Refutation of Sallé's Longstanding Conjecture

Benedetto Intrigila[1], Giulio Manzonetto[2,3], and Andrew Polonsky[2]

1 **Dipartimento di Ingegneria dell'Impresa, Università di Roma Tor Vergata, Rome, Italy**
2 **CNRS, IRIF, UMR 8243, Université Paris-Diderot, Sorbonne Paris Cité, F-75205 Paris, France**
3 **LIPN, UMR 7030, Université Paris 13, Sorbonne Paris Cité, F-93430, Villetaneuse, France**

Abstract

The λ-calculus possesses a strong notion of extensionality, called "the ω-rule", which has been the subject of many investigations. It is a longstanding open problem whether the equivalence obtained by closing the theory of Böhm trees under the ω-rule is strictly included in Morris's original observational theory, as conjectured by Sallé in the seventies. In a recent work, Breuvart *et al.* have shown that Morris's theory satisfies the ω-rule. In this paper we demonstrate that the two aforementioned theories actually coincide, thus disproving Sallé's conjecture.

1998 ACM Subject Classification F.4.1. Mathematical Logic

Keywords and phrases Lambda calculus, observational equivalence, Böhm trees, ω-rule.

Digital Object Identifier 10.4230/LIPIcs.FSCD.2017.20

Introduction

The problem of determining when two programs are equivalent is central in computer science. For instance, it is necessary to verify that the optimizations performed by a compiler actually preserve the meaning of the program. For λ-calculi, it has become standard to consider two λ-terms M and N as equivalent when they are *contextually equivalent* with respect to some fixed set $\mathcal{O}$ of observables [22]. This means that it is possible to plug either M or N into any context $C[-]$ without noticing any difference in the global behaviour: $C[M]$ produces a result belonging to $\mathcal{O}$ exactly when $C[N]$ does. The problem of working with this definition, is that the quantification over all possible contexts is difficult to handle. Therefore, many researchers undertook a quest for characterizing observational equivalences both semantically, by defining fully abstract denotational models, and syntactically, by comparing possibly infinite trees representing the programs executions.

The most famous observational equivalence is obtained by considering as observables the head normal forms, which are λ-terms representing stable amounts of information coming out of the computation. Introduced by Hyland [13] and Wadsworth [30], it has been ubiquitously studied in the literature [2, 11, 9, 25, 19, 4], since it enjoys many interesting properties. By definition, it corresponds to the extensional λ-theory $\mathcal{H}^*$ which is the greatest consistent sensible λ-theory [2, Thm. 16.2.6]. Semantically, it arises as the λ-theory of Scott's pioneering model $\mathcal{D}_\infty$ [27], a result which first appeared in [13] and [30]. Recently, Breuvart provided a characterization of all K-models that are fully abstract for $\mathcal{H}^*$ [4]. As shown in [2, Thm. 16.2.7], two λ-terms are equivalent in $\mathcal{H}^*$ exactly when their Böhm trees are equal up to countably many (possibly) infinite η-expansions.

However, the head normal forms are not the only reasonable choice of observables. For instance, the original extensional contextual equivalence defined by Morris in [22] arises by

2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017).
Editor: Dale Miller; Article No. 20; pp. 20:1–20:17
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

considering as observables the β-normal forms, that represent completely defined results. We denote by $\mathcal{H}^+$ the λ-theory corresponding to Morris's observational equivalence (using the notation of [20, 5], while it is denoted by $\mathscr{T}_{\mathrm{NF}}$ in [2]). The λ-theory $\mathcal{H}^+$ is sensible and distinct from $\mathcal{H}^*$, so $\mathcal{H}^+ \subsetneq \mathcal{H}^*$. Despite the fact that the equality in $\mathcal{H}^+$ has been the subject of fewer investigations, it has been characterized both semantically and syntactically. In [8], Coppo *et al.* proved that $\mathcal{H}^+$ corresponds to the λ-theory induced by a suitable filter model. More recently, Manzonetto and Ruoppolo introduced a simpler model of $\mathcal{H}^+$ living in the relational semantics [20] and Breuvart *et al.* provided necessary and sufficient conditions for a relational model to be fully abstract for $\mathcal{H}^+$ [5]. From a syntactic perspective, Hyland proved in [12] that two λ-terms are equivalent in $\mathcal{H}^+$ exactly when their Böhm trees are equal up to countably many η-expansions of finite size (cf. [25, §11.2] and [17]).

We have seen that both observational equivalences correspond to some extensional equalities between Böhm trees. A natural question is whether $\mathcal{H}^+$ can be generated just by adding the η-conversion to the λ-theory $\mathcal{B}$ equating all λ-terms having the same Böhm tree. The λ-theory $\mathcal{B}\eta$ so defined has been little studied in the literature, probably because it does not arise as an observational equivalence nor is induced by some known denotational model. In [2, Lemma 16.4.3], Barendregt shows that one η-expansion in a λ-term M can generate infinitely many finite η-expansions on its Böhm tree $\mathrm{BT}(M)$. In [2, Lemma 16.4.4], he exhibits two λ-terms that are equal in $\mathcal{H}^+$ but distinct in $\mathcal{B}\eta$, thus proving that $\mathcal{B}\eta \subsetneq \mathcal{H}^+$.

However, the λ-calculus also possesses another notion of extensionality, known as the *ω-rule*, which is strictly stronger than η-conversion. Such a rule has been studied by many researchers in connection with several λ-theories [16, 1, 23, 3, 15]. Formally, the ω-rule states that for all λ-terms M and N, $M = N$ whenever $MP = NP$ holds for all closed λ-terms P. A λ-theory $\mathcal{T}$ *satisfies the ω-rule* whenever it is closed under such a rule. Since this is such an impredicative rule, we can meaningfully wonder how the λ-theory $\mathcal{B}\omega$, obtained as the closure of $\mathcal{B}$ under the ω-rule, compares with the other λ-theories. As shown by Barendregt in [2, Lemma 16.4.4], $\mathcal{B}\eta$ does not satisfy the ω-rule, while $\mathcal{H}^*$ does [2, Thm. 17.2.17].

Therefore, the two possible scenarios are the following:

$$\mathcal{B}\eta \subsetneq \mathcal{B}\omega \subseteq \mathcal{H}^+ \subsetneq \mathcal{H}^* \qquad \text{or} \qquad \mathcal{B}\eta \begin{array}{c} \subsetneq \mathcal{B}\omega \subsetneq \\ \subsetneq \mathcal{H}^+ \subsetneq \end{array} \mathcal{H}^*.$$

In the seventies, Sallé was working with Coppo and Dezani on type systems for studying termination properties of λ-terms [26, 7]. In 1979, at the conference on λ-calculus that took place in Swansea, he conjectured that a strict inclusion $\mathcal{B}\omega \subsetneq \mathcal{H}^+$ holds. Such a conjecture was reported in the proof of [2, Thm. 17.4.16], but for almost fourty years no progress has been made in that direction. In 2016, the second and third authors with Breuvart and Ruoppolo proved that $\mathcal{H}^+$ satisfies the ω-rule [5], so $\mathcal{B}\omega \subseteq \mathcal{H}^+$. In this paper we demonstrate that the λ-theories $\mathcal{B}\omega$ and $\mathcal{H}^+$ actually coincide, thus disproving Sallé's conjecture.

To prove such a result we need to show that, whenever two λ-terms M and N are equal in $\mathcal{H}^+$, they are also equal in $\mathcal{B}\omega$. From [12], we know that in this case there is a Böhm tree T such that $\mathrm{BT}(M) \leq^\eta T \geq^\eta \mathrm{BT}(N)$, where $T' \leq^\eta T$ means that the Böhm tree T can be obtained from T' by performing countably many finite η-expansions. Thus, the Böhm trees of M, N are compatible and have a common "η-supremum" T.

Our proof can be divided into several steps:

1. We show that the aforementioned η-supremum T is λ-*definable*: there exists a λ-term P such that $\mathrm{BT}(P) = T$ (Proposition 38).
2. We apply the ω-rule to equate the Böhm tree of the *stream* (infinite list) $\langle\!\langle \eta \rangle\!\rangle$ containing all finite η-expansions of the identity, and the Böhm tree of the stream $\langle\!\langle \mathtt{I} \rangle\!\rangle$ containing infinitely many copies of the identity (Corollary 41).

3. We define a λ-term Ξ (Definition 34) taking as arguments the *codes* $\lceil\cdot\rceil$ of two λ-terms M_1, M_2 and a stream S, and such that, whenever $\mathrm{BT}(M_1) \leq^\eta \mathrm{BT}(M_2)$ holds:
   (*i*) $\mathrm{BT}(\Xi\lceil M_1\rceil\lceil M_2\rceil\langle\!\langle\eta\rangle\!\rangle) = \mathrm{BT}(M_2)$ (Lemma 36),
   (*ii*) $\mathrm{BT}(\Xi\lceil M_1\rceil\lceil M_2\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle) = \mathrm{BT}(M_1)$ (Lemma 37).
4. Summing up, if M, N are equal in $\mathcal{H}^+$, then by (1) there is a λ-term P such that $\mathrm{BT}(M) \leq^\eta \mathrm{BT}(P) \geq^\eta \mathrm{BT}(N)$. Since $\mathcal{B}\omega$ equates all λ-terms having the same Böhm tree, we obtain the following sequence of equalities:
$$\begin{array}{l} M =_{3(ii)} \Xi\lceil M\rceil\lceil P\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle =_{(2)} \Xi\lceil M\rceil\lceil P\rceil\langle\!\langle\eta\rangle\!\rangle =_{3(i)} P \\ N =_{3(ii)} \Xi\lceil N\rceil\lceil P\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle =_{(2)} \Xi\lceil N\rceil\lceil P\rceil\langle\!\langle\eta\rangle\!\rangle =_{3(i)} P \end{array}$$
   so M and N are equal in $\mathcal{B}\omega$ (Theorem 42).

The intuition behind $\Xi\lceil M\rceil\lceil N\rceil S$ is that, working on their codes, the λ-term Ξ computes the Böhm trees of M and N, compares them, and at every position applies to the "smaller" (less η-expanded) an element extracted from the stream S in the attempt of matching the structure of the "larger". When S contains all possible η-expansions each attempt succeeds, so $\Xi\lceil M\rceil\lceil N\rceil\langle\!\langle\eta\rangle\!\rangle$ computes the η-supremum of $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$. When S only contains the identity, each non-trivial attempt fails, and $\Xi\lceil M\rceil\lceil N\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle$ computes their η-infimum.

We announce that the technique developed can be also used to prove that two λ-terms M and N are equal in $\mathcal{B}\eta$ exactly when their Böhm trees are equal up to countably many η-expansions of *bounded size*. This result is beyond the scope of the present paper and omitted, but confirms the informal intuition about $\mathcal{B}\eta$ discussed by Barendregt in [2, §16.4].

Discussion

We build on the characterizations of $\mathcal{H}^+$ and $\mathcal{H}^*$ given by Hyland and Wadsworth [12, 13, 30] and subsequently improved by Lévy [17]. In Section 2 we give a uniform presentation of these preliminary results using the formulation given in [2, §19.2] for $\mathcal{H}^*$, that exploits the notion of Böhm-like trees, namely labelled trees that "look like" Böhm trees but might not be λ-definable. Böhm-like trees were introduced in [2] since at the time researchers were less familiar with the notion of coinduction, but they actually correspond to infinitary terms coinductively generated by the grammar of normal λ-terms possibly containing the constant $\perp$. It is worth mentioning that such characterizations of $\mathcal{H}^+$ and $\mathcal{H}^*$ have been recently rewritten by Severi and de Vries using the modern approach of infinitary rewriting [28, 29], and that we could have used their formulation instead.

A key ingredient in our proof is the fact that λ-terms can be encoded with natural numbers, and therefore with Church numerals, in an effective way. This is related to the theory of self-interpreters in λ-calculi, which is an ongoing subject of study [21, 10, 24, 6], and we believe that the present paper provides a nice illustration of the usefulness of such interpreters. As a presentation choice, we decided to use the encoding described in Barendregt's book [2, Def.6.5.6], even if it works for closed λ-terms only, because it is the most standard. However, our construction could be recast using any (effective) encoding, like the one proposed by Mogensen in [21] that works more generally for open terms.

Outline

After the preliminary Section 1, we review the main notions of extensional equalities on Böhm trees in Section 2, and the key results concerning the ω-rule in Section 3. In Section 4 we show how to build Böhm trees, and their η-supremum, starting from codes of λ-terms and streams of η-expansions. Finally, Section 5 is devoted to the proof of $\mathcal{B}\omega = \mathcal{H}^+$.

1 Preliminaries

1.1 The Lambda Calculus.

We generally use the notation of Barendregt's book [2] for λ-calculus. The set Λ of λ-terms over an infinite set Var of variables is defined by the following grammar:

$$\Lambda : \qquad M, N ::= x \mid \lambda x.M \mid MN \qquad (\text{for } x \in \text{Var})$$

The application associates to the left and has a higher precedence than λ-abstraction. For instance $\lambda xyz.xyz = \lambda x.(\lambda y.(\lambda z.((xy)z)))$. We write $MN^{\sim n}$ for $MN \cdots N$ (n times).

The set $\mathsf{fv}(M)$ of *free variables* of M and the *α-conversion* are defined as in [2, Ch. 1§2]. Hereafter, we consider λ-terms up to α-conversion. A λ-term M is called *closed* whenever $\mathsf{fv}(M) = \emptyset$ and we denote by Λ^o the set of all closed λ-terms.

The *β-reduction* is defined as usual $(\lambda x.M)N \to_\beta M[N/x]$ where $M[N/x]$ denotes the capture-free substitution of N for all free occurrences of x in M. We denote by $\mathtt{nf}_\beta(M)$ the β-normal form of M, if it exists. The *η-reduction* is given by $\lambda x.Mx \to_\eta M$ subject to the usual proviso $x \notin \mathsf{fv}(M)$. Given $\to_{\mathtt{R}}$, write $=_{\mathtt{R}}$ ($\twoheadrightarrow_{\mathtt{R}}$) for $\mathtt{R}$-conversion (multistep $\mathtt{R}$-reduction).

We will use the following notations for specific λ-terms:

$$\begin{array}{llll} \mathtt{I} = \mathtt{1}^0 = \lambda x.x, & \mathtt{1}^{n+1} = \lambda xz.x(\mathtt{1}^n z), & \mathtt{B} = \lambda fgx.f(gx), & \mathtt{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)), \\ \mathtt{K} = \lambda xy.x, & \mathtt{F} = \lambda xy.y, & \Omega = (\lambda x.xx)(\lambda x.xx), & \mathtt{J} = \mathtt{Y}(\lambda jxz.x(jz)), \end{array}$$

where $\mathtt{I}$ is the identity, $\mathtt{1}^n$ is a $\beta\eta$-expansion of $\mathtt{I}$, $\mathtt{B}$ is the composition combinator $M \circ N = \mathtt{B}MN$, $\mathtt{K}$ and $\mathtt{F}$ are the first and second projection, Ω is the paradigmatic looping λ-term, $\mathtt{Y}$ is Curry's fixed point and $\mathtt{J}$ is Wadsworth's combinator [30]. We denote by c_n the n-th Church numeral [2, Def. 6.4.4], by succ and pred the successor and predecessor, and by $\mathsf{ifz}(\mathrm{c}_n, M, N)$ the λ-term which is equal to M if $n = 0$ and is equal to N otherwise.

The *pairing* is encoded in λ-calculus by setting $[M, N] = \lambda y.yMN$ for $y \notin \mathsf{fv}(MN)$ [2, Def. 6.2.4].

▶ **Definition 1.** An enumeration of closed λ-terms $e = (M_0, M_1, M_2, \dots)$ is called *effective* (or *uniform* in [2, §8.2]) if there is $F \in \Lambda^o$ such that $F\mathrm{c}_n =_\beta M_n$.

Given an effective enumeration, we define (using $\mathtt{Y}$ like in [2, Def. 8.2.3]) the *sequence* $[M_n]_{n\in\mathbb{N}}$ as a single λ-term satisfying $[M_n]_{n\in\mathbb{N}} =_\beta [M_0, [M_{n+1}]_{n\in\mathbb{N}}]$. We often use the notations:

$$[M_n]_{n\in\mathbb{N}} = [M_0, [M_1, [M_2, \dots]]] = [M_0, M_1, M_2, \dots].$$

The *i-th projection* is $\pi_i = \lambda y.y\mathtt{F}^{\sim i}\mathtt{K}$ since $\pi_i[M_n]_{n\in\mathbb{N}} =_\beta M_i$.

▶ **Definition 2.** Starting from a sequence $S = [M_n]_{n\in\mathbb{N}}$ we can build a *stream* $\langle\!\langle S\rangle\!\rangle = \lambda\vec{x}.[M_n\vec{x}]_{n\in\mathbb{N}}$ having $\mathsf{P}_i = \lambda s\vec{x}.\pi_i(s\vec{x})$ as projection.

The difference between a sequence and a stream stands in their applicative behaviour: when applying $\langle\!\langle S\rangle\!\rangle$ to $\vec{P}$ all λ-terms in $\langle\!\langle S\rangle\!\rangle$ receive $\vec{P}$ as arguments, i.e., $\langle\!\langle S\rangle\!\rangle\vec{P} =_\beta [M_n\vec{P}]_{n\in\mathbb{N}}$.

1.2 Solvability and Böhm(-like) Trees

The λ-terms are classified into solvable and unsolvable, depending on their capability of interaction with the environment.

▶ **Definition 3.** A closed λ-term N is *solvable* if there are $\vec{P} \in \Lambda$ such that $N\vec{P} \twoheadrightarrow_\beta \mathtt{I}$. A λ-term M is *solvable* if its closure $\lambda\vec{x}.M$ is solvable. Otherwise M is called *unsolvable*.

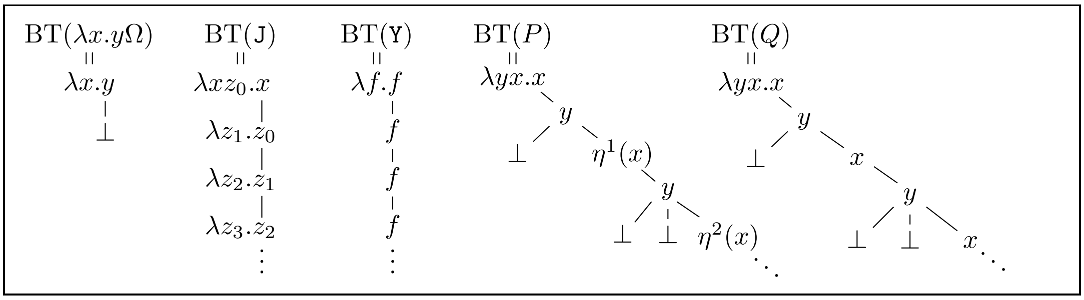

Figure 1 Some examples of Böhm trees, where $\eta^n(x) = \mathrm{BT}(\mathtt{1}^n x)$. We refer to [2, Lemma 16.4.4] for the definition of P, Q.

A λ-term M is in *head normal form* (*hnf*) if it has the shape $\lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ where either $x_j \in \vec{x}$ or it is free. If M has an hnf, it can be reached by *head reductions* $\twoheadrightarrow_h$, i.e. by repeatedly contracting its *head redex* $\lambda\vec{x}.\underline{(\lambda y.P)Q}M_1 \cdots M_k$. As shown by Wadsworth in [30], a λ-term M is solvable if and only if M has a head normal form. The typical example of an unsolvable is Ω. Any $M \in \Lambda^o$ can be turned into an unsolvable by applying enough Ω's.

▶ **Lemma 4.** *[2, Lemma 17.4.4] For all $M \in \Lambda^o$ there is $k \in \mathbb{N}$ such that $M\Omega^{\sim k}$ is unsolvable.*

▶ **Definition 5.** The *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M is defined coinductively as follows:

- if M is unsolvable then $\mathrm{BT}(M) = \perp$;
- if M is solvable and $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$ then:

$$\mathrm{BT}(M) = \begin{array}{c} \lambda x_1 \dots x_n.x_j \\ \diagup \qquad \diagdown \\ \mathrm{BT}(M_1) \quad \cdots \quad \mathrm{BT}(M_k) \end{array}$$

Notable examples of Böhm trees are given in Figure 1. Some of the results that we use were originally formulated for "Böhm-like" trees, so we recall their definition [2, Def. 10.1.12].

▶ **Definition 6.** A *Böhm-like* tree T is a partial function $T : \mathbb{N}^* \to \mathcal{L} \times \mathbb{N}$, where $\mathbb{N}^*$ is the set of finite sequences of natural numbers and $\mathcal{L} = \{\lambda\vec{x}.y \mid \vec{x}, y \in \mathrm{Var}\}$, such that $\mathrm{dom}(T)$ is closed under prefixes and for all positions $\sigma \in \mathrm{dom}(T)$ and $n \in \mathbb{N}$ if their concatenation $\sigma.n$ belongs to $\mathrm{dom}(T)$ then $n < \pi_2(T(\sigma))$ holds. We denote by $\mathbb{BT}$ the set of all Böhm-like trees.

Intuitively, we have $T(\sigma) = (\lambda\vec{x}.y, n)$ if the node of T in position σ is labelled with "$\lambda\vec{x}.y$" and has n (possibly undefined) children. Given a Böhm-like tree T, its *underlying naked tree* $|T|$ is given by $\{\langle\rangle\} \cup \{\sigma.k \in \mathbb{N}^* \mid \pi_2(T(\sigma)) = n \text{ and } k < n\}$. The positions $\sigma \in |T| - \mathrm{dom}(T)$ correspond to unsolvable λ-terms, so we write $T(\sigma) = \perp$.

By [2, Thm. 10.1.23], $T \in \mathbb{BT}$ is partial computable and $\mathsf{fv}(T)$ is finite if and only if there is a λ-term M such that $\mathrm{BT}(M) = T$.

We will systematically confuse finite (resp. infinite) Böhm-like trees $T \in \mathbb{BT}$ with the corresponding (infinitary) λ-terms and use the same notations.

▶ **Lemma 7.** *(cf. [2, 10.1.5(v)]) Let $M_i, N_i \in \Lambda$, for $i \in \mathbb{N}$. If for all $i \in \mathbb{N}$ we have* $\mathrm{BT}(M_i) = \mathrm{BT}(N_i)$ *then* $\mathrm{BT}([M_n]_{n\in\mathbb{N}}) = \mathrm{BT}([N_n]_{n\in\mathbb{N}})$.

1.3 Observational Equivalences and Lambda Theories

Observational equivalences and λ-theories become the main object of study when considering the computational equivalence more important than the process of computation.

▶ **Definition 8.** A *λ-theory* is a congruence on Λ (that is, an equivalence relation compatible with lambda abstraction and application) containing the β-conversion.

Given a λ-theory $\mathcal{T}$ we will write $\mathcal{T} \vdash M = N$, or simply $M =_{\mathcal{T}} N$, to express the fact that M and N are equal in $\mathcal{T}$. The set of all λ-theories, ordered by inclusion, forms quite a rich complete lattice, as shown by Lusin and Salibra in [18].

▶ **Definition 9.** A λ-theory $\mathcal{T}$ is called:
- *consistent* if it does not equate all λ-terms;
- *extensional* if it contains the η-conversion;
- *sensible* if it equates all unsolvable λ-terms.

We denote by $\boldsymbol{\lambda}$ the least λ-theory, by $\boldsymbol{\lambda}\eta$ the least extensional λ-theory, by $\mathcal{H}$ the least sensible λ-theory, by $\mathcal{B}$ the λ-theory equating all λ-terms having the same Böhm tree, and by $\mathcal{H}^*$ the (unique) greatest consistent sensible λ-theory.

The λ-theory $\mathcal{B}$ is sensible, thus we have $\boldsymbol{\lambda} \subsetneq \mathcal{H} \subsetneq \mathcal{B} \subsetneq \mathcal{H}^*$.

Given a λ-theory $\mathcal{T}$, we write $\mathcal{T}\eta$ for the least extensional λ-theory containing $\mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}'$ entails $\mathcal{T}\eta \subseteq \mathcal{T}'\eta$, we also have $\boldsymbol{\lambda}\eta \subseteq \mathcal{H}\eta \subseteq \mathcal{B}\eta \subseteq \mathcal{H}^*$ and actually all these inclusions turn out to be strict [2, Thm. 17.4.16].

▶ **Remark 10.** It is well known (see [2, Rem. 4.1.2]) that two λ-theories $\mathcal{T}, \mathcal{T}'$ that coincide on closed terms must be equal, hence we often focus on closed λ-terms.

Several interesting λ-theories are obtained via observational equivalences defined with respect to a set $\mathcal{O}$ of *observables*. Recall that a *context* $C[]$ is a λ-term with a *hole* denoted by $[]$. We write $C[M]$ for the λ-term obtained from $C[]$ by substituting M for the hole, possibly with capture of free variables in M.

Given $\mathcal{O} \subseteq \Lambda$, we write $M \in_\beta \mathcal{O}$ for $M \twoheadrightarrow_\beta M' \in \mathcal{O}$.

▶ **Definition 11.** Given a set $\mathcal{O} \subseteq \Lambda$, the *$\mathcal{O}$-observational equivalence* $\equiv^{\mathcal{O}}$ is defined by setting:

$$M \equiv^{\mathcal{O}} N \text{ if and only if } \forall C[](C[M] \in_\beta \mathcal{O} \iff C[N] \in_\beta \mathcal{O}).$$

We mainly focus on the following observational equivalences:
- Hyland/Wadsworth's observational equivalence $\equiv^{\mathtt{hnf}}$ is obtained by taking as $\mathcal{O}$ the set of head normal forms [13, 30].
- Morris's equivalence $\equiv^{\mathtt{nf}}$ is generated by taking as $\mathcal{O}$ the set of β-normal forms [22].

We will now see that $\equiv^{\mathtt{nf}}$ and $\equiv^{\mathtt{hnf}}$ have been characterized in terms of extensional equalities between Böhm trees.

2 Böhm Trees and Extensionality

We review three different notions of extensional equality between Böhm trees corresponding to the equality in $\mathcal{B}\eta, \mathcal{H}^+$ and $\mathcal{H}^*$. We start by analyzing the η-expansions of the identity.

2.1 The η-Expansions of The Identity

Let $\mathcal{I}^\eta \subseteq \Lambda$ be the set of finite η-expansions of the identity, that is $Q \in \mathcal{I}^\eta$ whenever $Q \twoheadrightarrow_{\beta\eta} \mathtt{I}$. The structural properties of such η-expansions have been analyzed in [14] (where this more liberal terminology is introduced). For instance, it is proved that $(\mathcal{I}^\eta, \circ, \mathtt{I})$ is an idempotent commutative monoid which is moreover closed under λ-calculus application.

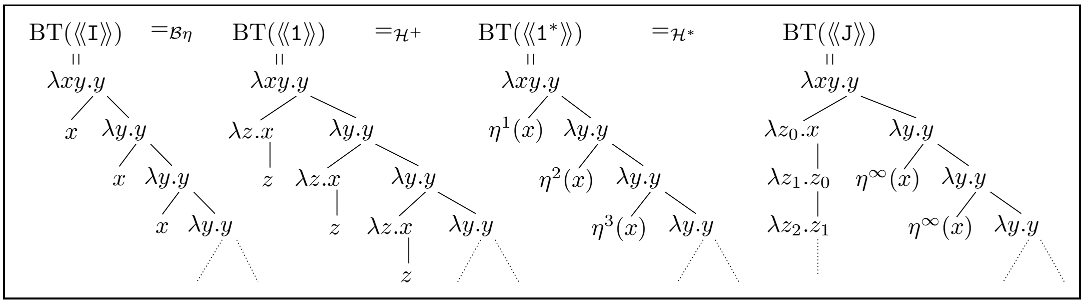

Figure 2 The Böhm trees of $\langle\!\langle \mathtt{I} \rangle\!\rangle, \langle\!\langle \mathtt{1} \rangle\!\rangle, \langle\!\langle \mathtt{1}^* \rangle\!\rangle$ and $\langle\!\langle \mathtt{J} \rangle\!\rangle$, where we set $\eta^n(x) = \mathrm{BT}(\mathtt{1}^n x)$ and $\eta^\infty(x) = \mathrm{BT}(\mathtt{J}x)$.

▶ **Lemma 12.** *For $Q \in \Lambda$, the following are equivalent:*
(i) $Q \in \mathcal{I}^\eta$, *i.e.* $Q \twoheadrightarrow_{\beta\eta} \mathtt{I}$,
(ii) $Q =_\beta \lambda y z_1 \dots z_m . y Q_1 \cdots Q_m$ *such that* $\lambda z_\ell . Q_\ell \in \mathcal{I}^\eta$,
(iii) $Q =_\beta \lambda y . Q'$ *such that* $Q' \twoheadrightarrow_{\beta\eta} y$.

There is a one-to-one correspondence between elements of $\mathcal{I}^\eta$ in β-normal forms and finite (unlabelled) trees [14]. Clearly, $\mathtt{1}^n \in \mathcal{I}^\eta$, every $Q \in \mathcal{I}^\eta$ is β-normalizing, $\mathtt{nf}_\beta(Q)$ is a closed λ-term and $\mathrm{BT}(Q)$ is finite and does not contain any occurrence of $\bot$.

▶ **Definition 13.** Given $Q \in \mathcal{I}^\eta$ its *depth* (resp. *branching number*) is the height (resp. maximal number of branching) of its Böhm tree. The *size* of Q is the maximum between its depth and its branching number.

There are also λ-terms, like Wadsworth's $\mathtt{J}$, that *look* like η-expansions of $\mathtt{I}$ but give rise to infinite computations:

$$\mathtt{J} =_\beta \lambda x z_0 . x(\mathtt{J} z_0) =_\beta \lambda x z_0 . x(\lambda z_1 . z_0(\mathtt{J} z_1)) =_\beta \cdots$$

The Böhm tree of $\mathtt{J}$ is an *infinite η-expansion* of the identity, a notion that is discussed in Section 2.4.

2.2 $\mathcal{B}\eta$: Countably Many η-Expansions of Bounded Size

Recall that $\mathcal{B}\eta$ is the least extensional λ-theory including $\mathcal{B}$. One might think that if $M =_{\mathcal{B}\eta} N$ then $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ differ because of finitely many η-expansions. In reality, one η-expansion of M can generate countably many η-expansions in its Böhm tree.

Consider, for instance, the following streams:

$$\langle\!\langle \mathtt{I} \rangle\!\rangle x = [x, x, x, \dots], \qquad \langle\!\langle \mathtt{1} \rangle\!\rangle x = [\mathtt{1}x, \mathtt{1}x, \mathtt{1}x, \dots], \qquad \langle\!\langle \mathtt{1}^* \rangle\!\rangle x = [\mathtt{1}^1 x, \mathtt{1}^2 x, \mathtt{1}^3 x, \dots].$$

whose Böhm trees are depicted in Figure 2. We have that $\langle\!\langle \mathtt{1} \rangle\!\rangle =_{\mathcal{B}} \mathtt{Y}(\lambda m x.[\lambda z.xz, mx]) \to_\eta \mathtt{Y}(\lambda m x.[x, mx]) =_{\mathcal{B}} \langle\!\langle \mathtt{I} \rangle\!\rangle$ thus $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1} \rangle\!\rangle$ are equated in $\mathcal{B}\eta$ despite the fact that their Böhm trees differ by infinitely many η-expansions. More precisely, $M \to_\eta N$ entails that $\mathrm{BT}(M)$ can be obtained from $\mathrm{BT}(N)$ by performing at most *one* η-expansion at every position (see [2, Lemma 16.4.3]). The proof technique that we develop in Section 4 allows to demonstrate that two λ-terms M and N are equated in $\mathcal{B}\eta$ exactly when their Böhm trees are equal up to countably many η-expansions whose sizes are bound by some natural number k.

In particular, no finite amount of η-expansions in $\langle\!\langle \mathtt{I} \rangle\!\rangle$ can turn its Böhm tree into $\mathrm{BT}(\langle\!\langle \mathtt{1}^* \rangle\!\rangle)$, which has infinitely many η-expansions *of increasing depth*.

▶ **Corollary 14.** $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1} \rangle\!\rangle$, *while* $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle \neq \langle\!\langle \mathtt{1}^* \rangle\!\rangle$.

2.3 $\mathcal{H}^+$: Countably Many Finite η-Expansions

Let $\mathcal{H}^+$ be the λ-theory corresponding to Morris's original observational equivalence $\equiv^{\mathtt{nf}}$ where the observables are the β-normal forms [22]. The λ-theory $\mathcal{H}^+$ has been studied both from a syntactic and semantic point of view in [8, 17, 5]. (The properties we present here can be found in [25, §11.2].) Two λ-terms having the same Böhm tree cannot be distinguished by any context $C[]$, so we have $\mathcal{B} \subseteq \mathcal{H}^+$. Since the η-reduction is strongly normalizable, a λ-term M has a β-normal form exactly when it has a $\beta\eta$-normal form, hence $\mathcal{H}^+$ is an extensional λ-theory. Therefore we have $\mathcal{B}\eta \subseteq \mathcal{H}^+$.

The question naturally arising is whether there are λ-terms different in $\mathcal{B}\eta$ that become equal in $\mathcal{H}^+$. It turns out that $\mathcal{H}^+ \vdash M = N$ holds exactly when $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ are equal up to countably many η-expansions of finite depth. A typical example of this situation is given by $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1}^* \rangle\!\rangle$. The next definition is coinductive on the Böhm-like trees.

▶ **Definition 15.** For all $T, T' \in \mathbb{BT}$, we have $T \leq^\eta T'$ if either $T = T' = \bot$, or $T = \lambda\vec{x}.x_j T_1 \cdots T_k$ and $T' = \lambda\vec{x}z_1 \dots z_m.x_j T'_1 \cdots T'_k Q_1 \cdots Q_m$, for $\vec{T}, \vec{T}' \in \mathbb{BT}$ and β-normal $\vec{Q} \in \Lambda$ such that $T_i \leq^\eta T'_i$ for all $i \leq k$, $z_\ell \notin \mathsf{fv}(x_j\vec{T}\vec{T}')$ and $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ for all $\ell \leq m$.

It is easy to check that $\mathrm{BT}(\langle\!\langle \mathtt{I} \rangle\!\rangle) \leq^\eta \mathrm{BT}(\langle\!\langle \mathtt{1}^* \rangle\!\rangle)$ holds.

▶ **Definition 16.** For $M, N \in \Lambda$, we write $M \leq^\eta N$ if and only if $\mathrm{BT}(M) \leq^\eta \mathrm{BT}(N)$.

Note that $M \leq^\eta N$ and $N \leq^\eta M$ entail $\mathrm{BT}(M) = \mathrm{BT}(N)$, so the equivalence corresponding to $\leq^\eta$ and capturing $=_{\mathcal{H}^+}$ needs to be defined in the following more subtle way.

▶ **Theorem 17** (Hyland [12], see also [17]). *For all $M, N \in \Lambda$, $\mathcal{H}^+ \vdash M = N$ if and only if there is a Böhm-like tree $T \in \mathbb{BT}$ such that $\mathrm{BT}(M) \leq^\eta T \geq^\eta \mathrm{BT}(N)$.*

So, in general, when $M =_{\mathcal{H}^+} N$, one may need to perform countably many η-expansions both in $\mathrm{BT}(M)$ and in $\mathrm{BT}(N)$ to equate them and find the common "supremum".

▶ **Corollary 18.** *$\mathcal{H}^+ \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle = \langle\!\langle \mathtt{1}^* \rangle\!\rangle$, while $\mathcal{H}^+ \vdash \mathtt{I} \neq \mathtt{J}$.*

2.4 $\mathcal{H}^*$: Countably Many Infinite η-Expansions

The theory $\mathcal{H}^*$ is, by far, the most well studied λ-theory. It corresponds to the observational equivalence $\equiv^{\mathtt{hnf}}$ where the observables are the head normal forms. It is also the maximal consistent sensible λ-theory [2, Thm. 16.2.6] and the theory of Scott's original λ-model $\mathcal{D}_\infty$ [27]. It is not difficult to check that $M \equiv^{\mathtt{nf}} N$ entails $M \equiv^{\mathtt{hnf}} N$, therefore $\mathcal{H}^+ \subseteq \mathcal{H}^*$.

Two λ-terms M, N are equated in $\mathcal{H}^*$ if their Böhm trees are equal up to countably many η-expansions of possibly infinite depth. The typical example is $\mathtt{I} =_{\mathcal{H}^*} \mathtt{J}$. However, $\mathtt{J}$ is not the only candidate: for every infinite (unlabelled) recursive tree T it is possible to define a λ-term $\mathtt{J}_T$ whose Böhm tree is an infinite η-expansion of the identity "following T" [5].

▶ **Definition 19.** For all $T, T' \in \mathbb{BT}$, we have $T \leq^\eta_\omega T'$ if either $T = T' = \bot$, or $T = \lambda\vec{x}.x_j T_1 \cdots T_k$ and $T' = \lambda\vec{x}z_1 \dots z_m.x_j T'_1 \cdots T'_k T''_1 \cdots T''_m$, for $\vec{T}, \vec{T}', \vec{T}'' \in \mathbb{BT}$ such that $T_i \leq^\eta_\omega T'_i$ for all $i \leq k$, $z_\ell \notin \mathsf{fv}(x_j\vec{T}\vec{T}')$ and $z_\ell \leq^\eta_\omega T''_\ell$ for all $\ell \leq m$.

E.g., we have $\mathrm{BT}(\langle\!\langle \mathtt{I} \rangle\!\rangle) \leq^\eta_\omega \mathrm{BT}(\langle\!\langle \mathtt{J} \rangle\!\rangle)$ where $\langle\!\langle \mathtt{J} \rangle\!\rangle$ is defined by $\langle\!\langle \mathtt{J} \rangle\!\rangle x = [\mathtt{J}x, \mathtt{J}x, \mathtt{J}x, \dots]$.

▶ **Theorem 20** (Hyland [13]/Wadsworth [30]). *For all $M, N \in \Lambda$, $\mathcal{H}^* \vdash M = N$ if and only if there is a Böhm-like tree $T \in \mathbb{BT}$ such that $\mathrm{BT}(M) \leq^\eta_\omega T \geq^\eta_\omega \mathrm{BT}(N)$.*

By Exercise 10.6.7 in [2], T can be always chosen to be the Böhm tree of some λ-term. As we will prove in Section 5, this property also holds for the tree T of Theorem 17.

▶ **Corollary 21.** *The streams $\langle\!\langle \mathtt{I} \rangle\!\rangle, \langle\!\langle \mathtt{1} \rangle\!\rangle, \langle\!\langle \mathtt{1}^* \rangle\!\rangle$ and $\langle\!\langle \mathtt{J} \rangle\!\rangle$ are all equal in $\mathcal{H}^*$. On the contrary, $\mathcal{B}\eta \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle \neq \langle\!\langle \mathtt{1}^* \rangle\!\rangle$ and $\mathcal{H}^+ \vdash \langle\!\langle \mathtt{1}^* \rangle\!\rangle \neq \langle\!\langle \mathtt{J} \rangle\!\rangle$, so we have $\mathcal{B}\eta \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$.*

3 The Omega Rule and Sallé's Conjecture

The ω-rule is a strong form of extensionality defined by:

$$(\omega) \qquad \forall P \in \Lambda^o . MP = NP \text{ entails } M = N.$$

Given a λ-theory $\mathcal{T}$ we denote its closure under the ω-rule by $\mathcal{T}\omega$. We say that $\mathcal{T}$ *satisfies the ω-rule*, written $\mathcal{T} \vdash \omega$, if $\mathcal{T} = \mathcal{T}\omega$. The ω-rule, and the question of which λ-theories satisfy it, has been extensively investigated by many authors [16, 1, 23, 3, 15]. The following lemma collects some results in [2, §4.1].

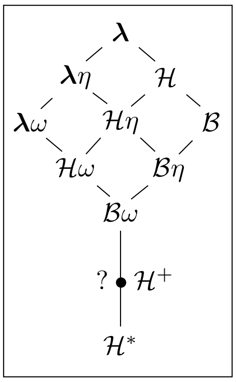

Fig. 3. Barendregt's kite.

▶ **Lemma 22.** *For all λ-theories $\mathcal{T}$, we have:*
(i) $\mathcal{T}\eta \subseteq \mathcal{T}\omega$,
(ii) $\mathcal{T} \subseteq \mathcal{T}'$ *entails* $\mathcal{T}\omega \subseteq \mathcal{T}'\omega$.

In general, because of the quantification over all $P \in \Lambda^o$, it can be difficult to understand what λ-terms different in $\mathcal{T}$ become equal in $\mathcal{T}\omega$, especially when $\mathcal{T}$ is extensional.

The picture in Figure 3, where $\mathcal{T}$ is above $\mathcal{T}'$ if $\mathcal{T} \subsetneq \mathcal{T}'$, is taken from [2, Thm. 17.4.16] and shows some facts about the λ-theories presented in Section 1.3 and the ω-rule.

The counterexample showing that $\boldsymbol{\lambda}\eta \nvdash \omega$ is based on complicated universal generators known as *Plotkin's terms* [2, Def. 17.3.26]. However, since these terms are unsolvable, they become useless when considering sensible λ-theories. We refer to [2, §17.4] for the proof of $\mathcal{H}\eta \nvdash \omega$ and rather discuss the validity of the ω-rule for λ-theories containing $\mathcal{B}$.

Let us consider two λ-terms P and Q whose Böhm trees are depicted in Figure 1. The Böhm trees of P and Q differ because of countably many finite η-expansions of increasing depth, therefore they are different in $\mathcal{B}\eta$ but equal in $\mathcal{H}^+$. This situation is analogous to what happens with $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \mathtt{1}^* \rangle\!\rangle$, indeed $P \leq^\eta Q$ holds. Perhaps surprisingly, P and Q can also be used to prove that $\mathcal{B}\eta \subsetneq \mathcal{B}\omega$ since $\mathcal{B}\omega \vdash P = Q$ holds. Indeed, by Lemma 4, for every $M \in \Lambda^o$, there exists k such that $M\Omega^k$ becomes unsolvable. By inspecting Figure 1, we notice that in $\mathrm{BT}(P)$ the variable y is applied to an increasing number of Ω's (represented in the tree by $\perp$). So, when substituting some $M \in \Lambda^o$ for y in $\mathrm{BT}(Py)$, there will be a level k of the tree where $M\Omega \cdots \Omega$ becomes $\perp$, thus cutting $\mathrm{BT}(PM)$ at level k. The same reasoning can be done for $\mathrm{BT}(QM)$. Therefore $\mathrm{BT}(PM)$ and $\mathrm{BT}(QM)$ only differ because of finitely many η-expansions. Since $\mathcal{B}\eta \subseteq \mathcal{B}\omega$, we conclude that $PM =_{\mathcal{B}\omega} QM$ and therefore by the ω-rule $P =_{\mathcal{B}\omega} Q$. Such an argument is due to Barendregt [2, Lemma 16.4.4].

The fact that $\mathcal{H}^* \vdash \omega$ is an easy consequence of its maximality. However, there are several direct proofs: see [2, §17.2] for a syntactic demonstration and [30] for a semantic one.

A natural question, raised by Barendregt in [2, Thm. 17.4.16], concerns the position of $\mathcal{H}^+$ in the picture of Figure 3. In the proof of that theorem, it is mentioned that Sallé formulated the following conjecture (represented in the diagram with a question mark).

▶ **Sallé's Conjecture.** $\mathcal{B}\omega \subsetneq \mathcal{H}^+$.

The longstanding open question whether $\mathcal{H}^+ \vdash \omega$ has been recently answered positively by Breuvart *et al.* in [5].

▶ **Theorem 23.** [5, Thm. 40] $\mathcal{H}^+ \vdash \omega$.

From this it follows that $\mathcal{B}\omega \subseteq \mathcal{H}^+$. In Theorem 42 we show that $\mathcal{B}\omega = \mathcal{H}^+$, thus disproving Sallé's conjecture.

4 Building Böhm Trees by Codes and Streams

The key step for proving $\mathcal{H}^+ = \mathcal{B}\omega$ is to show that the tree T of Theorem 17 giving the "η-supremum" of M, N can be chosen to be the Böhm tree of a λ-term P (Proposition 38). Intuitively, the λ-term P will inspect the structure of M, N looking at their codes and choose the correct η-expansion to apply from a suitable stream. We start by showing that the Böhm tree of a λ-term can be reconstructed from its code.

4.1 Building Böhm Trees by Codes

Let $\# : \Lambda \to \mathbb{N}$ be an effective one-to-one map, associating with every λ-term M its *code* $\#M$ (the Gödel number of M). The *quote* $\ulcorner M \urcorner$ of M is the corresponding numeral $\mathsf{c}_{\#M}$. We now recall some well established facts from [2, §8.1]. By [2, Thm. 8.1.6], there is a λ-term $\mathtt{E} \in \Lambda^o$ such that $\mathtt{E}\ulcorner M \urcorner = M$ for all $M \in \Lambda^o$. This is false in general for open λ-terms M.

▶ **Remark 24.** The following operations are effective:

- from $\#M$ compute $\#M'$ where $M \to_h M'$ (since the head-reduction is an effective reduction strategy),
- from $\#(\lambda\vec{x}.x_j M_1 \cdots M_k)$ compute $\#M_i$ for $i \leq k$,
- from $\#M$ compute $\#(\lambda x_1 \dots x_n.M)$ for $x_i \in \mathrm{Var}$.

From Remark 24 and Church's thesis, the following term Φ exists and can be defined using the fixed point combinator $\mathtt{Y}$.

▶ **Definition 25.** Let $\Phi \in \Lambda^o$ be such that for all $M \in \Lambda^o$:

- $\Phi\ulcorner M \urcorner = \lambda\underline{x}_1 \dots \underline{x}_n.\underline{x}_j(L_1\underline{x}_1 \cdots \underline{x}_n) \cdots (L_k\underline{x}_1 \cdots \underline{x}_n)$ where $L_i = \Phi\ulcorner \lambda\vec{x}.M_i \urcorner$ if $M \twoheadrightarrow_h \lambda x_1 \dots x_n.x_j M_1 \cdots M_k$.
- $\Phi\ulcorner M \urcorner$ is unsolvable whenever M is unsolvable.

(The $\underline{x}_i$ are underlined to stress the fact that they are fresh.)

The term Φ builds the Böhm tree of M from its code $\#M$. Notice that the closure $\lambda\vec{x}.M_i$ on the recursive calls is needed to obtain a closed term (since $M \in \Lambda^o$ entails $\mathsf{fv}(M_i) \subseteq \vec{x}$). In the definition above we use the fact that $\mathcal{B}$ is a λ-theory, so $\mathrm{BT}(\lambda\vec{x}.M) = \lambda\vec{x}.\mathrm{BT}(M)$ thus the free variables $\underline{\vec{x}}$ can be reapplied externally. This commutation property between $\mathrm{BT}(-)$ and λ-abstraction will be silently used when proving statements about Böhm trees of closed λ-terms, as below.

▶ **Lemma 26.** *For all $M \in \Lambda^o$, $\mathcal{B} \vdash \Phi\ulcorner M \urcorner = M$.*

Proof. If M is unsolvable then $\mathrm{BT}(\Phi\ulcorner M \urcorner) = \mathrm{BT}(M) = \bot$. Otherwise M is solvable, so we have $M \twoheadrightarrow_h \lambda\vec{x}.x_j M_1 \cdots M_k$ and $\Phi\ulcorner M \urcorner = \lambda\vec{x}.x_j(L_1\vec{x}) \cdots (L_k\vec{x})$ for $L_i = \Phi\ulcorner \lambda\vec{x}.M_i \urcorner$. We conclude since by coinductive hypothesis $\mathrm{BT}(\Phi\ulcorner \lambda\vec{x}.M_i \urcorner)\vec{x} = (\lambda\vec{x}.\mathrm{BT}(M_i))\vec{x} = \mathrm{BT}(M_i)$. ◀

4.2 η-Expanding Böhm Trees from Streams

The construction of Φ might look unimpressive in the sense that also the enumerator $\mathtt{E}$ enjoys the property $\mathrm{BT}(\mathtt{E}\ulcorner M \urcorner) = \mathrm{BT}(M)$ for all $M \in \Lambda^o$. However, $\mathtt{E}$ does not satisfy the recursive equation of Definition 25, which has the advantage of exposing the structure of the tree and, doing so, opens the way for altering the tree. For instance, it is possible to modify Definition 25 in order to obtain a λ-term Ψ which builds an η-expansion of $\mathrm{BT}(M)$ starting from the code of M and a stream of η-expansions of the identity (cf. Section 2.1).

▶ **Definition 27.** Let $\vec{\eta} = (\eta_0, \eta_1, \eta_2, \dots)$ be an effective enumeration of all closed η-expansions of $\mathtt{I}$, i.e., of the set $\mathcal{I}^\eta \cap \Lambda^o$. Define the corresponding stream $\langle\!\langle \eta \rangle\!\rangle x = [\eta_0 x, \eta_1 x, \eta_2 x, \dots]$.

From now on, we fix the enumeration $\vec{\eta}$ and the stream $\langle\!\langle \eta \rangle\!\rangle$.

In order to decide what η-expansion is applied at a certain position σ in $\mathrm{BT}(M)$, we use a function $f(\sigma) = n$ and extract the η-expansion of index n from $\langle\!\langle \eta \rangle\!\rangle$. Since f needs to be computable, we fix an effective encoding $\# : \mathbb{N}^* \to \mathbb{N}$ of all finite sequences and consider f computable "after coding".

Notice that, since $\#$ is effective, from the code $\#\sigma$ it is possible to compute the code $\#(\sigma.i)$ for all $i \in \mathbb{N}$, and *vice versa*. We denote by $\ulcorner \sigma \urcorner$ the corresponding numeral $\mathsf{c}_{\#\sigma}$.

In the following definition s is an arbitrary variable, but in practice we will always apply $\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner$ to some stream.

▶ **Definition 28.** Let $f : \mathbb{N}^* \to \mathbb{N}$ be a computable function, and $\Psi_f \in \Lambda^o$ be such that for all $M \in \Lambda^o$ and for all positions $\sigma \in \mathbb{N}^*$:

- $\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner s = \lambda \underline{x}_1 \dots \underline{x}_n . \mathtt{P}_{f(\sigma)} s (\underline{x}_j (L'_1 \vec{\underline{x}}) \cdots (L'_k \vec{\underline{x}}))$ where $L'_i = \Psi_f \ulcorner \lambda x_1 \dots x_n . M_i \urcorner \ulcorner \sigma.i \urcorner\, s$ if $M \twoheadrightarrow_h \lambda x_1 \dots x_n . x_j M_1 \cdots M_k$.
- $\Psi_f \ulcorner M \urcorner$ is unsolvable whenever M is unsolvable.

(Recall that $\mathtt{P}_i$ denotes the i-th projection for streams and the $\underline{x}_i$'s are fresh variables.)

The actual existence of such a Ψ_f follows from Remark 24, the effectiveness of the encodings, the computability of f and Church's thesis. We now verify that the λ-term $\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner$ when applied to the stream $\langle\!\langle \eta \rangle\!\rangle$ actually computes an η-expansion of $\mathrm{BT}(M)$ in the sense of Definition 15.

▶ **Lemma 29.** *Let $f : \mathbb{N}^* \to \mathbb{N}$ be a computable function. For all $M \in \Lambda^o$ and $\sigma \in \mathbb{N}^*$, $M \le^\eta \Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner \langle\!\langle \eta \rangle\!\rangle$.*

Proof. If M is unsolvable, $\mathrm{BT}(M) = \mathrm{BT}(\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner) = \bot$.

Otherwise $M \twoheadrightarrow_h \lambda \vec{x}. x_j M_1 \cdots M_k$. Thus, for $f(\sigma) = q$ and $\eta_q = \lambda y z_1 \cdots z_m . y Q_1 \cdots Q_m$ where $\lambda z_i . Q_i \in \mathcal{I}^\eta$ we have

$$\begin{aligned}\Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner \langle\!\langle \eta \rangle\!\rangle =_\beta \quad & \lambda \vec{x}. \mathtt{P}_q \langle\!\langle \eta \rangle\!\rangle (x_j (L'_1 \vec{x}) \cdots (L'_k \vec{x})) \\ =_\beta \quad & \lambda \vec{x}. \eta_q (x_j (L'_1 \vec{x}) \cdots (L'_k \vec{x})) =_\beta \lambda \vec{x} \vec{z}. x_j (L'_1 \vec{x}) \cdots (L'_k \vec{x}) Q_1 \cdots Q_m\end{aligned}$$

for $L'_i = \Psi_f \ulcorner \lambda \vec{x}. M_i \urcorner \ulcorner \sigma.i \urcorner \langle\!\langle \eta \rangle\!\rangle$. We conclude because by coinductive hypothesis $\mathrm{BT}(M_i) \le^\eta \mathrm{BT}(L'_i \vec{x})$. ◀

Since Ψ_f picks the η-expansion to apply from the input stream, we can retrieve the behaviour of Φ by applying $\langle\!\langle \mathtt{I} \rangle\!\rangle$.

▶ **Lemma 30.** *Let $f : \mathbb{N}^* \to \mathbb{N}$ be computable. For all $M \in \Lambda^o$ and $\sigma \in \mathbb{N}^*$, we have $\mathcal{B} \vdash M = \Psi_f \ulcorner M \urcorner \ulcorner \sigma \urcorner \langle\!\langle \mathtt{I} \rangle\!\rangle$.*

Proof sketch. As in the proof of Lemma 29, using the fact that $\mathtt{P}_q \langle\!\langle \mathtt{I} \rangle\!\rangle = \mathtt{I}$ for all $q \in \mathbb{N}$. ◀

4.3 Building the η-Supremum

Using similar techniques, we define a λ-term Ξ that builds from the codes of M, N and the stream $\langle\!\langle \eta \rangle\!\rangle$ the (smallest) η-supremum satisfying $M \le^\eta \Xi \ulcorner M \urcorner \ulcorner N \urcorner \langle\!\langle \eta \rangle\!\rangle \ge^\eta N$, if it exists, that is whenever M and N are compatible (Proposition 38). Intuitively, at every position σ, Ξ needs to compare the structure of M, N at σ and apply the correct η_i taken from $\langle\!\langle \eta \rangle\!\rangle$.

$$
\Xi_\iota\lceil M\rceil\lceil N\rceil s = \begin{cases}
\lambda\underline{x}_1\dots\underline{x}_n.\mathtt{P}_q s(\underline{x}_j(\Upsilon_1\underline{x}_1\cdots\underline{x}_n)\cdots(\Upsilon_k\underline{x}_1\cdots\underline{x}_n)) & \text{if } M \le_h N, \text{ with} \\
\text{where } \left[\begin{array}{l} q = \iota(\#(\mathtt{nf}_\beta(\lambda\underline{y}z_1\dots z_m.\underline{y}Q_1\cdots Q_m))) \\ \Upsilon_i := \Xi_\iota\lceil\lambda x_1\dots x_n.M_i\rceil\lceil\lambda x_1\dots x_n.N_i\rceil s \end{array}\right. & \begin{array}{l} M \twoheadrightarrow_h \lambda x_1\dots x_n.x_jM_1\cdots M_k \text{ and} \\ N \twoheadrightarrow_h \lambda x_1 ... x_n z_1 ... z_m.x_jN_1\cdots N_kQ_1\cdots Q_m; \end{array} \\
\Xi_\iota\lceil N\rceil\lceil M\rceil s & \text{if } N <_h M; \\
\Omega & \text{otherwise.}
\end{cases}
$$

Figure 4 The λ-term $\Xi_\iota \in \Lambda^o$ satisfies in $\mathcal{H}$ the recursive equation above.

Rather than proving that there exists a computable function $f : \mathbb{N}^* \to \mathbb{N}$ associating to every σ the corresponding η_i (which can be tedious) we use the following property of $\vec{\eta} = (\eta_0, \eta_1, \dots, \eta_i, \dots)$: since every closed η-expansion $Q \in \mathcal{I}^\eta$ is β-normalizable and the enumeration $\vec{\eta}$ is effective, it is possible to decide starting from the code $\#Q$ the index i of Q in $\vec{\eta}$. Moreover, it is possible to choose such an i minimal.

▶ **Lemma 31.** *There exists a computable function $\iota : \mathbb{N} \to \mathbb{N}$ such that, for all $M \in \Lambda^o$, if $M =_\beta \eta_i$ and $M \neq_\beta \eta_k$ for all $k < i$ then $\iota(\#M) = i$.*

Proof. Let $\delta(m, n)$ be the partial computable map satisfying for all normalizing $M, N \in \Lambda^o$: $\delta(\#M, \#N) = 0$ if M and N have the same β-normal form; $\delta(\#M, \#N) = 1$ otherwise. Then ι can be defined by setting $\iota(n) := \mu k.\delta(\#(\pi_k\vec{\eta}), n) = 0$. ◀

From now on we consider fixed such a function ι, which depends on the enumeration $\vec{\eta}$ generating the stream $\langle\!\langle \eta \rangle\!\rangle$.

▶ **Definition 32.** For $M, N \in \Lambda$, we define:

(i) $M \le_h N$ whenever $M \twoheadrightarrow_h \lambda\vec{x}.x_jM_1\cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}z_1\dots z_m.x_jN_1\cdots N_kQ_1\cdots Q_m$ with $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ for all $\ell \le m$,

(iii) $M \sim_h N$ if both $M \le_h N$ and $N \le_h M$ hold,

(ii) $M <_h N$ if $M \le_h N$ holds but $M \not\sim_h N$.

Whenever $M \le_h N$ holds, we say that N *looks like an η-expansion of M*. This does not necessarily mean that it actually is: for instance $\lambda z.x\mathtt{F}z \le_h \lambda z.x\mathtt{K}z$ since we do not require that $\mathtt{F} \le_h \mathtt{K}$ holds, and $z \le_h \lambda z.zz$ since we do not check that $z \notin \mathsf{fv}(\mathrm{BT}(x_j\vec{M}\vec{N}))$. Therefore, compared with $\le^\eta$ of Definition 15, the relation $\le_h$ is weaker since it lacks the coinductive calls and the occurrence check on z_ℓ. This is necessary to ensure the following semi-decidability property.

▶ **Remark 33.** The property $M \le_h N$ can be semi-decided:

- first head-reduce in parallel M, N until they reach a hnf,
- if both reductions terminate, compare the two hnfs and check whether they have the shape of Definition 32(i),
- then semi-decide whether $Q_\ell \twoheadrightarrow_{\beta\eta} z_\ell$ for all $\ell \le m$.

This procedure might fail to terminate when $M \not\le_h N$.

By Remarks 24 and 33, the fact that ι is computable (Lemma 31) and Church's thesis, the λ-term Ξ_ι below exists.

▶ **Definition 34.** Let $\iota : \mathbb{N} \to \mathbb{N}$ be the computable function of Lemma 31. We define $\Xi_\iota \in \Lambda^o$ such that for all $M, N \in \Lambda^o$ the recursive equation of Figure 4 is satisfied in $\mathcal{H}$.

There are some subtleties to discuss in the definition of Ξ_ι. The fact that $Q_\ell \twoheadrightarrow_{\beta\eta} z_\ell$ for all $\ell \le m$, although not explicitly written, is a consequence of $M \le_h N$. *A priori*

$\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ might be open (consider for instance $\lambda z_\ell.\mathtt{K}z_\ell y \to_\beta \mathtt{I}$) but its β-normal form is always a closed λ-term. This is the reason why we compute $\mathtt{nf}_\beta(\lambda y\vec{z}.y\vec{Q})$ before applying ι to its code. In particular, ι is defined on all the codes $\#(\mathtt{nf}_\beta(\lambda y\vec{z}.y\vec{Q}))$.

The following commutativity property follows from the second condition of Figure 4 and should be natural considering that $\Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\eta\rangle\!\rangle$ is supposed to compute the η-join of $\mathrm{BT}(M)$ and $\mathrm{BT}(N)$ which is a commutative operation.

▶ **Lemma 35.** *For all $M,N \in \Lambda^o$, we have $\mathcal{B} \vdash \Xi_\iota\lceil M\rceil\lceil N\rceil = \Xi_\iota\lceil N\rceil\lceil M\rceil$.*

Proof. We proceed by coinduction on their Böhm trees.

If M,N are unsolvable or neither $M \le_h N$ nor $N \le_h M$ holds, then $\mathrm{BT}(\Xi_\iota\lceil M\rceil\lceil N\rceil) = \mathrm{BT}(\Xi_\iota\lceil N\rceil\lceil M\rceil) = \bot$. The cases $M <_h N$ and $N <_h M$ follow by definition.

If $M \sim_h N$, then we have $M \twoheadrightarrow_h \lambda\vec{x}.x_j M_1\cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}.x_j N_1\cdots N_k$. Since $\mathtt{P}_q s \twoheadrightarrow_\beta \lambda y.sy\mathtt{F}^{\sim q}\mathtt{K}$ we have

$$\Xi_\iota\lceil M\rceil\lceil N\rceil s =_\beta \lambda\vec{x}.(\lambda y.sy\mathtt{F}^{\sim q}\mathtt{K})(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})) =_\beta \lambda\vec{x}.s(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\mathtt{F}^{\sim q}\mathtt{K}$$
$$\Xi_\iota\lceil N\rceil\lceil M\rceil s =_\beta \lambda\vec{x}.(\lambda y.sy\mathtt{F}^{\sim q}\mathtt{K})(x_j(\Upsilon'_1\vec{x})\cdots(\Upsilon'_k\vec{x})) =_\beta \lambda\vec{x}.s(x_j(\Upsilon'_1\vec{x})\cdots(\Upsilon'_k\vec{x}))\mathtt{F}^{\sim q}\mathtt{K}$$

where, for all $i \le k$, we have $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M_i\rceil\lceil\lambda\vec{x}.N_i\rceil s$ and $\Upsilon'_i = \Xi_\iota\lceil\lambda\vec{x}.N_i\rceil\lceil\lambda\vec{x}.M_i\rceil s$. We conclude since, by coinductive hypothesis, we have $\mathrm{BT}(\Upsilon_i\vec{x}) = \mathrm{BT}(\Upsilon'_i\vec{x})$ for all $i \le k$. ◀

Another property that we expect is that whenever $M \le^\eta N$ the λ-term $\Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\eta\rangle\!\rangle$ computes the Böhm tree of N.

▶ **Lemma 36.** *For all $M,N \in \Lambda^o$, if $M \le^\eta N$ then $\mathcal{B} \vdash \Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\eta\rangle\!\rangle = N$.*

Proof. By coinduction on their Böhm trees. If M,N are unsolvable then so is $\Xi_\iota\lceil M\rceil\lceil N\rceil$. Otherwise $M \le^\eta N$ implies $M \twoheadrightarrow_h \lambda\vec{x}.x_j M_1\cdots M_k$, $N \twoheadrightarrow_h \lambda\vec{x}z_1...z_m.x_j N_1\cdots N_k Q_1\cdots Q_m$ where each $z_\ell \notin \mathsf{fv}(\mathrm{BT}(x_j\vec{M}\vec{N}))$, $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ and $M_i \le^\eta N_i$. In particular $M \le_h N$ holds, so the first condition of Figure 4 applies.

From $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ it follows that $\lambda y\vec{z}.y\vec{Q} \in \mathcal{I}^\eta$, therefore $\iota(\#\mathtt{nf}_\beta(\lambda y\vec{z}.y\vec{Q})) = q$ for some index q. Setting $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M\rceil\lceil\lambda\vec{x}.N\rceil\langle\!\langle\eta\rangle\!\rangle$, easy calculations give:

$$\begin{aligned}\Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\eta\rangle\!\rangle &=_\beta \lambda\vec{x}.\mathtt{P}_q\langle\!\langle\eta\rangle\!\rangle(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ &=_\beta \lambda\vec{x}.(\lambda y\vec{z}.y\vec{Q})(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})) =_\beta \lambda\vec{x}\vec{z}.x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})Q_1\cdots Q_m\end{aligned}$$

We conclude as, by coinductive hypothesis, we have $\mathrm{BT}(\Upsilon_i) = \mathrm{BT}(\lambda\vec{x}.N_i)$ for all $i \le k$. ◀

Under the assumption $M \le^\eta N$ we can also use Ξ_ι to retrieve the Böhm tree of M by applying the stream $\langle\!\langle\mathtt{I}\rangle\!\rangle$. Since ι has been defined as depending on the enumeration $\vec{\eta}$, $\iota(\#Q)$ still provides an index q such that $\mathtt{P}_q\langle\!\langle\eta\rangle\!\rangle = Q$ but when applied to $\langle\!\langle\mathtt{I}\rangle\!\rangle$ it necessarily gives $\mathtt{P}_q\langle\!\langle\eta\rangle\!\rangle = \mathtt{I}$. This technique is analogous to the one used in Lemma 30.

▶ **Lemma 37.** *For all $M,N \in \Lambda^o$, if $M \le^\eta N$ then $\mathcal{B} \vdash \Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle = M$.*

Proof. We proceed by coinduction on their Böhm trees. If M,N are both unsolvable, then also $\Xi_\iota\lceil M\rceil\lceil N\rceil$ must be. Otherwise $M \le^\eta N$ implies $M \twoheadrightarrow_h \lambda\vec{x}.x_j M_1\cdots M_k$ and $N \twoheadrightarrow_h \lambda\vec{x}z_1\ldots z_m.x_j N_1\cdots N_k Q_1\cdots Q_m$ where each $z_\ell \notin \mathsf{fv}(\mathrm{BT}(x_j\vec{M}\vec{N}))$, $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ and $M_i \le^\eta N_i$. In particular $M \le_h N$ holds, so the first condition of Figure 4 applies.

From $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$ it follows that $\lambda y\vec{z}.y\vec{Q} \in \mathcal{I}^\eta$, therefore $\iota(\#\mathtt{nf}_\beta(\lambda y\vec{z}.y\vec{Q})) = q$ for some index q. Setting $\Upsilon_i = \Xi_\iota\lceil\lambda\vec{x}.M\rceil\lceil\lambda\vec{x}.N\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle$, easy calculations give:

$$\begin{aligned}\Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle\mathtt{I}\rangle\!\rangle &=_\beta \lambda\vec{x}.\mathtt{P}_q\langle\!\langle\mathtt{I}\rangle\!\rangle(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x}))\\ &=_\beta \lambda\vec{x}.\mathtt{I}(x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})) =_\beta \lambda\vec{x}.x_j(\Upsilon_1\vec{x})\cdots(\Upsilon_k\vec{x})\end{aligned}$$

We conclude since, by coinductive hypothesis, we have $\mathrm{BT}(\Upsilon_i) = \mathrm{BT}(\lambda\vec{x}.M_i)$ for all $i \le k$. ◀

5 $\mathcal{B}\omega = \mathcal{H}^+$ and Sallé's Conjecture is False

This section is devoted to prove that $\mathcal{B}\omega = \mathcal{H}^+$ holds (Theorem 42). As mentioned earlier, the first step is to show that the term Ξ_ι defined in the previous section, when applied to $\mathcal{H}^+$-equivalent terms, actually computes their η-supremum.

▶ **Proposition 38.** *For all $M, N \in \Lambda^o$, $\mathcal{H}^+ \vdash M = N$ iff $M \leq^\eta \Xi_\iota \lceil M \rceil \lceil N \rceil \langle\!\langle \eta \rangle\!\rangle \geq^\eta N$.*

Proof. ($\Leftarrow$) It follows directly from Theorem 17.

($\Rightarrow$) By Theorem 17, we known that there exists a Böhm-like tree $T \in \mathbb{BT}$ such that $\mathrm{BT}(M) \leq^\eta T \geq^\eta \mathrm{BT}(N)$. As usual, we proceed by coinduction on the Böhm(-like) trees.

If M or N is unsolvable then $\mathrm{BT}(M) = \mathrm{BT}(N) = T = \bot$.

Otherwise, from $\mathrm{BT}(M) \leq^\eta T \geq^\eta \mathrm{BT}(N)$ we have, say:

$$M \twoheadrightarrow_h \lambda\vec{x}.x_j M_1 \cdots M_k, \qquad N \twoheadrightarrow_h \lambda\vec{x}z_1 \ldots z_m.x_j N_1 \cdots N_k Q_1 \cdots Q_m,$$
$$T = \lambda\vec{x}z_1 \ldots z_m \ldots z_{m'}.x_j T_1 \cdots T_k Q'_1 \cdots Q'_m \cdots Q'_{m'},$$

such that $z_1, \ldots, z_{m'} \notin \mathsf{fv}(\mathrm{BT}(x_j M_1 \cdots M_k) T_1 \cdots T_k)$, $z_{m+1}, \ldots, z_{m'} \notin \mathsf{fv}(\mathrm{BT}(x_j N_1 \cdots N_k))$, $\mathrm{BT}(M_i) \leq^\eta T_i$ and $\mathrm{BT}(N_i) \leq^\eta T_i$ for all $i \leq k$, $Q_\ell \leq^\eta Q'_\ell$ for all $\ell \leq m$, and $\lambda z_{\ell'}.Q'_{\ell'} \in \mathcal{I}^\eta$ for all $\ell' > m$. By Lemma 12, $Q'_\ell \twoheadrightarrow_{\beta\eta} z_\ell$ so $Q_\ell \leq^\eta Q'_\ell$ entails $\lambda z_\ell.Q_\ell \in \mathcal{I}^\eta$, hence $M \leq_h N$. Setting $q = \iota(\#(\mathtt{nf}(\lambda x z_1 \ldots z_m.x Q_1 \cdots Q_m)))$ and $\Upsilon_i = \Xi_\iota \lceil \lambda\vec{x}.M \rceil \lceil \lambda\vec{x}.N \rceil \langle\!\langle \eta \rangle\!\rangle$, we obtain:

$$\begin{aligned}&\Xi_\iota \lceil M \rceil \lceil N \rceil \langle\!\langle \eta \rangle\!\rangle =_\beta \lambda\vec{x}.\mathsf{P}_q \langle\!\langle \eta \rangle\!\rangle (x_j(\Upsilon_1 \vec{x}) \cdots (\Upsilon_k \vec{x}))\\ &=_\beta \lambda\vec{x}.(\lambda x\vec{z}.x Q_1 \cdots Q_m)(x_j(\Upsilon_1 \vec{x}) \cdots (\Upsilon_k \vec{x})) =_\beta \lambda\vec{x}\vec{z}.x_j(\Upsilon_1 \vec{x}) \cdots (\Upsilon_k \vec{x}) Q_1 \cdots Q_m.\end{aligned}$$

This case follows from the coinductive hypotheses since, for all $i \leq k$, $\lambda\vec{x}.M_i \leq^\eta \Upsilon_i \geq^\eta \lambda\vec{x}.N_i$. The symmetric case $N <_h M$ is treated analogously, using Lemma 35. ◀

The second step towards the proof of Theorem 42 is to show that the streams $\langle\!\langle \mathtt{I} \rangle\!\rangle$ and $\langle\!\langle \eta \rangle\!\rangle$ are equated in $\mathcal{B}\omega$. To prove such a result, we are going to use the auxiliary streams:

- $\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega yx = [yx, y\Omega x, y\Omega^{\sim 2} x, y\Omega^{\sim 3} x, \ldots]$,
- $\langle\!\langle \eta \rangle\!\rangle^\Omega yx = [y(\eta_0 x), y\Omega(\eta_1 x), y\Omega^{\sim 2}(\eta_2 x), y\Omega^{\sim 3}(\eta_3 x), \ldots]$,

which are equal in $\mathcal{B}\omega$, for the same reason the λ-terms P, Q of Figure 1 are.

▶ **Lemma 39.** *$\mathcal{B}\omega \vdash \langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega = \langle\!\langle \eta \rangle\!\rangle^\Omega$.*

Proof. Let $M \in \Lambda^o$, by Lemma 4 there exists $k \in \mathbb{N}$ such that $M\Omega^{\sim k} =_{\mathcal{B}} \Omega$. So we have:

$$\begin{aligned}\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega M \quad &=_{\mathcal{B}} \quad \lambda x.[Mx, M\Omega x, \ldots, M\Omega^{\sim k-1} x, \Omega, \ldots]\\ &=_{\mathcal{B}} \quad \lambda x.[M(\mathtt{I}x), M\Omega(\mathtt{I}x), \ldots, M\Omega^{\sim k-1}(\mathtt{I}x), \Omega, \ldots]\\ &=_{\beta\eta} \quad \lambda x.[M(\eta_0 x), M\Omega(\eta_1 x), \ldots, M\Omega^{\sim k-1}(\eta_{k-1} x), \Omega, \ldots] =_{\mathcal{B}} \langle\!\langle \eta \rangle\!\rangle^\Omega M,\end{aligned}$$

where the third equality follows from $\mathtt{I} =_{\beta\eta} \eta_i$ for all $i \in \mathbb{N}$. Since M is an arbitrary closed λ-term, we can apply the ω-rule and conclude $\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega =_{\mathcal{B}\omega} \langle\!\langle \eta \rangle\!\rangle^\Omega$. ◀

As the variable y occurs in head-position in the terms of the streams $\langle\!\langle \eta \rangle\!\rangle^\Omega yx$ (resp. $\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega yx$), we can substitute for it a suitably modified projection that erases the Ω's and returns the n-th occurrence of x in $\langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega$ (resp. $\eta_n(x)$ in $\langle\!\langle \eta \rangle\!\rangle^\Omega$).

▶ **Lemma 40.** *There is a closed λ-term Eq such that $\mathsf{Eq}\,\mathsf{c}_n \langle\!\langle \mathtt{I} \rangle\!\rangle^\Omega =_{\mathcal{B}} \mathtt{I}$ and $\mathsf{Eq}\,\mathsf{c}_n \langle\!\langle \eta \rangle\!\rangle^\Omega =_{\mathcal{B}} \eta_n$.*

Proof. Let Eq be a λ-term satisfying the recursive equation

$$\mathsf{Eq}\, n\, s = \mathsf{ifz}(n, \lambda z.s\mathtt{I}z\mathtt{K}, \mathsf{Eq}\,(\mathsf{pred}\, n)\,(\lambda zw.s(\mathtt{K}z)w\mathtt{F})).$$

By induction on n, we show $\mathsf{Eq}\,\mathrm{c}_n(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) =_{\mathcal{B}} \eta_{n+k}$ for all $n,k \in \mathbb{N}$. Note that $\eta_i \in \mathcal{I}^\eta$ entails $\lambda z.\eta_i z =_\beta \eta_i$. If $n = 0$ then $\mathsf{Eq}\,\mathrm{c}_0(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}})$ β-reduces to

$$\begin{array}{rlrlrl} & \lambda z.(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}})\mathtt{I}z\mathtt{K} & =_\beta & \lambda z.[\Omega^{\sim i}(\eta_{i+k}z)]_{i\in\mathbb{N}}\mathtt{K} & & \\ =_\beta & \lambda z.\mathtt{K}(\eta_k z)[\Omega^{\sim i+1}(\eta_{i+k+1}z)]_{i\in\mathbb{N}} & =_\beta & \lambda z.\eta_k z & =_\beta & \eta_k. \end{array}$$

If $n > 0$ then we have $\mathsf{Eq}\,\mathrm{c}_n(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}}) =_\beta \mathsf{Eq}\,\mathrm{c}_{n-1}(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k+1}x)]_{i\in\mathbb{N}})$ $=_{(\text{by Ind. Hyp.})} \eta_{n-1+k+1} = \eta_{n+k}$. Indeed, easy calculations give:

$$\begin{array}{rlrl} & \lambda zw.(\lambda yx.[y\Omega^{\sim i}(\eta_{i+k}x)]_{i\in\mathbb{N}})(\mathtt{K}z)w\mathtt{F} & =_\beta & \lambda zw.[\mathtt{K}z(\eta_k w),[\mathtt{K}z\Omega^{\sim i+1}(\eta_{i+k+1}w)]_{i\in\mathbb{N}}]\mathtt{F} \\ =_{\mathcal{B}} & \lambda zw.[z,[z\Omega^{\sim i}(\eta_{i+k+1}w)]_{i\in\mathbb{N}}]\mathtt{F} & =_\beta & \lambda zw.\mathtt{F}z[z\Omega^{\sim i}(\eta_{i+k+1}w)]_{i\in\mathbb{N}} \\ =_\beta & \lambda zw.[z\Omega^{\sim i}(\eta_{i+k+1}w)]_{i\in\mathbb{N}} & =_\alpha & \lambda yx.[y\Omega^{\sim i}(\eta_{i+k+1}x)]_{i\in\mathbb{N}}. \end{array}$$

Analogous calculations show $\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega =_{\mathcal{B}} \mathtt{I}$. ◀

▶ **Corollary 41.** $\mathcal{B}\omega \vdash \langle\!\langle \mathtt{I}\rangle\!\rangle = \langle\!\langle \eta\rangle\!\rangle$.

Proof. By Lemmas 40, 7 and 39: $\langle\!\langle \mathtt{I}\rangle\!\rangle =_{\mathcal{B}} [\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle \mathtt{I}\rangle\!\rangle^\Omega]_{n\in\mathbb{N}} =_{\mathcal{B}\omega} [\mathsf{Eq}\,\mathrm{c}_n\langle\!\langle \eta\rangle\!\rangle^\Omega]_{n\in\mathbb{N}} =_{\mathcal{B}} \langle\!\langle \eta\rangle\!\rangle$. ◀

In Section 4.2 we have seen that, when $M \leq^\eta N$ holds, the λ-term $\Xi_\iota\lceil M\rceil\lceil N\rceil$ computes the Böhm tree of N from $\langle\!\langle \eta\rangle\!\rangle$ (Lemma 36) and the Böhm tree of M from $\langle\!\langle \mathtt{I}\rangle\!\rangle$ (Lemma 37), but now we have proved that $\langle\!\langle \eta\rangle\!\rangle =_{\mathcal{B}\omega} \langle\!\langle \mathtt{I}\rangle\!\rangle$. As a consequence, M and N are equal in $\mathcal{B}\omega$.

▶ **Theorem 42.** $\mathcal{B}\omega = \mathcal{H}^+$.

Proof. ($\subseteq$) By Lemma 22(ii), $\mathcal{B} \subseteq \mathcal{H}^+$ entails $\mathcal{B}\omega \subseteq \mathcal{H}^+\omega$. By Theorem 23 we have $\mathcal{H}^+\omega = \mathcal{H}^+$, so $\mathcal{B}\omega \subseteq \mathcal{H}^+$.

($\supseteq$) By Remark 10, it is enough to consider $M,N \in \Lambda^o$. If $\mathcal{H}^+ \vdash M = N$, then by Proposition 38 we have $M \leq^\eta P \geq^\eta N$ for $P = \Xi_\iota\lceil M\rceil\lceil N\rceil\langle\!\langle \eta\rangle\!\rangle$. Then we have:

$$\begin{array}{rlll} M & =_{\mathcal{B}} & \Xi_\iota\lceil M\rceil\lceil P\rceil\langle\!\langle \mathtt{I}\rangle\!\rangle & \text{by Lemma 37} \\ & =_{\mathcal{B}\omega} & \Xi_\iota\lceil M\rceil\lceil P\rceil\langle\!\langle \eta\rangle\!\rangle & \text{by Corollary 41} \\ & =_{\mathcal{B}} & P & \text{by Lemma 36} \\ & =_{\mathcal{B}} & \Xi_\iota\lceil N\rceil\lceil P\rceil\langle\!\langle \eta\rangle\!\rangle & \text{by Lemma 36} \\ & =_{\mathcal{B}\omega} & \Xi_\iota\lceil N\rceil\lceil P\rceil\langle\!\langle \mathtt{I}\rangle\!\rangle & \text{by Corollary 41} \\ & =_{\mathcal{B}} & N & \text{by Lemma 37} \end{array}$$

We conclude that $\mathcal{B}\omega \vdash M = N$. ◀

This theorem disproves Sallé's conjecture (page 9) and settles one of the few open problems left in Barendregt's book. The next theorem should substitute Theorem 17.4.16 in [2].

▶ **Theorem 43.** *The following diagram indicates all inclusion relations of the λ-theories involved (if $\mathcal{T}_1$ is above $\mathcal{T}_2$, then $\mathcal{T}_1 \subsetneq \mathcal{T}_2$):*

$$\begin{tikzcd}
 & & \boldsymbol{\lambda} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \boldsymbol{\lambda}\eta \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{H} \arrow[dl, dash] \arrow[dr, dash] & \\
\boldsymbol{\lambda}\omega \arrow[dr, dash] & & \mathcal{H}\eta \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{B} \arrow[dl, dash] \\
 & \mathcal{H}\omega \arrow[dr, dash] & & \mathcal{B}\eta \arrow[dl, dash] & \\
 & & \mathcal{B}\omega = \mathcal{H}^+ \arrow[d, dash] & & \\
 & & \mathcal{H}^* & &
\end{tikzcd}$$

Acknowledgments. The authors would like to thank Barendregt, Breuvart, Dezani-Ciancaglini, Lévy, Ronchi della Rocca and Ruoppolo for interesting discussions on $\mathcal{H}^+$.

References

1 Hendrik P. Barendregt. *Some extensional term models for combinatory logics and λ-calculi*. Ph.D. thesis, Utrecht Universiteit, the Netherlands, 1971.

2 Hendrik P. Barendregt. *The lambda calculus, its syntax and semantics*. Number 103 in Studies in Logic and the Foundations of Mathematics. North-Holland, second edition, 1984.

3 Hendrik P. Barendregt, Jan A. Bergstra, Jan Willem Klop, and Henri Volken. Degrees of sensible lambda theories. *Journal of Symbolic Logic*, 43(1):45–55, 1978.

4 Flavien Breuvart. On the characterization of models of $\mathcal{H}^*$. In Tom Henzinger and Dale Miller, editors, *CSL-LICS'14*, pages 24:1–24:10. ACM, 2014.

5 Flavien Breuvart, Giulio Manzonetto, Andrew Polonsky, and Domenico Ruoppolo. New results on Morris's observational theory. In Delia Kesner and Brigitte Pientka, editors, *Formal Structures for Computation and Deduction*, volume 52 of *LIPIcs*, pages 15:1–15:18. Schloss Dagstuhl, 2016.

6 Matt Brown and Jens Palsberg. Breaking through the normalization barrier: a self-interpreter for F-omega. In Rastislav Bodík and Rupak Majumdar, editors, *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2016*, pages 5–17. ACM, 2016.

7 Mario Coppo, Mariangiola Dezani-Ciancaglini, and Patrick Sallé. Functional characterization of some semantic equalities inside lambda-calculus. In Hermann A. Maurer, editor, *Automata, Languages and Programming*, volume 71 of *Lecture Notes in Computer Science*, pages 133–146. Springer, 1979.

8 Mario Coppo, Mariangiola Dezani-Ciancaglini, and Maddalena Zacchi. Type theories, normal forms and $\mathcal{D}_\infty$-lambda-models. *Information and Computation*, 72(2):85–116, 1987.

9 Pietro Di Gianantonio, Gianluca Franco, and Furio Honsell. Game semantics for untyped $\lambda\beta\eta$-calculus. In *Typed Lambda Calculi and Applications*, volume 1581 of *Lecture Notes in Computer Science*, pages 114–128. Springer, 1999.

10 Thomas Given-Wilson and Barry Jay. A combinatory account of internal structure. *J. Symb. Log.*, 76(3):807–826, 2011.

11 Xavier Gouy. *Étude des théories équationnelles et des propriétés algébriques des modèles stables du λ-calcul*. Thèse de doctorat, Université de Paris 7, 1995.

12 Martin Hyland. A survey of some useful partial order relations on terms of the λ-calculus. In *Lambda-Calculus and Computer Science Theory*, volume 37 of *Lecture Notes in Computer Science*, pages 83–95. Springer, 1975.

13 Martin Hyland. A syntactic characterization of the equality in some models for the λ-calculus. *Journal London Mathematical Society (2)*, 12(3):361–370, 1975/76.

14 Benedetto Intrigila and Monica Nesi. On structural properties of η-expansions of identity. *Inf. Proc. Lett.*, 87(6):327–333, 2003.

15 Benedetto Intrigila and Richard Statman. The omega rule is Π^0_2-hard in the $\lambda\beta$-calculus. In *Symposium on Logic in Computer Science (LICS 2004)*, pages 202–210. IEEE Computer Society, 2004.

16 Benedetto Intrigila and Richard Statman. The omega rule is Π^1_1-complete in the $\lambda\beta$-calculus. *Logical Methods in Computer Science*, 5(2), 2009.

17 Jean-Jacques Lévy. Approximations et arbres de Böhm dans le lambda-calcul. In Bernard Robinet, editor, *Lambda Calcul et Sémantique formelle des langages de programmation, Actes de la 6ème École de printemps d'Informatique théorique, La Châtre*, LITP-ENSTA, pages 239–257, 1978. (In French).

18 Stefania Lusin and Antonino Salibra. The lattice of λ-theories. *Journal of Logic and Computation*, 14(3):373–394, 2004.

19 Giulio Manzonetto. A general class of models of $\mathcal{H}^*$. In Rastislav Královic and Damian Niwinski, editors, *Mathematical Foundations of Computer Science 2009*, volume 5734 of *Lecture Notes in Computer Science*, pages 574–586. Springer, 2009.

20 Giulio Manzonetto and Domenico Ruoppolo. Relational graph models, Taylor expansion and extensionality. *Electronic Notes in Theoretical Computer Science*, 308:245–272, 2014.

21 Torben Æ. Mogensen. Efficient self-interpretations in lambda calculus. *J. Funct. Program.*, 2(3):345–363, 1992.

22 James H. Morris. *Lambda calculus models of programming languages*. PhD thesis, Massachusetts Institute of Technology, 1968.

23 Gordon D. Plotkin. The lambda-calculus is ω-incomplete. *Journal of Symbolic Logic*, 39(2):313–317, 1974.

24 Andrew Polonsky. Axiomatizing the Quote. In Marc Bezem, editor, *Computer Science Logic (CSL'11) - 25th International Workshop/20th Annual Conference of the EACSL*, volume 12 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 458–469, Dagstuhl, Germany, 2011. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

25 Simona Ronchi Della Rocca and Luca Paolini. *The Parametric λ-Calculus: a Metamodel for Computation*. EATCS Series. Springer, Berlin, 2004.

26 Patrick Sallé. Une extension de la théorie des types en λ-calcul. In Giorgio Ausiello and Corrado Böhm, editors, *Automata, Languages and Programming: Fifth Colloquium, Udine, Italy, July 17–21, 1978*, pages 398–410. Springer Berlin Heidelberg, 1978.

27 Dana S. Scott. Continuous lattices. In Lawvere, editor, *Toposes, Algebraic Geometry and Logic*, volume 274 of *Lecture Notes in Mathematics*, pages 97–136. Springer, 1972.

28 Paula Severi and Fer-Jan de Vries. An extensional Böhm model. In Sophie Tison, editor, *Rewriting Techniques and Applications: 13th International Conference, RTA 2002*, pages 159–173. Springer Berlin Heidelberg, 2002.

29 Paula Severi and Fer-Jan de Vries. The infinitary lambda calculus of the infinite η-Böhm trees. *Mathematical Structures in Computer Science*, 27(5):681–733, 2017.

30 Christopher P. Wadsworth. The relation between computational and denotational properties for Scott's $\mathcal{D}_\infty$-models of the lambda-calculus. *SIAM Journal of Computing*, 5(3):488–521, 1976.