Constructing differential categories and deconstructing categories of games

Jim Laird[1], Giulio Manzonetto[2], and Guy McCusker[1]

[1] Department of Computer Science, University of Bath, Bath, BA2 7AY, UK
[2] Radboud University, Intelligent Systems, Nijmegen, The Netherlands

Abstract. We present an abstract construction for building differential categories useful to model resource sensitive calculi, and we apply it to categories of games. In one instance, we recover a category previously used to give a fully abstract model of a nondeterministic imperative language. The construction exposes the differential structure already present in this model. A second instance corresponds to a new Cartesian differential category of games. We give a model of a Resource PCF in this category and show that it enjoys the finite definability property. Comparison with a relational semantics reveals that the latter also possesses this property and is fully abstract.

1 Introduction

An important aim in studying higher-order computation is to understand and control the way resources are used. One way to do this is by studying calculi designed to capture resource usage, and their denotational models. Two such calculi — the *differential λ-calculus* [7] of Ehrhard and Regnier and the *resource calculus* introduced by Tranquilli [14], inspired by the work of Boudol [5], are fundamentally related at the semantic level [15]: both may be interpreted using the notion of *differential category* introduced by Blute, Cockett and Seely [3]. In this paper, we study these concepts on both abstract and concrete levels. We give a construction of a differential category from any symmetric monoidal category, and use it to investigate the structure of newly discovered differential categories, relate them to existing examples, and to prove full abstraction results for *Resource PCF*, a typed programming language based on the resource calculus.

A potential source of differential categories, although not investigated hitherto, is *game semantics*: resource usage is represented rather explicitly in games and strategies. Indeed, we show that an existing games model of Idealized Algol with non-determinism, introduced by Harmer and McCusker [9] contains a differential Cartesian operator [4], and may therefore be used to interpret Resource PCF, although this interpretation contains non-definable finitary elements.

We then present the construction which we shall use to analyze differential categories. Its key step takes a symmetric monoidal category with countable biproducts, embeds it in its *Karoubi envelope* (idempotent splitting) and then constructs the *cofree cocommutative comonoid* on this category and a differential operator on the Kleisli category of the corresponding comonad. Since biproducts may be added to any category by free constructions, we have a way of embedding any symmetric monoidal (closed) category in a Cartesian (closed) differential category.

Although this construction is somewhat elaborate, it provides a useful tool for analyzing and relating more directly presented models. For example, applying it to the terminal (one object, one morphism) SMCC yields the key example of a differential category (and model of resource calculus [6]) based on the finite-multiset comonad on the category of sets and relations. We also show that our differential category of games embeds in one constructed from a simple symmetric monoidal category of games. By refining the strategies in these games to eliminate *history sensitive* behaviour, we obtain a constraint on strategies ($\sim$-closure) in our directly presented model of Resource PCF which corresponds to finite definability. Another useful observation is that any functor of symmetric monoidal categories lifts to one between the differential categories constructed from them. In particular, from the terminal functor we derive a functor from our category of games and $\sim$-closed strategies into the relational model which is shown to be *full*. From this we may deduce that the relational model of Resource PCF is *fully abstract*.

2 Differential categories

Differential categories were introduced by Blute, Cockett and Seely to formalize derivatives categorically. The authors started from monoidal categories [3], then extended the notion to Cartesian ones [4]; a further generalization to Cartesian closed categories has been made in [6] to model differential and resource λ-calculi.

Throughout this paper we will be working with categories whose hom-sets are endowed with the structure of a commutative monoid $(+, 0)$. We write the identity map on an object A as id_A. Composition is written using infix ; in diagram order. We use $\langle f, g\rangle$ to denote the pairing of maps $f : A \to B$ and $g : A \to C$, and π_0, π_1 for the corresponding projections. We elide all associativity and unit isomorphisms associated with monoidal categories.

Let $\mathbf{C}$ be a commutative-monoid-enriched symmetric monoidal category: this means that it is a symmetric monoidal category, and that composition and tensor preserve the commutative monoid structure on hom-sets, so that $(f + g); h = f; h + g; h$, $k; (f + g) = k; f + k; g$, $f; 0 = 0; f = 0$, $(f + g) \otimes h = f \otimes h + g \otimes h$ and $f \otimes 0 = 0$.

A *coalgebra modality* on $\mathbf{C}$ is a comonad $(!, \delta, \epsilon)$ such that each object $!A$ is equipped with a comonoid structure $\Delta_A : !A \longrightarrow !A \otimes !A$, $e_A : !A \longrightarrow I$. In addition to the associativity and unit equations for the comonoid, it should be the case that δ is a morphism of comonoids, that is, $\delta_A; e_{!A} = e_A$ and $\delta_A; \Delta_{!A} = \Delta_A; \delta_A \otimes \delta_A$.

Given such a structure, a *differential combinator* is a family of maps $D_{A,B} : \mathbf{C}(!A, B) \to \mathbf{C}(A \otimes !A, B)$, natural in A and B and respecting the commutative monoid structure of the hom-sets, satisfying the following four axioms.

- $D(e_A) = 0$,

Λ^r:	M, N	$::= x \mid \lambda x.M \mid MP \mid \mathrm{ifz}(M, M, M) \mid \mathrm{Fix}(M) \mid$ $\mid \quad \mathrm{succ}(M) \mid \mathrm{pred}(M) \mid \mathrm{zero}$	terms
Λ^b:	P	$::= [L_1, \dots, L_\ell, N_1^!, \dots, N_n^!]$	bags

(a) Grammar of terms, resources and bags

Evaluation contexts: $E(\!|-|\!) ::= (\!|-|\!) \mid EP \mid \lambda x.E \mid \mathrm{pred}(E) \mid \mathrm{succ}(E) \mid \mathrm{ifz}(E, M, N)$
Let contexts: $F(\!|-|\!) ::= (\!|-|\!) \mid (\lambda x.F)P$
Linear head reduction:

$$E(\!|F(\!|\lambda x.E'(\!|x|\!)|\!)(P \uplus [N])|\!) \to E(\!|F(\!|\lambda x.E'(\!|N|\!)|\!)P|\!)$$
$$E(\!|F(\!|\lambda x.E'(\!|x|\!)|\!)(P \uplus [N^!])|\!) \to E(\!|F(\!|\lambda x.E'(\!|N|\!)|\!)(P \uplus [N^!])|\!)$$
$$E(\!|F(\!|\lambda x.\underline{n}|\!)[N_1^!, \dots, N_k^!]|\!) \to E(\!|F(\!|\underline{n}|\!)|\!) \text{ for some } k \geq 0,$$
$$E(\!|\mathrm{ifz}(\mathrm{zero}, M, N)|\!) \to E(\!|M|\!) \qquad E(\!|\mathrm{ifz}(\mathrm{succ}(\underline{n}), M, N)|\!) \to E(\!|N|\!)$$
$$E(\!|\mathrm{pred}(\mathrm{succ}(\underline{n}))|\!) \to E(\!|\underline{n}|\!) \qquad E(\!|\mathrm{Fix}(M)|\!) \to E(\!|M[\mathrm{Fix}(M)^!]|\!)$$

(b) Operational semantics.

Fig. 1: Syntax and operational semantics of Resource PCF.

- $D(\Delta; f \otimes g) = (A \otimes \Delta); (D(f) \otimes g) + (A \otimes \Delta); \cong; (f \otimes D(g))$ where $f : !A \to B$, $g : !A \to C$ and $\cong$ is the appropriate symmetry map,
- $D(\epsilon_A; f) = (A \otimes e_A); f$,
- $D(\delta_A; !f; g) = (A \otimes \Delta_A); (D(f) \otimes (\delta_A; !f)); D(g)$ for $f : !A \to B$ and $g : !B \to C$.

A *differential category* is a commutative-monoid-enriched symmetric monoidal category with a coalgebra modality and a differential combinator. When the coalgebra modality is a linear exponential comonad, its Kleisli category is a *Cartesian differential category* whose differential combinators are denoted by $D^\times_{A,B} : \mathbf{C}(A, B) \to \mathbf{C}(A \times A, B)$. We refer to [4] for the general definition.

A *Cartesian-closed differential category* is a Cartesian differential category with closed structure, such that the operation of currying preserves the commutative monoid structure on hom-sets and for all $f : C \times A \to B$, $D^\times(\Lambda(f)) : C \times C \to (A \Rightarrow B)$ is equal to $\Lambda(\langle \pi_0 \times 0_A, \pi_1 \times \mathrm{id}_A \rangle; D^\times(f))$. The leading examples of such categories, studied in [6], are Ehrhard's category of finiteness spaces, and the category **MRel** of "multiset relations", which is the Kleisli category for the finite-multiset comonad on the category **Rel** of sets and relations.

3 Resource PCF and its models

We now describe a simply typed resource calculus which incorporates the constants of PCF, making it a prototypical resource-sensitive programming language. Resource PCF has two syntactic categories: terms, that are in functional position, and bags, that are in argument position and represent finite multisets of resources. Figure 1(a) gives the grammar generating the set Λ^r of *terms* and the set Λ^b of *bags* (whose union is denoted by $\uplus$) together with their typical metavariables. A *resource* can be linear (it must be used exactly once) or reusable (it can be used *ad libitum*) and in the latter case is decorated with a "!" superscript.

Terms of the form $\mathrm{succ}^n(\mathrm{zero})$ are denoted by $\underline{n}$. Hereafter terms and bags are considered up to α-equivalence, which is defined as usual in λ-calculus.

Types are generated by $A, B ::= \mathtt{nat} \mid A \to A$. *Environments* Γ are finite lists $x_1 : A_1; \cdots ; x_n : A_n$ assigning types to variables. *Typing rules* are straightforward to define; we say that M *has type* A *in* Γ when $\Gamma \vdash M : A$ is derivable.

The operational semantics is defined in Figure 1(b) via a *linear head reduction.* An equivalent presentation more in the style of [14] would also be possible. We say that a closed term M of ground type *converges*, written $M\Downarrow$, if M reduces to $\underline{k}$ for some $k \in \omega$. We denote by $C(\!|\cdot|\!)$ arbitrary contexts (e.g., in Theorem 13).

Resource PCF can be interpreted in any cpo-enriched Cartesian closed differential category having a weak natural number object. The interpretations of the constants and constructors of PCF are standard, leaving only the application, which is treated as follows, as in [6]. In every Cartesian closed differential category it is possible to define an operator $\star$ on morphisms $f : C \times A \to B$, $g : C \to A$ setting $f \star g := \langle\langle 0_C, g \circ \pi_0\rangle, \mathrm{id}\rangle; D^\times(f) : C \times A \to B$. Intuitively, $f \star g$ is obtained by force-feeding the 2^{nd} argument A of f with *one copy* of the result of g. The type is not modified because $f \star g$ may still depend on A. The weak natural number object is needed to interpret the natural numbers and the if-then-else, and the cpo-enrichment for the fixpoint operator.

The interpretation $[\![M]\!]_\Gamma : [\![\Gamma]\!] \to [\![A]\!]$ of $\Gamma \vdash M : A$ is defined as usual, except for the case of application where we set:

$$[\![M[\vec{L}, \vec{N}^!]]\!]_\Gamma = \langle \mathrm{id}, \textstyle\sum_{i=1}^n [\![N_i]\!]_\Gamma\rangle; ((\cdots(\Lambda^-([\![M]\!]_\Gamma) \star [\![L_1]\!]_\Gamma)\cdots) \star [\![L_\ell]\!]_\Gamma).$$

This definition is independent from the enumeration of the resources in the bag.

4 A differential category of games

Our first example of a differential category of games is the category introduced in [8, 9]. In this section we recall its definition, and show that it is a Cartesian closed differential category.

An *arena* A is a finite bipartite forest over two sets of moves, M_A^P and M_A^O with edge relation $\vdash$. We say that a move is *enabled* by its parent in the forest, and that root moves are *initial.* A *QA-arena* is an arena equipped with a labelling function that labels each move as a question (Q) or answer (A), such that every answer is the child of a question. We assume the standard notions of *justified sequence*, *views*, *P-* and *O-visibility* from the game semantics literature; see [11] for example. Given a justified sequence s, we say that an answer-move occurrence a *answers* the question occurrence q that justifies it. A justified sequence s satisfies *P-well-bracketing* if, for every prefix $s'a$ with a an answer move by P, the question that a answers is the rightmost O-question in the view $\ulcorner s' \urcorner$; call this the *pending question* at s'. A justified sequence is *complete* if every question is answered exactly once; we write $\mathsf{comp}(A)$ for the set of complete justified sequences of A.

Lemma 1. *If s is a complete justified sequence that satisfies P-visibility (resp. O-visibility), then s satisfies P-well-bracketing (resp. O-well-bracketing).*

Proof. A simple analysis of views shows that, if q is an O-question that is answered when some later O-question q' is pending, then when q' is answered, again a later O-question is pending. There can therefore be no O-question that is answered when it is not pending, because s is finite. $\square$

A sequence is *well-opened* if it contains exactly one initial O-move. A *strategy* for an arena A is a set of complete sequences in which O plays first, satisfying P-visibility (and, by Lemma 1, P-bracketing). Given a strategy σ, $\mathsf{wv}(\sigma)$ is the set of sequences in σ that are well-opened and satisfy O-visibility.

Given arenas A and B, we write $A \uplus B$ for the arena arising as the disjoint union of A and B, and $A^\perp$ for the arena A with O and P-moves interchanged. We can define a category $\mathbf{G}$ in which objects are arenas whose roots are all O-moves, and morphisms $A \to B$ are strategies on the arena $A^\perp \uplus B$. Composition of strategies is the usual "parallel composition plus hiding" construction, and identities are copycat strategies. As proved in [8, 9], this category is monoidal closed: disjoint union of arenas gives a tensor product, and exponentials are given by the arena $A \multimap B$, which consists of the arena B with a copy of $A^\perp$ attached below each initial move; duplication of $A^\perp$ is required to maintain the forest structure. Every object of $\mathbf{G}$ possesses a canonical comonoid structure, and the subcategory of comonoid homomorphisms is a Cartesian closed category $\mathbf{G}^\otimes$. These maps are those whose choice of move at any stage depends only on the current thread, that is, the subsequence of moves hereditarily justified by the initial O-move currently in view; it follows that such strategies are completely determined by the well-opened plays they contain.

The programming language Erratic Idealized Algol (EIA) is an applied typed λ-calculus with an appropriate stock of constants making it a higher-order imperative programming language with local state, consisting of variables in which natural numbers can be stored. The constants include an erratic choice operator `or` which encodes nondeterministic choice. As shown in [8], this programming language can be given denotational semantics in the category $\mathbf{G}^\otimes$. The interpretation of the imperative programming constants is as in the standard games model of Idealized Algol from [1], and the erratic choice operator is interpreted by union of strategies.

Theorem 2 (Full abstraction [8]). *The model of EIA in $\mathbf{G}^\otimes$ is sound, and moreover, for any type A:*

- *if s is a complete well-opened play of $[\![A]\!]$ satisfying visibility, there is a closed term M of type A such that $\mathsf{wv}([\![M]\!]) = \{s\}$;*
- *terms $M, N : A$ are contextually equivalent if and only if $\mathsf{wv}([\![M]\!]) = \mathsf{wv}([\![N]\!])$.*

We now exhibit the differential structure that $\mathbf{G}^\otimes$ possesses. Let s be a complete, well-opened play in $A^\perp \uplus B$ which contains at least one initial A-move. Say that a complete play s' in $A^\perp \uplus A^\perp \uplus B$ is a *derivative* of s if $\Delta; \{s'\} = \{s\}$ and s' contains one initial move in the left occurrence of $A^\perp$. We then define $D^\times(\sigma)$ as the strategy whose well-opened plays are

$$\{s' \in \mathsf{comp}(A^\perp \uplus A^\perp \uplus B) \mid s' \text{ is a derivative of some well-opened } s \in \sigma\}.$$

We can verify directly that this makes $\mathbf{G}^{\otimes}$ a Cartesian closed differential category; later we will see that this follows from a general construction. Because of the definability property of the model of EIA, it is reasonable to expect that the differential operator is programmable in EIA, and indeed it is. For terms of type $A \to \texttt{comm}$ (`comm` is the base type of *commands*) we can define the differential operator as follows (using appropriate syntactic sugar).

$$
\begin{array}{l}
\lambda f : A \to \texttt{comm}.\lambda a : A.\lambda a' : A.\,\texttt{new}\ b := \texttt{false} \\
\qquad\qquad\qquad\qquad\qquad \texttt{new}\ y := f((\texttt{if}\ b\ \texttt{then}\ (b := \texttt{true}; a)\ \texttt{else}\ a')\ \texttt{or}\ a')\ \texttt{in} \\
\qquad\qquad\qquad\qquad\qquad\quad \texttt{if}\ \neg b\ \texttt{then}\ y\ \texttt{else}\ \texttt{diverge}
\end{array}
$$

In any converging execution of this code, the argument a is supplied to f exactly once, though which call to f's argument receives a is chosen nondeterministically; all other calls to f's argument receive a'.

5 Constructing differential categories

We now describe a construction of models of intuitionistic linear logic that are also differential categories. The main ingredient is the construction of a category which possesses a comonad delivering cofree cocommutative comonoids; we begin with this step, before describing some preliminary steps that can readily construct appropriate categories.

Let $\mathbf{C}$ be a symmetric monoidal category enriched over sup-lattices, that is, over *idempotent* commutative monoids with *all* sums (we continue to use $(+, 0)$ for this monoid structure). Any product $A \times B$ in $\mathbf{C}$ is necessarily a *biproduct*, that is, it is also a coproduct and the canonical map $[\langle \mathrm{id}_A, 0\rangle, \langle 0, \mathrm{id}_B\rangle] : A \oplus B \to A \times B$ is an isomorphism. Similarly, every coproduct is a biproduct. Suppose that $\mathbf{C}$ has all *countable* biproducts, and that the monoidal structure distributes over them. We construct a differential structure on the Karoubi envelope $\mathcal{K}(\mathbf{C})$ (idempotent splitting) of $\mathbf{C}$. Recall that this category has as its objects pairs (A, f) where A is an object of $\mathbf{C}$ and $f : A \to A$ is an idempotent, and as its maps $(A, f) \to (B, g)$, those maps $h : A \to B$ from $\mathbf{C}$ such that $h = f; h; g$. This category inherits the monoidal structure, sup-lattice enrichment and biproducts from $\mathbf{C}$.

First, for any object A of $\mathbf{C}$, write $A^{\otimes n}$ to denote the n-fold tensor power of A. The *symmetric* tensor power A^n, if it exists, is the equaliser of the diagram

$$(A^{\otimes n}, f^{\otimes n}) \ \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots n!\ \text{permutations}}}\ (A^{\otimes n}, f^{\otimes n})$$

consisting of all $n!$ permutations from $A^{\otimes n}$ to itself.

In $\mathcal{K}(\mathbf{C})$ we can readily construct symmetric tensor powers, as follows. Given an object A of $\mathbf{C}$, define $\Theta_{A,n} : A^{\otimes n} \to A^{\otimes n}$ to be the sum of the $n!$ permutation maps. Straightforward calculation establishes the following.

Lemma 3. *For any object (A, f) of $\mathcal{K}(\mathbf{C})$, the following diagram is an equalizer.*

$$(A^{\otimes n}, f^{\otimes n}; \Theta_{A,n}) \xrightarrow{f^{\otimes n}; \Theta_{A,n}} (A^{\otimes n}, f^{\otimes n}) \ \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots n!\ \textit{permutations}}}\ (A^{\otimes n}, f^{\otimes n})$$

Moreover, these equalizer diagrams are preserved by tensor products.

One consequence of this is that there are maps $(A, f)^{m+n} \to (A, f)^m \otimes (A, f)^n$ whose underlying maps are given by $f^{\otimes m+n}; \Theta_{A,m+n}$, as one might expect.

These symmetric tensor powers will allow us to construct a coalgebra modality on $\mathcal{K}(\mathbf{C})$ as the free commutative comonoid. Recall that a *commutative comonoid* in a symmetric monoidal category is an object A together with maps $\Delta : A \to A \otimes A$ and $e : A \to I$ satisfying the obvious commutativity, associativity and unit diagrams; morphisms of comonoids are morphisms between the underlying objects that preserve the comonoid structure. Let $\mathcal{K}^{\otimes}(\mathbf{C})$ be the category of commutative comonoids and comonoid morphisms in $\mathcal{K}(\mathbf{C})$.

Lemma 4. *The forgetful functor $U : \mathcal{K}^{\otimes}(\mathbf{C}) \to \mathcal{K}(\mathbf{C})$ has a right adjoint, whose action on objects takes (A, f) to the biproduct $\bigoplus_{n \in \omega} (A, f)^n$, which we call $!(A, f)$.*

Proof. For any m and n, we have the map

$$\pi_{m+n}; f^{\otimes m+n}; \Theta_{A,m+n} : !(A, f) \longrightarrow (A, f)^m \otimes (A, f)^n.$$

Tupling all these gives us a map $!(A, f) \to \bigoplus_{m,n} (A, f)^m \otimes (A, f)^n$, and by distributivity of tensor over product, this gives a map $\Delta : !(A, f) \to !(A, f) \otimes !(A, f)$. We also have the map $\pi_0 : !(A, f) \to I$. It can readily be verified that these maps give $!(A, f)$ the structure of a comonoid. Moreover, it is the free comonoid on (A, f): if (B, g) is any commutative comonoid and $\alpha : (B, g) \to (A, f)$ any morphism, we construct a comonoid morphism $\alpha^{\dagger} : (B, g) \to !(A, f)$ as follows. The comultiplication $\Delta^n : (B, g) \to (B, g)^{\otimes n}$ equalizes all permutations, so the composition $\Delta^n; \alpha^{\otimes n}$ does too, yielding a map $(B, g) \to (A, f)^n$. Tupling all these maps gives us the required comonoid map $\alpha^{\dagger}$, and it is easily checked that this is the unique map such that $\alpha^{\dagger}; \pi_1 = \alpha$. $\square$

Composing these two adjoint functors yields a comonad $(!, \delta, \epsilon)$ on $\mathcal{K}(\mathbf{C})$.

Lemma 5. *The comonad $(!, \delta, \epsilon)$ is a coalgebra modality. In fact, it is a linear exponential comonad (also known as a storage modality).*

Proof. Showing that we have a coalgebra modality amounts to demonstrating that δ is a morphism of comonoids. In fact much more is true: it is routine to check that the forgetful functor $U : \mathcal{K}^{\otimes}(\mathbf{C}) \to \mathcal{K}(\mathbf{C})$ satisfies the conditions of Beck's monadicity theorem, which implies that the category of coalgebras is isomorphic to $\mathcal{K}^{\otimes}(\mathbf{C})$. Therefore $(!, \delta, \epsilon)$ is a linear exponential comonad. $\square$

The fact that the cofree commutative comonoid provides a linear exponential comonad may be considered folklore. The construction of this comonad along the lines given above follows the recipe in [13], though the use of Karoubi envelope to generate a category possessing the required equalizers seems to be new.

We are now in a position to construct a differential operator on $\mathcal{K}(\mathbf{C})$, making it into a differential category. The differential operator is given by precomposition with the *deriving transformation* $d : (A, f) \otimes !(A, f) \to !(A, f)$ defined as follows. For each n, the map $f^{\otimes n+1}; \Theta_{A,n+1}$ in $\mathbf{C}$ gives us a morphism

$$f^{\otimes n+1}; \Theta_{A,n+1} : (A, f) \otimes (A, f)^n \to (A, f)^{n+1}$$

and hence we obtain maps $\cong; \pi_n; f^{\otimes n+1}; \Theta_{A,n+1} : (A, f) \otimes !(A, f) \to (A, f)^{n+1}$ where $\cong$ is the distributivity map. Tupling all these gives us a morphism $(A, f) \otimes !(A, f) \to \bigoplus_n (A, f)^{n+1}$, and finally pairing this with $0 : (A, f) \otimes !(A, f) \to I$ gives the required map.

Theorem 6. *With the structure described above, $\mathcal{K}(\mathbf{C})$ is a sup-lattice-enriched differential category, and the Kleisli category $\mathcal{K}_!(\mathbf{C})$ a cpo-enriched Cartesian differential category. If $\mathbf{C}$ is monoidal closed (in the sup-lattice-enriched sense) then $\mathcal{K}_!(\mathbf{C})$ is a cpo-enriched Cartesian-closed differential category.*

Proof. That $\mathcal{K}(\mathbf{C})$ is a differential category is lengthy but straightforward to check. Sup-lattice enrichment follows directly from that of $\mathbf{C}$. The fact that $\mathcal{K}_!(\mathbf{C})$ is Cartesian differential follows from Proposition 3.2.1 of [4]. The Cartesian closure of $\mathcal{K}_!(\mathbf{C})$ is a well-known fact about linear exponential comonads. For the cpo-enrichment, it is enough to observe that the passage from $\alpha : !(A, f) \to (B, g)$ to $\alpha^\dagger : !(A, f) \to !(B, g)$ preserves directed suprema. □

Even when $\mathbf{C}$ is not monoidal closed, it is still possible to arrive at a Cartesian closed differential category when there are enough exponentials: if $\mathbf{C}$ has all exponentials $A \multimap R$ for some fixed object R, then the full subcategory of $\mathcal{K}_!(\mathbf{C})$ consisting of such R-exponentials is Cartesian closed, and also possesses a weak distributive coproduct structure given by the "lifted sum" $(\bigoplus_{i \in I}(!A_i \multimap R)) \multimap R$. In particular, $\bigoplus_{n \in \omega} R \multimap R$ is a weak natural numbers object.

To apply the construction above, we need a sup-lattice enriched symmetric monoidal category with countable distributive biproducts. Such categories can readily be manufactured via a series of free constructions.

Beginning with a symmetric monoidal category, one can construct its *sup-lattice-completion* as the category with the same objects, but whose maps $A \to B$ are sets of maps in the original category (cf. [10] VIII.2 exercise 5). This is a sup-lattice enriched category, with joins of maps given by unions, and monoidal structure inherited from the original category; closed structure is also inherited, if it exists. We denote the sup-lattice completion of a category $\mathbf{C}$ by $\mathbf{C}^+$.

Given a sup-lattice-enriched symmetric monoidal category, its *biproduct completion* (cf. [10] VIII.2 exercise 6) has as objects indexed sets $\{A_i \mid i \in I\}$ of objects in the original category, and as morphisms $\{A_i \mid i \in I\} \to \{B_j \mid j \in J\}$ *matrices* of morphisms, that is, for each i, j, a morphism $A_i \to B_j$. Composition is (potentially infinite) matrix multiplication; the infinite sums required for composition are the reason we require sup-lattice enrichment. The biproduct of a set of objects is given by the disjoint union of families. We write $\mathbf{BP}(\mathbf{C})$ for the biproduct completion of a category $\mathbf{C}$.

We will be interested in some categories which arise by performing these two constructions in sequence. Given a category $\mathbf{C}$, we denote by $\mathbf{FamRel}(\mathbf{C})$ the category whose objects are families $\{A_i \mid i \in I\}$ of objects of $\mathbf{C}$, and whose morphisms $\{A_i \mid i \in I\} \to \{B_j \mid j \in J\}$ are given by sets of triples (i, j, f) where $i \in I$, $j \in J$ and $f : A_i \to B_j$ in $\mathbf{C}$. Note that for a given i and j there may be no such triples in a morphism, or one, or many. It is easy to check that $\mathbf{FamRel}(\mathbf{C})$ is isomorphic to the category $\mathbf{BP}(\mathbf{C}^+)$.

A simple but central example begins with the terminal category $\mathbf{1}$. $\mathbf{FamRel}(\mathbf{1})$ is the category $\mathbf{Rel}$ of sets and relations. On the image of $\mathbf{Rel}$ in $\mathcal{K}(\mathbf{Rel})$, $!$ is the finite-multiset comonad, and we therefore find $\mathbf{MRel}$ embedded in $\mathcal{K}_!(\mathbf{Rel})$ as a sub-Cartesian-differential-category.

6 Analysis of $\mathbf{G}^{\otimes}$

In this section we apply some of the constructions developed above to reconstruct $\mathbf{G}^{\otimes}$ and discover its differential structure as an instance of our construction. We begin by defining a new category $\mathbf{EG}$ of *exhausting games*.

Given a finite arena A, a *path* is a non-repeating enumeration of all moves, respecting the order given by the edge relation in the arena — that is, a traversal of the forest — such that the first move is by O and moves alternate polarity thereafter. Note that every move in a path has a unique justifier earlier in the path. An *exhausting strategy* on A is a set of even-length paths that satisfy P-visibility; if A has an odd number of moves, the only strategy is the empty set. The category $\mathbf{EG}$ has finite O-rooted arenas as objects and exhausting strategies on $A^{\perp} \uplus B$ as maps from A to B, with composition and identities as usual. Again, disjoint union of arenas gives a monoidal structure; and if B has a single root, then the arena $A \multimap B$ is an exponential, so $\mathbf{EG}$ has all R-exponentials, where R is the arena with a single move belonging to O.

It is clear that $\mathbf{EG}$ is sup-lattice enriched: unions of strategies are strategies, and composition preserves unions. We may therefore form its biproduct completion, to obtain the structure we require to construct a differential category as in Section 5. We write $\mathcal{K}(\mathbf{BP}(\mathbf{EG}))$ for the differential category so constructed, and $\mathcal{K}_!(\mathbf{BP}(\mathbf{EG}))$ for its Kleisli category, which is a Cartesian differential category. The full subcategory of R-exponentials is Cartesian closed and has a weak natural numbers object. We shall now show how to recover $\mathbf{G}^{\otimes}$ as a subcategory of this.

Let A be any QA-arena, and consider a non-repeating (justified) sequence s of pairs (a, n) where a is a move of A and n is a natural number. Taking left projection on such a sequence gives a justified sequence in A, which we call $\hat{s}$; we say that s is a tagging of $\hat{s}$, and write $\mathsf{tcomp}(A)$ for the set of all taggings of complete well-opened plays in A.

Let $s \in \mathsf{tcomp}(A)$ be a tagging of the complete play $\hat{s}$. We can define an arena $\|s\|$ whose moves are the elements of s and whose edge relation is precisely the justification structure of s. Thus s becomes a path of $\|s\|$. If t is a tagged sequence such that $\hat{t} = \hat{s}$, there is an isomorphism between the moves of $\|s\|$ and $\|t\|$ which maps the n-th move of s to the n-th move of t. The free monoid extension induces an isomorphism ϕ between paths in $\|s\|$ and those in $\|t\|$, and in turn an isomorphism in $\mathbf{EG}$, given by the strategy

$$\phi_{\|s\|,\|t\|} = \{u \in \|s\|^{\perp} \uplus \|t\| \mid \hat{u} \in \mathrm{id}_A, \phi(u \restriction \|s\|) = u \restriction \|t\|\}.$$

Let A^* be the family of arenas $\{\|s\| \mid s \in \mathsf{tcomp}(A)\}$. We define a morphism $\phi_A : A^* \to A^*$ in the biproduct completion of $\mathbf{EG}$ as the "matrix" with entries given by

$$(\phi_A)_{s,t} = \begin{cases} \phi_{\|s\|,\|t\|}, & \text{if } \hat{s} = \hat{t} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Our embedding of $\mathbf{G}^{\otimes}$ in $\mathcal{K}_!(\mathbf{BP}(\mathbf{EG}))$ maps an arena A to (A^*, ϕ_A). The action on morphisms is slightly trickier to describe; we begin by analysing complete, well-opened plays in $A^{\perp} \uplus B$. Such a play consists of an interleaving of a complete, well-opened play in B with a number of complete, well-opened plays in A. Suppose the play s is an interleaving of plays $s_1, \dots, s_n$ in A and s' in B. Let u be any tagging of s, yielding taggings $u_1, \dots, u_n$ and u' of the projections s_1, $\dots$, s_n and s'. This tagging induces a morphism $(A^*)^{\otimes n} \to B^*$ in the biproduct completion of $\mathbf{EG}$, as follows.

The family $(A^*)^{\otimes n}$ consists of arenas $\|s'_1\| \otimes \cdots \otimes \|s'_n\|$ for $s'_i \in \mathsf{tcomp}(A)$. If $\|s'_i\| = \|u_i\|$ for each i and $\|s'\| = \|u'\|$, then the singleton u is a morphism

$$\{u\} : \|s'_1\| \otimes \cdots \otimes \|s'_n\| \to \|s'\|$$

and hence, taking all such taggings and inserting empty strategies everywhere else in the matrix, we have a morphism $(A^*)^{\otimes n} \to B^*$.

Given a map $\sigma : A \to B$ in $\mathbf{G}^{\otimes}$, let σ_n be the set of well-opened plays in σ with n initial A-moves. Each $s \in \sigma_n$ induces a morphism $(A^*)^{\otimes n} \to B^*$ in $\mathbf{EG}$ as above, so we can define $\Phi(\sigma_n) : (A^*)^{\otimes n} \to B^*$ to be the sum of all these morphisms. The copairing of the $\Phi(\sigma_n)$ gives us a map

$$[\Phi(\sigma_n) \mid n \in \omega] : \bigoplus_{n \in \omega} (A^*)^{\otimes n} \to B^*.$$

By construction, for each $\Phi(\sigma_n)$ we have $\Phi(\sigma_n) = \phi_A^{\otimes n}; \Theta_{A^*,n}; \Phi(\sigma_n); \phi_B$, so $[\Phi(\sigma_n) \mid n \in \omega]$ is a morphism from $!(A^*, \phi_A)$ to (B^*, ϕ_B) in $\mathcal{K}(\mathbf{BP}(\mathbf{EG}))$.

Proposition 7. *The construction defined above gives a full and faithful product-preserving functor from $\mathbf{G}^{\otimes}$ to $\mathcal{K}_!(\mathbf{BP}(\mathbf{EG}))$.*

7 Some refined categories of games

The category $\mathbf{G}^{\otimes}$ is clearly rich enough to interpret Resource PCF. Indeed, it is richer, as the model of EIA demonstrates, and the model of Resource PCF in $\mathbf{G}^{\otimes}$ is far from being fully abstract. In this section we develop a new Cartesian closed differential category of games, by applying our general construction to a refined games model, and arrive at a model of Resource PCF which possesses the finite definability property.

Let A be any arena. We define an equivalence relation $\sim$ on the paths of A as the smallest equivalence relation such that $s \cdot o \cdot p \cdot o' \cdot p' \cdot t \sim s \cdot o' \cdot p' \cdot o \cdot p \cdot t$ where o, o' are O-moves and p, p' are P-moves. We call a path *safe* if, whenever $s = s' \cdot o \cdot p \cdot o' \cdot p' \cdot t$ and o justifies p', p' justifies o'. The $\sim$ relation captures a notion of causal independence similar to that of Melliès [12], and allows us to refine our games model to obtain definability for Resource PCF.

A $\sim$-*strategy* σ on an arena A is a set of safe paths that is $\sim$-closed, that is, if $s \in \sigma$ and $s \sim t$ then $t \in \sigma$. A $\sim$-strategy σ is *deterministic* if it is non-empty, and the longest common prefix of any $s, t \in \sigma$ has even length.

Lemma 8. *If σ is a $\sim$-strategy and $s \in \sigma$ then s satisfies P-visibility.*

Proof. Let $s = s_1 \cdot p \cdot s_2$, for some P-move p. By permuting moves we can find $s_1' \cdot p \cdot s_2' \sim s$ such that $s_1' = \ulcorner s_1 \urcorner$. The justifier of p must appear in $s_1' = \ulcorner s_1 \urcorner$. $\square$

Given a path s in an arena A, write $\widetilde{s}$ for the equivalence class of s under $\sim$.

Lemma 9. *For any safe path s of A, $\widetilde{s}$ is a deterministic $\sim$-strategy, and any deterministic $\sim$-strategy is of the form $\widetilde{s}$ for some safe path s.*

We can now build two categories: $\mathbf{EG}_\sim$ has O-rooted arenas as objects and deterministic $\sim$-strategies on $A^\perp \uplus B$ as maps $A \to B$; $\mathbf{EG}_\sim^+$ has the same objects, but its maps $A \to B$ are arbitrary $\sim$-strategies on $A^\perp \uplus B$. $\mathbf{EG}_\sim^+$ is therefore the subcategory of $\mathbf{EG}$ consisting of $\sim$-closed strategies.

We are ready to construct a Cartesian differential category, starting with $\mathbf{EG}_\sim$. The first step is to take its sup-lattice completion; a consequence of Lemma 9 is that this is exactly $\mathbf{EG}_\sim^+$, justifying our choice of nomenclature.

Lemma 10. $\mathbf{EG}_\sim^+$ *is the sup-lattice completion of* $\mathbf{EG}_\sim$*.*

Nevertheless, it is convenient to take the first two steps of the construction together, working with $\mathbf{FamRel}(\mathbf{EG}_\sim)$. Our construction gives us a comonad ! on $\mathcal{K}(\mathbf{FamRel}(\mathbf{EG}_\sim))$, such that the Kleisli category $\mathcal{K}_!(\mathbf{FamRel}(\mathbf{EG}_\sim))$ is a Cartesian differential category. Though $\mathbf{EG}_\sim$ is not monoidal closed, it has all R-exponentials, so the full subcategory of $\mathcal{K}_!(\mathbf{FamRel}(\mathbf{EG}_\sim))$ comprising the arenas with a single root is a Cartesian closed differential category.

As before, we may give a direct definition of a category of games which is a sub-Cartesian-closed-differential-category of this one. Let $\mathbf{G}_\sim$ be the subcategory of $\mathbf{G}$ consisting of $\sim$-closed strategies. Again taking the subcategory of comonoid homomorphisms, we arrive at a Cartesian closed differential category $\mathbf{G}_\sim^\otimes$. Just as in Section 6, we can define a full and faithful functor from $\mathbf{G}_\sim^\otimes$ into $\mathcal{K}_!(\mathbf{FamRel}(\mathbf{EG}_\sim))$ which preserves the Cartesian closed differential structure.

8 Analysing models of Resource PCF

Our constructions show that each of $\mathcal{K}_!(\mathbf{FamRel}(\mathbf{EG}_\sim))$, $\mathcal{K}_!(\mathbf{BP}(\mathbf{EG}))$ and $\mathcal{K}_!(\mathbf{FamRel}(\mathbf{1}))$ is a cpo-enriched differential Cartesian category with enough exponentials to interpret Resource PCF; and indeed we have identified full subcategories $\mathbf{G}^\otimes$, $\mathbf{G}_\sim^\otimes$ and $\mathbf{MRel}$ which are cpo-enriched Cartesian closed differential categories containing all the objects needed to interpret Resource PCF soundly. However, for $\mathbf{G}_\sim^\otimes$ and $\mathbf{MRel}$, there is more to be said.

Consider those arenas for which there exists a Q/A-labelling such that every question enables a unique answer — this is a constraint on the shapes of the trees, rather than additional structure. We write $\mathbf{EG}_{\sim}^{QA}$ for the full subcategory of $\mathbf{EG}_{\sim}$ consisting of such arenas, and note that $\mathbf{G}_{\sim}^{\otimes}$ embeds in $\mathcal{K}_{!}(\mathbf{FamRel}(\mathbf{EG}_{\sim}^{QA}))$ by construction.

Lemma 11. *For every such arena, the set of safe paths is non-empty.*

Proof. We construct a safe path by induction on partial paths. Begin with any root node. Having constructed a partial path s, consider the pending question in s. If it enables any questions that do not appear in s, extend s with one of them. Otherwise, extend s with the unique answer of the pending question. If there is no pending question, extend s with any question enabled by one of the answers in the P-view of s, if one exists, or an unplayed root node, if one exists. If no such moves exist, s is a safe path. $\square$

Corollary 12. *The unique functor $\top : \mathbf{EG}_{\sim}^{QA} \to \mathbf{1}$ is full. (This amounts to the fact that the set of safe paths of $A^{\perp} \uplus B$ is non-empty.)*

This full functor extends through our constructions to a full functor from $\mathcal{K}_{!}(\mathbf{FamRel}(\mathbf{EG}_{\sim}^{QA}))$ to $\mathcal{K}_{!}(\mathbf{FamRel}(\mathbf{1}))$. Moreover, the only idempotents we make use of in the Karoubi envelope have the form $\sum_{f \in G} f$ where G is some group of automorphisms. In the case of $\mathbf{Rel}$, these idempotents are equivalence relations, and an object $(A, \simeq)$ in $\mathcal{K}(\mathbf{Rel})$ is isomorphic to $(A/\simeq, \mathrm{id}_A)$. The part of the Karoubi envelope that is used in our constructions is therefore equivalent to $\mathbf{Rel}$ itself, with the comonad being the usual finite-multiset comonad, and the Kleisli category being $\mathbf{MRel}$. We therefore obtain a full functor from $\mathbf{G}_{\sim}^{\otimes}$ to $\mathbf{MRel}$ which preserves all the relevant structure.

This functor may be described concretely as follows. Given a complete justified sequence s on a QA-arena, write $|s|$ for the underlying multiset of moves of s, partially ordered by the justification relation. The functor sends an arena A to the set of all such pomsets, which we call the *positions* of A. If s is a well-opened complete justified sequence on $A^{\perp} \uplus B$, $|s|$ is a pair consisting of a multiset of positions of A and a position of B. The functor sends a map $A \to B$ to the set of positions of its sequences. This is essentially the "time-forgetting" map of [2], which here is functorial because of $\sim$-closure.

Theorem 13. *The models of Resource PCF in $\mathbf{G}_{\sim}^{\otimes}$ and $\mathbf{MRel}$ have the finite definability property, and the model in $\mathbf{MRel}$ is fully abstract.*

Proof. For $\mathbf{G}_{\sim}^{\otimes}$, a straightforward induction on the size of strategies, following the steps in the definability proof for the innocent strategy model of PCF. $\sim$-closure ensures that strategies are insensitive to the order in which O-moves are made. Definability for $\mathbf{MRel}$ follows from the fullness of the positional collapse of $\mathbf{G}_{\sim}^{\otimes}$ onto $\mathbf{MRel}$. For full abstraction of $\mathbf{MRel}$, let M and N be closed terms of type A. If $[\![M]\!] \neq [\![N]\!]$, wlog there is some $a \in [\![M]\!] \setminus [\![N]\!]$. By finite definability, the relation $\{(a, 0)\} : A \to \mathtt{nat}$ is the denotation of some term $x : A \vdash C(\!|x|\!) : \mathtt{nat}$. Therefore $[\![C(\!|M|\!)]\!] = [\![\mathrm{zero}]\!]$ while $[\![C(\!|N|\!)]\!] = \emptyset$, so $C(\!|M|\!) \Downarrow$ but $C(\!|N|\!) \not\Downarrow$. $\square$

Acknowledgements. Research supported in part by NWO Project 612.000.936 CALMOC and by UK EPSRC grant EP/HO23097.

References

1. S. Abramsky and G. McCusker. Linearity, sharing and state: a fully abstract game semantics for Idealized Algol with active expressions. In P. W. O'Hearn and R. D. Tennent, editors, *Algol-like Languages*, pages 297–329 of volume 2. Birkhaüser, 1997.
2. P. Baillot, V. Danos, T. Ehrhard, and L. Regnier. Timeless games. In M. Nielsen and W. Thomas, editors, *Computer Science Logic: 11th International Workshop Proceedings*, Lecture Notes in Computer Science, pages 56–77. Springer-Verlag, 1998.
3. R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Differential categories. *Mathematical. Structures in Comp. Sci.*, 16:1049–1083, December 2006.
4. R. F. Blute, J. R. B. Cockett, and R. A. G. Seely. Cartesian differential categories. *Theory and Applications of Categories*, 22(23):622–672, 2009.
5. G. Boudol. The lambda-calculus with multiplicities (abstract). In E. Best, editor, *CONCUR*, volume 715 of *Lecture Notes in Computer Science*, pages 1–6. Springer, 1993.
6. A. Bucciarelli, T. Ehrhard, and G. Manzonetto. Categorical models for simply typed resource calculi. *Electronic Notes in Theoretical Computer Science*, 265:213 – 230, 2010. Proceedings of the 26th Conference on the Mathematical Foundations of Programming Semantics (MFPS 2010).
7. T. Ehrhard and L. Regnier. The differential lambda-calculus. *Theor. Comput. Sci.*, 309:1–41, December 2003.
8. R. Harmer. *Games and full abstraction for nondeterministic languages*. PhD thesis, University of London, 1999.
9. R. Harmer and G. McCusker. A fully abstract game semantics for finite nondeterminism. In *Proceedings, Fourteenth Annual IEEE Symposium on Logic in Computer Science*, pages 422–430, 1999.
10. S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, Berlin, 1971.
11. G. McCusker. *Games and Full Abstraction for a Functional Metalanguage with Recursive Types*. Distinguished Dissertations in Computer Science. Springer-Verlag, 1998.
12. P.-A. Melliès. Asynchronous games 2: the true concurrency of innocence. *Theor. Comput. Sci.*, 358:200–228, August 2006.
13. P.-A. Melliès, N. Tabareau, and C. Tasson. An explicit formula for the free exponential modality of linear logic. In S. Albers, A. Marchetti-Spaccamela, Y. Matias, S. Nikoletseas, and W. Thomas, editors, *Automata, Languages and Programming*, volume 5556 of *Lecture Notes in Computer Science*, pages 247–260. Springer Berlin / Heidelberg, 2009.
14. M. Pagani and P. Tranquilli. Parallel reduction in resource lambda-calculus. In Z. Hu, editor, *APLAS*, volume 5904 of *LNCS*, pages 226–242, 2009.
15. P. Tranquilli. Intuitionistic differential nets and lambda-calculus. *Theoretical Computer Science*, In Press, Corrected Proof:–, 2010.