additional coherence condition with respect to dig.

In order that $\mathscr{L}$ is also a denotational model of the bang calculus we need a further assumption:

$$\text{we assume that } \textit{the unique morphism in } \mathscr{L}(0,\top) \textit{ is an iso} \ \text{ (to simplify, assume just } 0=\top). \tag{1}$$

From (1) it follows that for any two objects X and Y there is a morphism $0_{X,Y}:=\mathsf{i}\,\mathsf{t}\in\mathscr{L}(X,Y)$ where t is the unique morphism $X\to\top$ and i is the unique morphism $0\to Y$. It turns out that this specific zero morphism satisfies the identities $f\,0_{X,Y}=0_{X,Z}=0_{Y,Z}g$ for all $f\in\mathscr{L}(Y,Z)$ and $g\in\mathscr{L}(X,Y)$. Assumption (1) is satisfied by many models of LL, like relational model [6], finiteness spaces [10], Scott model [12], (hyper-)coherence [15, 9] and probabilistic coherence spaces [8], all models based on Indexed LL [6].

A model of the bang calculus is any object $\mathscr{U}$ of $\mathscr{L}$ satisfying the identity $\mathscr{U}\cong!\mathscr{U}\,\&\,(!\mathscr{U}\multimap\mathscr{U})$ (we assume this iso to be an equality). Note that this entails both $!\mathscr{U}\lhd\mathscr{U}$ and $!\mathscr{U}\multimap\mathscr{U}\lhd\mathscr{U}$.

Given a term T and a repetition-free list of variables $\vec{x}=(x_1,\dots,x_n)$ which contains all the free variables of T, we can define a morphism $[\![T]\!]_{\vec{x}}\in\mathscr{L}((!\mathscr{U})^{\otimes n},\mathscr{U})$, the *denotational semantics* (or *interpretation*) of T (where $(!\mathscr{U})^{\otimes k}:=\bigotimes_{i=1}^k!\mathscr{U}$). The definition is by induction on $T\in!\Lambda$:

- $[\![x_i]\!]_{\vec{x}}:=\mathsf{w}_{\mathscr{U}}^{\otimes i-1}\otimes\ \mathsf{der}_{\mathscr{U}}\otimes\mathsf{w}_{\mathscr{U}}^{\otimes n-i}$ where $\mathsf{w}_{\mathscr{U}}\in\mathscr{L}(!\mathscr{U},1)$ is the weakening and we keep implicit the monoidality isos $1\otimes\mathscr{U}\simeq\mathscr{U}$,
- $[\![\lambda y\,S]\!]_{\vec{x}}:=\langle 0_{(!\mathscr{U})^{\otimes n},!\mathscr{U}},\mathsf{cur}\,([\![S]\!]_{\vec{x},y})\rangle$, where we assume without loss of generality $y\notin\{x_1,\dots,x_n\}$,
- $[\![\langle S\rangle R]\!]_{\vec{x}}:=\mathsf{ev}\,(\mathsf{pr}_2\,[\![S]\!]_{\vec{x}}\otimes\mathsf{pr}_1\,[\![R]\!]_{\vec{x}})\,\mathsf{c}$, where $\mathsf{c}\in\mathscr{L}((!\mathscr{U})^{\otimes n},(!\mathscr{U})^{\otimes n}\otimes(!\mathscr{U})^{\otimes n})$ is the contraction,
- $[\![S^!]\!]_{\vec{x}}:=\langle([\![S]\!]_{\vec{x}})^!,0_{(!\mathscr{U})^{\otimes n},!\mathscr{U}\multimap\mathscr{U}}\rangle$, for $([\![S]\!]_{\vec{x}})^!=!([\![S]\!]_{\vec{x}})\,\mathsf{h}$ where $\mathsf{h}\in\mathscr{L}((!\mathscr{U})^{\otimes n},!((!\mathscr{U})^{\otimes n}))$ is the coalgebra structure map of $(!\mathscr{U})^{\otimes n}$ (see [13]),
- $[\![\mathsf{der}\,S]\!]_{\vec{x}}:=\mathsf{der}_{\mathscr{U}}\ \mathsf{pr}_1\,[\![S]\!]_{\vec{x}}$.

Theorem 13 (Invariance, [14])**.** *Let $T,S\in!\Lambda$ and $\vec{x}$ be a repetition-free list of variables which contains all free variables of T and S. If $T\simeq_{\mathsf{b}}S$ then $[\![T]\!]_{\vec{x}}=[\![S]\!]_{\vec{x}}$.*

The proof of Thm. 13 uses crucially the fact that $[\![R^!]\!]_{\vec{x}}$ is a coalgebra morphism, see [13].

The general notion of denotational model for the bang calculus presented here and obtained from *any* denotational model $\mathscr{L}$ of LL satisfying the assumption (1) above is a particular case of Moggi's semantics of computations based on monads [22, 23], if one keeps in mind that the functor "!" defines a strong monad on the Kleisli category $\mathscr{L}_!$ of $\mathscr{L}$.

Call-by-name. A model of the CbN λ-calculus is a reflexive object in a cartesian closed category. The category $\mathscr{L}$ being $*$-autonomous, its Kleisli $\mathscr{L}_!$ over the comonad $(!,\mathsf{dig},\mathsf{der})$ is cartesian closed. The category $\mathscr{L}_!$ (whose objects are the same as $\mathscr{L}$ and morphisms are given by $\mathscr{L}_!(A,B)=\mathscr{L}(!A,B)$) has composition $f\circ g$ defined as $f\,!g\,\mathsf{dig}$ and identities A given by der_A. In $\mathscr{L}_!$, products $A\,\&\,B$ are preserved, with projections $\pi_i := \mathsf{pr}_i\,\mathsf{der}_{!(A\&B)}$ ($i\in\{1,2\}$); the exponential object $A\Rightarrow B$ is $!A\multimap B$ (this is the semantic counterpart of Girard's CbN translation) and has an evaluation morphism $\mathsf{Ev}=\mathsf{ev}(\mathsf{der}_{!A\multimap B}\otimes\mathsf{id}_{!A})(\mathsf{m}^2)^{-1}\in\mathscr{L}(!((!A\multimap B)\,\&\,!A),B)$. This defines an exponentiation since for all $f\in\mathscr{L}(!(C\,\&\,A),B)$ there is a unique morphism $\Lambda(f)=\mathsf{cur}(f\,\mathsf{m}^2)\in\mathscr{L}(!C,!A\multimap B)$ satisfying $\mathsf{Ev}\circ\langle\Lambda(f),A\rangle=f$.

The identity $\mathscr{U}=!\mathscr{U}\;\&\;(!\mathscr{U}\multimap\mathscr{U})$ entails $!\mathscr{U}\multimap\mathscr{U}\lhd\mathscr{U}$ in $\mathscr{L}$ via $\mathsf{lam}:=\langle 0_{!\mathscr{U}\multimap\mathscr{U},!\mathscr{U}},\mathsf{id}_{!\mathscr{U}\multimap\mathscr{U}}\rangle\in\mathscr{L}(!\mathscr{U}\multimap\mathscr{U},\mathscr{U})$ and $\mathsf{app}:=\mathsf{pr}_2\in\mathscr{L}(\mathscr{U},!\mathscr{U}\multimap\mathscr{U})$, since $\mathsf{app}\;\mathsf{lam}=\mathsf{id}_{!\mathscr{U}\multimap\mathscr{U}}$. So, $\mathscr{U}$ is a reflexive object (*i.e.* $!\mathscr{U}\multimap\mathscr{U}\lhd\mathscr{U}$) in $\mathscr{L}_!$ via $\mathsf{app}_\mathsf{n}:=\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\,!\mathsf{app}\in\mathscr{L}_!(\mathscr{U},!\mathscr{U}\multimap\mathscr{U})$ and $\mathsf{lam}_\mathsf{n}:=\mathsf{der}_{\mathscr{U}}\,!\mathsf{lam}\in\mathscr{L}_!(!\mathscr{U}\multimap\mathscr{U},\mathscr{U})$, since $\mathsf{app}_\mathsf{n}\circ\mathsf{lam}_\mathsf{n}=!\mathscr{U}\multimap\mathscr{U}$. Therefore, the interpretation of a λ-term t can be defined as usual as a morphism $|t|^n_{\vec{x}}\in\mathscr{L}_!(\mathscr{U}^k,\mathscr{U})$, with $\vec{x}=(x_1,\dots,x_k)$ such that $\mathsf{fv}(t)\subseteq\{x_1,\dots,x_k\}$:

$$|x_i|^n_{\vec{x}}:=\pi_i^k,\qquad |\lambda y\,t|^n_{\vec{x}}:=\mathsf{lam}_\mathsf{n}\circ\Lambda(|t|^n_{\vec{x},y}),\qquad |ts|^n_{\vec{x}}:=\mathsf{Ev}\circ\langle\mathsf{app}_\mathsf{n}\circ|t|^n_{\vec{x}},|s|^n_{\vec{x}}\rangle.$$

Summing up, the object $\mathscr{U}$ provides both a model of the bang calculus and a model of the CbN λ-calculus. The relation between the two is elegant: the semantics $|t|^n$ in the CbN model of the λ-calculus of a λ-term t decomposes into the semantics $[\![\cdot]\!]$ in the model of the bang calculus of the CbN translation t^n of t.

Theorem 14 (Factorization of any CbN semantics)**.** *For every λ-term t, $[\![t^\mathsf{n}]\!]_{\vec{x}}=|t|^n_{\vec{x}}$ (up to Seely's isos).*

Proof. Below we use $\cong$ to transform a morphism $f\in\mathscr{L}((!\mathscr{U})^{\otimes k},\mathscr{U})$ into a morphism $g\in\mathscr{L}_!(\mathscr{U}^k,\mathscr{U})$ using Seely's isos (where $\mathscr{U}^k:=\&_{i=1}^k\mathscr{U}$). We proceed by induction on $t\in\Lambda$.

- If $t:=x_i$ then $t^\mathsf{n}=x_i$, so we have $[\![t^\mathsf{n}]\!]_{\vec{x}}=[\![x_i]\!]_{\vec{x}}=\mathsf{w}_{\mathscr{U}}^{\otimes i-1}\otimes\,\mathsf{der}_{\mathscr{U}}\otimes\mathsf{w}_{\mathscr{U}}^{\otimes k-i}\cong\mathsf{pr}_i^k\,\mathsf{der}_{\mathscr{U}^k}=|x_i|^n_{\vec{x}}$.
- If $t:=\lambda y\,s$ then $t^\mathsf{n}=\lambda y\,s^\mathsf{n}$.

$$\begin{aligned}
[\![\lambda y\,s^\mathsf{n}]\!]_{\vec{x}}&=\langle 0_{(!\mathscr{U})^{\otimes k},!\mathscr{U}},\mathsf{cur}\,([\![s^\mathsf{n}]\!]_{\vec{x},y})\rangle\\
&\cong\langle 0_{(!\mathscr{U})^{\otimes k},!\mathscr{U}},\mathsf{cur}(|s|^n_{\vec{x},y}\mathsf{m}^2)\rangle\\
&=\mathsf{der}_{\mathscr{U}}\,!(\langle 0_{(!\mathscr{U})^{\otimes k},!\mathscr{U}},\mathsf{cur}(|s|^n_{\vec{x},y}\mathsf{m}^2)\rangle)\,\mathsf{dig}_{\mathscr{U}^k}\\
&=\mathsf{der}_{\mathscr{U}}\,!\langle 0_{!\mathscr{U}\multimap\mathscr{U},!\mathscr{U}},\mathsf{id}_{!\mathscr{U}\multimap\mathscr{U}}\rangle\,!\mathsf{cur}(|s|^n_{\vec{x},y}\mathsf{m}^2)\,\mathsf{dig}_{\mathscr{U}^k}\\
&=(\mathsf{der}_{\mathscr{U}}\,!\mathsf{lam})\circ\Lambda(|s|^n_{\vec{x},y})=|\lambda y\,s|^n_{\vec{x}}
\end{aligned}$$

- If $t:=sr$ then $t^\mathsf{n}=\langle s^\mathsf{n}\rangle(r^\mathsf{n})^!$.

$$\begin{aligned}
[\![\langle s^\mathsf{n}\rangle(r^\mathsf{n})^!]\!]_{\vec{x}}&=\mathsf{ev}((\mathsf{pr}_2\,[\![s^\mathsf{n}]\!]_{\vec{x}})\otimes(\mathsf{pr}_1\,[\![(r^\mathsf{n})^!]\!]_{\vec{x}}))\,\mathsf{c}\\
&=\mathsf{ev}((\mathsf{pr}_2\,[\![s^\mathsf{n}]\!]_{\vec{x}})\otimes(\mathsf{pr}_1\,\langle([\![r^\mathsf{n}]\!]_{\vec{x}})^!,0_{(!\mathscr{U})^{\otimes k},!\mathscr{U}\multimap\mathscr{U}}\rangle))\,\mathsf{c}\\
&=\mathsf{ev}((\mathsf{pr}_2\,[\![s^\mathsf{n}]\!]_{\vec{x}})\otimes([\![r^\mathsf{n}]\!]_{\vec{x}})^!)\,\mathsf{c}\\
&\cong\mathsf{ev}((\mathsf{app}\,|s|^n_{\vec{x}})\otimes(!|r|^n_{\vec{x}}\,\mathsf{dig}_{\mathscr{U}^k}))\,\mathsf{c}\\
&=\mathsf{ev}(\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\,!(\mathsf{app}\,|s|^n_{\vec{x}})\,\mathsf{dig}_{\mathscr{U}^k})\otimes(!|r|^n_{\vec{x}}\,\mathsf{dig}_{\mathscr{U}^k}))\,\mathsf{c}\\
&=\mathsf{ev}(\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\otimes\mathsf{id}_{!\mathscr{U}})\,(!(\mathsf{app}\,|s|^n_{\vec{x}})\otimes!|r|^n_{\vec{x}})(\mathsf{dig}_{\mathscr{U}^k}\otimes\mathsf{dig}_{\mathscr{U}^k})\,\mathsf{c}\\
&=\mathsf{ev}(\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\otimes\mathsf{id}_{!\mathscr{U}})\mathsf{m}^2!\langle\mathsf{app}\,|s|^n_{\vec{x}},|r|^n_{\vec{x}}\rangle\,\mathsf{dig}_{\mathscr{U}^k}\\
&=\mathsf{Ev}\circ\langle\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\,!(\mathsf{app}\;|s|^n_{\vec{x}})\,\mathsf{dig}_{\mathscr{U}^k},|r|^n_{\vec{x}}\rangle\\
&=\mathsf{Ev}\circ\langle\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\;!\mathsf{app}\,!|s|^n_{\vec{x}}\,\mathsf{dig}_{\mathscr{U}^k},|r|^n_{\vec{x}}\rangle\\
&=\mathsf{Ev}\circ\langle(\mathsf{der}_{!\mathscr{U}\multimap\mathscr{U}}\;!\mathsf{app})\circ|s|^n_{\vec{x}},|r|^n_{\vec{x}}\rangle=|sr|^n_{\vec{x}}
\end{aligned}$$

□

Thm. 14 is a powerful result: it says not only that *every* LL based model of the bang calculus is also a model of the CbN λ-calculus, but also that the CbN semantics of any λ-term in such a model *always* naturally factorizes into the CbN translation of the λ-term and its semantics in the bang calculus.

Call-by-value. Following [26, 11], models of the CbV λ-calculus can be defined using Girard's "boring" CbV translation of the intuitionistic implication into LL. It is enough to find an object X in $\mathscr{L}$ satisfying $!X \multimap !X \lhd X$ (or equivalently, $!(X \multimap X) \lhd X$).[4] This is the case for our object $\mathscr{U}$ (the model of the bang calculus) since $!\mathscr{U} \multimap \mathscr{U} \lhd \mathscr{U}$ and $!\mathscr{U} \lhd \mathscr{U}$ entail $!\mathscr{U} \multimap !\mathscr{U} \lhd \mathscr{U}$ (by the variance of $!\mathscr{U} \multimap \_$) via the morphisms $\mathsf{lam}_{\mathsf{v}} = \langle 0_{!\mathscr{U}\multimap!\mathscr{U},!\mathscr{U}}, \mathsf{cur}(\langle \mathsf{ev}, 0_{(!\mathscr{U}\multimap!\mathscr{U})\otimes!\mathscr{U},\mathscr{U}}\rangle)\rangle \in \mathscr{L}(!\mathscr{U} \multimap !\mathscr{U}, \mathscr{U})$ and $\mathsf{app}_{\mathsf{v}} = \mathsf{cur}(\mathsf{pr}_1\ \mathsf{ev})\mathsf{pr}_2 \in \mathscr{L}(\mathscr{U}, !\mathscr{U} \multimap !\mathscr{U})$. As in [11], we can then define the interpretation of a λ-term t as a morphism $|t|^{v}_{\vec{x}} \in \mathscr{L}((!\mathscr{U})^{\otimes k}, !\mathscr{U})$, with $\vec{x} = (x_1, \dots, x_k)$ such that $\mathsf{fv}(t) \subseteq \{x_1, \dots, x_k\}$:

$$|x_i|^{v}_{\vec{x}} = \mathsf{w}_{\mathscr{U}}^{\otimes i-1} \otimes \mathsf{id}_{!\mathscr{U}} \otimes \mathsf{w}_{\mathscr{U}}^{\otimes k-i}, \qquad |\lambda y\, t|^{v}_{\vec{x}} = (\mathsf{lam}_{\mathsf{v}}\ \mathsf{cur}(|t|^{v}_{\vec{x},y}))^{!}, \qquad |ts|^{v}_{\vec{x}} = \mathsf{ev}((\mathsf{app}_{\mathsf{v}}\, |t|^{v}_{\vec{x}}) \otimes (|s|^{v}_{\vec{x}}))\,\mathsf{c}\,.$$

We now have two possible ways of interpreting the CbV λ-calculus in our model $\mathscr{U}$: either by translating a λ-term t into $t^{\mathsf{v}} \in !\Lambda$ and then compute $[\![t^{\mathsf{v}}]\!]$, or by computing directly $|t|^{v}$. It is natural to wonder whether the two interpretations $[\![t^{\mathsf{v}}]\!]$ and $|t|^{v}$ are related, and in what way. In [14] the authors conjectured that, at least in the case of a particular relational model $\mathscr{U}$ satisfying $\mathscr{U} = !\mathscr{U} \cup (!\mathscr{U} \times \mathscr{U}) = !\mathscr{U}\ \&\ (!\mathscr{U} \multimap \mathscr{U})$, the two interpretations coincide. We show that the situation is actually more complicated than expected.

The *relational model* $\mathscr{U}$ introduced in [14] admits the following concrete description as a type system. The set $\mathscr{U}$ of *types* and the set $!\mathscr{U}$ of *finite multisets* over $\mathscr{U}$ are defined by mutual induction as follows:

$$\text{(set: } \mathscr{U}) \quad \alpha, \beta, \gamma ::= a \mid a \multimap \alpha \qquad\qquad \text{(set: } !\mathscr{U}) \quad a, b, c ::= [\alpha_1, \dots, \alpha_k] \quad \text{for any } k \geq 0. \tag{2}$$

Environments Γ are functions from variables to $!\mathscr{U}$ whose *support* $\mathsf{supp}(\Gamma) = \{x \mid \Gamma(x) \neq [\,]\}$ is finite. We write $x_1 : a_1, \dots, x_n : a_n$ for the environment Γ satisfying $\Gamma(x_i) = a_i$ and $\Gamma(y) = [\,]$ for $y \notin \vec{x}$. The multiset union $a + b$ is extended to environments pointwise, namely $(\Gamma + \Delta)(x) = \Gamma(x) + \Delta(x)$.

On the one hand, the relational model $\mathscr{U}$ for the CbV λ-calculus inteprets a λ-term t using $|\cdot|^{v}$, which gives $|t|^{v}_{\vec{x}} = \{(\Gamma, \beta) \mid \Gamma \vdash_v t : \beta\}$ where $\vdash_v$ is the type system below (note that if $\Gamma \vdash_v t : \beta$ then $\beta \in !\mathscr{U}$):

$$\frac{}{x : a \vdash_v x : a}\ \text{ax} \qquad \frac{\Gamma \vdash_v t : [a \multimap b] \qquad \Delta \vdash_v s : a}{\Gamma + \Delta \vdash_v ts : b}\ \text{app} \qquad \frac{(\Gamma_i, y : a_i \vdash_v t : b_i)_{1 \leq i \leq k} \quad k \geq 0}{\sum_{i=1}^{k} \Gamma_i \vdash_v \lambda y\, t : [a_1 \multimap b_1, \dots, a_k \multimap b_k]}\ \text{lam}$$

On the other hand, the relational model $\mathscr{U}$ for the bang calculus interprets a term $T \in !\Lambda$ using $[\![\cdot]\!]$, which gives $[\![T]\!]_{\vec{x}} = \{(\Gamma, \beta) \mid \Gamma \vdash_! T : \beta\}$ where $\vdash_!$ is the type system defined as follows:

$$\frac{}{x : [\alpha] \vdash_! x : \alpha}\ ax_! \qquad \frac{\Gamma \vdash_! T : a \multimap \beta \qquad \Delta \vdash_! S : a}{\Gamma + \Delta \vdash_! \langle T \rangle S : \beta}\ @ \qquad \frac{(\Gamma_i \vdash_! T : \beta_i)_{1 \leq i \leq k} \quad k \geq 0}{\sum_{i=1}^{k} \Gamma_i \vdash_! T^{!} : [\beta_1, \dots, \beta_k]}\ !$$

$$\frac{\Gamma \vdash_! T : [\alpha]}{\Gamma \vdash_! \mathrm{der}\, T : \alpha}\ \text{der} \qquad \frac{\Gamma, x : a \vdash_! T : \beta}{\Gamma \vdash_! \lambda x T : a \multimap \beta}\ \lambda$$

In $\mathscr{U}$ (seen as the relational model for the bang calculus) what is the interpretation $[\![t^{\mathsf{v}}]\!]_{\vec{x}}$ of the CbV translation t^{v} of a λ-term t? Easy calculations show that in the type system $\vdash_!$ the rules below — the ones needed to interpret terms of the form t^{v} for some λ-term t — can be derived:

$$\frac{}{x : a \vdash_! x^{!} : a}\ \text{ax} \qquad \frac{\Gamma \vdash_! t^{\mathsf{v}} : [a \multimap \beta] \qquad \Delta \vdash_! s^{\mathsf{v}} : a}{\Gamma + \Delta \vdash_! \langle \mathrm{der}\, t^{\mathsf{v}} \rangle s^{\mathsf{v}} : \beta}\ \text{app} \qquad \frac{(\Gamma_i, y : a_i \vdash_! t^{\mathsf{v}} : \beta_i)_{1 \leq i \leq k} \quad k \geq 0}{\sum_{i=1}^{k} \Gamma_i \vdash_! (\lambda y\, t^{\mathsf{v}})^{!} : [a_1 \multimap \beta_1, \dots, a_k \multimap \beta_k]}\ \text{lam} \tag{3}$$

[4] This approach is compatible with other notions of model such as Moggi's one [22, 23], since the functor "!" defines a strong monad on the Kleisli category $\mathscr{L}_!$. The reflexive object $!X \multimap !X \lhd X$ (or equivalently, $!(X \multimap X) \lhd X$) is the CbV version in $\mathscr{L}$ of the reflexive object $X \Rightarrow X \lhd X$ in a cartesian closed category, in accordance with Girard's CbV decomposition of the arrow.

Intuitively, the type system $\vdash_v$ is obtained from the restriction of $\vdash_!$ to the image of $(\cdot)^\vee$ by substituting arbitrary types β with multisets b of types. So, given a λ-term t, the two interpretations $|t|^v_{\vec{x}}$ and $[\![t^\vee]\!]_{\vec{x}}$ can be different: for $\alpha \in \mathscr{U} \smallsetminus !\mathscr{U}$ (*e.g.* take $\alpha = [\,] \multimap [\,]$), one has $[([a \multimap \alpha] + a) \multimap \alpha] \in [\![(\lambda xxx)^\vee]\!] \smallsetminus |\lambda xxx|^v$.

Proposition 15 (Relational semantics for CbV)**.** *In the relational model $\mathscr{U}$, $|t|^v_{\vec{x}} \subseteq [\![t^\vee]\!]_{\vec{x}}$ for any λ-term t, with $\vec{x} = (x_1, \ldots, x_k)$ such that $\mathsf{fv}(t) \subseteq \{x_1, \ldots, x_k\}$. There exists a closed λ-term s such that $|s|^v \neq [\![s^\vee]\!]$.*

Proof. We have just shown that $|s|^v \neq [\![s^\vee]\!]$ for $s = \lambda xxx$. To prove that $|t|^v_{\vec{x}} \subseteq [\![t^\vee]\!]_{\vec{x}}$, it is enough to show, by induction on $t \in \Lambda$, that $x_1 : a_1, \ldots, x_n : a_k \vdash_! t^\vee : \beta$ is derivable whenever $x_1 : a_1, \ldots, x_k : a_k \vdash_v t : \beta$ is.

If t is a variable, then $t = x_i$ for some $1 \leq i \leq k$, and $t^\vee = x_i^!$. All the derivations for t in the type system $\vdash_v$ are of the form $\dfrac{}{x_i : a \vdash_v x_i : a}\,\mathrm{ax}$ for any $a \in !\mathscr{U}$, and in the type system $\vdash_!$, according to (3), $\dfrac{}{x : a \vdash_! x^! : a}\,\mathrm{ax}$ is derivable.

If $t = sr$, then $t^\vee = \langle \mathrm{der}\, s^\vee \rangle r^\vee$ and all the type derivations for t in the type system $\vdash_v$ are of the form

$$\frac{\Gamma \vdash_v s : [a \multimap b] \qquad \Delta \vdash_v r : a}{\Gamma + \Delta \vdash_v sr : b}\,\mathrm{app} \qquad \text{for any } a, b \in !\mathscr{U}.$$

By *i.h.*, $\Gamma \vdash_! s^\vee : [a \multimap \beta]$ and $\Delta \vdash_! s^\vee : a$ are derivable in the type system $\vdash_!$, hence the following derivation is derivable in the type system $\vdash_!$, according to (3) since $!\mathscr{U} \subseteq \mathscr{U}$

$$\frac{\Gamma \vdash_! s^\vee : [a \multimap b] \qquad \Delta \vdash_! r^\vee : a}{\Gamma + \Delta \vdash_! \langle \mathrm{der}\, s^\vee \rangle r^\vee : b}\,\mathrm{app}.$$

If $t = \lambda y s$, then $t^\vee = (\lambda y s^\vee)^!$ and all the type derivations for t in the type system $\vdash_v$ are of the form

$$\frac{(\Gamma_i, y : a_i \vdash_v s : b_i)_{1 \leq i \leq k} \quad k \geq 0}{\sum_{i=1}^k \Gamma_i \vdash_v \lambda y\, s : [a_1 \multimap b_1, \ldots, a_k \multimap b_k]}\,\mathrm{lam} \qquad \text{for any } a_1, b_1, \ldots, a_k, b_k \in !\mathscr{U}.$$

By *i.h.*, $\Gamma_i, y : a_i \vdash_! s^\vee : b_i$ is derivable in the type system $\vdash_!$ for all $1 \leq i \leq k$, hence the following derivation is derivable in the type system $\vdash_!$, according to (3) since $!\mathscr{U} \subseteq \mathscr{U}$

$$\frac{(\Gamma_i, y : a_i \vdash_! s^\vee : b_i)_{1 \leq i \leq k} \quad k \geq 0}{\sum_{i=1}^k \Gamma_i \vdash_! (\lambda y\, s^\vee)^! : [a_1 \multimap b_1, \ldots, a_k \multimap b_k]}\,\mathrm{lam}.$$ □

The example above of $|s|^v \neq [\![s^\vee]\!]$ shows also that in general neither $[\![t^\vee]\!]_{\vec{x}} = \langle |t|^v, 0 \rangle_{\vec{x}}$ nor $\mathsf{pr}_1 [\![t^\vee]\!]_{\vec{x}} = |t|^v_{\vec{x}}$ hold in relational semantics.[5] We conjecture that, for any λ-term t, $|t|^v_{\vec{x}}$ can be obtained from $[\![t^\vee]\!]_{\vec{x}}$ by iterating the application of pr_1 to $[\![\cdot]\!]$ along the structure of t, but how to express this formally and categorically for a generic model $\mathscr{U}$ of the bang calculus? Usually in these situations one defines a logical relation between the two interpretations, but this is complicated by the fact that we are in the untyped setting so there is no type hierarchy to base our induction. We plan to investigate whether the (syntactic) logical relations introduced by Pitts in [24] can give an inspiration to define semantic logical relations in the untyped setting. Another source of inspiration might be the study of other concrete LL based models of the CbV λ-calculus, such as Scott domains and coherent semantics [26, 11].

[5] Relational semantics interprets terms in the object $\mathscr{U}$ — defined in (2), where $a \multimap \alpha$ denotes the ordered pair (a, α) — of the category **Rel** of sets and relations. The cartesian product & is the disjoint union, with the empty set as terminal and initial object $\top = 0$, so that the zero morphism $0_{X,Y}$ for any objects X and Y is the empty relation and the projection pr_i is the obvious selection. Therefore, in relational semantics, $\langle |t|^v_{\vec{x}}, 0 \rangle = |t|^v_{\vec{x}}$ and $\mathsf{pr}_1 [\![t^\vee]\!]_{\vec{x}} = [\![t^\vee]\!]_{\vec{x}}$ for any λ-term t.

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A Technical appendix: omitted proofs

The enumeration of propositions, theorems, lemmas already stated in the body of the article is unchanged.

A.1 Omitted proofs and remarks of Section 2

Let $n \in \mathbb{N}$ and $T, S_1, \ldots, S_n \in !\Lambda$ and $x_1, \ldots, x_n \in \mathscr{V}ar$. The *substitution of* $S_1, \ldots, S_n$ *for* $x_1, \ldots, x_n$ *in* T is the term $T\{S_1/x_1, \ldots, S_n/x_n\}$ defined by induction on $T \in !\Lambda$ as follows:

$$\begin{aligned}
x_i\{S_1/x_1, \ldots, S_n/x_n\} &:= S_i \text{ for all } 1 \leq i \leq n \\
y\{S_1/x_1, \ldots, S_n/x_n\} &:= y \text{ with } y \notin \{x_1, \ldots, x_n\} \\
(\lambda y T)\{S_1/x_1, \ldots, S_n/x_n\} &:= \lambda y T\{S_1/x_1, \ldots, S_n/x_n\} \text{ with } y \notin \bigcup_{i=1}^{n} \mathsf{fv}(S_i) \cup \{x_i\} \\
(\langle T \rangle R)\{S_1/x_1, \ldots, S_n/x_n\} &:= \langle T\{S_1/x_1, \ldots, S_n/x_n\} \rangle R\{S_1/x_1, \ldots, S_n/x_n\} \\
(T^!)\{S_1/x_1, \ldots, S_n/x_n\} &:= (T\{S_1/x_1, \ldots, S_n/x_n\})^! \\
(\mathrm{der}\, T)\{S_1/x_1, \ldots, S_n/x_n\} &:= \mathrm{der}(T\{S_1/x_1, \ldots, S_n/x_n\})
\end{aligned}$$

Lemma 16 (Composition of substitution)**.** *Let* $T, S, R \in !\Lambda$ *and* $x, y \in \mathscr{V}ar$*. If* $y \notin \mathsf{fv}(R) \cup \{x\}$ *then* $T\{S/y\}\{R/x\} = T\{R/x\}\{S\{R/x\}/y\}$.

Proof. By straightforward induction on $T \in !\Lambda$. □

Remark 17 (Reflexive-transitive closure of parallel ℓ-reduction). It is immediate to check that $\Rightarrow_\ell$ is reflexive and $\to_\ell \subseteq \Rightarrow_\ell \subseteq \to_\ell^*$, whence $\Rightarrow_\ell^* = \to_\ell^*$.

Lemma 18 (Substitution Lemma for parallel ℓ-reduction)**.** *Let* $T, S, R, Q \in !\Lambda$ *and* $x \in \mathscr{V}ar$*. If* $T \Rightarrow_\ell S$ *and* $R \Rightarrow_\ell Q$*, then* $T\{R/x\} \Rightarrow_\ell S\{Q/x\}$.

Proof. By induction on the derivation of $T \Rightarrow_\ell S$. Let us consider its last rule r.

If $\mathsf{r} = \mathsf{var}$, then there are two sub-cases:

- either $T = S = y \neq x$ and then $T\{R/x\} = y = S\{Q/x\}$, therefore $T\{R/x\} \Rightarrow_\ell S\{Q/x\}$ by applying the rule var;
- or $T = x = S$ and then $T\{R/x\} = R \Rightarrow_\ell Q = S\{Q/x\}$ by hypothesis.

If $\mathsf{r} = !$, then $T = T'^! \Rightarrow_\ell S'^! = S$ with $T' \Rightarrow_\ell S'$. By induction hypothesis, $T'\{R/x\} \Rightarrow_\ell S'\{R/x\}$ and hence $S = Q^! \Rightarrow_\ell (R^*)^! = T^*$ (apply the rule !).

The cases where $\mathsf{r} = \lambda$ and $\mathsf{r} = \mathrm{der}$ are analogous to the previous one.

If $\mathsf{r} = \ell$, then $T = \langle \lambda y T_1 \rangle T_2^! \Rightarrow_\ell S_1\{S_2/y\} = S$ with $T_1 \Rightarrow_\ell S_1$ and $T_2 \Rightarrow_\ell S_2$ as conclusions of sub-derivations. We can suppose without loss of generality that $y \notin \mathsf{fv}(R) \cup \{x\}$. By induction hypothesis, $T_1\{R/x\} \Rightarrow_\ell S_1\{R/x\}$ and $T_2\{R/x\} \Rightarrow_\ell S_2\{R/x\}$, so $T\{R/x\} = \langle \lambda y T_1\{R/x\} \rangle (T_2\{R/x\})^! \Rightarrow_\ell S_1\{R/x\}\{S_2\{R/x\}/y\} = S_1\{S_2/y\}\{R/x\} = S\{R/x\}$ by applying the rule ℓ and Lemma 16.

Finally, if $\mathsf{r} = @$, then there are two subcases:

- either $T = \langle R \rangle Q$ and $S = \langle R' \rangle Q'$ with $R \Rightarrow_\ell R'$, $Q \Rightarrow_\ell Q'$, and $R \neq \lambda x P$ or $Q \notin !\Lambda_!$; by induction hypothesis, $R' \Rightarrow_\ell R^*$ and $Q' \Rightarrow_\ell Q^*$, therefore $S = \langle R' \rangle Q' \Rightarrow_\ell \langle R^* \rangle Q^* = T^*$ (apply the rule @);
- or $T = \langle \lambda x R \rangle T^!$ and $S = \langle \lambda x Q \rangle S^!$ with $R \Rightarrow_\ell Q$ and $T^! \Rightarrow_\ell S^!$; by induction hypothesis, $Q \Rightarrow_\ell R^*$ and $S^! \Rightarrow_\ell T'^{!*}$, hence $S = \langle \lambda x Q \rangle S^! \Rightarrow_\ell R^*\{T'^{!*}/x\} = T^*$ by applying the rule ℓ. □

For any term T, we denote by T^* the term obtained by reducing *all* ℓ-redexes in T simultaneously. Formally, T^* is defined by induction on $T \in !\Lambda$ as follows:

$$x^* := x \qquad (\lambda x T)^* := \lambda x T^* \qquad (T^!)^* := (T^*)^! \qquad (\mathrm{der}\, T)^* := \mathrm{der}(T^*)$$
$$(\langle T\rangle S)^* := \langle T^*\rangle S^* \text{ if } T \neq \lambda x R \text{ or } S \notin !\Lambda_! \qquad (\langle \lambda x T\rangle S^!)^* := T^*\{S^*/x\}$$

Lemma 2 (Development)**.** *Let $T, S \in !\Lambda$. If $T \Rightarrow_\ell S$, then $S \Rightarrow_\ell T^*$.* *See p. 5*

Proof. By induction on the derivation of $T \Rightarrow_\ell S$. Let us consider its last rule r.

If $\mathsf{r} = \mathsf{var}$, then $T = x = S$ and $T^* = x$, thus $S \Rightarrow_\ell T^*$ (apply the rule var).

If $\mathsf{r} = !$, then $T = R^! \Rightarrow_\ell Q^! = S$ with $R \Rightarrow_\ell Q$ as premise. By induction hypothesis, $Q \Rightarrow_\ell R^*$ and hence $S = Q^! \Rightarrow_\ell (R^*)^! = T^*$ (apply the rule !).

The cases where $\mathsf{r} = \lambda$ and $\mathsf{r} = \mathsf{der}$ are analogous to the previous one.

If $\mathsf{r} = \ell$, then $T = \langle \lambda x R\rangle T_1^! \Rightarrow_\ell Q\{S_1/x\} = S$ with $R \Rightarrow_\ell Q$ and $T_1 \Rightarrow_\ell S_1$ as conclusions of sub-derivations. By induction hypothesis, $Q \Rightarrow_\ell R^*$ and $S_1 \Rightarrow_\ell T_1^*$, hence $S = Q\{S_1/x\} \Rightarrow_\ell R^*\{T_1^*/x\} = T^*$, according to Lemma 18.

Finally, if $\mathsf{r} = @$, then there are two subcases:

- either $T = \langle R\rangle Q \Rightarrow_\ell \langle R'\rangle Q' = S$ with $R \Rightarrow_\ell R'$ and $Q \Rightarrow_\ell Q'$ as premises, and $R \neq \lambda x P$ or $Q \notin !\Lambda_!$; by *i.h.*, $R' \Rightarrow_\ell R^*$ and $Q' \Rightarrow_\ell Q^*$, therefore $S = \langle R'\rangle Q' \Rightarrow_\ell \langle R^*\rangle Q^* = T^*$ (apply the rule @);
- or $T = \langle \lambda x R\rangle T_1^! \Rightarrow_\ell \langle \lambda x Q\rangle S_1^! = S$ with $R \Rightarrow_\ell Q$ and $T_1 \Rightarrow_\ell S_1$ as conclusions of sub-derivations; by *i.h.*, $Q \Rightarrow_\ell R^*$ and $S_1 \Rightarrow_\ell T_1^*$, so $S = \langle \lambda x Q\rangle S_1^! \Rightarrow_\ell R^*\{T_1^*/x\} = T^*$ (apply the rule ℓ). □

Lemma 2 is the key ingredient to prove the confluence of $\to_\ell$ (Lemma 3.4), indeed it entails that $\Rightarrow_\ell$, and hence $\to_\ell^*$ by Rmk. 17, are strongly confluent.

Lemma 19 (Substitution Lemma for reductions)**.** *Let $T, T', R \in !\Lambda$, $x \in \mathcal{V}ar$.*

1. *For any $\mathsf{r} \in \{\ell, \mathsf{d}, \mathsf{b}, \ell_\mathsf{g}, \mathsf{d}_\mathsf{g}, \mathsf{b}_\mathsf{g}\}$, if $T \to_\mathsf{r} T'$ then $T\{R/x\} \to_\mathsf{r} T'\{R/x\}$.*
2. *For any $\mathsf{r} \in \{\ell, \mathsf{d}, \mathsf{b}\}$, if $T \to_\mathsf{r} T'$ then $R\{T/x\} \to_\mathsf{r}^* R\{T'/x\}$.*

Proof. 1. By induction on $T \in !\Lambda$.

First, let us consider the cases where $\mathsf{r} \in \{\ell_\mathsf{g}, \mathsf{d}_\mathsf{g}, \mathsf{b}_\mathsf{g}\}$. T cannot be a box because boxes are r-normal. Therefore, there are only the following cases:

- *Root-step*, *i.e.* $T \mapsto_\mathsf{r} T'$ where $\mathsf{r} \in \{\ell, \mathsf{d}, \mathsf{b}\}$. Since $\mapsto_\mathsf{b} = \mapsto_\mathsf{d} \cup \mapsto_\ell$, there are only two subcases to consider:
  - $T := \mathrm{der}(T'^!) \mapsto_\mathsf{d} T'$, so $T\{R/x\} = \mathrm{der}(T'\{R/x\}^!) \to_\mathsf{d} T'\{R/x\}$.
  - $T := \langle \lambda y Q\rangle S^! \mapsto_\ell Q\{S/y\} =: T'$ and we can suppose without loss of generality that $y \notin \mathsf{fv}(R) \cup \{x\}$. Note that $S^!\{R/x\} = (S\{R/x\})^! \in !\Lambda_!$, so $T\{R/x\} = \langle \lambda y Q\{R/x\}\rangle (S\{R/x\})^! \to_\ell Q\{R/x\}\{S\{R/x\}/y\} = Q\{S/y\}\{R/x\} = T'\{R/x\}$.
- *Dereliction*, *i.e.* $T := \mathrm{der}\, S \to_\mathsf{r} \mathrm{der}\, S' =: T'$ with $S \to_\mathsf{r} S'$: by induction hypothesis, $S\{R/x\} \to_\mathsf{r} S'\{R/x\}$ and hence $T\{R/x\} = \mathrm{der}(S\{R/x\}) \to_\mathsf{r} \mathrm{der}(S'\{R/x\}) = T'\{R/x\}$.
- *Abstraction*, *i.e.* $T := \lambda y S \to_\mathsf{r} \lambda y S' =: T'$ with $S \to_\mathsf{r} S'$: we can suppose without loss of generality that $y \notin \mathsf{fv}(R) \cup \{x\}$. By *i.h.*, $S\{R/x\} \to_\mathsf{r} S'\{R/x\}$ and hence $T\{R/x\} = \lambda y (S\{R/x\}) \to_\mathsf{r} \lambda y (S'\{R/x\}) = T'\{R/x\}$.
- *Application Right*, *i.e.* $T := \langle Q\rangle S \to_\mathsf{r} \langle Q\rangle S' =: T'$ with $S \to_\mathsf{r} S'$. By *i.h.*, $S\{R/x\} \to_\mathsf{r} S'\{R/x\}$ and hence $T\{R/x\} = \langle Q\{R/x\}\rangle (S\{R/x\}) \to_\mathsf{r} \langle Q\{R/x\}\rangle (S'\{R/x\}) = T'\{R/x\}$.

- *Application Left*, *i.e.* $T := \langle S\rangle Q \to_{\mathsf{r}} \langle S'\rangle Q =: T'$ with $S \to_{\mathsf{r}} S'$: analogous to the previous case.

Now, let us consider the case where $\mathsf{r} \in \{\ell, !, \mathsf{b}\}$. The poof is perfectly analogous to the one above, the only novelty is a new case:

- *Box*, *i.e.* $T := S^! \to_{\mathsf{r}} S'^! =: T'$ with $S \to_{\mathsf{r}} S'$: by induction hypothesis, $S'\{R/x\} \to_{\mathsf{r}} S'\{R/x\}$ and hence $T\{R/x\} = (S\{R/x\})^! \to_{\mathsf{r}} (S'\{R/x\})^! = T'\{R/x\}$.

2. Concerning $\mathsf{r} = \ell$, from $T \to_\ell T'$ it follows that $T \Rightarrow_\ell T'$ by Remark 17. According to Lemma 18 (since $R \Rightarrow_\ell R$ by reflexivity of $\Rightarrow_\ell$), $R\{T/x\} \Rightarrow_\ell R\{T'/x\}$ and hence $R\{T/x\} \to_\ell^* R\{T'/x\}$ by Remark 17 again.

   Concerning $\mathsf{r} = \mathsf{d}$, the proof is by straightforward induction on $R \in !\Lambda$.

   Concerning $\mathsf{r} = \mathsf{b}$, we have just proved that $R\{T'/x\} \to_\ell^* R\{T'/x\}$ and $R\{T/x\} \to_{\mathsf{d}}^* R\{T'/x\}$, therefore we are done because $\to_{\mathsf{b}} = \to_\ell \cup \to_{\mathsf{d}}$. □

The next lemma lists a series of properties of ℓ-, ℓ_{g}-, d- and d_{g}-reductions that will be used to prove strong confluence of $\to_{\mathsf{b}_{\mathsf{g}}}$ and confluence of $\to_{\mathsf{b}}$ (Prop. 4).

See p. 5 **Lemma 3** (Basic properties of reductions)**.**

1. $\to_{\ell_{\mathsf{g}}}$ *is strongly confluent (i.e.* ${}_{\ell_{\mathsf{g}}}\!\leftarrow \cdot \to_{\ell_{\mathsf{g}}} \subseteq (\to_{\ell_{\mathsf{g}}} \cdot {}_{\ell_{\mathsf{g}}}\!\leftarrow) \cup =$*).*
2. $\to_{\mathsf{d}_{\mathsf{g}}}$ *and* $\to_{\mathsf{d}}$ *are strongly confluent (separately).*
3. $\to_{\mathsf{d}_{\mathsf{g}}}$ *and* $\to_{\ell_{\mathsf{g}}}$ *strongly commute (i.e.* ${}_{\mathsf{d}_{\mathsf{g}}}\!\leftarrow \cdot \to_{\ell_{\mathsf{g}}} \subseteq \to_{\ell_{\mathsf{g}}} \cdot {}_{\mathsf{d}_{\mathsf{g}}}\!\leftarrow$*).*
   $\to_{\mathsf{d}}$ *quasi-strongly commutes over* $\to_\ell$ *(i.e.* ${}_{\mathsf{d}}\!\leftarrow \cdot \to_\ell \subseteq \to_\ell \cdot {}_{\mathsf{d}}^*\!\leftarrow$*).*
   $\to_{\mathsf{d}}$ *and* $\to_\ell$ *commute (i.e.* ${}_{\mathsf{d}}^*\!\leftarrow \cdot \to_\ell^* \subseteq \to_\ell^* \cdot {}_{\mathsf{d}}^*\!\leftarrow$*).*
4. $\to_\ell$ *is confluent.*

Proof. 1. Let $T, T_1, T_2 \in !\Lambda$ be such that $T_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} T_2$ and $T_1 \neq T_2$: we prove by induction on $T \in !\Lambda$ that $T_1 \to_{\ell_{\mathsf{g}}} T' \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T_2$ for some $T' \in !\Lambda$. Since $\to_{\ell_{\mathsf{g}}}$ does not reduce under !, there are only the following cases:

- *Root-step for* $T \to_{\ell_{\mathsf{g}}} T_1$ and *non-root-step for* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T := \langle \lambda x S\rangle R^! \mapsto_\ell S\{R/x\} =: T_1$ and $T \not\mapsto_\ell T_2$. Note that $R^!$ is ℓ_{g}-normal since $\to_{\ell_{\mathsf{g}}}$ does not reduce under !, hence the fact that $T \to_{\ell_{\mathsf{g}}} T_2$ via a non-root-step implies that $T_2 := \langle \lambda x Q\rangle R^!$ and $S \to_{\ell_{\mathsf{g}}} Q$. By Lemma 19.1, $T_1 \to_{\ell_{\mathsf{g}}} Q\{R/x\} \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T_2$.
- *Dereliction for both* $T \to_{\ell_{\mathsf{g}}} T_1$ *and* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T_1 := \mathrm{der} S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} \mathrm{der} S_2 =: T_2$ with $T := \mathrm{der} S$ and $S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S \to_{\ell_{\mathsf{g}}} S_2$. By induction hypothesis, there exists $S' \in !\Lambda$ such that $S_1 \to_{\ell_{\mathsf{g}}} S' \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S_2$, whence $T_1 \to_{\ell_{\mathsf{g}}} \mathrm{der} S' \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T_2$.
- *Abstraction for both* $T \to_{\ell_{\mathsf{g}}} T_1$ *and* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T_1 := \lambda x S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} \lambda x S_2 =: T_2$ with $T := \lambda x S$ and $S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S \to_{\ell_{\mathsf{g}}} S_2$: analogous to the previous case.
- *Application left for both* $T \to_{\ell_{\mathsf{g}}} T_1$ *and* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T_1 := \langle S_1\rangle R \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} \langle S_2\rangle R =: T_2$ with $T := \langle S\rangle R$ and $S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S \to_{\ell_{\mathsf{g}}} S_2$. By induction hypothesis, there exists $S' \in !\Lambda$ such that $S_1 \to_{\ell_{\mathsf{g}}} S' \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S_2$, whence $T_1 \to_{\ell_{\mathsf{g}}} \langle S'\rangle R \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T_2$.
- *Application right for both* $T \to_{\ell_{\mathsf{g}}} T_1$ *and* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T_1 := \langle R\rangle S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} \langle R\rangle S_2 =: T_2$ with $T := \langle R\rangle S$ and $S_1 \,{}_{\ell_{\mathsf{g}}}\!\leftarrow S \to_{\ell_{\mathsf{g}}} S_2$: analogous to the previous case.
- *Application left for* $T \to_{\ell_{\mathsf{g}}} T_1$ and *Application Right* $T \to_{\ell_{\mathsf{g}}} T_2$, *i.e.* $T := \langle S\rangle R$ and $T_1 := \langle S'\rangle R \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} \langle S\rangle R' =: T_2$ with $S \to_{\ell_{\mathsf{g}}} S'$ and $R \to_{\ell_{\mathsf{g}}} R'$. Then, $T_1 \to_{\ell_{\mathsf{g}}} \langle S'\rangle R' \,{}_{\ell_{\mathsf{g}}}\!\leftarrow T_2$.

2. We prove that $\to_{\mathsf{d}}$ is strongly confluent; the proof of strong confluence of $\to_{\mathsf{d_g}}$ is analogous (and without the cases box and root-step). Let $T, T_1, T_2 \in !\Lambda$ be such that $T_1 \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} T_2$ and $T_1 \neq T_2$: we prove by induction on $T \in !\Lambda$ that $T_1 \to_{\mathsf{d}} T' \,{}_{\mathsf{d}}\!\leftarrow T_2$ for some $T' \in !\Lambda$. Cases:
   - *Root-step for $T \to_{\mathsf{d}} T_1$* and *non-Root-step for $T \to_{\mathsf{d}} T_2$, i.e.* $T := \mathrm{der}(T_1^!) \mapsto_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} \mathrm{der}(S^!) =: T_2$ with $T_1 \to_{\mathsf{d}} S$. Then, $T_1 \to_{\mathsf{d}} S \,{}_{\mathsf{d}}\!\leftarrow T_2$.
   - *Dereliction for both $T \to_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} T_2$, i.e.* $T := \mathrm{der}\, S$ and $T_1 := \mathrm{der}\, S_1 \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} \mathrm{der}\, S_2 =: T_2$ with $S_1 \,{}_{\mathsf{d}}\!\leftarrow S \to_{\mathsf{d}} S_2$. By induction hypothesis, there exists $S' \in !\Lambda$ such that $S_1 \to_{\mathsf{d}} S' \,{}_{\mathsf{d}}\!\leftarrow S_2$, whence $T_1 \to_{\mathsf{d}} \mathrm{der}\, S' \,{}_{\mathsf{d}}\!\leftarrow T_2$.
   - *Abstraction for both $T \to_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} T_2$, i.e.* $T := \lambda x S$ and $T_1 := \lambda x S_1 \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} \lambda x S_2 =: T_2$ with $S_1 \,{}_{\mathsf{d}}\!\leftarrow S \to_{\mathsf{d}} S_2$: analogous to the previous case.
   - *Box for both $T \to_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} T_2$, i.e.* $T_1 := S_1^! \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} S_2^! =: T_2$ with $T := S^!$ and $S_1 \,{}_{\mathsf{d}}\!\leftarrow S \to_{\mathsf{d}} S_2$: analogous to the previous case.
   - *Application Left for both $T \to_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} T_2$, i.e.* $T := \langle S \rangle R$ and $T_1 := \langle S_1 \rangle R \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} \langle S_2 \rangle R =: T_2$ with $S_1 \,{}_{\mathsf{d}}\!\leftarrow S \to_{\mathsf{d}} S_2$. By induction hypothesis, there exists $S' \in !\Lambda$ such that $S_1 \to_{\mathsf{d}} S' \,{}_{\mathsf{d}}\!\leftarrow S_2$, whence $T_1 \to_{\mathsf{d}} \langle S' \rangle R \,{}_{\mathsf{d}}\!\leftarrow T_2$.
   - *Application Right for both $T \to_{\mathsf{d}} T_1$ and $T \to_{\mathsf{d}} T_2$, i.e.* $T := \langle R \rangle S$ and $T_1 := \langle R \rangle S_1 \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} \langle R \rangle S_2 =: T_2$ with $S_1 \,{}_{\mathsf{d}}\!\leftarrow S \to_{\mathsf{d}} S_2$: analogous to the previous case.
   - *Application Left for $T \to_{\mathsf{d}} T_1$* and *Application Right for $T \to_{\mathsf{d}} T_2$, i.e.* $T := \langle S \rangle R$ and $T_1 := \langle S' \rangle R \,{}_{\mathsf{d}}\!\leftarrow T \to_{\mathsf{d}} \langle S \rangle R' =: T_2$ with $S \to_{\mathsf{d}} S'$ and $R \to_{\mathsf{d}} R'$. Then, $T_1 \to_{\mathsf{d}} \langle S' \rangle R' \,{}_{\mathsf{d}}\!\leftarrow T_2$.
3. First we show that $\to_{\ell_{\mathsf{g}}}$ and $\to_{\mathsf{d_g}}$ *strongly commute*. Let $T, T_1, T_2 \in !\Lambda$ be such that $T_1 \,{}_{\mathsf{d_g}}\!\leftarrow T \to_{\ell_{\mathsf{g}}} T_2$: we prove by induction on $T \in !\Lambda$ that there exists $T' \in !\Lambda$ such that $T_1 \to_{\ell_{\mathsf{g}}} T' \,{}_{\mathsf{d_g}}\!\leftarrow T_2$. Since $\to_{\ell_{\mathsf{g}}}$ and $\to_{\mathsf{d_g}}$ do not reduce under !, the only interesting case is the following:
   - *Root-step for $T \to_{\ell_{\mathsf{g}}} T_2$* and *non-Root-step for $T \to_{\mathsf{d_g}} T_1$, i.e.* $T := \langle \lambda x S \rangle R^! \mapsto_{\ell_{\mathsf{g}}} S\{R/x\} =: T_2$ and $T \to_{\mathsf{d_g}} \langle \lambda x S' \rangle R^! =: T_1$ with $S \to_{\mathsf{d_g}} S'$ (recall that $R^!$ is $\mathsf{d_g}$-normal, so $T \to_{\mathsf{d_g}} T_1$ implies $S \to_{\mathsf{d_g}} S'$ by necessity). By Lemma 19.1, $T_1 \to_{\ell_{\mathsf{g}}} S'\{R/x\} \,{}_{\mathsf{d_g}}\!\leftarrow T_2$.

   The other cases are analogous to the ones in the proof of Lemma 3.1 not involving root-steps, paying attention that now the induction hypothesis is different.

   Now we show that $\to_{\mathsf{d}}$ *quasi-strongly commutes* over $\to_\ell$, *i.e.* ${}_{\mathsf{d}}\!\leftarrow \cdot \to_\ell \subseteq \to_\ell \cdot {}^*_{\mathsf{d}}\!\leftarrow$. This implies that $\to_\ell$ and $\to_{\mathsf{d}}$ *commute*. Let $T, T_1, T_2 \in !\Lambda$ be such that $T_1 \,{}_{\mathsf{d}}\!\leftarrow T \to_\ell T_2$: we prove by induction on $T \in !\Lambda$ that $T_1 \to_\ell T' \,{}^*_{\mathsf{d}}\!\leftarrow T_2$ for some $T' \in !\Lambda$. The only interesting cases are the following:
   - *Root-step for $T \to_{\mathsf{d}} T_1$* and *non-Root-step for $T \to_\ell T_2$, i.e.* $T := \mathrm{der}(T_1^!) \mapsto_{\mathsf{d}} T_1$ and $T \to_\ell \mathrm{der}(T'^!) =: T_2$ with $T_1 \to_\ell T'$. Then, $T_1 \to_\ell T' \,{}_{\mathsf{d}}\!\leftarrow T_2$.
   - *Root-step for $T \to_\ell T_2$* and *non-Root-step for $T \to_{\mathsf{d}} T_1$*. There are two subcases:
     - $T := \langle \lambda x S \rangle R^! \to_{\mathsf{d}} \langle \lambda x S' \rangle R^! =: T_1$ and $T \mapsto_\ell S\{R/x\} =: T_2$ with $S \to_{\mathsf{d}} S'$. Then, $T_1 \to_\ell S'\{R/x\} \,{}_{\mathsf{d}}\!\leftarrow T_2$ by Lemma 19.1.
     - $T := \langle \lambda x S \rangle R^! \to_{\mathsf{d}} \langle \lambda x S \rangle R'^! =: T_1$ and $T \mapsto_\ell S\{R/x\} =: T_2$ with $R \to_{\mathsf{d}} R'$. Then, $T_1 \to_\ell S\{R'/x\} \,{}^*_{\mathsf{d}}\!\leftarrow T_2$ according to Lemma 19.2.

   The other cases are analogous to the ones in the proof of Lemma 3.2 not involving root-steps, paying attention that now the inductive hypothesis is different.
4. Lemma 2 entails that $\Rightarrow_\ell$ is strongly confluent: indeed, if $S \,\ell\!\Leftarrow T \Rightarrow_\ell R$ then $S \Rightarrow_\ell T^* \,\ell\!\Leftarrow R$. Therefore, $\Rightarrow_\ell$ is confluent, that is $\Rightarrow_\ell^*$ is strongly confluent. But $\Rightarrow_\ell^* = \to_\ell^*$ according to Rmk. 17. So, $\to_\ell$ is confluent. □

See p. 5 **Proposition 4** (Strong confluence of $\to_{\mathsf{b_g}}$ and confluence of $\to_{\mathsf{b}}$)**.**

1. *The reduction $\to_{\mathsf{b_g}}$ is strongly confluent, i.e.* ${}_{\mathsf{b_g}}\!\leftarrow \cdot \to_{\mathsf{b_g}} \subseteq (\to_{\mathsf{b_g}} \cdot {}_{\mathsf{b_g}}\!\leftarrow) \cup =$.
2. *The reduction $\to_{\mathsf{b}}$ is confluent:* ${}^{*}_{\mathsf{b}}\!\leftarrow \cdot \to^{*}_{\mathsf{b}} \subseteq \to^{*}_{\mathsf{b}} \cdot {}^{*}_{\mathsf{b}}\!\leftarrow$.

Proof.

1. Since $\to_{\mathsf{d_g}}$ and $\to_{\ell_{\mathsf{g}}}$ are strongly confluent (Lemmas 3.1-2) and strongly commute (Lemma 3.3), it follows immediately that $\to_{\mathsf{d_g}} \cup \to_{\ell_{\mathsf{g}}}$ is strongly confluent, where by definition $\to_{\mathsf{b_g}} = \to_{\mathsf{d_g}} \cup \to_{\ell_{\mathsf{g}}}$.
2. Since $\to_{\mathsf{d}}$ and $\to_{\ell}$ are confluent (Lemmas 3.2 and 3.4) and commute (Lemma 3.3), $\to_{\mathsf{d}} \cup \to_{\ell}$ is confluent by Hindley-Rosen Lemma [4, Lemma 3.3.5], where by definition $\to_{\mathsf{b}} = \to_{\mathsf{d}} \cup \to_{\ell}$. □

A.2 Omitted proofs and remarks of Section 3

See p. 10 **Lemma 11** (Properties of the forgetful translation $(\cdot)^{\dagger}$)**.**

1. $(\cdot)^{\dagger}$ *is a right-inverse of* $(\cdot)^{\mathsf{v}}$*: For every* $t \in \Lambda$*, one has* $t^{\mathsf{v}\dagger} = t$.
2. $(\cdot)^{\dagger}$ *preserves substitution:* $M\{U/x\} \in !\Lambda^{\mathsf{v}}$ *with* $(M\{U/x\})^{\dagger} = M^{\dagger}\{U^{\dagger}/x\}$*, and* $U'\{U/x\} \in !\Lambda^{\mathsf{v}}_{v}$ *with* $(U'\{U/x\})^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\}$*, for any* $M \in !\Lambda^{\mathsf{v}}$ *and* $U, U' \in !\Lambda^{\mathsf{v}}_{v}$.
3. $(\cdot)^{\dagger}$ *maps* b*-reduction into* β^{v}*-reduction: For any* $M \in !\Lambda^{\mathsf{v}}$*,* $U \in !\Lambda^{\mathsf{v}}_{v}$ *and* $T, S \in !\Lambda$*, if* $M \to_{\mathsf{b}} T$ *then* $T \in !\Lambda^{\mathsf{v}}$ *and* $M^{\dagger} \to^{=}_{\beta^{v}} T^{\dagger}$*; if* $U \to_{\mathsf{b}} S$ *then* $S \in !\Lambda^{\mathsf{v}}_{v}$ *and* $U^{\dagger} \to^{=}_{\beta^{v}} S^{\dagger}$.

Proof. Statements of Lemmas 11.2-3 proved here are slightly stronger than in p. 10, to get the right *i.h.*

1. Proof by induction on $t \in \Lambda$. Cases:
   - *Variable, i.e.* $t := x$, then $t^{\mathsf{v}} = x^{!}$ and hence $t^{\mathsf{v}\dagger} = x = t$.
   - *Abstraction, i.e.* $t := \lambda x s$ for some $s \in \Lambda$, then $t^{\mathsf{v}} = (\lambda x s^{\mathsf{v}})^{!}$. By *i.h.*, $s^{\mathsf{v}\dagger} = s$ and hence $t^{\mathsf{v}\dagger} = \lambda x s^{\mathsf{v}\dagger} = \lambda x s = t$.
   - *Application, i.e.* $t := sr$ with $s, r \in \Lambda$, then $t^{\mathsf{v}} = \langle \mathrm{der} s^{\mathsf{v}} \rangle r^{\mathsf{v}}$. By *i.h.*, $s^{\mathsf{v}\dagger} = s$ and $R^{\mathsf{v}\dagger} = R$, so $t^{\mathsf{v}\dagger} = s^{\mathsf{v}\dagger} r^{\mathsf{v}\dagger} = sr = t$.
2. Proof by mutual induction on $M \in !\Lambda^{\mathsf{v}}$ and $U' \in !\Lambda^{\mathsf{v}}_{v}$. Cases:
   - *Variable, i.e.* $U' := y$. There are two subcases:
     - either $y = x$ and then $U'\{U/x\} = U \in !\Lambda^{\mathsf{v}}_{v}$ and $U'^{\dagger} = x$, hence $(U'\{U/x\})^{\dagger} = U^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\}$;
     - or $y \neq x$ and then $U'\{U/x\} = y \in !\Lambda^{\mathsf{v}}_{v}$ and $U'^{\dagger} = y$, thus $(U'\{U/x\})^{\dagger} = y = U'^{\dagger}\{U^{\dagger}/x\}$.
   - *Abstraction, i.e.* $U' := \lambda y N$. We can suppose without loss of generality that $y \notin \mathsf{fv}(U) \cup \{x\}$, hence $U'\{U/x\} = \lambda y N\{U/x\} \in !\Lambda^{\mathsf{v}}_{v}$ (since $N\{U/x\} \in !\Lambda^{\mathsf{v}}_{v}$ by *i.h.*) and $U'^{\dagger} = \lambda y N^{\dagger}$. By *i.h.*, $(N\{U/x\})^{\dagger} = N^{\dagger}\{U^{\dagger}/x\}$. So, $(U'\{U/x\})^{\dagger} = \lambda y (N\{U/x\})^{\dagger} = \lambda y N^{\dagger}\{U^{\dagger}/x\} = U'^{\dagger}\{U^{\dagger}/x\}$.
   - *Box, i.e.* $M := U'^{!}$. Then, $M\{U/x\} = (U'\{U/x\})^{!} \in !\Lambda^{\mathsf{v}}$ (since $U'\{U/x\} \in !\Lambda^{\mathsf{v}}_{v}$ by *i.h.*) and $M^{\dagger} = U'^{\dagger}$. By *i.h.*, $(U'\{U/x\})^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\}$. Therefore, $(M\{U/x\})^{\dagger} = (U'\{U/x\})^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\} = M^{\dagger}\{U^{\dagger}/x\}$.
   - *Application without top-level dereliction, i.e.* $M := \langle U' \rangle N$. So, $M\{U/x\} = \langle U'\{U/x\} \rangle N\{U/x\} \in !\Lambda^{\mathsf{v}}$ (as $U'\{U/x\} \in !\Lambda^{\mathsf{v}}_{v}$ and $N\{U/x\} \in !\Lambda^{\mathsf{v}}$ by *i.h.*) and $M^{\dagger} = U'^{\dagger} N^{\dagger}$. By *i.h.*, $(U'\{U/x\})^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\}$ and $(N\{U/x\})^{\dagger} = N^{\dagger}\{U^{\dagger}/x\}$. Therefore, $(M\{U/x\})^{\dagger} = (U'\{U/x\})^{\dagger}(N\{U/x\})^{\dagger} = U'^{\dagger}\{U^{\dagger}/x\} N^{\dagger}\{U^{\dagger}/x\} = M^{\dagger}\{U^{\dagger}/x\}$.

- *Application with top-level dereliction*, *i.e.* $M := \langle \mathrm{der} N\rangle L$. Then, $M^\dagger = N^\dagger L^\dagger$ and $M\{U/x\} = \langle \mathrm{der} N\{U/x\}\rangle L\{U/x\} \in !\Lambda^{\mathsf{v}}$ (since $N\{U/x\}, L\{U/x\} \in !\Lambda^{\mathsf{v}}$). By induction hypothesis, $(N\{U/x\})^\dagger = N^\dagger\{U^\dagger/x\}$ and $(L\{U/x\})^\dagger = L^\dagger\{U^\dagger/x\}$. Therefore,

$$(M\{U/x\})^\dagger = (N\{U/x\})^\dagger (L\{U/x\})^\dagger = N^\dagger\{U^\dagger/x\}L^\dagger\{U^\dagger/x\} = M^\dagger\{U^\dagger/x\}.$$

3. Proof by mutual induction on $M \in !\Lambda^{\mathsf{v}}$ and $U \in !\Lambda^{\mathsf{v}}_v$. Since there is neither $M = \mathrm{der} R$ nor $U = \mathrm{der} R$, there are only the following cases:
   - *Root-step*, *i.e.* $M \mapsto_{\mathsf{b}} T$. As $\mapsto_{\mathsf{b}} = \mapsto_{\mathsf{d}} \cup \mapsto_{\ell}$, there are two cases to consider. The case $M \mapsto_{\mathsf{d}} T$ is impossible since there is no $M = \mathrm{der} R \in !\Lambda^{\mathsf{v}}$. It remains only one case for the root-step: $M := \langle \lambda x N\rangle U^! \mapsto_\ell N\{U/x\} =: T$. Notice that $U^\dagger$ is λ-value, *i.e.* a variable or an abstraction. Then, by Lemma 11.2, $T \in !\Lambda^{\mathsf{v}}$ and $M^\dagger = (\lambda x N^\dagger)U^\dagger \mapsto_{\beta^v} N^\dagger\{U^\dagger/x\} = (N\{U/x\})^\dagger = T^\dagger$.
   - *Box*, *i.e.* $M := U^! \to_{\mathsf{b}} S^! =: T$ with $U \to_{\mathsf{b}} S$. By *i.h.*, $S \in !\Lambda^{\mathsf{v}}_v$ and $U^\dagger \to^{=}_{\beta^v} S^\dagger$. Hence, $M^\dagger = U^\dagger \to^{=}_{\beta^v} S^\dagger = T^\dagger$.
   - *Application left*, *i.e.* $M := \langle R\rangle N \to_{\mathsf{b}} \langle R'\rangle N =: T$ with $R \to_{\mathsf{b}} R'$. There are three subcases:
     - either $R := \mathrm{der}(U^!) \mapsto_{\mathsf{d}} U =: R'$, and then $T = \langle U\rangle N \in !\Lambda^{\mathsf{v}}$ with $M^\dagger = (\mathrm{der}(U^!))^\dagger N^\dagger = U^\dagger N^\dagger = T^\dagger$ (in particular, $M^\dagger \to^{=}_{\beta^v} T^\dagger$);
     - or $R := \mathrm{der} L \to_{\mathsf{b}} \mathrm{der} S =: R'$ with $L \to_{\mathsf{b}} S$, and then $S \in !\Lambda^{\mathsf{v}}$ with $L^\dagger \to^{=}_{\beta^v} S^\dagger$ by *i.h.*, so $T = \langle \mathrm{der} S\rangle R \in !\Lambda^{\mathsf{v}}$ and $M^\dagger = L^\dagger N^\dagger \to^{=}_{\beta^v} S^\dagger N^\dagger = T^\dagger$;
     - or $R := U \to_{\mathsf{b}} S =: R'$ with $U \to_{\mathsf{b}} S$, and then $S \in !\Lambda^{\mathsf{v}}_v$ and $U^\dagger \to^{=}_{\beta^v} S^\dagger$ by *i.h.*, thus $T = \langle S\rangle N \in !\Lambda^{\mathsf{v}}$ and $M^\dagger = U^\dagger N^\dagger \to_{\beta^v} S^\dagger N^\dagger = T^\dagger$.
   - *Application right*, *i.e.* $M := \langle R\rangle N \to_{\mathsf{b}} \langle R\rangle S =: T$ where $N \to_{\mathsf{b}} S$ and either $R := U$ or $R := \mathrm{der} L$. By *i.h.*, $S \in !\Lambda^{\mathsf{v}}$ and $N^\dagger \to^{=}_{\beta^v} S^\dagger$. Therefore, either $T = \langle U\rangle S \in !\Lambda^{\mathsf{v}}$ and $M^\dagger = U^\dagger N^\dagger \to^{=}_{\beta^v} U^\dagger S^\dagger = T^\dagger$, or $T = \langle \mathrm{der} L\rangle S \in !\Lambda^{\mathsf{v}}$ and $M^\dagger = L^\dagger N^\dagger \to^{=}_{\beta^v} L^\dagger S^\dagger = T^\dagger$.
   - *Abstraction*, *i.e.* $U := \lambda x N \to_{\mathsf{b}} \lambda x R =: S$ with $N \to_{\mathsf{b}} R$. By *i.h.*, $R \in !\Lambda^{\mathsf{v}}$ and $N^\dagger \to^{=}_{\beta^v} R^\dagger$. Hence, $S \in !\Lambda^{\mathsf{v}}_v$ and $U^\dagger = \lambda x N^\dagger \to^{=}_{\beta^v} \lambda x R^\dagger = S^\dagger$. □