DEGREES OF EXTENSIONALITY IN THE THEORY OF BÖHM TREES AND SALLÉ'S CONJECTURE.

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ABSTRACT. The main observational equivalences of the untyped λ-calculus have been characterized in terms of extensional equalities between Böhm trees. It is well known that the λ-theory $\mathcal{H}^*$, arising by taking as observables the head normal forms, equates two λ-terms whenever their Böhm trees are equal up to countably many possibly infinite η-expansions. Similarly, two λ-terms are equal in Morris's original observational theory $\mathcal{H}^+$, generated by considering as observable the β-normal forms, whenever their Böhm trees are equal up to countably many finite η-expansions.

The λ-calculus also possesses a strong notion of extensionality called *the ω-rule*, which has been the subject of many investigations. It is a longstanding open problem whether the equivalence $\mathcal{B}\omega$ obtained by closing the theory of Böhm trees under the ω-rule is strictly included in $\mathcal{H}^+$, as conjectured by Sallé in the seventies. In this paper we demonstrate that the two aforementioned theories actually coincide, thus disproving Sallé's conjecture.

The inclusion $\mathcal{B}\omega \subseteq \mathcal{H}^+$ is a consequence of the fact that the λ-theory $\mathcal{H}^+$ satisfies the ω-rule. This result follows from a weak form of separability enjoyed by λ-terms whose Böhm trees only differ because of some infinite η-expansions, a property which is proved through a refined version of the famous Böhm-out technique.

The inclusion $\mathcal{H}^+ \subseteq \mathcal{B}\omega$ follows from the fact that whenever two λ-terms are observationally equivalent in $\mathcal{H}^+$ their Böhm trees have a common "η-supremum" that can be λ-defined starting from a stream (infinite sequence) of η-expansions of the identity. It turns out that in $\mathcal{B}\omega$ such a stream is equal to the stream containing infinitely many copies of the identity, a peculiar property that actually makes the two theories collapse.

The proof technique we develop for proving the latter inclusion is general enough to provide as a byproduct a new characterization, based on bounded η-expansions, of the least extensional equality between Böhm trees. Together, these results provide a taxonomy of the different degrees of extensionality in the theory of Böhm trees.

Key words and phrases: λ-calculus, λ-theories, observational equivalences, Böhm trees, ω-rule, Sallé's conjecture, Böhm-out technique.

LOGICAL METHODS IN COMPUTER SCIENCE

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Introduction

The problem of determining when two programs are equivalent is central in computer science. For instance, it is necessary to verify that the optimizations performed by a compiler actually preserve the meaning of the program. For λ-calculi, it has become standard to consider two λ-terms M and N as equivalent when they are *contextually equivalent* with respect to some fixed set $\mathcal{O}$ of observables [32]. This means that it is possible to plug either M or N into any context $C[]$ without noticing any difference in the global behaviour: $C[M]$ produces a result belonging to $\mathcal{O}$ exactly when $C[N]$ does. The problem of working with this definition, is that the quantification over all possible contexts is difficult to handle. Therefore, many researchers undertook a quest for characterizing observational equivalences both semantically, by defining fully abstract denotational models, and syntactically, by comparing possibly infinite trees representing possible program executions.

The most famous observational equivalence is obtained by considering as observables the head normal forms, which are λ-terms representing stable amounts of information coming out of the computation. Introduced by Hyland [19] and Wadsworth [41], it has been ubiquitously studied in the literature [3, 17, 15, 36, 29, 6], since it enjoys many interesting properties. By definition, it corresponds to the extensional λ-theory $\mathcal{H}^*$ which is the greatest consistent sensible λ-theory [3, Thm. 16.2.6]. Semantically, it arises as the λ-theory of Scott's pioneering model $\mathcal{D}_\infty$ [38], a result which first appeared in [19] and [41], independently. More recently, Breuvart provided in [6] a characterization of all K-models that are fully abstract for $\mathcal{H}^*$. As shown in [3, Thm. 16.2.7], two λ-terms are equivalent in $\mathcal{H}^*$ exactly when their Böhm trees are equal up to countably many possibly infinite η-expansions.

However, the head normal forms are not the only reasonable choice of observables. For instance, the original extensional contextual equivalence defined by Morris in [32] arises by considering as observables the β-normal forms, which represent completely defined results. We denote by $\mathcal{H}^+$ the λ-theory corresponding to Morris's observational equivalence[1]. The λ-theory $\mathcal{H}^+$ is sensible and distinct from $\mathcal{H}^*$, so we have $\mathcal{H}^+ \subsetneq \mathcal{H}^*$. Despite the fact that the equality in $\mathcal{H}^+$ has been the subject of fewer investigations, it has been characterized both semantically and syntactically. In [13], Coppo *et al.* proved that $\mathcal{H}^+$ corresponds to the λ-theory induced by a suitable filter model. More recently, Manzonetto and Ruoppolo introduced a simpler model of $\mathcal{H}^+$ living in the relational semantics [30] and Breuvart *et al.* provided necessary and sufficient conditions for a relational model to be fully abstract for $\mathcal{H}^+$ [7]. From a syntactic perspective, Hyland proved in [18] that two λ-terms are equivalent in $\mathcal{H}^+$ exactly when their Böhm trees are equal up to countably many η-expansions of finite depth (see also [36, §11.2] and [27]).

We have seen that both observational equivalences correspond to some extensional equalities between Böhm trees. A natural question is whether $\mathcal{H}^+$ can be generated just by adding η-conversion to the λ-theory $\mathcal{B}$ equating all λ-terms having the same Böhm tree. The λ-theory $\mathcal{B}\eta$ so defined has been little studied in the literature, probably because it does not arise as an observational equivalence nor is induced by some known denotational model. In [3, Lemma 16.4.3], Barendregt shows that one η-expansion in a λ-term M can generate infinitely many finite η-expansions on its Böhm tree $\mathrm{BT}(M)$. In [3, Lemma 16.4.4], he exhibits two λ-terms that are equal in $\mathcal{H}^+$ but distinct in $\mathcal{B}\eta$, thus proving that $\mathcal{B}\eta \subsetneq \mathcal{H}^+$.

[1] The notation $\mathcal{H}^+$ has been introduced in [30], while the same theory is denoted $\mathscr{T}_{\mathrm{NF}}$ in [3] and $\mathbf{N}$ in [36].

However, the λ-calculus also possesses another notion of extensionality, known as the *ω-rule*, which is strictly stronger than η-conversion. Such a rule has been studied by many researchers in connection with several λ-theories [23, 1, 34, 4, 22]. Formally, the ω-rule states that for all λ-terms M and N, $M = N$ whenever $MP = NP$ holds for all closed λ-terms P. A λ-theory $\mathcal{T}$ *satisfies the ω-rule* whenever it is closed under such a rule. Since this is such an impredicative rule, we can meaningfully wonder how the λ-theory $\mathcal{B}\omega$, obtained as the closure of $\mathcal{B}$ under the ω-rule, compares with the other λ-theories. As shown by Barendregt in [3, Lemma 16.4.4], $\mathcal{B}\eta$ does not satisfy the ω-rule, while $\mathcal{H}^*$ does [3, Thm. 17.2.8(i)].

Therefore, the two possible scenarios are the following:

$$\mathcal{B}\eta \subsetneq \mathcal{B}\omega \subseteq \mathcal{H}^+ \subsetneq \mathcal{H}^* \qquad \text{or} \qquad \mathcal{B}\eta \begin{array}{c} \subsetneq \mathcal{B}\omega \subsetneq \\ \subsetneq \mathcal{H}^+ \subsetneq \end{array} \mathcal{H}^*.$$

In the seventies, Sallé was working with Coppo and Dezani on type systems for studying termination properties of λ-terms [37, 12]. In 1979, at the conference on λ-calculus that took place in Swansea, he conjectured that a strict inclusion $\mathcal{B}\omega \subsetneq \mathcal{H}^+$ holds. Such a conjecture was reported in the proof of [3, Thm. 17.4.16], but for almost fourty years no progress has been made in that direction. In this paper we demonstrate that the λ-theories $\mathcal{B}\omega$ and $\mathcal{H}^+$ actually coincide, thus disproving Sallé's conjecture. We now give an outline of the proof, discuss the results we need, the techniques we develop and the underlying ideas.

$\mathcal{B}\omega \subseteq \mathcal{H}^+$. The fact that the λ-theory $\mathcal{B}\omega$ is included in $\mathcal{H}^+$ follows immediately if one can prove that $\mathcal{H}^+$ satisfies the ω-rule. To this aim one can observe that when demonstrating the analogous statement for $\mathcal{H}^*$, a key step is to show that, given two closed λ-terms M and N such that the former has a head normal form, while the latter does not, it is possible to find a *non-empty* applicative context $[]P_1 \cdots P_k$ constituted by closed λ-terms that preserves such a property (namely, that $M\vec{P}$ has a head normal form, while $N\vec{P}$ does not). This follows from Wadsworth's characterization of λ-terms having a head normal form in terms of solvability: it is indeed possible to find $\vec{P}$ such that $M\vec{P}$ is equal to the identity $\mathtt{I}$, while $N\vec{P}$ is unsolvable and $\vec{P}$ can be chosen of any length by adding copies of $\mathtt{I}$ at the end.

To prove that $\mathcal{H}^+$ satisfies the ω-rule, we need to show something similar, namely that when M has a β-normal form while N does not, we can find a non-empty context $[]\vec{P}$ preserving this property. Interestingly, it is sufficient to prove this for λ-terms M, N that are equated in the λ-theory $\mathcal{H}^*$; in other words we need to perform a detailed analysis of the equations in $\mathcal{H}^* - \mathcal{H}^+$. We show that when two closed λ-terms M, N are equal in $\mathcal{H}^*$, but different in $\mathcal{H}^+$, their Böhm trees are similar but there exists a (possibly virtual) position σ where they differ because of an infinite η-expansion of a variable x, and such an η-expansion follows the structure of some computable infinite tree T. Thanks to a refined Böhm-out technique, we prove that it is always possible to extract such a difference by defining a suitable applicative context $[]\vec{P}$ that sends M into the identity and N into some infinite η-expansion of the identity (Theorem 4.16). This provides a separability theorem in the spirit of [18, 11, 14] but the notion of separability that we consider is weaker since it arises from Morris's observability. We then prove that applying the identity $\mathtt{I}$ to an infinite η-expansion of $\mathtt{I}$, one still gets a (possibly different) infinite η-expansion of $\mathtt{I}$. From this closure property we obtain that also in this case the length of the discriminating context $[]\vec{P}$ can be chosen arbitrarily by adding copies of $\mathtt{I}$ at the end. Once this property has been established, the fact that $\mathcal{H}^+$ satisfies the ω-rule follows (Theorem 4.19).

$\mathcal{H}^+ \subseteq \mathcal{B}\omega$. To prove such a result we need to show that, whenever two λ-terms M and N are equal in $\mathcal{H}^+$, they are also equal in $\mathcal{B}\omega$. From [18], we know that in this case there is a Böhm tree U such that $\mathrm{BT}(M) \leq^\eta U \geq^\eta \mathrm{BT}(N)$, where $V \leq^\eta U$ means that the Böhm tree U can be obtained from V by performing countably many finite η-expansions. Thus, the Böhm trees of M, N are compatible and have a common "η-supremum" U.

Our proof can be divided into several steps:

(1) We show that the aforementioned η-supremum U is λ-*definable*: there exists a λ-term P such that $\mathrm{BT}(P) = U$ (Proposition 6.1).

(2) We apply the ω-rule to equate the Böhm tree of the λ-term $[\eta_n]_{n\in\mathbb{N}}$ representing a *stream* (infinite list) containing all finite η-expansions of the identity, and the Böhm tree of the stream $[\mathtt{I}]_{n\in\mathbb{N}}$ containing infinitely many copies of the identity (Corollary 6.5).

(3) We define a λ-term Ξ (Definition 5.13) taking as arguments the *codes* $\lceil\cdot\rceil$ of two λ-terms M_1, M_2 and a stream S, and such that, whenever $\mathrm{BT}(M_1) \leq^\eta \mathrm{BT}(M_2)$ holds:
   (i) $\mathrm{BT}(\Xi\lceil M_1\rceil\lceil M_2\rceil[\eta_n]_{n\in\mathbb{N}}) = \mathrm{BT}(M_2)$ (Lemma 5.15),
   (ii) $\mathrm{BT}(\Xi\lceil M_1\rceil\lceil M_2\rceil[\mathtt{I}]_{n\in\mathbb{N}}) = \mathrm{BT}(M_1)$ (Lemma 5.16).

(4) Summing up, if M, N are equal in $\mathcal{H}^+$, then by (1) there is a λ-term P such that $\mathrm{BT}(M) \leq^\eta \mathrm{BT}(P) \geq^\eta \mathrm{BT}(N)$. Since $\mathcal{B}\omega$ equates all λ-terms having the same Böhm tree, we obtain the following sequence of equalities:
$$\begin{array}{lllllll} M & =_{(3)(ii)} & \Xi\lceil M\rceil\lceil P\rceil[\mathtt{I}]_{n\in\mathbb{N}} & =_{(2)} & \Xi\lceil M\rceil\lceil P\rceil[\eta_n]_{n\in\mathbb{N}} & =_{(3)(i)} & P \\ N & =_{(3)(ii)} & \Xi\lceil N\rceil\lceil P\rceil[\mathtt{I}]_{n\in\mathbb{N}} & =_{(2)} & \Xi\lceil N\rceil\lceil P\rceil[\eta_n]_{n\in\mathbb{N}} & =_{(3)(i)} & P \end{array}$$
so M and N are equal in $\mathcal{B}\omega$ (Theorem 6.6).

The intuition behind $\Xi\lceil M\rceil\lceil N\rceil S$ is that, working on their codes, the λ-term Ξ computes the Böhm trees of M and N, compares them, and at every position applies to the "smaller" (less η-expanded) an element extracted from the stream S in the attempt of matching the structure of the "larger" (more η-expanded). If the stream S contains all possible η-expansions then each attempt succeeds, so $\Xi\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}}$ computes the η-supremum of $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$. If S only contains infinitely many copies of the identity, each non-trivial attempt fails, and $\Xi\lceil M\rceil\lceil N\rceil[\mathtt{I}]_{n\in\mathbb{N}}$ computes their η-infimum.

A characterization of $\mathcal{B}\eta$. The technique that we develop for η-expanding Böhm trees in a controlled way is powerful enough to open the way for a characterization of $\mathcal{B}\eta$ as well. More precisely, we prove that two λ-terms M and N are equal in $\mathcal{B}\eta$ exactly when their Böhm trees are equal up to countably many η-expansions of *bounded size* (Theorem 7.16). Indeed, in this case M and N admit an η-supremum U obtained from their Böhm trees by performing at every position σ at most n η-expansions, each having size bounded by n. (In this context, the *size* of an η-expansion is not the actual size of its tree but rather the maximum between its height and its maximal number of branchings.) It turns out that, when exploiting our λ-term $\Xi\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}}$ to compute the η-supremum, it only relies on a *finite* portion of the input stream $[\eta_n]_{n\in\mathbb{N}}$. Since in $\mathcal{B}\eta$ any finite sequence $[\eta_1, \dots, \eta_k]$ is equal to the sequence $[\mathtt{I}, \dots, \mathtt{I}]$ and $\Xi\lceil M\rceil\lceil N\rceil[\mathtt{I}, \dots, \mathtt{I}]$ actually computes the η-infimum, we obtain once again that the η-supremum and the η-infimum collapse. We can therefore proceed as in the proof sketched above for $\mathcal{B}\omega$ and conclude that M and N are equal in $\mathcal{B}\eta$.

Discussion. We build on the characterizations of $\mathcal{H}^+$ and $\mathcal{H}^*$ given by Hyland and Wadsworth [18, 19, 41] and subsequently improved by Lévy [27]. In Section 2 we give a uniform presentation of these preliminary results using the formulation given in [3, §19.2] for $\mathcal{H}^*$, that exploits the notion of Böhm-like trees, namely labelled trees that "look like" Böhm trees but might not be λ-definable. Böhm-like trees were introduced in [3] since at the time researchers were less familiar with the notion of coinduction, but they actually correspond to infinitary terms coinductively generated by the grammar of β-normal forms possibly containing the constant $\bot$. It is worth mentioning that such characterizations of $\mathcal{H}^+$ and $\mathcal{H}^*$ have been recently rewritten by Severi and de Vries using the modern approach of infinitary rewriting [39, 40], and that we could have used their formulation instead.

A key ingredient in our proof of $\mathcal{B}\omega \subseteq \mathcal{H}^+$ is the fact that λ-terms can be encoded as natural numbers, and therefore as Church numerals, in an effective way. This is related to the theory of self-interpreters in λ-calculi, which is an ongoing subject of study [31, 16, 35, 9], and we believe that the present paper provides a nice illustration of the usefulness of such interpreters. As a presentation choice, we decided to use the encoding described in Barendregt's book [3, Def. 6.5.6], even if it works for closed λ-terms only, because it is the most standard. However, our construction could be recast using any (effective) encoding, like the one proposed by Mogensen in [31] that works more generally for open terms.

Disclaimer. The present paper is a long version of the extended abstract [20] published in the Proceedings of the Second International Conference on Formal Structures for Computation and Deduction (FSCD) 2017. The primary goal of this article is to describe the mathematical context where Sallé's conjecture has arisen, and provide a self-contained treatment of its refutation. Besides giving more detailed proofs and examples, we provide some original results only announced in [20], like the characterization of $\mathcal{B}\eta$ in terms of an extensional equality between Böhm trees up to bounded η-expansions.

Notice that a proof-sketch of the fact that $\mathcal{H}^+$ satisfies the ω-rule previously appeared in a conference paper written by the second and third authors in collaboration with Breuvart and Ruoppolo [7]. Since the topic of that paper is mainly semantical we decided – in agreement with them – to exploit the present article to provide the missing details concerning Morris's separability theorem. The semantic results contained in [7] are the subject of a different article [8] presenting more broadly the class of relational graph models and their properties.

Outline. The structure of the present paper is the following. Section 1 contains the preliminaries, mainly concerning the untyped λ-calculus — we present its syntax and recall several well-established properties. In Section 2, we review the main notions of extensional equalities on Böhm trees and provide a few paradigmatic examples. The key results concerning the ω-rule in connection with several λ-theories are presented in Section 3; we conclude the section by stating Sallé's conjecture. Section 4 is devoted to study the structural properties of the set of η-expansions of the identity, introduce our version of the Böhm-out technique and present the weak separability theorem for Morris's observability. As a consequence, we get that $\mathcal{H}^+$ satisfies the ω-rule. In Section 5 we show how to build Böhm trees, and their η-supremum and η-infimum, starting from the codes of λ-terms and streams of η-expansions of the identity. Section 6 is devoted to the actual proof of the refutation of Sallé's conjecture. In Section 7 we provide a characterization of $\mathcal{B}\eta$ in terms of a notion of equality on Böhm trees up to *bounded* η-expansions.

1. Preliminaries

We review some basic notions and introduce some notations that will be used in the rest of the paper.

1.1. **Coinduction.**

Throughout this paper, we often consider possibly infinite trees as coinductive objects and perform coinductive reasoning. Here we recall some basic facts and introduce some terminology, but we mainly assume that the reader is familiar with these concepts. If that is not the case, we suggest the following tutorials on the subject [24, 25].

A *coinductive structure*, also known as *coinductive datatype*, is the greatest fixed point over a grammar, or equivalently the final coalgebra over the corresponding signature. We also consider *coinductive relations*, that are the greatest relations over such coinductive structures that respect the structural constraints. A coinductive proof that two elements of the structures stand in relation to one another is given by an infinite derivation tree, which is a coinductive structure itself.

Since structural coinduction has been around for decades and many efforts have been made within the community to explain why it should be used as innocently as structural induction, in our proofs we will not reassert the coinduction principle every time it is used. Borrowing the terminology from [25], we say that we apply the "coinductive hypothesis" whenever the coinduction principle is applied. The idea is that one can appeal to the coinductive hypothesis as long as there has been progress in producing the nodes of a tree and there is no further analysis of the subtrees. We believe that this style of mathematical writing greatly improves the readability of our proofs without compromising their correctness — the suspicious reader can study [25] where it is explained how this informal terminology actually corresponds to a formal application of the coinduction principle.

1.2. **Sequences, Trees and Encodings.**

We let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}^*$ be the set of all finite sequences over $\mathbb{N}$. Given a sequence $\sigma = \langle n_1, \dots, n_k \rangle$ and $n \in \mathbb{N}$ we write $\sigma.n$ for the sequence $\langle n_1, \dots, n_k, n \rangle$. Given two sequences $\sigma, \tau \in \mathbb{N}^*$ we write $\sigma \star \tau$ for their concatenation. We will denote the empty sequence by ε.

We consider fixed an effective (bijective) encoding of all finite sequences of natural numbers $\# : \mathbb{N}^* \to \mathbb{N}$. In particular, we assume that for all $\sigma \in \mathbb{N}^*$ and $n \in \mathbb{N}$ the code $\#(\sigma.n)$ is computable from $\#\sigma$ and n.

Definition 1.1. An *(unlabelled) tree* is a partial function $T : \mathbb{N}^* \to \mathbb{N}$ such that $\mathrm{dom}(T)$ is closed under prefixes and, for all $\sigma \in \mathrm{dom}(T), n \in \mathbb{N}$ we have $\sigma.n \in \mathrm{dom}(T) \iff n < T(\sigma)$. The *subtree of* T *at* σ is the tree $T{\restriction}_\sigma$ defined by setting $T{\restriction}_\sigma(\tau) = T(\sigma \star \tau)$ for all $\tau \in \mathbb{N}^*$.

The elements of $\mathrm{dom}(T)$ are called *positions*. Notice that, in our definition of a tree T, $T(\sigma)$ provides the number of children of the node at position σ; therefore we have $T(\sigma) = 0$ whenever the position σ corresponds to a leaf.

Definition 1.2. A tree T is called: *recursive* if the function T is partial recursive (after coding $\mathbb{N}^*$ using $\#(-)$); *finite* if $\mathrm{dom}(T)$ is finite; *infinite* if it is not finite.

We denote by $\mathbb{T}_{\mathsf{rec}}$ (resp. $\mathbb{T}^{\infty}_{\mathsf{rec}}$) the set of all (infinite) recursive trees.

1.3. **The Lambda Calculus.** We generally use the notation of Barendregt's first book [3] for the untyped λ-calculus. Let us fix a denumerable set Var of variables.

The set Λ of λ-*terms* (over Var) is defined inductively by the following grammar:

$$\Lambda : \qquad M, N, P, Q \ ::= \ x \ \mid \ \lambda x.M \ \mid \ MN \qquad \text{where } x \in \mathrm{Var}$$

We suppose that the application associates to the left and has higher precedence than the λ-abstraction. For instance, we write $\lambda x.\lambda y.\lambda z.xyz$ for the λ-term $\lambda x.(\lambda y.(\lambda z.((xy)z)))$. We write $\lambda\vec{x}.M$ as an abbreviation for $\lambda x_1.\dots\lambda x_n.M$, $M\vec{N}$ for $MN_1\cdots N_n$, $MN^{\sim n}$ for $MN\cdots N$ and finally $M^n(N)$ for $M(M(\cdots(MN)))$ (n times).

The set $\mathsf{FV}(M)$ of *free variables of* M and α-conversion are defined as in [3, Ch. 1§2]. Given $M, N \in \Lambda$ and $x \in \mathrm{Var}$ we denote by $M[N/x]$ the capture-free substitution of N for all free occurrences of x in M. From now on, λ-terms are considered up to α-*conversion*.

Definition 1.3. We say that a λ-term M is *closed* whenever $\mathsf{FV}(M) = \emptyset$. In this case, M is also called *a combinator*. We denote by Λ^o the set of all combinators.

The λ-calculus is a higher order term rewriting system and several *notions of reduction* can be considered. Let us consider an arbitrary notion of reduction $\to_{\mathtt{R}}$. The *multistep* $\mathtt{R}$-*reduction* $\twoheadrightarrow_{\mathtt{R}}$ is obtained by taking its reflexive and transitive closure. The $\mathtt{R}$-*conversion* $=_{\mathtt{R}}$ is defined as the reflexive, transitive and symmetric closure of $\to_{\mathtt{R}}$. We denote by $\mathtt{nf}_{\mathtt{R}}(M)$ the $\mathtt{R}$-normal form of M, if it exists, and by $\mathtt{NF}_{\mathtt{R}}$ the set of all λ-terms in $\mathtt{R}$-normal form.

In this paper we are mainly concerned with the study of β- and η- reductions.

Definition 1.4.

- The β-*reduction* $\to_\beta$ is the contextual closure of the rule:
$$(\beta) \qquad (\lambda x.M)N \to M[N/x].$$
- The η-*reduction* $\to_\eta$ is the contextual closure of the rule:
$$(\eta) \qquad \lambda x.Mx \to M \text{ where } x \notin \mathsf{FV}(M).$$

We denote by $\to_{\beta\eta}$ the notion of reduction obtained from the union $\to_\beta \cup \to_\eta$, and by $\twoheadrightarrow_{\beta\eta}$ (resp. $=_{\beta\eta}$) the corresponding multistep reduction (resp. conversion). A λ-term M in β-normal form has the shape $M = \lambda\vec{x}.yM_1\cdots M_k$ where each M_i is in β-normal form.

Concerning specific combinators, we use the following notations (for $n \in \mathbb{N}$):

$$\begin{array}{lll}
\mathtt{I} = \mathtt{1}^0 = \lambda x.x, & \mathtt{1}^{n+1} = \lambda xz.x(\mathtt{1}^n z), & \mathtt{B} = \lambda fgx.f(gx), \\
\mathtt{K} = \lambda xy.x, & \mathtt{F} = \lambda xy.y, & \Omega = (\lambda x.xx)(\lambda x.xx), \\
\mathtt{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)), & & \mathtt{J} = \mathtt{Y}(\lambda jxz.x(jz)),
\end{array}$$

where $=$ denotes syntactic equality (up to α-conversion). The λ-term $\mathtt{I}$ represents the identity, $\mathtt{1}^n$ is a $\beta\eta$-expansion of the identity, $\mathtt{B}$ is the composition combinator $M \circ N = \mathtt{B}MN$, $\mathtt{K}$ and $\mathtt{F}$ are the first and second projections, Ω is the paradigmatic looping λ-term, $\mathtt{Y}$ is Curry's fixed point combinator and $\mathtt{J}$ is the combinator defined by Wadsworth in [41].

We denote by $\mathrm{c}_n = \lambda fz.f^n(z)$ the n-th *Church numeral* [3, Def. 6.4.4], by succ and pred the successor and predecessor, and by $\mathsf{ifz}(\mathrm{c}_n, M, N)$ the λ-term β-convertible to M if $n = 0$ and β-convertible to N otherwise. For $\sigma \in \mathbb{N}^*$, we denote by $\lceil\sigma\rceil$ the numeral $\mathrm{c}_{\#\sigma}$.

We encode the n-*tuple* $(M_1, \dots, M_n) \in \Lambda^n$ in λ-calculus by setting $[M_1, \dots, M_n] = \lambda y.yM_1\cdots M_n$ with $y \notin \mathsf{FV}(M_i)$. Thus, $[M_1, M_2]$ represents the *pair* (M_1, M_2) [3, Def. 6.2.4].

Definition 1.5. An enumeration of closed λ-terms $e = (M_0, M_1, M_2, \dots)$ is called *effective* (or *uniform* in [3, §8.2]) if there exists $F \in \Lambda^o$ such that $F\mathrm{c}_n =_\beta M_n$.

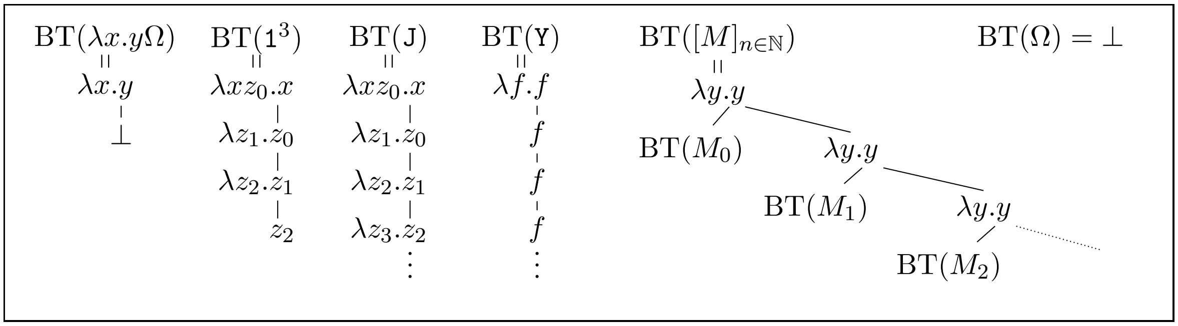


Figure 1: Some examples of Böhm trees.

Given an effective enumeration e as above, it is possible[2] to define the *stream* $[M_n]_{n\in\mathbb{N}}$ as a single λ-term satisfying $[M_n]_{n\in\mathbb{N}} =_\beta [M_0, [M_{n+1}]_{n\in\mathbb{N}}]$. We often use the notation:

$$[M_n]_{n\in\mathbb{N}} = [M_0, [M_1, [M_2, \dots]]].$$

The *i-th projection* associated with a stream is $\pi_i = \lambda y.y\mathtt{F}^{\sim i}\mathtt{K}$ since $\pi_i[M_n]_{n\in\mathbb{N}} =_\beta M_i$.

1.4. **Solvability.** The λ-terms are classified into solvable and unsolvable, depending on their capability of interaction with the environment.

Definition 1.6. A closed λ-term N is *solvable* if $NP_1 \cdots P_n =_\beta \mathtt{I}$ for some $P_1, \dots, P_n \in \Lambda$. A λ-term M is *solvable* if its closure $\lambda\vec{x}.M$ is solvable. Otherwise M is called *unsolvable*.

It was first noticed by Wadsworth in [41] that solvable terms can be characterised in terms of "head normalizability". The *head reduction* $\to_h$ is the notion of reduction obtained by contracting the *head redex* $(\lambda y.P)Q$ in a λ-term having shape $\lambda\vec{x}.(\lambda y.P)QM_1 \cdots M_k$. A λ-term M *is in head normal form* (*hnf*, for short) if it has the form $\lambda x_1 \dots x_n.yM_1 \cdots M_k$. The *principal hnf of M* is the hnf (if it exists) obtained from M by head reduction, i.e. $M \twoheadrightarrow_h \mathrm{phnf}(M)$; it is unique since $\to_h$ is deterministic. A λ-term M has a head normal form if and only if $\mathrm{phnf}(M)$ exists. We denote by $\mathtt{HNF}$ the set of all head normal forms.

Theorem 1.7. *A λ-term M is solvable if and only if M has a head normal form.*

The typical example of an unsolvable is Ω. The following result appears as Lemma 17.4.4 in [3] and shows that any $M \in \Lambda^o$ can be turned into an unsolvable by applying enough Ω's.

Lemma 1.8. *For all $M \in \Lambda^o$ there exists $k \in \mathbb{N}$ such that $M\Omega^{\sim k}$ is unsolvable.*

1.5. **Böhm(-like) trees.** The Böhm trees were introduced by Barendregt [2], who named them after Böhm since their structure arises from the proof of the homonymous theorem [5]. They are coinductively defined labelled trees representing the execution of a λ-term [26].

Definition 1.9. The *Böhm tree of a λ-term M* is the coinductive structure defined by:
- if M is unsolvable then $\mathrm{BT}(M) = \bot$;
- if M is solvable and $M \twoheadrightarrow_h \lambda x_1 \dots x_n.yM_1 \cdots M_k$ then:

$$\mathrm{BT}(M) = \begin{array}{c} \lambda x_1 \dots x_n.y \\ \mathrm{BT}(M_1) \ \cdots \ \mathrm{BT}(M_k) \end{array}$$

[2]One needs to use the fixed point combinator $\mathtt{Y}$, as shown in [3, Def. 8.2.3].

In Figure 1, we provide some examples of Böhm trees of notable λ-terms. Comparing the Böhm trees of $\mathtt{1}^3$ and $\mathtt{J}$ we note that both look like η-expansions of the identity, but the former actually is, while the latter gives rise to infinite computations. Terms like $\mathtt{J}$ are called "infinite η-expansions" of the identity. A simple inspection of the Böhm tree of $[M_n]_{n\in\mathbb{N}}$ should convince the reader that the following lemma holds.

Lemma 1.10. *(cf.* [3, 10.1.5(v)]*) Let $M_i, N_i \in \Lambda$, for $i \in \mathbb{N}$.*

$$\forall i \in \mathbb{N}\ .\ \mathrm{BT}(M_i) = \mathrm{BT}(N_i) \Rightarrow \mathrm{BT}([M_n]_{n\in\mathbb{N}}) = \mathrm{BT}([N_n]_{n\in\mathbb{N}}).$$

Many results from the literature are expressed by exploiting, directly as in [3, Ch. 10] or indirectly as in [36, Ch. 11], the more general notion of "Böhm-like" trees.

Definition 1.11. The set $\mathbb{BT}$ of *Böhm-like trees* is coinductively generated by:

$$\mathbb{BT}: \qquad U, V \ ::=_{\text{coind}}\ \bot \mid \lambda x_1 \dots x_n.yU_1 \cdots U_k \qquad \text{for } k, n \geq 0.$$

In [3, Def. 10.1.12], Böhm-like trees are labelled trees defined as partial functions mapping positions $\sigma \in \mathbb{N}^*$ to pairs $(\lambda\vec{x}.y, k)$ where k is the number of children of the node labelled $\lambda\vec{x}.y$ at σ. In a personal communication, Barendregt told us that the reason is twofold: on the one side coinduction was not as well understood at the time he wrote that book as it is now; on the other side speaking of functions makes easier to define "partial computable" trees. Indeed, not all Böhm-like trees U arise as a Böhm tree of a λ-term M: by [3, Thm. 10.1.23], this is the case only when U is partial computable and $\mathsf{FV}(U)$ is finite.

Notation 1.12. Given $U \in \mathbb{BT}$ we denote by $\lfloor U \rfloor$ its *underlying naked tree*, namely the (unlabelled) tree T having the same structure as U.

1.6. **The Lattice of Lambda Theories.**

Inequational theories and λ-theories become the main object of study when one considers the computational equivalence between λ-terms as being more important than the process of computation itself.

A relation $\mathtt{R}$ between λ-terms is *compatible* if it is compatible with lambda abstraction and application. We say that $\mathtt{R}$ is a *congruence* if it is a compatible equivalence relation.

Definition 1.13. An *inequational theory* is any compatible preorder on Λ containing the β-conversion. A λ-*theory* is any congruence on Λ containing the β-conversion.

Given an inequational theory $\mathcal{T}$, we write $\mathcal{T} \vdash M \sqsubseteq N$ or $M \sqsubseteq_{\mathcal{T}} N$ when M is smaller than or equal to N in $\mathcal{T}$. For a λ-theory $\mathcal{T}$, $\mathcal{T} \vdash M = N$ and $M =_{\mathcal{T}} N$ stand for $(M, N) \in \mathcal{T}$.

The set of all λ-theories, ordered by set-theoretical inclusion, forms a complete lattice having a quite rich mathematical structure, as shown by Lusin and Salibra in [28].

Definition 1.14. A λ-theory (resp. inequational theory) $\mathcal{T}$ is called:

- *consistent* if it does not equate all λ-terms, *inconsistent* otherwise;
- *extensional* if it contains η-conversion;
- *sensible* if it equates all unsolvable λ-terms.

Notation 1.15. We denote by

- $\boldsymbol{\lambda}$ the least λ-theory,
- $\boldsymbol{\lambda}\eta$ the least extensional λ-theory,
- $\mathcal{H}$ the least sensible λ-theory,
- $\mathcal{B}$ the λ-theory equating all λ-terms having the same Böhm tree.

It is easy to check that the λ-theory $\mathcal{B}$ is sensible, but not extensional.

Definition 1.16. We let $\mathcal{T}\eta$ be the least extensional λ-theory containing the λ-theory $\mathcal{T}$.

Remark 1.17. It is well known that two λ-theories $\mathcal{T}, \mathcal{T}'$ that coincide on closed terms must be equal. Hence we often focus on equalities between closed λ-terms.

Several interesting λ-theories are obtained through observational preorders that are defined with respect to some set $\mathcal{O}$ of *observables*.

Definition 1.18. A *context* $C[]$ is a λ-term containing some occurrences of a "*hole*", namely an algebraic variable denoted by $[]$. A context is called a *head context* if it has the shape $(\lambda x_1 \dots x_n.[])P_1 \cdots P_k$ for some $k, n \geq 0$ and $\vec{P} \in \Lambda$. A head context is called *applicative* if it has the shape $[]P_1 \cdots P_k$ and *closed* if all the P_i's are closed.

Given a context $C[]$ and a λ-term M, we write $C[M]$ for the λ-term obtained from $C[]$ by substituting M for the hole, possibly with capture of free variables in M.

Definition 1.19. Let $\mathcal{O} \subseteq \Lambda$.

- The $\mathcal{O}$-*observational preorder* $\sqsubseteq^{\mathcal{O}}$ is defined as follows:
$$M \sqsubseteq^{\mathcal{O}} N \iff \forall C[]\ .\ \big(C[M] \in_\beta \mathcal{O} \ \Rightarrow\ C[N] \in_\beta \mathcal{O}\big).$$
where $M \in_\beta \mathcal{O}$ means that there exists $M' \in \Lambda$ such that $M \twoheadrightarrow_\beta M' \in \mathcal{O}$.
- The $\mathcal{O}$-*observational equivalence* $\equiv^{\mathcal{O}}$ is defined by setting $M \equiv^{\mathcal{O}} N$ if and only if both $M \sqsubseteq^{\mathcal{O}} N$ and $N \sqsubseteq^{\mathcal{O}} M$ hold.

It is easy to check that $M \equiv^{\mathcal{O}} N$ if and only if $\forall C[]\ .\ (C[M] \in_\beta \mathcal{O} \iff C[N] \in_\beta \mathcal{O})$.

In the present paper we focus on the following observational preorders and equivalences.

Definition 1.20.

- $\mathcal{H}^*$: Hyland/Wadsworth's observational preorder $\sqsubseteq_{\mathcal{H}^*}$ and equivalence $=_{\mathcal{H}^*}$ are generated by taking as $\mathcal{O}$ the set $\mathtt{HNF}$ of head normal forms [19, 41].
- $\mathcal{H}^+$: Morris's observational preorder $\sqsubseteq_{\mathcal{H}^+}$ and equivalence $=_{\mathcal{H}^+}$ are generated by taking as $\mathcal{O}$ the set $\mathtt{NF}_\beta$ of β-normal forms [32].

These observational preorders and equivalences are easily proved to be inequational and λ-theories, respectively. (For the general case, one needs some hypotheses on the set $\mathcal{O}$ of observables, as discussed in [33].) The λ-theory $\mathcal{H}^*$ can be characterized as the (unique) maximal consistent sensible λ-theory [3, Thm. 16.2.6]. The proof extends easily to $\sqsubseteq_{\mathcal{H}^*}$ which is maximal consistent among sensible inequational theories [8, Lemma 2.3]. It follows that $\mathcal{H}^*$ is extensional and $\mathcal{H}^+ \subseteq \mathcal{H}^*$. As η-reduction is strongly normalizing, a λ-term has a β-normal form if and only if it has a $\beta\eta$-normal form, therefore $\mathcal{H}^+$ is extensional as well.

For both $\mathcal{H}^*$ and $\mathcal{H}^+$ appropriate Context Lemmas describe the kind of contexts it is sufficient to consider for separating two λ-terms (cf. [41, Lemma 6.1] and [14, Lemma 1.2]).

Lemma 1.21 (Context Lemma)**.** *For $\mathcal{O} = \mathtt{NF}_\beta$ or $\mathcal{O} = \mathtt{HNF}$, the following are equivalent:*

(1) $M \not\sqsubseteq^{\mathcal{O}} N$,

(2) *there exists a closed head context $C[]$ such that $C[M] \in_\beta \mathcal{O}$, while $C[N] \notin_\beta \mathcal{O}$.*

In particular, if M and N are closed then $C[]$ can be chosen closed and applicative.

The quantification over all possible (applicative) contexts makes these definitions difficult to handle in practice. That is the reason why researchers have looked for characterizations of $\mathcal{H}^*$ and $\mathcal{H}^+$ in terms of extensional equalities between Böhm trees.

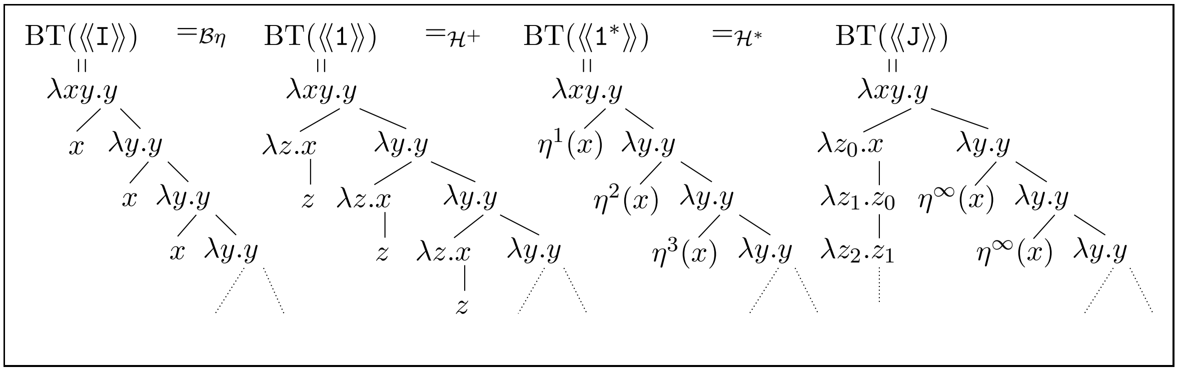


Figure 2: The Böhm trees of $\langle\!\langle \mathtt{I} \rangle\!\rangle, \langle\!\langle \mathtt{1} \rangle\!\rangle, \langle\!\langle \mathtt{1}^* \rangle\!\rangle$ and $\langle\!\langle \mathtt{J} \rangle\!\rangle$, where we set $\eta^n(x) = \mathrm{BT}(\mathtt{1}^n x)$ and $\eta^\infty(x) = \mathrm{BT}(\mathtt{J}x)$.

2. Böhm Trees and Extensionality

We review three different notions of extensional equality between Böhm trees corresponding to the equality in $\mathcal{B}\eta, \mathcal{H}^+$ and $\mathcal{H}^*$.

The following streams, whose Böhm trees are depicted in Figure 2, will be used as running examples in the rest of the section:

$$\langle\!\langle \mathtt{I} \rangle\!\rangle x = [x, [x, [x, \dots]]], \qquad \langle\!\langle \mathtt{1} \rangle\!\rangle x = [\mathtt{1}x, [\mathtt{1}x, [\mathtt{1}x, \dots]]],$$
$$\langle\!\langle \mathtt{1}^* \rangle\!\rangle x = [\mathtt{1}^1 x, [\mathtt{1}^2 x, [\mathtt{1}^3 x, \dots]]], \qquad \langle\!\langle \mathtt{J} \rangle\!\rangle x = [\mathtt{J}x, [\mathtt{J}x, [\mathtt{J}x, \dots]]].$$

2.1. $\mathcal{B}\eta$: Countably Many η-Expansions of Bounded Size.

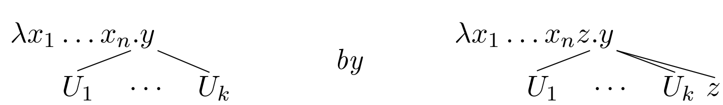
Recall that $\mathcal{B}\eta$ is the least extensional λ-theory including $\mathcal{B}$. Except for some lemmas in [3, §16.4], the λ-theory $\mathcal{B}\eta$ has been mostly neglected in the literature, probably because it does not arise as the theory of any known denotational model. Perhaps, one might be led to think that $M =_{\mathcal{B}\eta} N$ entails that the Böhm trees of M and N differ because of finitely many η-expansions. In reality, one η-expansion of M can generate countably many η-expansions in its Böhm tree.

A typical example of this situation is the following:

$$\langle\!\langle \mathtt{1} \rangle\!\rangle =_{\mathcal{B}} \mathtt{Y}(\lambda mx.[\lambda z.xz, mx]) \to_\eta \mathtt{Y}(\lambda mx.[x, mx]) =_{\mathcal{B}} \langle\!\langle \mathtt{I} \rangle\!\rangle$$

thus $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1} \rangle\!\rangle$ are equated in $\mathcal{B}\eta$ despite the fact that their Böhm trees differ by infinitely many η-expansions. More precisely, $M \to_\eta N$ entails that $\mathrm{BT}(M)$ can be obtained from $\mathrm{BT}(N)$ by performing at most *one* η-expansion at every position.

Lemma 2.1. [3, Lemma 16.4.3] *If $M \to_\eta N$, then $\mathrm{BT}(M)$ is obtained from $\mathrm{BT}(N)$ by replacing in the latter some*



possibly infinitely often (without changing the new variable z).

In particular, no finite amount of η-expansions in $\langle\!\langle \mathtt{I} \rangle\!\rangle$ can turn its Böhm tree into $\mathrm{BT}(\langle\!\langle \mathtt{1}^* \rangle\!\rangle)$, which has infinitely many η-expansions of increasing size.

Corollary 2.2. $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1} \rangle\!\rangle$*, while* $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle \neq \langle\!\langle \mathtt{1}^* \rangle\!\rangle$.

2.2. $\mathcal{H}^+$: **Countably Many Finite η-Expansions.** By definition, $\mathcal{H}^+$ is the λ-theory corresponding to Morris's original observational equivalence where the observables are the β-normal forms [32]. The λ-theory $\mathcal{H}^+$ and its inequational version have been studied both from a syntactic and from a semantic point of view. We refer to [18, 27, 13] for some standard literature and to [30, 7] for more recent work. The properties that we present here can be found in [36, §11.2].

Two λ-terms having the same Böhm tree cannot be distinguished by any context $C[]$, so we have $\mathcal{B} \subseteq \mathcal{H}^+$. Since $\mathcal{H}^+$ is an extensional λ-theory, we also have $\mathcal{B}\eta \subseteq \mathcal{H}^+$.

The question naturally arising is whether there are λ-terms different in $\mathcal{B}\eta$ that become equal in $\mathcal{H}^+$. It turns out that $\mathcal{H}^+ \vdash M = N$ holds exactly when $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ are equal up to countably many η-expansions of finite size. A typical example of this situation is given by the streams $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1}^* \rangle\!\rangle$.

Definition 2.3. Given two Böhm-like trees U and V, we define coinductively the relation $U \le^\eta V$ expressing the fact that V is a *finitary η-expansion* of U. We let $\le^\eta$ be the greatest relation between Böhm-like trees such that $U \le^\eta V$ entails that

- either $U = V = \bot$,
- or, for some $k, m, n \ge 0$:
$$U = \lambda x_1 \dots x_n.yU_1 \cdots U_k \ \text{ and } \ V = \lambda x_1 \dots x_n z_1 \dots z_m.yV_1 \cdots V_k Q_1 \cdots Q_m$$
where $\vec{z} \notin \mathsf{FV}(yU_1 \cdots U_k)$, $U_j \le^\eta V_j$ for all $j \le k$ and $Q_i \twoheadrightarrow_\eta z_i$ for all $i \le m$.

It is easy to check that $\mathrm{BT}(\langle\!\langle \mathtt{I} \rangle\!\rangle) \le^\eta \mathrm{BT}(\langle\!\langle \mathtt{1}^* \rangle\!\rangle)$ holds.

Notation 2.4. For $M, N \in \Lambda$, we write $M \le^\eta N$ if and only if $\mathrm{BT}(M) \le^\eta \mathrm{BT}(N)$.

Note that $M \le^\eta N$ and $N \le^\eta M$ entail $\mathrm{BT}(M) = \mathrm{BT}(N)$, therefore the equivalence corresponding to $\le^\eta$ and capturing $=_{\mathcal{H}^+}$ needs to be defined in a more subtle way.

Theorem 2.5 (Hyland [18], see also [27])**.** *For all $M, N \in \Lambda$, we have $\mathcal{H}^+ \vdash M = N$ if and only if there exists a Böhm-like tree $U \in \mathbb{BT}$ such that* $\mathrm{BT}(M) \le^\eta U \ge^\eta \mathrm{BT}(N)$*.*

This means that in general, when $M =_{\mathcal{H}^+} N$, one may need to perform countably many η-expansions of finite size both in $\mathrm{BT}(M)$ and in $\mathrm{BT}(N)$ to find the common "η-supremum".

Corollary 2.6. *$\mathcal{H}^+ \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1}^* \rangle\!\rangle$, while $\mathcal{H}^+ \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle \neq \langle\!\langle \mathtt{J} \rangle\!\rangle$.*

2.3. $\mathcal{H}^*$: **Countably Many Infinite η-Expansions.** The λ-theory $\mathcal{H}^*$ and its inequational version are, by far, the most well studied theories of the untyped λ-calculus [3, 17, 15, 36, 29, 6]. Recall that $\mathcal{H}^*$ corresponds to the observational equivalence where the observables are the head normal forms. Two λ-terms M, N are equated in $\mathcal{H}^*$ if their Böhm trees are equal up to countably many η-expansions of possibly infinite depth. The typical example is $\mathtt{I} =_{\mathcal{H}^*} \mathtt{J}$.

Definition 2.7. Given two Böhm-like trees U and V, we define coinductively the relation $U \le^\eta_\omega V$ expressing the fact that V is a *possibly infinite η-expansion* of U. We let $\le^\eta_\omega$ be the greatest relation between Böhm-like trees such that $U \le^\eta_\omega V$ entails that

- either $U = V = \bot$,
- or (for some $k, m, n \ge 0$):
$$U = \lambda x_1 \dots x_n.yU_1 \cdots U_k \ \text{ and } \ V = \lambda x_1 \dots x_n z_1 \dots z_m.yV_1 \cdots V_k V'_1 \cdots V'_m$$
where $\vec{z} \notin \mathsf{FV}(yU_1 \cdots U_k)$, $U_j \le^\eta_\omega V_j$ for all $j \le k$ and $z_i \le^\eta_\omega V'_i$ for all $i \le m$.

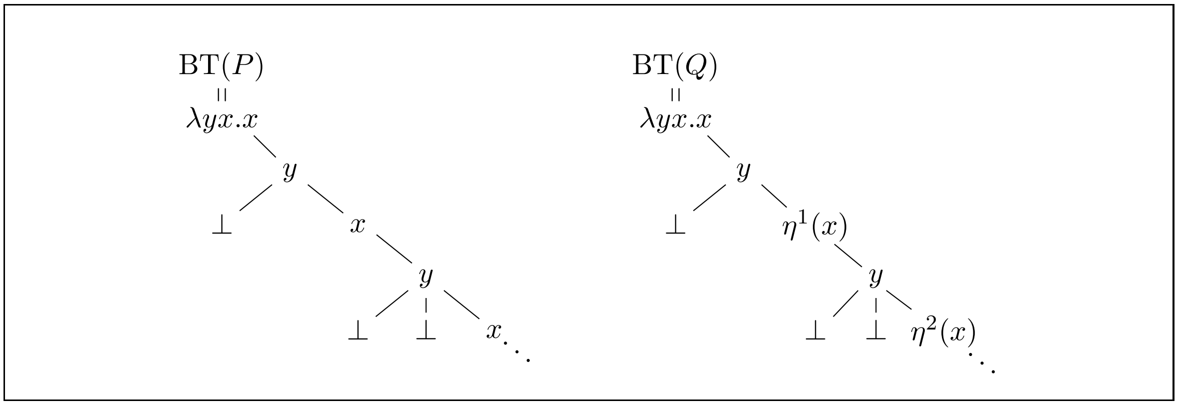


Figure 3: The Böhm trees of the λ-terms P, Q from [3, Lemma 16.4.4].

For instance, we have $\mathrm{BT}(\langle\!\langle \mathtt{I} \rangle\!\rangle) \leq_\omega^\eta \mathrm{BT}(\langle\!\langle \mathtt{J} \rangle\!\rangle)$.

Notation 2.8. For $M, N \in \Lambda$, we write $M \leq_\omega^\eta N$ if and only if $\mathrm{BT}(M) \leq_\omega^\eta \mathrm{BT}(N)$.

Definition 2.9. Given two Böhm-like trees U and V we write $U \leq_\perp V$ whenever U results from V by replacing some subtrees by $\perp$ (possibly infinitely many).

Theorem 2.10 (Hyland [19]/Wadsworth [41])**.** *For all $M, N \in \Lambda$, we have that:*

(i) $\mathcal{H}^* \vdash M \sqsubseteq N$ *if and only if there exist two Böhm-like trees $U, V \in \mathbb{BT}$ such that* $\mathrm{BT}(M) \leq_\omega^\eta U \leq_\perp V \geq_\omega^\eta \mathrm{BT}(N)$.

(ii) $\mathcal{H}^* \vdash M = N$ *if and only if there exists a Böhm-like tree $U \in \mathbb{BT}$ such that* $\mathrm{BT}(M) \leq_\omega^\eta U \geq_\omega^\eta \mathrm{BT}(N)$.

Exercise 10.6.7 in Barendregt's book [3] consists in showing that the η-supremum U in item (ii) above can always be chosen to be the Böhm tree of some λ-term. As we will prove in Section 6, this property also holds for the Böhm-like tree U of Theorem 2.5.

Corollary 2.11. $\langle\!\langle \mathtt{I} \rangle\!\rangle, \langle\!\langle \mathtt{1} \rangle\!\rangle, \langle\!\langle \mathtt{1}^* \rangle\!\rangle, \langle\!\langle \mathtt{J} \rangle\!\rangle$ *are all equal in $\mathcal{H}^*$. However, $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle \neq \langle\!\langle \mathtt{1}^* \rangle\!\rangle$ and $\mathcal{H}^+ \vdash \langle\!\langle \mathtt{1}^* \rangle\!\rangle \neq \langle\!\langle \mathtt{J} \rangle\!\rangle$, so we have $\mathcal{B}\eta \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$.*

3. The Omega Rule and Sallé's Conjecture

The λ-calculus possesses a strong form of extensionality which is known as the *ω-rule* [3, Def. 4.1.10]. The ω-rule has been extensively investigated in the literature by many authors [23, 1, 34, 4, 22]. Intuitively the ω-rule mimics the definition of functional equality, namely it states that two λ-terms M and N are equal whenever they coincide on every closed argument P.

Definition 3.1. The *ω-rule* is given by:

$$(\omega) \qquad (\forall P \in \Lambda^o \,.\, MP = NP) \text{ entails } M = N.$$

We write (ω^0) for the ω-rule restricted to $M, N \in \Lambda^o$.

Definition 3.2. We say that a λ-theory $\mathcal{T}$ *satisfies the ω-rule*, written $\mathcal{T} \vdash \omega$, if for all $P \in \Lambda^o$ $\mathcal{T} \vdash MP = NP$ entails $\mathcal{T} \vdash M = N$.

Given a λ-theory $\mathcal{T}$ we denote by $\mathcal{T}\omega$ the closure of $\mathcal{T}$ under the rule (ω). By definition, $\mathcal{T} \vdash \omega$ entails $\mathcal{T}\omega = \mathcal{T}$. The lemma below collects some basic results from [3, §4.1].

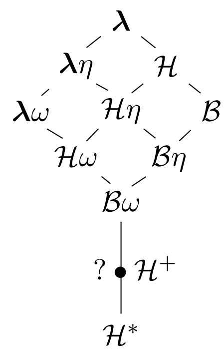
Lemma 3.3. *For all λ-theories $\mathcal{T}$, we have:*

(i) $\mathcal{T}\eta \subseteq \mathcal{T}\omega$,
(ii) $\mathcal{T} \vdash \omega$ *if and only if* $\mathcal{T} \vdash \omega^0$,
(iii) $\mathcal{T} \subseteq \mathcal{T}'$ *entails* $\mathcal{T}\eta \subseteq \mathcal{T}'\eta$ *and* $\mathcal{T}\omega \subseteq \mathcal{T}'\omega$.

In general, because of the quantification over all $P \in \Lambda^o$, it can be difficult to understand which λ-terms different in $\mathcal{T}$ become equal in $\mathcal{T}\omega$, especially when $\mathcal{T}$ is extensional.

Theorem 17.4.16 from [3], that we reproduce below, contains one of the main diagrams of the book and shows all the relationships among the different λ-theories under consideration.

Theorem 3.4. [3, Thm. 17.4.16] *The following diagram indicates all possible inclusion relations of the λ-theories involved (if $\mathcal{T}_1$ is above $\mathcal{T}_2$, then $\mathcal{T}_1 \subsetneq \mathcal{T}_2$):*



The picture above is known as "Barendregt's kite" because of its kite shaped structure. Since $\boldsymbol{\lambda} \subsetneq \mathcal{B} \subsetneq \mathcal{H} \subsetneq \mathcal{H}^*$ and $\mathcal{H}^*$ is maximal sensible, we have both $\boldsymbol{\lambda}\eta \subseteq \mathcal{H}\eta \subseteq \mathcal{B}\eta \subseteq \mathcal{H}^*$ and $\boldsymbol{\lambda}\omega \subseteq \mathcal{H}\omega \subseteq \mathcal{B}\omega \subseteq \mathcal{H}^*$ by Lemma 3.3(iii). All these inclusions turn out to be strict.

The counterexample showing that $\boldsymbol{\lambda}\eta \nvdash \omega$ is based on complicated universal generators known as *Plotkin terms* [3, Def. 17.3.26]. However, since these terms are unsolvable, they become useless when considering sensible λ-theories. We refer to [3, §17.4] for the proof of $\mathcal{H}\eta \nvdash \omega$ and rather discuss the validity of the ω-rule for λ-theories containing $\mathcal{B}$.

Let us consider two λ-terms P and Q whose Böhm trees are depicted in Figure 3. The Böhm trees of P and Q differ because of countably many finite η-expansions of increasing size, therefore they are different in $\mathcal{B}\eta$ but equal in $\mathcal{H}^+$. This situation is analogous to what happens with $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1}^* \rangle\!\rangle$; indeed, $P \leq^\eta Q$ holds. Perhaps surprisingly, P and Q can also be used to prove that $\mathcal{B}\eta \subsetneq \mathcal{B}\omega$ since $\mathcal{B}\omega \vdash P = Q$ holds. Indeed, by Lemma 1.8, for every $M \in \Lambda^o$, there exists k such that $M\Omega^{\sim k}$ becomes unsolvable. By inspecting Figure 3, we notice that in $\mathrm{BT}(P)$ the variable y is applied to an increasing number of Ω's (represented in the tree by $\bot$). So, when substituting some $M \in \Lambda^o$ for y in $\mathrm{BT}(Py)$, there will be a level k of the tree where $M\Omega \cdots \Omega$ becomes $\bot$, thus cutting $\mathrm{BT}(PM)$ at level k. The same reasoning can be done for $\mathrm{BT}(QM)$. Therefore $\mathrm{BT}(PM)$ and $\mathrm{BT}(QM)$ only differ because of finitely many η-expansions. Since $\mathcal{B}\eta \subseteq \mathcal{B}\omega$, we conclude that $PM =_{\mathcal{B}\omega} QM$ and therefore by the ω-rule $P =_{\mathcal{B}\omega} Q$. Such an argument is due to Barendregt [3, Lemma 16.4.4].

The fact that $\mathcal{H}^* \vdash \omega$ is an easy consequence of its maximality. However, there are several direct proofs: see [3, §17.2] for a syntactic demonstration and [41] for a semantic one.

A natural question, raised by Barendregt in [3, Thm. 17.4.16], concerns the position of $\mathcal{H}^+$ in the kite. In the proof of that theorem, it is mentioned that Sallé formulated the following conjecture (represented in the diagram with a question mark).

Conjecture 3.5 (Sallé's Conjecture)**.** $\mathcal{B}\omega \subsetneq \mathcal{H}^+$.

4. $\mathcal{H}^+$ Satisfies the ω-Rule

We start by proving that $\mathcal{H}^+$ satisfies the ω-rule (Theorem 4.19), a result from which it follows that $\mathcal{B}\omega \subseteq \mathcal{H}^+$.

4.1. η-Expansions of the Identity.

We need to analyze more thoroughly the possibly infinite η-expansions of the identity that will play a key role in the rest of the paper. The structural properties of the finite η-expansions of the identity have been analyzed in [21].

Definition 4.1. We say that $Q \in \Lambda$ *is an η-expansion of the identity* whenever $\mathtt{I} \leq_\omega^\eta Q$. We let $\mathcal{I}_\omega^\eta$ be the set of all η-expansions of the identity.

As a matter of terminology, we are slightly abusing notation when saying that $Q \in \mathcal{I}_\omega^\eta$ is an η-expansion of the identity, since $Q \twoheadrightarrow_\eta \mathtt{I}$ does not hold in general. We use this terminology because we are silently considering $\mathcal{B}$-equivalence classes of λ-terms.

Definition 4.2. We say that $Q \in \mathcal{I}_\omega^\eta$ is a *finite* (resp. *infinite*) η-expansion of the identity whenever $\mathrm{BT}(Q)$ is a finite (infinite) tree. We let $\mathcal{I}^\eta$ (resp. $\mathcal{I}_\infty^\eta$) be the set of finite (infinite) η-expansions of the identity.

Clearly, $\mathcal{I}_\omega^\eta = \mathcal{I}^\eta \cup \mathcal{I}_\infty^\eta$ and this decomposition actually gives a partition. Concerning finite η-expansions of the identity, we have that $\mathtt{1}^n \in \mathcal{I}^\eta$ for all $n \in \mathbb{N}$, every $Q \in \mathcal{I}^\eta$ is β-normalizing, $\mathtt{NF}_\beta(Q)$ is a closed λ-term and $\mathrm{BT}(Q)$ is not only finite but also $\perp$-free.

Lemma 4.3. *For $Q \in \Lambda$, the following are equivalent:*
- (i) $Q \in \mathcal{I}^\eta$,
- (ii) $Q =_\beta \lambda y z_1 \ldots z_m . y Q_1 \cdots Q_m$ *such that* $\lambda z_i . Q_i \in \mathcal{I}^\eta$ *for all* $i \leq m$,
- (iii) $Q =_\beta \lambda y . Q'$ *such that* $Q' \twoheadrightarrow_{\beta\eta} y$,
- (iv) $Q \twoheadrightarrow_{\beta\eta} \mathtt{I}$.

In [21], it is shown that there exists a bijection between β-equivalence classes of λ-terms in $\mathcal{I}^\eta$ and finite (unlabelled) trees. It is also proved that $(\mathcal{I}^\eta, \circ, \mathtt{I})$ is an idempotent commutative monoid which is moreover closed under λ-calculus application. We generalize some of these properties to encompass the infinite η-expansions of the identity, like $\mathtt{J}$.

Theorem 4.4. *There is a one-to-one correspondence between $\mathbb{T}_{\mathsf{rec}}$ and $\mathcal{I}_\omega^\eta/_\mathcal{B}$.*

Proof. On the one side, consider any $Q \in \mathcal{I}_\omega^\eta$. By [3, Thm. 10.1.23], $\mathrm{BT}(Q)$ is computable and so is its underlying naked tree $\lfloor \mathrm{BT}(Q) \rfloor$. Since $\mathtt{I} \leq_\infty^\eta \mathrm{BT}(Q)$ entails that $\mathrm{BT}(Q)$ cannot have any occurrences of $\perp$, we conclude that $\mathrm{dom}(\lfloor \mathrm{BT}(Q) \rfloor)$ is recursive. By construction, for all $Q, Q' \in \mathcal{I}_\omega^\eta$, we have that $\mathcal{B} \vdash Q \neq Q'$ entails $\lfloor \mathrm{BT}(Q) \rfloor \neq \lfloor \mathrm{BT}(Q') \rfloor$.

On the other side, consider any $T \in \mathbb{T}_{\mathsf{rec}}$. Recall from Section 1.2 that $\#\sigma$ denotes the code of $\sigma \in \mathbb{N}^*$ and $\lceil \sigma \rceil$ the corresponding Church numeral $\mathsf{c}_{\#\sigma}$. Using the combinator $\mathtt{Y}$, define a λ-term $X \in \Lambda^o$ satisfying the following recursive equation (for all $\sigma \in \mathrm{dom}(T)$):

$$X \lceil \sigma \rceil =_\beta \lambda x z_1 \ldots z_m . x (X \lceil \sigma.0 \rceil z_1) \cdots (X \lceil \sigma.m-1 \rceil z_m) \text{ where } m = T(\sigma). \tag{4.1}$$

The existence of such a λ-term follows from the fact that T is recursive, the effectiveness of the encoding $\#$ and Church's Thesis. We prove by coinduction that for all $\sigma \in \mathrm{dom}(T)$, $X \lceil \sigma \rceil$ is an infinite η-expansion of the identity having $T{\restriction_\sigma}$ as underlying naked tree. Indeed, $X \lceil \sigma \rceil$ is β-convertible to the λ-term of Equation 4.1. By coinductive hypothesis we get for all $i < T(\sigma)$ that $\mathtt{I} \leq_\omega^\eta \mathrm{BT}(X \lceil \sigma.i \rceil)$ and $\lfloor \mathrm{BT}(X \lceil \sigma.i \rceil) \rfloor = T{\restriction_{\sigma.i}}$. From this, we conclude that $\mathtt{I} \leq_\infty^\eta \mathrm{BT}(X \lceil \sigma \rceil)$ and $\lfloor \mathrm{BT}(X \lceil \sigma \rceil) \rfloor = T{\restriction_\sigma}$. So, the λ-term associated with T is $\mathtt{J}_T = X \lceil \varepsilon \rceil$. By construction, for all $T, T' \in \mathbb{T}_{\mathsf{rec}}^\infty$, we have that $T \neq T'$ entails $\mathcal{B} \vdash \mathtt{J}_T \neq \mathtt{J}_{T'}$. $\square$

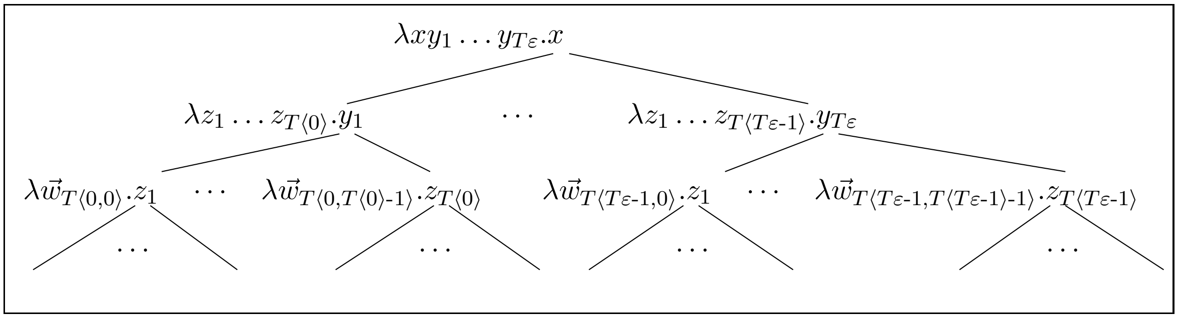


Figure 4: The Böhm-like tree of an infinite η-expansion of $\mathtt{I}$ following $T \in \mathbb{T}_{\mathsf{rec}}$. To lighten the notations we write $T\sigma$ for $T(\sigma)$ and $\lambda\vec{w}_n$ for $\lambda w_1 \dots w_n$.

Notation 4.5. Given $T \in \mathbb{T}_{\mathsf{rec}}$, we denote by $\mathtt{J}_T$ the corresponding λ-term in $\mathcal{I}^\eta_\omega$ whose Böhm tree is shown in Figure 4. We say that $\mathtt{J}_T$ is *an η-expansion of the identity following T*.

Lemma 4.6.

(i) *The set $\mathcal{I}^\eta_\omega$ is closed under composition.*
Moreover, $Q_1 \circ Q_2 \in \mathcal{I}^\eta_\infty$ whenever one of the Q_i's belongs to $\mathcal{I}^\eta_\infty$.

(ii) *If $Q \in \mathcal{I}^\eta_\infty$ then $Q\,\mathtt{I} \in \mathcal{I}^\eta_\infty$.*

Proof. (i) Let $Q, Q' \in \mathcal{I}^\eta_\omega$, we prove by coinduction that $x \leq^\eta_\omega (Q \circ Q')x$ which is equivalent to $\mathtt{I} \leq^\eta_\omega Q \circ Q'$. By definition, we have $Q =_\beta \lambda x z_1 \dots z_n.x(Q_1 z_1) \cdots (Q_n z_n)$ with $z_i \leq^\eta_\omega Q_i z_i$ for $i \leq n$, and $Q' =_\beta \lambda x z_1 \dots z_m.x(Q'_1 z_1) \cdots (Q'_m z_m)$ with $z_j \leq^\eta_\omega Q'_j z_j$ for $j \leq m$.

Consider $n \leq m$, the other case being analogous. Easy calculations give:

$$\begin{array}{lcl} Q \circ Q' & =_\beta & \lambda x z_1 \dots z_n.(Q'x)(Q_1 z_1) \cdots (Q_n z_n) \\ & =_\beta & \lambda x z_1 \dots z_m.x(Q'_1(Q_1 z_1)) \cdots (Q'_n(Q_n z_n))(Q'_{n+1} z_{n+1}) \cdots (Q'_m z_m) \end{array}$$

Since $Q'_i(Q_i z_i) =_\beta (Q'_i \circ Q_i) z_i$ for every $i \leq n$ we get from the coinductive hypothesis that $z_i \leq^\eta_\omega (Q'_i \circ Q_i) z_i$. Together with the hypotheses $z_j \leq^\eta_\omega Q'_j z_j$ for $j \leq m$, this allows to conclude $x \leq^\eta_\omega (Q \circ Q')x$. Note that when Q or Q' belongs to $\mathcal{I}^\eta_\infty$, then at least one among $Q_1, \dots, Q_n, Q'_1, \dots Q'_m$ must have an infinite Böhm tree. If $\mathrm{BT}(Q'_j)$ is infinite for some $j > n$ then it is immediate that $Q \circ Q' \in \mathcal{I}^\eta_\infty$, otherwise it follows from the coinductive hypothesis.

(ii) We prove that $\mathtt{I} \leq^\eta_\omega Q\mathtt{I}$ holds and in the meanwhile we check that $\mathrm{BT}(Q\mathtt{I})$ is infinite. By definition, $Q =_\beta \lambda x z_0 \dots z_n.x(Q_0 z_0) \cdots (Q_n z_n)$ with $z_i \leq^\eta_\omega Q_i z_i$, for all $i \leq n$, and $\mathrm{BT}(Q_j)$ is infinite for some index j. Easy calculations give $Q\mathtt{I} =_\beta \lambda z_0 \dots z_n.(Q_0 z_0) \cdots (Q_n z_n)$. Now, if $Q_0 =_\beta \mathtt{I}$ then $j > 0$ and we the result follows immediately. Otherwise $Q_0 =_\beta \lambda z_0 y_1 \dots y_m.z_0(Q'_1 y_1) \cdots (Q'_m y_m)$ with, say, $m \geq n$ and $y_j \leq^\eta_\omega Q'_j y_j$ for all $j \leq m$.

Therefore, in this case we have:

$$\begin{array}{lcl} Q\mathtt{I} & =_\beta & \lambda z_0 \dots z_n.(\lambda y_1 \dots y_m.z_0(Q'_1 y_1) \cdots (Q'_m y_m))(Q_1 z_1) \cdots (Q_n z_n) \\ & =_\beta & \lambda z_0 \dots z_n.z_0(Q'_1(Q_1 z_1)) \cdots (Q'_n(Q_n z_n))(Q'_{n+1} z_{n+1}) \cdots (Q'_m z_m) \end{array}$$

Notice that $Q'_i(Q_i z_i) =_\beta (Q'_i \circ Q_i) z_i$ for every $i \leq n$ so by applying (i) we get $Q'_i \circ Q_i \in \mathcal{I}^\eta_\omega$ and even $Q'_i \circ Q_i \in \mathcal{I}^\eta_\infty$ if one among these Q'_i, Q_i has an infinite Böhm tree. Otherwise $\mathrm{BT}(Q'_j)$ must be infinite for some $j > n$. In both cases we conclude $Q\,\mathtt{I} \in \mathcal{I}^\eta_\infty$. □

We notice that the properties above generalize to the following proposition, that however we do not prove since it is beyond the scope of the present paper.

Proposition 4.7. *The triple $(\mathcal{I}^\eta_\omega, \circ, \mathtt{I})$ is an idempotent commutative monoid closed under application. Moreover, $\mathcal{I}^\eta_\infty$ is a two-sided ideal in this monoid.*

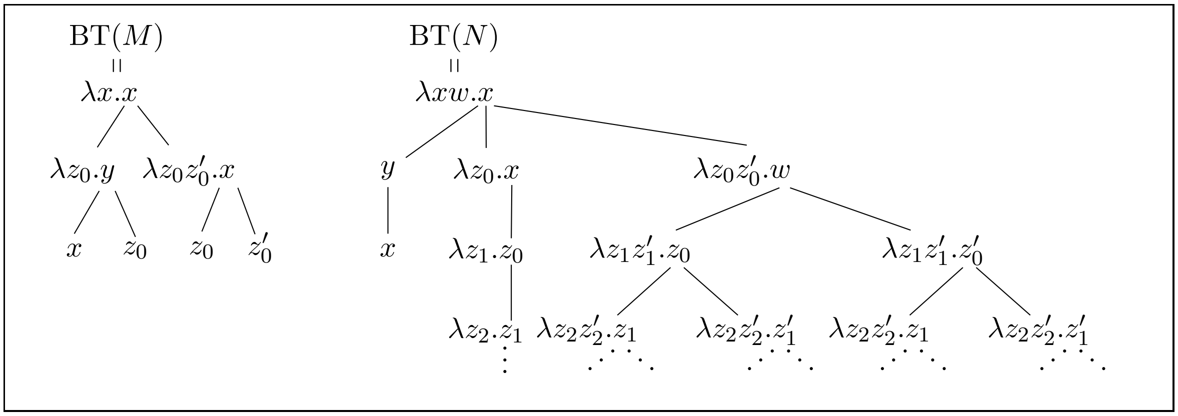


Figure 5: Two λ-terms M, N such that $M \in \mathtt{NF}_\beta$, $\mathcal{H}^* \vdash M = N$, but $\mathcal{H}^+ \vdash M \not\sqsubseteq N$.

4.2. **A Morris-Style Separation Theorem.** The *Böhm-out* technique [5, 3, 36] aims to build a context which extracts (an instance of) the subterm of a λ-term M at position σ. It is used for separating two λ-terms M, N provided that their structure is sufficiently different, depending on the notion of *separation* under consideration. When $\mathcal{H}^* \vdash M = N$ the λ-terms M, N are not *semi-separable*[3] and no Böhm-out technique has been developed in that context. We show that also in this case, under the hypothesis $\mathcal{H}^+ \vdash M \not\sqsubseteq N$, this difference can be Böhmed-out via an appropriate head context thus providing a weak separation result.

Definition 4.8. Given $M \in \Lambda$ and $\sigma \in \mathbb{N}^*$, we define the *subterm of M at σ (relative to its Böhm tree)*, written M_σ, as follows:
(1) $M_\varepsilon = M$,
(2) $M_{\langle i\rangle\star\sigma} = (M_{i+1})_\sigma$ whenever $\mathrm{phnf}(M) = \lambda x_1 \dots x_n.yM_1 \cdots M_k$ and $i < k$,
(3) M_σ is undefined, otherwise.

Notice that $M_{\langle 0\rangle}$, when defined, corresponds to the first argument of the phnf of M. This explains the apparent mismatch in (2) between the indices of $M_{\langle i\rangle\star\sigma}$ and $(M_{i+1})_\sigma$.

Definition 4.9. We say that $\sigma \in \mathbb{N}^*$ is a *Morris separator for M, N*, written $\sigma : M \not\sqsubseteq N$, if there exists $i > 0$ such that, for some $p \geq i$, we have:

$$M_\sigma =_\beta \lambda x_1 \dots x_n.yM_1 \cdots M_m \quad \text{and} \quad N_\sigma =_\beta \lambda x_1 \dots x_{n+p}.yN_1 \cdots N_{m+p}$$

where $\mathcal{B} \vdash N_{m+i} = \mathtt{J}_T x_{n+i}$ for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$.

Remark 4.10. $\sigma : M \not\sqsubseteq N$ and $\sigma = \langle k\rangle \star \tau$ entail $\tau : M_{k+1} \not\sqsubseteq N_{k+1}$.

Intuitively, a Morris separator for two λ-terms M, N that are equal in $\mathcal{H}^*$ is a common position belonging to their Böhm trees and witnessing the fact that $\mathcal{H}^+ \vdash M \not\sqsubseteq N$. We will start by considering the case where M has a β-normal form, which entails that $\mathrm{BT}(M)$ is finite and $\bot$-free. Since $\mathcal{H}^* \vdash M = N$, by Theorem 2.10(ii) also $\mathrm{BT}(N)$ is $\bot$-free; moreover, at every common position σ, M_σ and N_σ have *similar hnf's*, which means that the number of lambda abstractions and applications can be matched via suitable η-expansions. Note that $\mathrm{BT}(M)$ might have η-expansions that are not present in $\mathrm{BT}(N)$. As $\mathcal{H}^+ \vdash M \not\sqsubseteq N$, the Böhm tree of N must have infinite subtrees of the form $\mathrm{BT}(\mathtt{J}_T x)$ for some $x \in \mathrm{Var}, T \in \mathbb{T}^\infty_{\mathsf{rec}}$.

[3]Using the terminology of [3].

Example 4.11. Consider the λ-terms M, N whose Böhm trees are depicted in Figure 5. This example admits two Morris separators:

- The empty sequence ε is a separator since $\mathcal{B} \vdash N_{\langle 2\rangle} = \mathtt{J}_{T_2} w$ where T_2 is the complete binary tree.
- The sequence $\langle 1, 0\rangle$ is a separator because $\mathcal{B} \vdash N_{\langle 1,0,0\rangle} = \mathtt{J}_{T_1} z_1$ where T_1 is the complete unary tree (i.e., $\mathcal{B} \vdash \mathtt{J}_{T_1} = \mathtt{J}$).

Proposition 4.12. *Let $M, N \in \Lambda$ be such that $M \in \mathtt{NF}_\beta$, N does not have a β-normal form and $\mathcal{H}^* \vdash M = N$. Then there exists a position $\sigma \in \mathbb{N}^*$ such that $\sigma : M \not\sqsubseteq N$.*

Proof. We proceed by structural induction on the β-normal form of M.

Base case: $\mathtt{nf}_\beta(M) = \lambda x_1 \dots x_n.y$ for $n \geq 0$. By Theorem 2.10(ii) there is $U \in \mathbb{BT}$ such that $\mathrm{BT}(M) \leq^\eta_\omega U \geq^\eta_\omega \mathrm{BT}(N)$. Since $\mathtt{nf}_\beta(M)$ is η-normal, this entails $\mathtt{nf}_\beta(M) \leq^\eta_\omega N$. Then $N =_\beta \lambda x_1 \dots x_{n+p}.yQ_1 \cdots Q_p$ with $x_{n+j} \leq^\eta_\omega Q_j$ for all $j \leq p$. As N does not have a β-normal form, we have that $p > 0$ and there is an index $0 < i \leq p$ such that $\lambda x_{n+i}.Q_i \in \mathcal{I}^\eta_\infty$. By Theorem 4.4, there is $T \in \mathbb{T}^\infty_{\mathsf{rec}}$ such that $\mathcal{B} \vdash \lambda x_{n+i}.Q_i = \mathtt{J}_T$, hence $\mathcal{B} \vdash Q_i = \mathtt{J}_T x_{n+i}$. Therefore the empty sequence ε is a Morris separator for M, N.

Induction: $\mathtt{nf}_\beta(M) = \lambda x_1 \dots x_n.yM_1 \cdots M_m$ for $m > 0$, where each M_i is β-normal. By Theorem 2.10(ii), there is a Böhm-like tree U such that $\mathtt{nf}_\beta(M) \leq^\eta_\omega U \geq^\eta_\omega \mathrm{BT}(N)$, say:

$$U = \lambda x_1 \dots x_{n+p} z_1 \dots z_k.yU_1 \cdots U_{m+p} V_1 \cdots V_k$$

and $N =_\beta \lambda x_1 \dots x_{n+p}.yN_1 \cdots N_{m+p}$ for some integer p and $k \geq 0$. Assume $p \geq 0$, the other case being analogous. We also have $M_j \leq^\eta_\omega U_j \geq^\eta_\omega \mathrm{BT}(N_j)$ for $j \leq m$, $x_{m+j} \leq^\eta_\omega U_{m+j} \geq^\eta_\omega \mathrm{BT}(N_{m+j})$ for $j \leq p$ and finally $z_\ell \leq^\eta_\omega V_\ell$ for $\ell \leq k$. By Theorem 2.10(ii), this entails that $\mathcal{H}^* \vdash M_j = N_j$ for $j \leq m$ and $\mathcal{H}^* \vdash x_{m+j} = N_{m+j}$ for $j \leq p$. By hypothesis, there is an index $i \leq m + p$ such that N_i does not have a β-normal form. There are two subcases:

(1) $0 < i \leq m$. By the inductive hypothesis there exists $\sigma : M_i \not\sqsubseteq N_i$, so the Morris separator we are looking for is the position $\langle i - 1\rangle \star \sigma$.
(2) $i > m$. By Theorem 4.4, there exists $T \in \mathbb{T}^\infty_{\mathsf{rec}}$ such that $\mathcal{B} \vdash \lambda x_i.N_i = \mathtt{J}_T$, hence $\mathcal{B} \vdash N_i = \mathtt{J}_T x_i$ and the Morris separator is ε. □

Definition 4.13. The following two combinators constitute the main ingredients to build our Böhm-out context:

$$\mathtt{U}^n_k = \lambda x_1 \dots x_n.x_k, \qquad \mathtt{P}_n = \lambda x_1 \dots x_n.[x_1, \dots, x_n].$$

where we recall that $[M_1, \dots, M_n] = \lambda z.zM_1 \cdots M_n$ for some fresh variable z.

The combinator $\mathtt{U}^n_k$ is called *the projector* and the combinator $\mathtt{P}_n$ *the tupler* since they enjoy the following properties.

Lemma 4.14. *Let $k \geq n \geq 0$ and $X_1, \dots, X_n, Y_1, \dots, Y_{k-n} \in \Lambda^o$.*

(1) $(\mathtt{P}_k X_1 \cdots X_n)Y_1 \cdots Y_{k-n} =_\beta [X_1, \dots, X_n, Y_1, \dots, Y_{k-n}]$,
(2) $[X_1, \dots, X_n]\, \mathtt{U}^n_i =_\beta X_i$.

When $\mathtt{U}^n_k$ is substituted for y in $yM_1 \cdots M_n$, it extracts an instance of the subterm M_k. Let us consider the λ-term N whose Böhm tree is depicted in Figure 5. The applicative context $[]\mathtt{U}^2_1$ extracts from N the subterm yx where x is replaced by $\mathtt{U}^2_1$. The idea of the Böhm-out technique is to replace every variable along the path σ with the correct projector.

An issue arises when the same variable occurs several times in σ and we must select different children in these occurrences. For example, to extract $N_{\langle 1,0\rangle}$ from the N in Figure 5, the first occurrence of x should be replaced by $\mathtt{U}^3_2$, the second by $\mathtt{U}^1_1$ (which is actually $\mathtt{I}$).

The problem was originally solved by Böhm in [5] by first replacing the occurrences of the same variables along the path by different variables using the tupler, and then replacing each variable by the suitable projector. In the example under consideration, the applicative context $[]\mathtt{P}_3\Omega\mathtt{U}_2^3\mathtt{U}_1^1\Omega\Omega\mathtt{U}_1^3$ extracts from N the instance of $N_{\langle 1,0\rangle}$ where z_0 is replaced by $\mathtt{I}$.

Obviously, finite η-differences can be destroyed during the process of Böhming out. In contrast, we show that infinite η-differences can always be preserved. In the following lemma we simply write $[\mathtt{P}_k/\vec{y}]$ for the sequence of substitutions $[\mathtt{P}_k/y_1]\cdots[\mathtt{P}_k/y_n]$.

Lemma 4.15 (Böhm-out). *Let $M, N \in \Lambda$ be such that $\mathcal{H}^* \vdash M = N$, let $\vec{y} = \mathsf{FV}(MN)$ and $\sigma : M \not\sqsubseteq N$. Then for all $k \in \mathbb{N}$ large enough, there are combinators $\vec{X} \in \Lambda^o$ such that*

$$M[\mathtt{P}_k/\vec{y}]\vec{X} =_\beta \mathtt{I} \text{ and } N[\mathtt{P}_k/\vec{y}]\vec{X} =_\mathcal{B} \mathtt{J}_T$$

for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$.

Proof. We proceed by induction on σ.

Base case $\sigma = \varepsilon$. Then there exists $i > 0$ such that, for some $p \geq i$, we have:

$$M =_\beta \lambda x_1 \dots x_n.yM_1 \cdots M_m \text{ and } N =_\beta \lambda x_1 \dots x_{n+p}.yN_1 \cdots N_{m+p}$$

where $\mathcal{B} \vdash N_{m+i} = \mathtt{J}_T x_{n+i}$. For any $k \geq n + m + p$ let us set

$$\vec{X} = \mathtt{P}_k^{\sim n}\mathtt{I}^{\sim p}\Omega^{\sim k-m-p}\,\mathtt{U}^k_{m+i},$$

where we recall that $M^{\sim n}$ denotes the sequence of λ-terms containing n copies of M.

We split into cases depending on whether y is free or $y = x_j$ for some $j \leq n$. We consider the former case, as the latter is analogous. On the one side we have:

$$\begin{array}{ll}
(\lambda x_1 \dots x_n.yM_1 \cdots M_m)[\mathtt{P}_k/\vec{y}]\vec{X} = & \\
(\lambda x_1 \dots x_n.\mathtt{P}_k M'_1 \cdots M'_m)\vec{X} =_\beta & \text{where } M'_\ell = M_\ell\,[\mathtt{P}_k/\vec{y}], \\
(\mathtt{P}_k M''_1 \cdots M''_m)\mathtt{I}^{\sim p}\Omega^{\sim k-m-p}\,\mathtt{U}^k_{m+i} =_\beta & \text{where } M''_\ell = M'_\ell\,[\mathtt{P}_k/\vec{x}], \\
(\lambda z.zM''_1 \cdots M''_m\mathtt{I}^{\sim p}\Omega^{\sim k-m-p})\mathtt{U}^k_{m+i} =_\beta \mathtt{I} & \text{by Lemmas 4.14(1) and 4.14(2).}
\end{array}$$

On the other side, we have:

$$\begin{array}{ll}
(\lambda x_1 \dots x_{n+p}.yN_1 \cdots N_{m+p})[\mathtt{P}_k/\vec{y}]\vec{X} = & \\
(\lambda x_1 \dots x_{n+p}.\mathtt{P}_k N'_1 \cdots N'_{m+p})\vec{X} =_\beta & \text{for } N'_\ell = N_\ell\,[\mathtt{P}_k/\vec{y}], \\
(\lambda x_{n+1} \dots x_{n+p}.\mathtt{P}_k N''_1 \cdots N''_{m+p})\mathtt{I}^{\sim p}\Omega^{\sim k-m-p}\,\mathtt{U}^k_{m+i} =_\beta & \text{for } N''_\ell = N'_\ell\,[\mathtt{P}_k/x_1,\dots,x_n], \\
(\mathtt{P}_k N'''_1 \cdots N'''_{m+p})\Omega^{\sim k-m-p}\,\mathtt{U}^k_{m+i} =_\beta & \text{for } N'''_\ell = N''_\ell\,[\mathtt{I}/x_{n+1},\dots,x_{n+p}], \\
(\lambda z.zN'''_1 \cdots N'''_{m+p}\Omega^{\sim k-m-p})\,\mathtt{U}^k_{m+i} =_\beta & \text{by Lemma 4.14(1),} \\
N'''_{m+i} = (\mathtt{J}_T x_{n+i})[\mathtt{I}/x_{n+i}] = \mathtt{J}_T\mathtt{I} & \text{by Lemma 4.14(2),} \\
=_\mathcal{B} \mathtt{J}_{T'} \text{ for some } T' \in \mathbb{T}^\infty_{\mathsf{rec}} & \text{by Thm. 4.4 and Lemma 4.6(ii).}
\end{array}$$

Induction case $\sigma = \langle i\rangle \star \sigma'$.
Since $\mathcal{H}^* \vdash M = N$, we can apply Theorem 2.10(ii) as in the proof of Proposition 4.12 to show that M, N have similar hnf's, namely, for $n - m = n' - m'$ and $i + 1 \leq \min\{m, m'\}$ we have:

$$M =_\beta \lambda x_1 \dots x_n.yM_1 \cdots M_m \text{ and } N =_\beta \lambda x_1 \dots x_{n'}.yN_1 \cdots N_{m'}$$

where $\mathcal{H}^* \vdash M_j = N_j$ for all $j \leq \min\{m, m'\}$ and either y is free or $y = x_j$ for $j \leq \min\{n, n'\}$. Suppose that, say, $n \leq n'$. Then there is $p \geq 0$ such that $n' = n + p$ and $m' = m + p$. Since $\mathcal{H}^* \vdash M_{i+1} = N_{i+1}$ and $\sigma' : M_{i+1} \not\sqsubseteq N_{i+1}$ we apply the induction hypothesis and get, for any k' large enough, $\vec{Y} \in \Lambda^o$ such that

$$M_{i+1}[\mathtt{P}_{k'}/\vec{y},\vec{x}]\vec{Y} =_\beta \mathtt{I} \text{ and } N_{i+1}[\mathtt{P}_{k'}/\vec{y},\vec{x}\,] =_\mathcal{B} \mathtt{J}_T \text{ for some } T \in \mathbb{T}^\infty_{\mathsf{rec}}.$$

For any $k \geq \max\{k', n+m+p\}$, let us set

$$\vec{X} = \mathtt{P}_k^{\sim n+p}\Omega^{\sim k-m-p}\,\mathtt{U}_{i+1}^k\vec{Y}.$$

We suppose that y is free, the other case being analogous. On the one side we have:

$$\begin{array}{ll}
(\lambda x_1 \dots x_n.yM_1 \cdots M_m)[\mathtt{P}_k/\vec{y}\,]\,\vec{X} = & \\
(\lambda x_1 \dots x_n.\mathtt{P}_k M_1' \cdots M_m')\mathtt{P}_k^{\sim n+p}\Omega^{\sim k-m-p}\,\mathtt{U}_{i+1}^k\vec{Y} =_\beta & \text{where } M_\ell' = M_\ell\,[\mathtt{P}_k/\vec{y}\,], \\
(\mathtt{P}_k M_1'' \cdots M_m'')\mathtt{P}_k^{\sim p}\Omega^{\sim k-m-p}\,\mathtt{U}_{i+1}^k\vec{Y} =_\beta & \text{where } M_\ell'' = M_\ell'\,[\mathtt{P}_k/\vec{x}\,], \\
(\lambda z.zM_1'' \cdots M_m''\mathtt{P}_k^{\sim p}\Omega^{\sim k-m-p})\,\mathtt{U}_{i+1}^k\vec{Y} =_\beta & \text{by Lemma 4.14(1)}, \\
M_{i+1}''\vec{Y} = M_{i+1}[\mathtt{P}_k/\vec{y},\vec{x}\,]\vec{Y} =_\beta & \text{by Lemma 4.14(2)}, \\
\mathtt{I} & \text{by the induction hypothesis.}
\end{array}$$

On the other side, we have:

$$\begin{array}{ll}
(\lambda x_1 \dots x_{n+p}.yN_1 \cdots N_{m+p})[\mathtt{P}_k/\vec{y}\,]\vec{X} = & \\
(\lambda x_1 \dots x_{n+p}.\mathtt{P}_k N_1' \cdots N_{m+p}')\vec{X} =_\beta & \text{where } N_\ell' = N_\ell\,[\mathtt{P}_k/\vec{y}\,], \\
(\mathtt{P}_k N_1'' \cdots N_{m+p}'')\Omega^{\sim k-m-p}\,\mathtt{U}_{i+1}^k\vec{Y} =_\beta & \text{where } N_\ell'' = N_\ell'\,[\mathtt{P}_k/\vec{x}\,], \\
(\lambda z.zN_1'' \cdots N_{m+p}''\Omega^{\sim k-m-p})\,\mathtt{U}_{i+1}^k\vec{Y} =_\beta & \text{by Lemma 4.14(1)}, \\
N_{i+1}''\vec{Y} = N_{i+1}[\mathtt{P}_k/\vec{y},\vec{x}\,]\vec{Y} =_\mathcal{B} & \text{by Lemma 4.14(2)}, \\
\mathtt{J}_T & \text{by the induction hypothesis.}
\end{array}$$

□

Thanks to the Context Lemma we obtain the Morris-separation result in its full generality.

Theorem 4.16 (Morris Separation)**.**
Let $M, N \in \Lambda$ be such that $\mathcal{H}^* \vdash M = N$ while $\mathcal{H}^+ \vdash M \not\sqsubseteq N$. There is a closed head context $C[]$ such that

$$C[M] =_{\beta\eta} \mathtt{I} \qquad \textit{and} \qquad C[N] =_\mathcal{B} \mathtt{J}_T \textit{ for some } T \in \mathbb{T}^\infty_{\mathsf{rec}}.$$

When $M, N \in \Lambda^o$ the context $C[]$ can be chosen applicative.

Proof. By Lemma 1.21 for $\mathcal{H}^+$, we get that $\mathcal{H}^+ \vdash M \not\sqsubseteq N$ entails the existence of a closed head context $C_2[]$ such that $C_2[M] \in_\beta \mathtt{NF}_\beta$, while $C_2[N]$ does not have a β-normal form. From $\mathcal{H}^* \vdash M = N$ we obtain $\mathcal{H}^* \vdash C_2[M] = C_2[N]$. Therefore, by Proposition 4.12 there exists a Morris separator $\sigma : C_2[M] \not\sqsubseteq C_2[N]$. We can now apply Lemma 4.15, and get a head context $C_1[] = (\lambda\vec{y}.[])\vec{\mathtt{P}}_k\vec{X}$, where $\vec{y} = \mathsf{FV}(C_2[M]C_2[N])$, and such that $C_1[C_2[M]] =_\beta \mathtt{I}$ and $C_1[C_2[N]] =_\mathcal{B} \mathtt{J}_T$ for some $T \in \mathbb{T}^\infty_{\mathsf{rec}}$. Hence the closed head context $C[]$ we are looking for is actually $C_1[C_2[]]$. When $M, N \in \Lambda^o$, all the contexts can be chosen applicative. □

4.3. **The λ-Theory $\mathcal{B}\omega$ is Included in $\mathcal{H}^+$.** The fact that $\mathcal{H}^+$ satisfies the ω-rule is an easy consequence of the Morris Separation Theorem. Without loss of generality we can focus on (ω^0), namely the restriction of ω to closed λ-terms.

Lemma 4.17. *Let $M, N \in \Lambda^o$ be such that $M \in_\beta \mathtt{NF}_\beta$, while N does not have a β-normal form. Then, there exist $k \geq 1$ and combinators $Z_1, \dots, Z_k \in \Lambda^o$ such that*

- *$MZ_1 \cdots Z_k \in_\beta \mathtt{NF}_\beta$ while*
- *$NZ_1 \cdots Z_k$ does not have a β-normal form.*

Proof. By hypothesis $\mathcal{H}^+ \vdash M \not\sqsubseteq N$. There are two possible cases.

Case $\mathcal{H}^* \vdash M \sqsubseteq N$. Since $M \in \mathtt{NF}_\beta$, Theorem 2.10(i) actually entails $\mathcal{H}^* \vdash M = N$. Therefore, by Theorem 4.16 there are $Z_1, \dots, Z_k \in \Lambda^o$ and $T \in \mathbb{T}^\infty_{\mathsf{rec}}$ such that $MZ_1 \cdots Z_k =_\beta \mathtt{I}$ and $NZ_1 \cdots Z_k =_\mathcal{B} \mathtt{J}_T$. If $k = 0$ just take $Z_1 = \mathtt{I}$ and conclude since $\mathtt{J}_T\mathtt{I} =_\mathcal{B} \mathtt{J}_{T'}$ for some $T' \in \mathbb{T}^\infty_{\mathsf{rec}}$ by Theorem 4.4 and Lemma 4.6(ii).

Case $\mathcal{H}^* \vdash M \not\sqsubseteq N$. By the usual semi-separability theorem, there are $Z_1, \dots, Z_k \in \Lambda^o$ such that $M\vec{Z} =_\beta \mathtt{I}$ and $N\vec{Z} =_\mathcal{H} \Omega$. When $k = 0$ we can take again $Z_1 = \mathtt{I}$ since $\Omega\mathtt{I} =_\mathcal{H} \Omega$. $\square$

Lemma 4.18. *Let $M, N \in \Lambda^o$. If $\forall Z \in \Lambda^o, MZ =_{\mathcal{H}^+} NZ$, then*

$$\forall \vec{P} \in \Lambda^o (M\vec{P} \in_\beta \mathtt{NF}_\beta \iff N\vec{P} \in_\beta \mathtt{NF}_\beta).$$

Proof. Suppose that for all $Z, \vec{Q} \in \Lambda^o$, $MZ\vec{Q} \in_\beta \mathtt{NF}_\beta$ if and only if $NZ\vec{Q} \in_\beta \mathtt{NF}_\beta$. We show by induction on the length k of $\vec{P} \in \Lambda^o$ that $M\vec{P} \in_\beta \mathtt{NF}_\beta$ if and only if $N\vec{P} \in_\beta \mathtt{NF}_\beta$.

Base: $k = 0$. Since the contrapositive holds by Lemma 4.17.

Induction: $k > 0$. It follows trivially from the induction hypothesis. $\square$

As a consequence, we get the main result of this section.

Theorem 4.19. *$\mathcal{H}^+$ satisfies the ω-rule, therefore $\mathcal{B}\omega \subseteq \mathcal{H}^+$.*

Proof. Lemma 4.18 shows that $\mathcal{H}^+ \vdash \omega^0$, which is equivalent to $\mathcal{H}^+ \vdash \omega$ by Lemma 3.3(ii). By Lemma 3.3(iii), $\mathcal{B} \subseteq \mathcal{H}^+$ entails $\mathcal{B}\omega \subseteq \mathcal{H}^+\omega$ and from $\mathcal{H}^+ \vdash \omega$ we get $\mathcal{H}^+\omega = \mathcal{H}^+$. $\square$

5. Building Böhm Trees by Codes and Streams

Now that we have shown $\mathcal{B}\omega \subseteq \mathcal{H}^+$, we focus our efforts to prove the converse inclusion. The key step in our proof is to show that when $\mathcal{H}^+ \vdash M = N$ holds the Böhm-like tree U of Theorem 2.5 giving the "η-supremum" of $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ can always be chosen to be the Böhm tree of a λ-term P (Proposition 6.1). Intuitively, the λ-term P that we construct is going to inspect the structure of M and N looking at their codes and retrieve the correct η-expansion to apply from a suitable stream.

5.1. **Building Böhm Trees by Codes.** We start by showing that the Böhm tree of a λ-term can be reconstructed from its code.

Definition 5.1.

- Let us fix an effective one-to-one encoding $\# : \Lambda \to \mathbb{N}$ associating with every λ-term M its *code* $\#M$ (the Gödel number of M).
- The *quote* $\ulcorner M \urcorner$ of M is the corresponding numeral $\mathrm{c}_{\#M}$.

We recall some well established facts from [3, §8.1]. By [3, Thm. 8.1.6], there is a λ-term $\mathtt{E} \in \Lambda^o$ such that $\mathtt{E}\ulcorner M \urcorner = M$ for all $M \in \Lambda^o$. This is false in general for open λ-terms M.

Remark 5.2. The following operations are effective:

- from $\#M$ compute $\#M'$ where $M \to_h M'$ (since the head-reduction is an effective reduction strategy),
- from $\#(\lambda\vec{x}.yM_1 \cdots M_k)$ compute $\#M_i$ for $i \leq k$,
- from $\#M$ compute $\#(\lambda x_1 \dots x_n.M)$ for $x_1, \dots, x_n \in \mathrm{Var}$.

From Remark 5.2 and Church's thesis, the following combinator Φ exists and can be defined using the fixed point combinator $\mathtt{Y}$.

Definition 5.3. Let $\Phi \in \Lambda^o$ be such that for all $M \in \Lambda^o$:

- if $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$, then
$$\Phi\lceil M\rceil = \lambda \underline{x}_1 \dots \underline{x}_n.\underline{x}_j(\Phi\lceil \lambda\vec{x}.M_1\rceil \underline{x}_1 \cdots \underline{x}_n) \cdots (\Phi\lceil \lambda\vec{x}.M_k\rceil \underline{x}_1 \cdots \underline{x}_n)$$
where the $\underline{x}_i$ are underlined to stress the fact that they are fresh variables.
- $\Phi\lceil M\rceil$ is unsolvable whenever M is unsolvable.

The term Φ builds the Böhm tree of M from its code $\#M$. Notice that the closure $\lambda\vec{x}.M_i$ on the recursive calls is needed to obtain a closed term (since $M \in \Lambda^o$ entails $\mathsf{FV}(M_i) \subseteq \vec{x}$). In the definition above we use the fact that $\mathcal{B}$ is a λ-theory, so $\mathrm{BT}(\lambda\vec{x}.M) = \lambda\vec{x}.\mathrm{BT}(M)$ thus the free variables $x_1, \dots, x_n$ can be reapplied externally. This commutation property between $\mathrm{BT}(-)$ and λ-abstraction will be silently used when proving statements about Böhm trees of closed λ-terms, as below.

Lemma 5.4. *For all $M \in \Lambda^o$, we have $\mathcal{B} \vdash \Phi\lceil M\rceil = M$.*

Proof. We prove by coinduction that $\mathrm{BT}(\Phi\lceil M\rceil) = \mathrm{BT}(M)$.

If M is unsolvable then $\mathrm{BT}(\Phi\lceil M\rceil) = \mathrm{BT}(M) = \perp$.

Otherwise M is solvable, therefore we have $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ and $\Phi\lceil M\rceil = \lambda x_1 \dots x_n.x_j(\Phi\lceil \lambda\vec{x}.M_1\rceil x_1 \cdots x_n) \cdots (\Phi\lceil \lambda\vec{x}.M_k\rceil x_1 \cdots x_n)$. We conclude since by coinductive hypothesis we have $\mathrm{BT}(\Phi\lceil \lambda\vec{x}.M_i\rceil)\vec{x} = (\lambda\vec{x}.\mathrm{BT}(M_i))\vec{x} = \mathrm{BT}(M_i)$ for every $i \leq k$. $\square$

5.2. **η-Expanding Böhm Trees from Streams.** The construction of Φ might look unimpressive in the sense that also the enumerator $\mathtt{E}$ enjoys the property $\mathrm{BT}(\mathtt{E}\lceil M\rceil) = \mathrm{BT}(M)$ for all $M \in \Lambda^o$. However, $\mathtt{E}$ does not satisfy the recursive equation of Definition 5.3, which has the advantage of exposing the structure of the Böhm tree and, doing so, opens the way for altering the tree. For instance, it is possible to modify Definition 5.3 in order to obtain a λ-term Ψ which builds a finitary η-expansion of $\mathrm{BT}(M)$ starting from the code of M and a stream of finite η-expansions of the identity.

Definition 5.5.

(1) Let $[\mathtt{I}]_{n\in\mathbb{N}}$ be the stream containing infinitely many copies of $\mathtt{I}$:
$$[\mathtt{I}]_{n\in\mathbb{N}} = [\mathtt{I}, [\mathtt{I}, [\mathtt{I}, \dots]]]$$
(2) Let $\vec{\eta} = (\eta_0, \eta_1, \eta_2, \dots)$ be an effective enumeration of all (closed) finite η-expansions of the identity, i.e. $\eta_i \in \mathcal{I}^\eta \cap \Lambda^o$ for all $i \in \mathbb{N}$. Define the corresponding stream
$$[\eta_n]_{n\in\mathbb{N}} = [\eta_0, [\eta_1, [\eta_2, \dots]]].$$

Notice that, from now on, we consider the enumeration $\vec{\eta}$ and the stream $[\eta_n]_{n\in\mathbb{N}}$ fixed.

In order to decide what η-expansion is applied at a certain position σ in $\mathrm{BT}(M)$, we use a function $f(\sigma) = k$ and extract the η-expansion of index k from $[\eta_n]_{n\in\mathbb{N}}$ using the projection π_k. Since f needs to be λ-definable we consider f computable "after coding". We recall that an effective encoding $\# : \mathbb{N}^* \to \mathbb{N}$ of all finite sequences of natural numbers has been fixed in Section 1.2 and that we denote by $\lceil\sigma\rceil$ the corresponding numeral $\mathrm{c}_{\#\sigma}$.

In the following definition s is an arbitrary variable, but in practice we will always apply $\Psi_f\lceil M\rceil\lceil\sigma\rceil$ to some stream of combinators.

Definition 5.6. Let $f : \mathbb{N}^* \to \mathbb{N}$ be a computable function, and $\Psi_f \in \Lambda^o$ be such that for all $M \in \Lambda^o$ and $\sigma \in \mathbb{N}^*$:

- if $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ then
$$\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner s = \lambda \vec{\underline{x}}.\pi_{f(\sigma)} s \left(\underline{x}_j (L_1\, \underline{x}_1 \cdots \underline{x}_n) \cdots (L_k\, \underline{x}_1 \cdots \underline{x}_n)\right)$$
where $L_i = \Psi_f \ulcorner \lambda \vec{x}.M_i \urcorner \ulcorner \sigma.i \urcorner\, s$ for all $i \leq k$ and π_n denotes the n-th projection for streams,
- $\Psi_f \ulcorner M \urcorner$ is unsolvable whenever M is unsolvable.

The actual existence of such a Ψ_f follows from Remark 5.2, the effectiveness of the encodings, the computability of f and Church's thesis. We now verify that the λ-term $\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner$ when applied to the stream $[\eta_n]_{n\in\mathbb{N}}$ actually computes a finitary η-expansion of $\mathrm{BT}(M)$ in the sense of Definition 2.3.

Lemma 5.7. *Let $f : \mathbb{N}^* \to \mathbb{N}$ be a computable function. For all $M \in \Lambda^o$ and $\sigma \in \mathbb{N}^*$:*
$$M \leq^\eta \Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\eta_n]_{n\in\mathbb{N}}.$$

Proof. We prove by coinduction that $\mathrm{BT}(M) \leq^\eta \mathrm{BT}(\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\eta_n]_{n\in\mathbb{N}})$.

If M is unsolvable, then $\mathrm{BT}(M) = \mathrm{BT}(\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\eta_n]_{n\in\mathbb{N}}) = \bot$.

Otherwise $M \twoheadrightarrow_h \lambda \vec{x}.x_j M_1 \cdots M_k$. Thus, for $f(\sigma) = q$ and $\eta_q = \lambda y z_1 \cdots z_m.y Q_1 \cdots Q_m$ where $\lambda z_i.Q_i \in \mathcal{I}^\eta$ for all $i \leq m$ we have:
$$\begin{aligned}\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\eta_n]_{n\in\mathbb{N}} &=_\beta \lambda \vec{x}.\pi_q [\eta_n]_{n\in\mathbb{N}} (x_j (L_1 \vec{x}) \cdots (L_k \vec{x}))\\ &=_\beta \lambda \vec{x}.\eta_q (x_j (L_1 \vec{x}) \cdots (L_k \vec{x}))\\ &=_\beta \lambda \vec{x} z_1 \dots z_m.x_j (L_1 \vec{x}) \cdots (L_k \vec{x}) Q_1 \cdots Q_m\end{aligned}$$
where $L_i = \Psi_f \ulcorner \lambda \vec{x}.M_i \urcorner \ulcorner \sigma.i \urcorner [\eta_n]_{n\in\mathbb{N}}$ for all $i \leq k$. We conclude because by coinductive hypothesis we get for each $i \leq k$ that $\mathrm{BT}(M_i) \leq^\eta \mathrm{BT}(L_i \vec{x})$ holds. □

Since Ψ_f picks the η-expansion to apply from the input stream, we can retrieve the behaviour of Φ by applying $[\mathtt{I}]_{n\in\mathbb{N}}$.

Lemma 5.8. *Let $f : \mathbb{N}^* \to \mathbb{N}$ be computable. For all $M \in \Lambda^o$ and $\sigma \in \mathbb{N}^*$, we have*
$$\mathcal{B} \vdash M = \Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\mathtt{I}]_{n\in\mathbb{N}}.$$

Proof. We prove by coinduction that $\mathrm{BT}(M) = \mathrm{BT}(\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\mathtt{I}]_{n\in\mathbb{N}})$.

If M is unsolvable, then $\mathrm{BT}(M) = \mathrm{BT}(\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\mathtt{I}]_{n\in\mathbb{N}}) = \bot$.

Otherwise $M \twoheadrightarrow_h \lambda \vec{x}.x_j M_1 \cdots M_k$. Since $\pi_q [\mathtt{I}]_{n\in\mathbb{N}} =_\beta \mathtt{I}$ for all $q \in \mathbb{N}$ we have:
$$\begin{aligned}\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner [\mathtt{I}]_{n\in\mathbb{N}} &=_\beta \lambda \vec{x}.\pi_{f(\sigma)} [\mathtt{I}]_{n\in\mathbb{N}} (x_j (L_1 \vec{x}) \cdots (L_k \vec{x}))\\ &=_\beta \lambda \vec{x}.\mathtt{I} (x_j (L_1 \vec{x}) \cdots (L_k \vec{x}))\\ &=_\beta \lambda \vec{x}.x_j (L_1 \vec{x}) \cdots (L_k \vec{x})\end{aligned}$$
where $L_i = \Psi_f \ulcorner \lambda \vec{x}.M_i \urcorner \ulcorner \sigma.i \urcorner [\eta_n]_{n\in\mathbb{N}}$ for all $i \leq k$. We conclude because by coinductive hypothesis we get for each $i \leq k$ that $\mathrm{BT}(M_i) = \mathrm{BT}(L_i \vec{x})$ holds. □

We would like to draw the attention to the fact that the λ-term Ψ_f does not inspect the structure of the input stream to select one of its elements, it blindly relies on the function f. This is the key property that opens the way to modify the behaviour of Ψ_f by merely changing the input stream as in Lemma 5.8. We mention this fact since the λ-term Ξ that we define in the next section satisfies a similar property and we exploit it in a crucial way.

5.3. **Building the η-Supremum.** Using similar techniques, we now define a λ-term Ξ that builds from the codes of M, N and the stream $[\eta_n]_{n\in\mathbb{N}}$ the (smallest) η-supremum satisfying

$$M \le^\eta \Xi\ulcorner M\urcorner\ulcorner N\urcorner[\eta_n]_{n\in\mathbb{N}} \ge^\eta N$$

when such an η-supremum exists, that is whenever M and N are compatible (Proposition 6.1). Intuitively, at every common position σ, the λ-term Ξ needs to compare the structure of the subterms of M, N at σ and apply the correct expansion η_i taken from $[\eta_n]_{n\in\mathbb{N}}$.

Rather than proving that there exists a computable function $f : \mathbb{N}^* \to \mathbb{N}$ associating to every σ the corresponding η_i (which can be tedious) we use the following property of $\vec{\eta} = (\eta_0, \eta_1, \dots, \eta_i, \dots)$: since every closed η-expansion $Q \in \mathcal{I}^\eta$ is β-normalizable and the enumeration $\vec{\eta}$ is effective, it is possible to decide starting from the code $\#Q$ the index i of Q in $\vec{\eta}$. Moreover, it is possible to choose such an index i minimal.

Lemma 5.9. *There exists a computable function $\iota : \mathbb{N} \to \mathbb{N}$ such that, for all $M \in \Lambda^o$, if $M =_\beta \eta_i$ and $M \neq_\beta \eta_k$ for all $k < i$ then $\iota(\#M) = i$.*

Proof. Let $\delta(m, n)$ be the partial computable map satisfying for all normalizing $M, N \in \Lambda^o$:

$$\delta(\#M, \#N) = \begin{cases} 0 & \text{if } M \text{ and } N \text{ have the same } \beta\text{-normal form,} \\ 1 & \text{otherwise.} \end{cases}$$

Then ι can be defined as $\iota(n) := \mu k.\delta\big(\#(\pi_k \circ \vec{\eta}), n\big) = 0$. $\square$

From now on we consider fixed such a function ι, which depends on the enumeration $\vec{\eta}$ generating the stream $[\eta_n]_{n\in\mathbb{N}}$.

Definition 5.10. For $M, N \in \Lambda$, we define:

(i) $M \le_h N$ whenever $M \twoheadrightarrow_h \lambda\vec{x}.yM_1 \cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}z_1 \dots z_m.yN_1 \cdots N_k Q_1 \cdots Q_m$ with $\lambda z_i.Q_i \in \mathcal{I}^\eta$ for all $i \le m$,
(ii) $M \sim_h N$ if both $M \le_h N$ and $N \le_h M$ hold,
(iii) $M <_h N$ if $M \le_h N$ holds but $M \not\sim_h N$.

Whenever $M \le_h N$ holds, we say that N *looks like an η-expansion of M*. This does not necessarily mean that it actually is, as shown by the following examples.

Examples 5.11. We have:
- $\lambda z.x\mathtt{F}z \le_h \lambda z.x\mathtt{K}z$ since we do not require that $\mathtt{F} \le_h \mathtt{K}$ holds;
- $z \le_h \lambda z.zz$ since in Definition 5.10(i) we do not check that $z_i \notin \mathsf{FV}(\mathrm{BT}(y\vec{M}\vec{N}))$.

Therefore, compared with the relation $\le^\eta$ of Definition 2.3, the relation $\le_h$ is weaker since it lacks the coinductive calls and the occurrence check on the z_i's. This is necessary to ensure the following semi-decidability property.

Remark 5.12. The property $M \le_h N$ can be semi-decided by the following procedure:
- first head-reduce in parallel M, N until they reach a head normal form,
- if both reductions terminate, compare the two hnf's and check whether they have the shape of Definition 5.10(i),
- then semi-decide whether $Q_i \twoheadrightarrow_{\beta\eta} z_i$ for all $i \le m$.

This procedure might fail to terminate when $M \not\le_h N$.

By Remarks 5.2 and 5.12, the fact that ι is computable (Lemma 5.9) and Church's thesis, the λ-term Ξ_ι below exists.

Definition 5.13. Let $\iota : \mathbb{N} \to \mathbb{N}$ be the computable function from Lemma 5.9. We define a combinator $\Xi_\iota \in \Lambda^o$ such that for all $M, N \in \Lambda^o$:

(1) if $M \leq_h N$ with

$$M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k \text{ and } N \twoheadrightarrow_h \lambda x_1 \dots x_n z_1 \dots z_m.x_j N_1 \cdots N_k Q_1 \cdots Q_m$$

then

$$\Xi_\iota \lceil N \rceil \lceil M \rceil s = \lambda \underline{x}_1 \dots \underline{x}_n.\pi_q s\big(\underline{x}_j(\Upsilon_1 \underline{x}_1 \cdots \underline{x}_n) \cdots (\Upsilon_k \underline{x}_1 \cdots \underline{x}_n)\big)$$

where $q = \iota\big(\#\mathtt{NF}_\beta(\lambda \underline{y} z_1 \dots z_m.\underline{y} Q_1 \cdots Q_m)\big)$ and $\Upsilon_i = \Xi_\iota \lceil \lambda x_1 \dots x_n.M_i \rceil \lceil \lambda x_1 \dots x_n.N_i \rceil s$.

(2) if $N <_h M$ then $\Xi_\iota \lceil N \rceil \lceil M \rceil s = \Xi_\iota \lceil M \rceil \lceil N \rceil s$,

(3) otherwise, $\Xi_\iota \lceil N \rceil \lceil M \rceil s$ is unsolvable.

In the definition of Ξ_ι there are some subtleties that deserve to be discussed. The fact that $Q_i \twoheadrightarrow_{\beta\eta} z_i$ for all $i \leq m$, although not explicitly written, is a consequence of $M \leq_h N$. *A priori* $\lambda z_i.Q_i \in \mathcal{I}^\eta$ might be open (consider for instance $\lambda z_i.\mathtt{K} z_i y \to_\beta \mathtt{I}$) but its β-normal form is always a closed λ-term. This is the reason why we compute $\mathtt{NF}_\beta(\lambda y \vec{z}.y\vec{Q})$ before applying ι to its code. In particular, ι is defined on all the codes $\#(\mathtt{NF}_\beta(\lambda y \vec{z}.y\vec{Q}))$.

The following commutativity property follows from the second condition of Definition 5.13 and should be natural considering the fact that $\Xi_\iota \lceil M \rceil \lceil N \rceil [\eta_n]_{n\in\mathbb{N}}$ is supposed to compute the η-join of $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$, which is a commutative operation.

Lemma 5.14. *For all $M, N \in \Lambda^o$, we have:*

$$\mathcal{B} \vdash \Xi_\iota \lceil M \rceil \lceil N \rceil = \Xi_\iota \lceil N \rceil \lceil M \rceil.$$

Proof. We prove by coinduction that $\mathrm{BT}(\Xi_\iota \lceil M \rceil \lceil N \rceil) = \mathrm{BT}(\Xi_\iota \lceil N \rceil \lceil M \rceil)$.

If M, N are unsolvable or neither $M \leq_h N$ nor $N \leq_h M$ holds, then $\mathrm{BT}(\Xi_\iota \lceil M \rceil \lceil N \rceil) = \mathrm{BT}(\Xi_\iota \lceil N \rceil \lceil M \rceil) = \bot$.

The cases $M <_h N$ and $N <_h M$ follow directly from the definition.

If $M \sim_h N$, then we have $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ and $N \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j N_1 \cdots N_k$. Since $\pi_q s \twoheadrightarrow_\beta s\, \mathtt{F}^{\sim q}\, \mathtt{K}$ we have

$$\begin{aligned} \Xi_\iota \lceil M \rceil \lceil N \rceil s &=_\beta \lambda \vec{x}.s\, \mathtt{F}^{\sim q}\, \mathtt{K}(x_j(\Upsilon_1 \vec{x}) \cdots (\Upsilon_k \vec{x})) \\ \Xi_\iota \lceil N \rceil \lceil M \rceil s &=_\beta \lambda \vec{x}.s\, \mathtt{F}^{\sim q}\, \mathtt{K}(x_j(\Upsilon'_1 \vec{x}) \cdots (\Upsilon'_k \vec{x})) \end{aligned}$$

where, for all $i \leq k$, we have $\Upsilon_i = \Xi_\iota \lceil \lambda \vec{x}.M_i \rceil \lceil \lambda \vec{x}.N_i \rceil s$ and $\Upsilon'_i = \Xi_\iota \lceil \lambda \vec{x}.N_i \rceil \lceil \lambda \vec{x}.M_i \rceil s$. We conclude since, by coinductive hypothesis, we get $\mathrm{BT}(\Upsilon_i \vec{x}) = \mathrm{BT}(\Upsilon'_i \vec{x})$ for all $i \leq k$. □

Another property that we expect is that whenever $M \leq^\eta N$ the λ-term $\Xi_\iota \lceil M \rceil \lceil N \rceil [\eta_n]_{n\in\mathbb{N}}$ computes the Böhm tree of N.

Lemma 5.15. *For all $M, N \in \Lambda^o$, if $M \leq^\eta N$ then*

$$\mathcal{B} \vdash \Xi_\iota \lceil M \rceil \lceil N \rceil [\eta_n]_{n\in\mathbb{N}} = N.$$

Proof. We prove by coinduction that $\mathrm{BT}(\Xi_\iota \lceil M \rceil \lceil N \rceil [\eta_n]_{n\in\mathbb{N}}) = \mathrm{BT}(N)$.

If M, N are both unsolvable, then also $\Xi_\iota \lceil M \rceil \lceil N \rceil$ must be and their Böhm trees are $\bot$.

Otherwise, we have $M \twoheadrightarrow_h \lambda \vec{x}.x_j M_1 \cdots M_k$ and $N \twoheadrightarrow_h \lambda \vec{x} z_1 \dots z_m.x_j N_1 \cdots N_k Q_1 \cdots Q_m$ where each $z_\ell \notin \mathsf{FV}(\mathrm{BT}(x_j \vec{M}\vec{N}))$, $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ for all $\ell \leq m$ and $M_i \leq^\eta N_i$ for all $i \leq k$. In particular $M \leq_h N$ holds, so the first condition of Definition 5.13 applies.

From $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ it follows that $\lambda y\vec{z}.y\vec{Q} \in \mathcal{I}^\eta$, therefore $\iota(\#\mathtt{NF}_\beta(\lambda y\vec{z}.y\vec{Q})) = q$ for some index q. Setting $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M\rceil\lceil\lambda\vec{x}.N\rceil[\eta_n]_{n\in\mathbb{N}}$, easy calculations give:

$$\begin{array}{rcl}\Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} & =_\beta & \lambda\vec{x}.\pi_q[\eta_n]_{n\in\mathbb{N}}(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ & =_\beta & \lambda\vec{x}.(\lambda y\vec{z}.y\vec{Q})(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ & =_\beta & \lambda\vec{x}\vec{z}.x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})Q_1\cdots Q_m\end{array}$$

We conclude since, by coinductive hypothesis, we have $\mathrm{BT}(\Upsilon_i) = \mathrm{BT}(\lambda\vec{x}.N_i)$ for all $i \le k$. □

Under the assumption $M \le^\eta N$ we can also use Ξ_ι to retrieve the Böhm tree of M by applying the stream $[\mathtt{I}]_{n\in\mathbb{N}}$. Since ι has been defined in function of the enumeration $\vec{\eta}$, $\iota(\#Q)$ still provides an index q such that $\pi_q[\eta_n]_{n\in\mathbb{N}} =_\beta Q$ but when applied to $[\mathtt{I}]_{n\in\mathbb{N}}$ it necessarily gives $\pi_q[\mathtt{I}]_{n\in\mathbb{N}} = \mathtt{I}$. This technique is analogous to the one used in Lemma 5.8.

Lemma 5.16. *For all $M, N \in \Lambda^o$, if $M \le^\eta N$ then*

$$\mathcal{B} \vdash \Xi_\iota\lceil M\rceil\lceil N\rceil[\mathtt{I}]_{n\in\mathbb{N}} = M.$$

Proof. We prove by coinduction that $\mathrm{BT}(\Xi_\iota\lceil M\rceil\lceil N\rceil[\mathtt{I}]_{n\in\mathbb{N}}) = \mathrm{BT}(M)$.

If M, N are both unsolvable, then also $\Xi_\iota\lceil M\rceil\lceil N\rceil$ must be and their Böhm trees are $\bot$.

Otherwise, $M \twoheadrightarrow_h \lambda\vec{x}.x_jM_1\cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}z_1\ldots z_m.x_jN_1\cdots N_kQ_1\cdots Q_m$ where each $z_\ell \notin \mathsf{FV}(\mathrm{BT}(x_j\vec{M}\vec{N}))$, $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ for all $\ell \le m$ and $M_i \le^\eta N_i$ for all $i \le k$. In particular $M \le_h N$ holds, so the first condition of Definition 5.13 applies.

From $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ it follows that $\lambda y\vec{z}.y\vec{Q} \in \mathcal{I}^\eta$, therefore $\iota(\#\mathtt{NF}_\beta(\lambda y\vec{z}.y\vec{Q})) = q$ for some index q. Setting $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M\rceil\lceil\lambda\vec{x}.N\rceil[\mathtt{I}]_{n\in\mathbb{N}}$, easy calculations give:

$$\begin{array}{rcl}\Xi_\iota\lceil M\rceil\lceil N\rceil[\mathtt{I}]_{n\in\mathbb{N}} & =_\beta & \lambda\vec{x}.\pi_q[\mathtt{I}]_{n\in\mathbb{N}}(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ & =_\beta & \lambda\vec{x}.\mathtt{I}(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ & =_\beta & \lambda\vec{x}.x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})\end{array}$$

We conclude since, by coinductive hypothesis, we have $\mathrm{BT}(\Upsilon_i) = \mathrm{BT}(\lambda\vec{x}.M_i)$ for all $i \le k$. □

6. $\mathcal{B}\omega = \mathcal{H}^+$ and Sallé's Conjecture is False

This section is devoted to prove that $\mathcal{B}\omega = \mathcal{H}^+$ holds (Theorem 6.6). As mentioned earlier, the first step is to show that the term Ξ_ι defined in the previous section, when applied to $\mathcal{H}^+$-equivalent λ-terms, actually computes their η-supremum.

Proposition 6.1. *For all $M, N \in \Lambda^o$,*

$$\mathcal{H}^+ \vdash M = N \iff M \le^\eta \Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} \ge^\eta N$$

Proof. ($\Leftarrow$) It follows directly from Theorem 2.5.

($\Rightarrow$) By Theorem 2.5, we known that there exists a Böhm-like tree $U \in \mathbb{BT}$ such that $\mathrm{BT}(M) \le^\eta U \ge^\eta \mathrm{BT}(N)$. As usual, we proceed by coinduction on the Böhm(-like) trees.

If M or N is unsolvable then $\mathrm{BT}(M) = \mathrm{BT}(N) = U = \bot$.

Otherwise, from $\mathrm{BT}(M) \le^\eta U \ge^\eta \mathrm{BT}(N)$ we have, say:

$$M \twoheadrightarrow_h \lambda\vec{x}.x_jM_1\cdots M_k, \qquad N \twoheadrightarrow_h \lambda\vec{x}z_1\ldots z_m.x_jN_1\cdots N_kQ_1\cdots Q_m,$$
$$U = \lambda\vec{x}z_1\ldots z_m\ldots z_{m'}.x_jU_1\cdots U_kQ'_1\cdots Q'_m\cdots Q'_{m'},$$

such that

- $z_1,\ldots,z_{m'} \notin \mathsf{FV}(x_j\mathrm{BT}(M_1)\cdots\mathrm{BT}(M_k)\,U_1\cdots U_k)$,
- $z_{m+1},\ldots,z_{m'} \notin \mathsf{FV}(x_j\mathrm{BT}(N_1)\cdots\mathrm{BT}(N_k))$,

- $\mathrm{BT}(M_i) \leq^\eta U_i$ and $\mathrm{BT}(N_i) \leq^\eta U_i$ for all $i \leq k$,
- $Q_\ell \leq^\eta Q'_\ell$ for all $\ell \leq m$,
- and $\lambda z_{\ell'}.Q'_{\ell'} \in \mathcal{I}^\eta$ for all $\ell' > m$.

By Lemma 4.3, we have $Q'_\ell \twoheadrightarrow_{\beta\eta} z_\ell$ so $Q_\ell \leq^\eta Q'_\ell$ entails $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$, hence M looks like an η-expansion of N. (That is, $M \leq_h N$ holds.)

Put $q = \iota\big(\#(\mathtt{nf}(\lambda y z_1 \dots z_m.yQ_1 \cdots Q_m))\big)$, $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M\rceil\lceil\lambda\vec{x}.N\rceil[\eta_n]_{n\in\mathbb{N}}$. We have

$$\begin{array}{rl}\Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} =_\beta & \lambda\vec{x}.\pi_q[\eta_n]_{n\in\mathbb{N}}(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ =_\beta & \lambda\vec{x}.(\lambda y\vec{z}.yQ_1\cdots Q_m)(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ =_\beta & \lambda\vec{x}\vec{z}.x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})Q_1\cdots Q_m.\end{array}$$

This case follows by coinductive hypotheses since, for all $i \leq k$, $\lambda\vec{x}.M_i \leq^\eta \Upsilon_i \geq^\eta \lambda\vec{x}.N_i$.

The symmetric case $N <_h M$ is treated analogously, using Lemma 5.14 to apply the coinductive hypotheses. □

The second step towards the proof of Theorem 6.6 is to show that the streams $[\mathtt{I}]_{n\in\mathbb{N}}$ and $[\eta_n]_{n\in\mathbb{N}}$ are equated in $\mathcal{B}\omega$. To prove such a result, we are going to use the following auxiliary streams.

Definition 6.2. Using the fixed point operator $\mathtt{Y}$, define the streams:

- $\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega\, yx = [yx, [y\Omega x, [y\Omega^{\sim 2}x, [y\Omega^{\sim 3}x, \dots]]]]$,
- $\langle\!\langle \eta\rangle\!\rangle^\Omega\, yx = [y(\eta_0 x), [y\Omega(\eta_1 x), [y\Omega^{\sim 2}(\eta_2 x), [y\Omega^{\sim 3}(\eta_3 x), \dots]]]]$.

The streams $\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega$ and $\langle\!\langle \eta\rangle\!\rangle^\Omega$ are equal in $\mathcal{B}\omega$, for the same reason the λ-terms P, Q of Figure 3 are. The formal reasoning is the following.

Lemma 6.3. *$\mathcal{B}\omega \vdash \langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega = \langle\!\langle \eta\rangle\!\rangle^\Omega$.*

Proof. Let $M \in \Lambda^o$, by Lemma 1.8 there exists $k \in \mathbb{N}$ such that $M\Omega^{\sim k} =_\mathcal{B} \Omega$. So we have:

$$\begin{array}{rll}\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega M & =_\mathcal{B} & \lambda x.[Mx, [M\Omega x, [\dots, [M\Omega^{\sim k-1}x, [\Omega, \dots]]]]]\\ & =_\mathcal{B} & \lambda x.[M(\mathtt{I}x), [M\Omega(\mathtt{I}x), [\dots, [M\Omega^{\sim k-1}(\mathtt{I}x), [\Omega, \dots]]]]]\\ & =_{\beta\eta} & \lambda x.[M(\eta_0 x), [M\Omega(\eta_1 x), [\dots, [M\Omega^{\sim k-1}(\eta_{k-1}x), [\Omega, \dots]]]]]\\ & =_\mathcal{B} & \langle\!\langle \eta\rangle\!\rangle^\Omega M,\end{array}$$

where the third equality follows from $\mathtt{I} =_{\beta\eta} \eta_i$ for all $i \in \mathbb{N}$. Since M is an arbitrary closed λ-term, we can apply the ω-rule and conclude that $\mathcal{B}\omega \vdash \langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega = \langle\!\langle \eta\rangle\!\rangle^\Omega$. □

As the variable y occurs in head-position in the λ-terms occurring in the stream $\langle\!\langle \eta\rangle\!\rangle^\Omega yx$ (resp. $\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega yx$), we can substitute for it a suitably modified projection that erases the Ω's and returns the n-th occurrence of x in $\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega$ (resp. $(\eta_n x)$ in $\langle\!\langle \eta\rangle\!\rangle^\Omega$).

Lemma 6.4. *There exists a closed λ-term Eq such that, for all $n \in \mathbb{N}$:*

$$\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega =_\mathcal{B} \mathtt{I}, \qquad \mathsf{Eq}\,\mathrm{c}_n\langle\!\langle \eta\rangle\!\rangle^\Omega =_\mathcal{B} \eta_n.$$

Proof. Let Eq be a λ-term satisfying the recursive equation

$$\mathsf{Eq}\,n\,s = \mathsf{ifz}(n, \lambda z.s\mathtt{I}z\mathtt{K}, \mathsf{Eq}\,(\mathsf{pred}\,n)\,(\lambda zw.s(\mathtt{K}z)w\mathtt{F})).$$

By induction on n, we show

$$\mathsf{Eq}\,\mathrm{c}_n(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) =_\mathcal{B} \eta_{n+k}$$

for all $n, k \in \mathbb{N}$. Note that $\eta_i \in \mathcal{I}^\eta$ entails $\lambda z.\eta_i z =_\beta \eta_i$.

If $n = 0$ then

$$\begin{array}{ll} & \mathsf{Eq}\,\mathrm{c}_0(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) \\ =_\beta & \lambda z.(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}})\mathtt{I}z\mathtt{K} \\ =_\beta & \lambda z.[\Omega^{\sim i}(\eta_{i+k}z)]_{i\in\mathbb{N}}\mathtt{K} \\ =_\beta & \lambda z.\mathtt{K}\Omega^{\sim 0}(\eta_{0+k}z)[\Omega^{\sim i+1}(\eta_{i+1+k}z)]_{i\in\mathbb{N}} \\ =_\beta & \lambda z.\eta_k z \quad =_\beta \quad \eta_k. \end{array}$$

If $n = n' + 1$, then IH yields $\mathsf{Eq}\,\mathrm{c}_{n'}(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) = \eta_{n'+k}$ for all k. Thus

$$\begin{array}{ll} & \mathsf{Eq}\mathrm{c}_{n'+1}(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) \\ =_\beta & \mathsf{Eq}\mathrm{c}_{n'}(\lambda zw.\left(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}\right)(\mathtt{K}z)w\mathtt{F}) \\ =_\beta & \mathsf{Eq}\mathrm{c}_{n'}(\lambda zw.\left[(\mathtt{K}z)\Omega^{\sim 0}(\eta_{0+k}w),[(\mathtt{K}z)\Omega^{\sim i+1}(\eta_{i+1+k}w)]_{i\in\mathbb{N}}\right]\mathtt{F}) \\ =_\beta & \mathsf{Eq}\mathrm{c}_{n'}(\lambda zw.[z\Omega^{\sim i}(\eta_{i+1+k}w)_{i\in\mathbb{N}}]) \\ =_\alpha & \mathsf{Eq}\mathrm{c}_{n'}(\lambda yx.[y\Omega^{\sim i}(\eta_{i+(k+1)}x)]_{i\in\mathbb{N}}) \quad =_{\mathrm{IH}} \quad \eta_{k+1}. \end{array}$$

Analogous calculations show $\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle\mathtt{I}\rangle\!\rangle^\Omega =_\mathcal{B} \mathtt{I}$. □

Corollary 6.5. *$\mathcal{B}\omega \vdash [\mathtt{I}]_{n\in\mathbb{N}} = [\eta_n]_{n\in\mathbb{N}}$.*

Proof. From Lemmas 6.4, 1.10 and 6.3 we get:

$$[\mathtt{I}]_{n\in\mathbb{N}} =_\mathcal{B} [\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle\mathtt{I}\rangle\!\rangle^\Omega]_{n\in\mathbb{N}} =_{\mathcal{B}\omega} [\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle\eta\rangle\!\rangle^\Omega]_{n\in\mathbb{N}} =_\mathcal{B} [\eta_n]_{n\in\mathbb{N}}.$$

□

In Section 5.2 we have seen that, when $M \leq^\eta N$ holds, the λ-term $\Xi_\iota\lceil M\rceil\lceil N\rceil$ computes the Böhm tree of N from $[\eta_n]_{n\in\mathbb{N}}$ (Lemma 5.15) and the Böhm tree of M from $[\mathtt{I}]_{n\in\mathbb{N}}$ (Lemma 5.16), but now we have proved that $[\eta_n]_{n\in\mathbb{N}} =_{\mathcal{B}\omega} [\mathtt{I}]_{n\in\mathbb{N}}$. As a consequence, we get that M and N are equal in $\mathcal{B}\omega$.

Theorem 6.6. *$\mathcal{B}\omega = \mathcal{H}^+$.*

Proof. ($\subseteq$) This inclusion was shown in Theorem 4.19.

($\supseteq$) By Remark 1.17, it is enough to consider $M, N \in \Lambda^o$. If $\mathcal{H}^+ \vdash M = N$, then by Proposition 6.1 we have $M \leq^\eta P \geq^\eta N$ for $P = \Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}}$. Then we have:

$$\begin{array}{rlll} M & =_\mathcal{B} & \Xi_\iota\lceil M\rceil\lceil P\rceil[\mathtt{I}]_{n\in\mathbb{N}} & \text{by Lemma 5.16} \\ & =_{\mathcal{B}\omega} & \Xi_\iota\lceil M\rceil\lceil P\rceil[\eta_n]_{n\in\mathbb{N}} & \text{by Corollary 6.5} \\ & =_\mathcal{B} & P & \text{by Lemma 5.15} \\ & =_\mathcal{B} & \Xi_\iota\lceil N\rceil\lceil P\rceil[\eta_n]_{n\in\mathbb{N}} & \text{by Lemma 5.15} \\ & =_{\mathcal{B}\omega} & \Xi_\iota\lceil N\rceil\lceil P\rceil[\mathtt{I}]_{n\in\mathbb{N}} & \text{by Corollary 6.5} \\ & =_\mathcal{B} & N & \text{by Lemma 5.16} \end{array}$$

We conclude that $\mathcal{B}\omega \vdash M = N$. □

This theorem constitutes a refutation of Sallé's conjecture and settles one of the few open problems left in Barendregt's book [3].

7. A Characterization of $\mathcal{B}\eta$

As discussed in Section 2.1, $\mathcal{B}\eta$ equates strictly less than $\mathcal{B}\omega$, and by Theorem 6.6 than $\mathcal{H}^+$. However, to the best of our knowledge, in the literature there is no formal characterization of $\mathcal{B}\eta$ in terms of some extensional equality between Böhm trees. The only property shown in [3, §16.4] concerning the interaction between $\mathrm{BT}(-)$ and $\to_\eta$ is Lemma 2.1.

In this section we show that the approach described in Section 5.3 is general enough to prove that $\mathcal{B}\eta \vdash M = N$ holds exactly when $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ are equal up to countably many η-expansions having *uniformly bounded size* (Theorem 7.16).

7.1. Bounded η-Expansions of Böhm Trees.

We start by defining the *size* of a finite η-expansion Q of the identity. Intuitively, the size of Q is the maximum between its height and its width (namely, the maximal number of branching in its Böhm tree).

Definition 7.1. The *size* of $Q \in \mathcal{I}^\eta$ is defined inductively as follows:

$$|Q| = \begin{cases} 0 & \text{if } Q \twoheadrightarrow_\beta \mathtt{I}, \\ \max\{m, \max_{i\le m}\{|\lambda z_i.Q_i|\} + 1\} & \text{if } Q \twoheadrightarrow_\beta \lambda y z_1 \dots z_m.yQ_1 \cdots Q_m. \end{cases}$$

For $Q, Q' \in \mathcal{I}^\eta$, we have that $Q =_\beta Q'$ entails $|Q| = |Q'|$ since the size $|-|$ is determined by their β-normal forms. It is easy to verify that for each $n \in \mathbb{N}$ the size of $\mathtt{1}^n$ is $|\mathtt{1}^n| = n$.

Definition 7.2. For $p \in \mathbb{N}$, let $\mathcal{I}^\eta_p = \{Q \in \mathcal{I}^\eta : |Q| < p\}$ be the set of η-expansions of $\mathtt{I}$ whose size is bounded by p.

We have $\mathcal{I}^\eta_0 = \emptyset$, $\mathcal{I}^\eta_1 = \{\mathtt{I}\}$, $\mathcal{I}^\eta_2 = \{\mathtt{I}, \lambda yz.yz\}$, and so on. Therefore $\mathcal{I}^\eta_p \subseteq \mathcal{I}^\eta_{p+1}$ and the sequence $\left(\mathcal{I}^\eta_p\right)_{p\in\mathbb{N}}$ is increasing.

The following lemma will be useful for studying the behaviour of Ξ_ι, when applied to the codes of λ-terms whose Böhm trees differ because of bounded η-expansions.

Lemma 7.3. *Let $m, p \in \mathbb{N}$. If $m \le p$ and $\lambda z_i.Q_i \in \mathcal{I}^\eta_p$ holds for all $i \le m$, then we have that $\lambda y z_1 \dots z_m.yQ_1 \cdots Q_m \in \mathcal{I}^\eta_{p+1}$.*

Proof. Since $m \le p$ and $|\lambda z_i.Q_i| < p$ for each $i \le m$, it follows that $|\lambda y z_1 \dots z_m.yQ_1 \cdots Q_m| = \max\{m, \max\{|\lambda z_1.Q_1|, \dots, |\lambda z_m.Q_m|\} + 1\} < p + 1$. □

Now that we have formalized when $Q \in \mathcal{I}^\eta$ has size bounded by a certain p, we specify when two Böhm-like trees U, V are such that V is an η-expansion of U bounded by p.

Definition 7.4. For all $p \in \mathbb{N}$, we define the greatest relation $\le^\eta_p$ between Böhm-like trees such that $U \le^\eta_p V$ entails that:

- either $U = V = \bot$,
- or for some $m \le p$

$$U = \lambda \vec{x}.yU_1 \cdots U_k \text{ and } V = \lambda \vec{x} z_1 \dots z_m.yV_1 \cdots V_k Q_1 \cdots Q_m,$$

where $z_\ell \notin \mathsf{FV}(yU_1 \cdots U_k V_1 \cdots V_k)$, $\lambda z_i.Q_i \in \mathcal{I}^\eta_p$ for all $i \le m$ and $U_j \le^\eta_p V_j$ for all $j \le k$.

Remark that in the definition above we verify not only that the size of each $\lambda z_i.Q_i$ is bounded by p, but also that their number m is. Notice the asymmetry between the strict bound $|\lambda z_i.Q_i| < p$ and the bound $m \le p$, which arises naturally from Lemma 7.3.

Notation 7.5. For $M, N \in \Lambda$ and $p \in \mathbb{N}$, we write $M \le^\eta_p N$ whenever $\mathrm{BT}(M) \le^\eta_p \mathrm{BT}(N)$.

The next lemma follows straightforwardly from the definition.

Lemma 7.6. *For $M, N \in \Lambda$ and $p \in \mathbb{N}$. If $M \leq^\eta_p N$ then:*
(i) $M \leq^\eta N$,
(ii) $\mathcal{B} \vdash M = M'$ *entails* $M' \leq^\eta_p N$,
(iii) *for all* $p' \geq p$, *we have* $M \leq^\eta_{p'} N$.

We need a couple of technical lemmas. The first one exhibits the interaction between $\leq^\eta_p$ and the size $|-|$, the intuition being that $\leq^\eta_p$ can increase the size of $Q \in \mathcal{I}^\eta$ by at most p.

Lemma 7.7. *Given $p, p' \in \mathbb{N}$, we have that $\lambda y.Q \in \mathcal{I}^\eta_p$ and $Q \leq^\eta_{p'} Q'$ imply $\lambda y.Q' \in \mathcal{I}^\eta_{p+p'}$.*

Proof. We proceed by induction on the structure of $\mathtt{nf}_\beta(Q)$.

From $\lambda y.Q \in \mathcal{I}^\eta_p$ we get $\mathtt{nf}_\beta(Q) = \lambda z_1 \dots z_m.yQ_1 \cdots Q_m$ for some $m < p$ and Q_i's such that $|\lambda z_i.Q_i| < p$. Since $Q \leq^\eta_{p'} Q'$ holds we have $\mathtt{nf}_\beta(Q') = \lambda z_1 \dots z_m w_1 \dots w_{m'}.yQ'_1 \cdots Q'_{m+m'}$ where $m' \leq p'$, $Q_i \leq^\eta_{p'} Q'_i$ for all $i \leq m$ and $\lambda w_j.Q'_j \in \mathcal{I}^\eta_{p'}$ for all $j > m$. Therefore $m + m' < p + p'$, $|\lambda w_j.Q'_j| < p'$ for all $j > m$ and, by inductive hypothesis, $|\lambda z_i.Q'_i| < p + p'$ for all $i \leq m$. We conclude that $|\lambda y.Q'| < p + p'$, which entails $\lambda y.Q' \in \mathcal{I}^\eta_{p+p'}$. □

The next lemma is devoted to show that the relation $\leq^\eta_p$ enjoys the following "weighted" transitivity property.

Lemma 7.8. *For all $M, N \in \Lambda$, $p_1, p_2 \in \mathbb{N}$ we have that $M \leq^\eta_{p_1} P$ and $P \leq^\eta_{p_2} N$ entail $M \leq^\eta_{p_1+p_2} N$.*

Proof. We proceed by coinduction on their Böhm trees.

If one among M, N, P is unsolvable, then all their Böhm trees are $\perp$ and we are done.

Otherwise from $M \leq^\eta_{p_1} P$ we get

$$M \twoheadrightarrow_h \lambda\vec{x}.yM_1 \cdots M_k, \text{ and } P \twoheadrightarrow_h \lambda\vec{x}z_1 \dots z_{m_1}.yP_1 \cdots P_k Q_1 \cdots Q_{m_1}$$

where $\vec{z} \notin \mathsf{FV}(\mathrm{BT}(x_j\vec{M}\vec{P}))$, $m_1 \leq p_1$, $\lambda z_i.Q_i \in \mathcal{I}^\eta_{p_1}$ for $i \leq m_1$ and $M_j \leq^\eta_{p_1} P_j$ for $j \leq k$.

From $P \leq^\eta_{p_2} N$ we obtain (for $\vec{w} \notin \mathsf{FV}(\mathrm{BT}(y\vec{P}\vec{Q}\vec{N}\vec{Q}')))$:

$$N \twoheadrightarrow_h \lambda\vec{x}z_1 \dots z_{m_1}w_1 \dots w_{m_2}.yN_1 \cdots N_k Q'_1 \cdots Q'_{m_1} Q''_1 \cdots Q''_{m_2}$$

with $m_2 \leq p_2$, $P_j \leq^\eta_{p_2} N_j$ for $j \leq k$, $Q_i \leq^\eta_{p_2} Q'_i$ for $i \leq m_1$ and $\lambda w_\ell.Q''_\ell \in \mathcal{I}^\eta_{p_2}$ for $\ell \leq m_2$. We notice that $m_1 \leq p_1$ and $m_2 \leq p_2$ imply $m_1 + m_2 \leq p_1 + p_2$, and that $\mathcal{I}^\eta_{p_1} \cup \mathcal{I}^\eta_{p_2} \subseteq \mathcal{I}^\eta_{p_1+p_2}$. From $\lambda z_i.Q_i \in \mathcal{I}^\eta_{p_1}$ and $Q_i \leq^\eta_{p_2} Q'_i$ we obtain by Lemma 7.7 that $\lambda z_i.Q'_i \in \mathcal{I}^\eta_{p_1+p_2}$ for $i \leq m_1$. Since, for all $j \leq k$, $M_j \leq^\eta_{p_1} P_j$ and $P_j \leq^\eta_{p_2} N_j$ we conclude that $M_j \leq^\eta_{p_1+p_2} N_j$ holds by applying the coinductive hypotheses. □

The following is an easy corollary of Lemma 2.1.

Corollary 7.9. *If $M \twoheadrightarrow_\eta N$ then there exists a bound $p \in \mathbb{N}$ such that $N \leq^\eta_p M$.*

Proof. We perform an induction loading and prove that if the reduction $M \twoheadrightarrow_\eta N$ has length p, then $N \leq^\eta_p M$ holds.

If $p = 0$ then $\mathrm{BT}(M) = \mathrm{BT}(N)$ so we have $N \leq^\eta_0 M$.

If $p > 0$ then $M \twoheadrightarrow_\eta N'$ in $p-1$ steps and $N' \to_\eta N$. From Lemma 2.1, $N \leq^\eta_1 N'$ since at every position σ of their Böhm trees the lengths of $x_1, \dots, x_n$ and $x_1, \dots, x_n z$ differ at most by 1, and $\lambda z.z \in \mathcal{I}^\eta_1$. By induction hypothesis we have $N' \leq^\eta_{p-1} M$, so we conclude by applying Lemma 7.8. □

7.2. **The Behaviour of Ξ_ι on Bounded η-Expansions.** Recall that in Definition 5.5 we have fixed an effective enumeration $\vec{\eta} = (\eta_0, \eta_1, \dots)$ of the set $\mathcal{I}^\eta$, together with the corresponding stream $[\eta_n]_{n\in\mathbb{N}} = [\eta_0, [\eta_1, [\eta_2, \dots]]]$. Moreover, we consider fixed a map ι satisfying the properties of Lemma 5.9.

As shown in Section 6, given two λ-terms M and N whose Böhm trees differ because of countably many η-expansions, the λ-term $\Xi_\iota\lceil M\rceil\lceil N\rceil$ of Definition 5.13 builds their η-supremum from the stream $[\eta_n]_{n\in\mathbb{N}}$ (Proposition 6.1). We now prove that when the size of such η-expansions is bounded by p, then the Böhm trees of M, N also differ from $\mathrm{BT}(\Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}})$ because of η-expansions bounded by p.

Lemma 7.10. *Let $M, N, P \in \Lambda^o$. If $M \geq^\eta_p P \leq^\eta_p N$, then:*

$$M \leq^\eta_p \Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} \geq^\eta_p N.$$

Proof. We proceed by coinduction on their Böhm trees.

If P is unsolvable, then also M, N and $\Xi_\iota\lceil M\rceil\lceil N\rceil$ are unsolvable and we are done.

Otherwise $P \twoheadrightarrow_h \lambda\vec{x}.x_j P_1 \cdots P_k$ and from $P \leq^\eta_p M$ we obtain for some $m \leq p$:

$$M \twoheadrightarrow_h \lambda\vec{x}z_1 \dots z_m.x_j M_1 \cdots M_k Q_1 \cdots Q_m$$

where $P_i \leq^\eta_p M_i$ for $i \leq k$ and $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta_p$ for $\ell \leq m$. Similarly, from $P \leq^\eta_p N$ we get, say,

$$N \twoheadrightarrow_h \lambda\vec{x}z_1 \dots z_m w_1 \dots w_{m'}.x_j N_1 \cdots N_k Q'_1 \cdots Q'_{m+m'}$$

where $m + m' \leq p$, $P_i \leq^\eta_p N_i$ for $i \leq k$, $\lambda z_\ell.Q'_\ell \in \mathcal{I}^\eta_p$ for $\ell \leq m$ and $\lambda w_\ell.Q'_{m+\ell} \in \mathcal{I}^\eta_p$ for $\ell \leq m'$, so we have $M \leq_h N$. (Notice that the case $N <_h M$ is symmetrical, one just needs to apply Lemma 5.14.) Before going further, we need to prove the following claim.

Claim 7.11. Let $Q, Q' \in \Lambda$, $\vec{z} \supseteq \mathsf{FV}(QQ')$. If $\lambda z_\ell.Q, \lambda z_\ell.Q' \in \mathcal{I}^\eta_p$ for some index ℓ then $Q \leq^\eta_p \Xi_\iota\lceil\lambda\vec{z}.Q\rceil\lceil\lambda\vec{z}.Q'\rceil[\eta_n]_{n\in\mathbb{N}}\,\vec{z} \geq^\eta_p Q'$.

Subproof. Since Q, Q' are normalizing, we assume they are in β-normal form and proceed by structural induction. We have, say:

$$Q = \lambda\vec{y}.z_\ell Q_1 \cdots Q_m \quad Q' = \lambda\vec{y}\vec{w}.z_\ell Q'_1 \cdots Q'_{m+m'}$$

where $m + m' \leq p$, $\lambda y_i.Q_i, \lambda y_i.Q'_i \in \mathcal{I}^\eta_p$ for $i \leq m$ and $\lambda w_j.Q'_j \in \mathcal{I}^\eta_p$ for $j > m$.

We split into two subcases.

- If $m = 0$ then $Q = z_\ell$ and $\lambda\vec{z}.z_\ell \leq^\eta_p \lambda\vec{z}.Q'$, so by Lemma 5.15 we get that $\mathcal{B} \vdash \lambda\vec{z}.Q' = \Xi_\iota\lceil\lambda\vec{z}.Q\rceil\lceil\lambda\vec{z}.Q'\rceil[\eta_n]_{n\in\mathbb{N}}$ and the case follows by applying the variables $\vec{z}$ to both sides.
- Otherwise $m > 0$, so for $q = \iota(\#(\lambda x\vec{w}.xQ'_{m+1} \cdots Q'_{m+m'}))$ we have:

$$\begin{aligned}
\Xi_\iota\lceil\lambda\vec{z}.Q\rceil\lceil\lambda\vec{z}.Q'\rceil[\eta_n]_{n\in\mathbb{N}}\vec{z} &=_\beta \lambda\vec{y}.\pi_q[\eta_n]_{n\in\mathbb{N}}(z_\ell(\Upsilon_1\vec{z}\vec{y}) \cdots (\Upsilon_m\vec{z}\vec{y}))\\
&=_\beta \lambda\vec{y}.(\lambda x\vec{w}.xQ'_{m+1} \cdots Q'_{m+m'})(z_\ell(\Upsilon_1\vec{z}\vec{y}) \cdots (\Upsilon_m\vec{z}\vec{y}))\\
&=_\beta \lambda\vec{y}w.z_\ell(\Upsilon_1\vec{z}\vec{y}) \cdots (\Upsilon_m\vec{z}\vec{y})Q'_{m+1} \cdots Q'_{m+m'}
\end{aligned}$$

  where $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{z}\vec{y}.Q_i\rceil\lceil\lambda\vec{z}\vec{y}.Q'_i\rceil[\eta_n]_{n\in\mathbb{N}}$ and the case follows by induction hypothesis, thus concluding the proof of the claim. □

We now continue with the main proof of the lemma.

For $q = \iota(\#(\mathtt{NF}_\beta(\lambda y\vec{w}.yQ'_{m+1}\cdots Q'_{m+m'})))$, we have that:

$$\begin{array}{rcl} \Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} & =_\beta & \lambda\vec{x}\vec{z}.\pi_q[\eta_n]_{n\in\mathbb{N}}(x_j(\Upsilon_1\,\vec{x}\vec{z})\cdots(\Upsilon_k\,\vec{x}\vec{z})(\Upsilon'_1\,\vec{x}\vec{z})\cdots(\Upsilon'_m\,\vec{x}\vec{z})) \\ & =_\beta & \lambda\vec{x}\vec{z}\vec{w}.x_j(\Upsilon_1\vec{x}\vec{z})\cdots(\Upsilon_k\vec{x}\vec{z})(\Upsilon'_1\vec{x}\vec{z})\cdots(\Upsilon'_m\vec{x}\vec{z})Q'_{m+1}\cdots Q'_{m+m'} \end{array}$$

where $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}\vec{z}.M_i\rceil\lceil\lambda\vec{x}\vec{z}.N_i\rceil[\eta_n]_{n\in\mathbb{N}}$ for $i \leq k$, and $\Upsilon'_\ell = \Xi_\iota\lceil\lambda\vec{x}\vec{z}.Q_\ell\rceil\lceil\lambda\vec{x}\vec{z}.Q'_\ell\rceil[\eta_n]_{n\in\mathbb{N}}$ for $\ell \leq m$. By applying Claim 7.11, we get $Q_\ell \leq^\eta_p \Upsilon'_\ell\vec{x}\vec{z} \geq^\eta_p Q'_\ell$. By coinductive hypotheses, we get $M_i \leq^\eta_p \Upsilon_i\vec{x}\vec{z} \geq^\eta_p N_i$. Since $m \leq m + m' \leq p$ and $Q'_{m+\ell} \in \mathcal{I}^\eta_p$ we conclude that $M \leq^\eta_p \Xi_\iota\lceil M\rceil\lceil N\rceil[\eta_n]_{n\in\mathbb{N}} \geq^\eta_p N$ holds. □

7.3. **A New Characterization of the Equality in $\mathcal{B}\eta$.** We are now ready to provide a characterization of the equality $M =_{\mathcal{B}\eta} N$ in terms of equality between $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ up to bounded η-expansions. The key idea we exploit is the fact that, under these hypotheses, $\Xi_\iota\lceil M\rceil\lceil N\rceil$ only depends on a finite restriction of its input stream S and that all finite restrictions of $[\eta_n]_{n\in\mathbb{N}}$ and $[\mathtt{I}]_{n\in\mathbb{N}}$ of the same length are $\beta\eta$-convertible with each other (since they are finite).

Definition 7.12. Let $S = [S_i]_{i\in\mathbb{N}}$ be a stream of λ-terms. For $n \in \mathbb{N}$, define the *n-truncation of S* as the following sequence:

$$S\upharpoonright_n \;=\; [S_0, [S_1, [\cdots, [S_n, \Omega]\cdots]]]$$

It is easy to check that the i-th projection π_i defined for S also works on $S\upharpoonright_n$ for all $i \leq n$. Notice that the Ω at the end of the sequence $S\upharpoonright_n$ does not have a profound meaning: we just need to have an $n+1$-component since $\pi_n = \lambda y.y\mathtt{F}^{\sim n}\mathtt{K}$ needs something to erase in that position.

We have seen in Lemma 5.9 that $\iota(\#Q)$ corresponds to the *smallest* index i such that Q occurs in $[\eta_n]_{n\in\mathbb{N}}$. The following property is a consequence of such a minimality condition.

Lemma 7.13. *Let $p \in \mathbb{N}$. There exists an index n such that for every closed λ-term $Q \in \mathcal{I}^\eta_p$ we have $\iota(\#Q) \leq n$.*

Proof. Since the size of each $Q \in \mathcal{I}^\eta_p$ is bounded by p, the set $\mathtt{nf}_\beta(\mathcal{I}^\eta_p) = \{\mathtt{nf}_\beta(Q) : Q \in \mathcal{I}^\eta_p\}$ is finite. By Lemma 5.9, $\iota(\#Q)$ gives the smallest index i such that $Q =_\beta \eta_i$, therefore the set $\iota[\mathcal{I}^\eta_p] = \{\iota(\#Q) : Q \in \mathcal{I}^\eta_p \cap \Lambda^o\}$ is finite. We conclude by considering $n = \max(\iota[\mathcal{I}^\eta_p])$. □

As a corollary we get that if $M \leq^\eta_p N$ holds, then $\Xi_\iota\lceil M\rceil\lceil N\rceil$ only uses a finite portion of its input stream.

Corollary 7.14. *For all $M, N \in \Lambda^o$, if $M \leq^\eta_p N$ then there exists an index $n \in \mathbb{N}$ such that for every stream $S = [S_i]_{i\in\mathbb{N}}$ we have:*

$$\mathcal{B} \vdash \Xi_\iota\lceil M\rceil\lceil N\rceil S = \Xi_\iota\lceil M\rceil\lceil N\rceil(S\upharpoonright_n).$$

Proof. By Lemma 7.13, there exists n such that for every $Q \in \mathcal{I}^\eta_{p+1} \cap \Lambda^o$ we have $\iota(\#Q) \leq n$. We prove the statement by coinduction for that particular n.

If M or N are unsolvable, than so is $\Xi_\iota\lceil M\rceil\lceil N\rceil$ and we are done.

Otherwise, we have $M \twoheadrightarrow_h \lambda\vec{x}.x_jM_1\cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}\vec{z}.x_jN_1\cdots N_kQ_1\cdots Q_m$ for $m \leq p$ and $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta_p$ for all $\ell \leq m$. By Lemma 7.3, the λ-term defined as $Q =$

$\mathtt{nf}_\beta(\lambda y\vec{z}.yQ_1\cdots Q_m)$ belongs to the set $\mathcal{I}^\eta_{p+1}$. Hence, for some index $q = \iota(\#Q) \leq n$, we have on the one side:

$$\begin{aligned}\Xi_\iota\lceil M\rceil\lceil N\rceil S &=_\beta \lambda\vec{x}.\pi_q S(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ &=_\beta \lambda\vec{x}.S_q(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\end{aligned}$$

where $\Upsilon_i = \Xi_\iota\lceil M_i\rceil\lceil N_i\rceil S$. On the other side, we have:

$$\begin{aligned}\Xi_\iota\lceil M\rceil\lceil N\rceil(S\restriction_n) &=_\beta \lambda\vec{x}.\pi_q(S\restriction_n)(x_j(\Upsilon'_1\vec{x})\cdots(\Upsilon'_k\vec{x}))\\ &=_\beta \lambda\vec{x}.S_q(x_j(\Upsilon'_1\vec{x})\cdots(\Upsilon'_k\vec{x}))\end{aligned}$$

where $\Upsilon'_i = \Xi_\iota\lceil M_i\rceil\lceil N_i\rceil(S\restriction_n)$. We can conclude since, by coinductive hypothesis, $\mathrm{BT}(\Upsilon_i) = \mathrm{BT}(\Upsilon'_i)$ holds for all $i \leq k$. $\square$

Since $M =_{\mathcal{B}\eta} N$ holds exactly when there is an alternating sequence of shape $M =_\mathcal{B} M_0 =_\eta M_1 =_\mathcal{B} \cdots =_\eta M_k =_\mathcal{B} N$, "zig-zag" sequences like the one in the following lemma naturally arise (for instance, in the proof of Theorem 7.16).

Lemma 7.15. *Let $M_1,\dots,M_{k+1} \in \Lambda^o$ and let $p \in \mathbb{N}$. If, for some $N_i \in \Lambda^o$, there is a zig-zag sequence such that:*

$$\begin{array}{ccccccccccccccccc} M_1 & & & & M_2 & & & \cdots & M_{k-1} & & & & M_k & & & & M_{k+1}\\ & \geq^\eta_p & & \leq^\eta_p & & \geq^\eta_p & & \leq^\eta_p & & \geq^\eta_p & & \leq^\eta_p & & \geq^\eta_p & & \leq^\eta_p & \\ & & N_1 & & & & N_2 & \cdots & & & N_{k-1} & & & & N_k & & \end{array}$$

then there exists $P \in \Lambda^o$ such that $M_1 \leq^\eta_{kp} P \geq^\eta_{kp} M_k$.

Proof. We proceed by induction on k.

Case $k = 0$. Trivial, just take $P = M_1$.

Case $k > 0$. By applying Lemma 7.10 to each pair M_i and M_{i+1} we get, setting $N'_i = \Xi_\iota\lceil M_i\rceil\lceil M_{i+1}\rceil[\eta_n]_{n\in\mathbb{N}}$, the sequence:

$$M_1 \leq^\eta_p N'_1 \geq^\eta_p M_2 \cdots \leq^\eta_p N'_k \geq^\eta_p M_{k+1}.$$

As the subsequence $N'_1 \geq^\eta_p M_2 \cdots \leq^\eta_p N'_k$ is shorter and satisfies the hypotheses of the lemma, we get from the inductive hypothesis a λ-term $P \in \Lambda^o$ such that $N'_1 \leq^\eta_{(k-1)p} P \geq^\eta_{(k-1)p} N'_k$. We conclude by Lemma 7.8 since $(k-1)p + p = kp$. $\square$

We are now able to provide the following characterization of $\mathcal{B}\eta$, which constitutes the second main result of the paper.

Theorem 7.16. *For all $M, N \in \Lambda^o$, the following are equivalent:*

(1) *$\mathcal{B}\eta \vdash M = N$,*
(2) *there exist $P \in \Lambda^o$ and $p \in \mathbb{N}$ such that $M \leq^\eta_p P \geq^\eta_p N$.*

Proof. ($1 \Rightarrow 2$) Since $\mathcal{B}\eta$ is the join of two congruences, namely $=_\mathcal{B}$ and $=_\eta$, we have that $\mathcal{B}\eta \vdash M = N$ holds if and only if there are $M_0,\dots,M_k \in \Lambda^o$ such that $M =_\mathcal{B} M_0 =_\eta M_1 =_\mathcal{B} \cdots =_\eta M_k =_\mathcal{B} N$ (cf. [10, Thm. 4.6]). Since η-reduction is Church-Rosser and the M_j's are closed, for even indices i (as odd indices correspond to $=_\mathcal{B}$ steps), we have $M_i =_\eta M_{i+1}$ if and only if $M_i \twoheadrightarrow_\eta N_i \;{}_\eta\!\twoheadleftarrow M_{i+1}$ for some $N_i \in \Lambda^o$. By Corollary 7.9 there are p_i, q_i such that $M_i \geq^\eta_{p_i} N_i \leq^\eta_{q_i} M_{i+1}$. By Lemma 7.6(iii), setting $p' = \max_i\{p_i, q_i\}$ we get $M_i \geq^\eta_{p'} N_i \leq^\eta_{p'} M_{i+1}$. By applying Lemma 7.6(ii) we can get rid of the equality $=_\mathcal{B}$ and obtain the sequence:

$$M \geq^\eta_{p'} N_1 \leq^\eta_{p'} M_2 \geq^\eta_{p'} N_2 \cdots N_{k-1} \leq^\eta_{p'} N.$$

Therefore this implication follows from Lemma 7.15.

($2 \Rightarrow 1$) We assume that $M \leq_p^\eta P \geq_p^\eta N$ holds. From Lemma 7.6(i), we obtain $M \leq^\eta P \geq^\eta N$ as well. Therefore (for some $k, k' \in \mathbb{N}$):

$$\begin{aligned}
M &=_{\mathcal{B}} \Xi_\iota \lceil M \rceil \lceil P \rceil [\mathtt{I}]_{n\in\mathbb{N}} && \text{by Lemma 5.16} \\
&=_{\mathcal{B}} \Xi_\iota \lceil M \rceil \lceil P \rceil (([\mathtt{I}]_{n\in\mathbb{N}})\restriction_k) && \text{by Corollary 7.14} \\
&=_{\beta\eta} \Xi_\iota \lceil M \rceil \lceil P \rceil (([\eta_n]_{n\in\mathbb{N}})\restriction_k) && \text{by } \beta\eta\text{-conversion} \\
&=_{\mathcal{B}} \Xi_\iota \lceil M \rceil \lceil P \rceil [\eta_n]_{n\in\mathbb{N}} && \text{by Corollary 7.14} \\
&=_{\mathcal{B}} P && \text{by Lemma 5.15} \\
&=_{\mathcal{B}} \Xi_\iota \lceil N \rceil \lceil P \rceil [\eta_n]_{n\in\mathbb{N}} && \text{by Lemma 5.15} \\
&=_{\mathcal{B}} \Xi_\iota \lceil N \rceil \lceil P \rceil (([\eta_n]_{n\in\mathbb{N}})\restriction_{k'}) && \text{by Corollary 7.14} \\
&=_{\beta\eta} \Xi_\iota \lceil N \rceil \lceil P \rceil (([\mathtt{I}]_{n\in\mathbb{N}})\restriction_{k'}) && \text{by } \beta\eta\text{-conversion} \\
&=_{\mathcal{B}} \Xi_\iota \lceil N \rceil \lceil P \rceil [\mathtt{I}]_{n\in\mathbb{N}} && \text{by Corollary 7.14} \\
&=_{\mathcal{B}} N && \text{by Lemma 5.16}
\end{aligned}$$

We conclude that $\mathcal{B}\eta \vdash M = N$. □

This result confirms informal intuition about $\mathcal{B}\eta$ discussed in [3, §16.4].

8. Conclusion

Refutation of Sallé's Conjecture provides the final picture of relationships between the classical lambda theories. The next theorem should substitute Theorem 17.4.16 in [3].

Theorem 8.1. *The following diagram indicates all possible inclusion relations of the λ-theories involved (if $\mathcal{T}_1$ is above $\mathcal{T}_2$, then $\mathcal{T}_1 \subsetneq \mathcal{T}_2$):*

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \boldsymbol{\lambda} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \boldsymbol{\lambda}\eta \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{H} \arrow[dl, dash] \arrow[dr, dash] & \\
\boldsymbol{\lambda}\omega \arrow[dr, dash] & & \mathcal{H}\eta \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{B} \arrow[dl, dash] \\
 & \mathcal{H}\omega \arrow[dr, dash] & & \mathcal{B}\eta \arrow[dl, dash] & \\
 & & \mathcal{B}\omega = \mathcal{H}^+ \arrow[d, dash] & & \\
 & & \mathcal{H}^* & &
\end{tikzcd}$$

Fortuitously, the technique of constructing the η-supremum of two terms effectively from their codes also yields a characterization of equality in the theory $\mathcal{B}\eta$. Together these results illuminate a "spectrum" of degrees of extensionality in the theory of Böhm trees:

Theory	Syntactic characterization	Sample equality
$\mathcal{B}$	$\mathrm{BT}(M) = \mathrm{BT}(N)$	$\langle\!\langle \mathtt{I} \rangle\!\rangle = \Phi\lceil \langle\!\langle \mathtt{I} \rangle\!\rangle \rceil$
$\mathcal{B}\eta$	$\mathrm{BT}(M) \leq_p^\eta U \geq_p^\eta \mathrm{BT}(N)$	$\langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1} \rangle\!\rangle$
$\mathcal{B}\omega$	$\mathrm{BT}(M) \leq^\eta U \geq^\eta \mathrm{BT}(N)$	$\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega = \langle\!\langle \eta \rangle\!\rangle^\Omega$
$\mathcal{H}^+$	$\mathrm{BT}(M) \leq^\eta U \geq^\eta \mathrm{BT}(N)$	$\langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1}^* \rangle\!\rangle$
$\mathcal{H}^*$	$\mathrm{BT}(M) \leq_\omega^\eta U \geq_\omega^\eta \mathrm{BT}(N)$	$\langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{J} \rangle\!\rangle$

$$\mathcal{B} \subsetneq \mathcal{B}\eta \subsetneq \mathcal{B}\omega = \mathcal{H}^+ \subsetneq \mathcal{H}^*$$

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