# A general class of models of $\mathcal{H}^*$

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**Abstract.** We recently introduced an extensional model of the pure $\lambda$-calculus living in a cartesian closed category of sets and relations. In this paper, we provide sufficient conditions for categorical models living in arbitrary cpo-enriched cartesian closed categories to have $\mathcal{H}^*$, the maximal consistent sensible $\lambda$-theory, as their equational theory. Finally, we prove that our relational model fulfils these conditions.

**Keywords:** $\lambda$-calculus, non-well-pointed categories, $\lambda$-theories, relational semantics, Approximation Theorem.

## Introduction

The first model of $\lambda$-calculus, namely $\mathscr{D}_\infty$, was postulated by Scott in 1969 in the category of complete lattices and continuous functions. After Scott's $\mathscr{D}_\infty$, a large number of models have been introduced in various categories of domains. For example, the *continuous semantics* [15] is given in the cartesian closed category (ccc, for short) whose objects are complete partial orders and morphisms are Scott continuous functions. The *stable semantics* [3] and the *strongly stable semantics* [5] are refinements of the continuous semantics which have been introduced to capture the notion of 'sequential' continuous function.

Although these semantics are very rich (in each of them it is possible to build up $2^{\aleph_0}$ models having pairwise distinct $\lambda$-theories) they are also hugely *incomplete*: there is a continuum of $\lambda$-theories that cannot be presented as equational theories of continuous, stable, or strongly stable models (see [14]). For this reason, researchers are today shifting their attention towards less canonical structures and categories [11, 7, 13, 1, 12]. This is also due to a widespread growing interest in two branches of computer science which are strongly related to the semantics of $\lambda$-calculus: *game semantics* and *linear logic*. The categories arising in these fields are often *non-standard* since they can have morphisms which are not functions and/or they can be non-well-pointed.

At the moment, there is a lack of general methods for a uniform treatment of models living in non-standard semantics. For instance, the classic method for turning a categorical model into a $\lambda$-model asked for well-pointed categories [2, Sec. 5.5], whilst, in collaboration with Bucciarelli and Ehrhard, we have recently shown that such a requirement was unnecessary [6]. In the same paper we have also built an extensional model $\mathscr{D}$ of $\lambda$-calculus living in a (highly) non-well-pointed ccc of sets and relations, which has been previously studied as a semantic framework for linear logic [8, 4]. We conjectured that $\mathscr{D}$ can be seen as a "relational version" of Scott's $\mathscr{D}_\infty$ and, hence, that its equational theory is the maximal consistent sensible $\lambda$-theory $\mathcal{H}^*$ (just like for $\mathscr{D}_\infty$). Unfortunately, the classic methods to characterize the equational theory of a model are not directly applicable to our model $\mathscr{D}$, since it lives in a non-well-pointed category.

In the present paper, we provide sufficient conditions for categorical models living in possibly non-well-pointed, but cpo-enriched, ccc's to have $\mathcal{H}^*$ as their equational theory. The idea of the proof is that we want to find a class of models (as large as possible) satisfying an Approximation Theorem. More precisely, we want to be able to characterize the interpretation of a $\lambda$-term $M$ as the least upper bound of the interpretations of its approximants. These approximants are particular terms of an auxiliary calculus, due to Wadsworth [16], and called here the *labelled $\lambda\bot$-calculus*, which is strongly normalizable and Church-Rosser.

Then we define the "well stratifiable $\bot$-models", and we show that they model Wadsworth's calculus and satisfy the Approximation Theorem. As a consequence, we get that every well stratifiable $\bot$-model $\mathscr{U}$ equates all $\lambda$-terms having the same Böhm tree; in particular, $\mathscr{U}$ is sensible, i.e., it equates all unsolvable $\lambda$-terms. Finally we prove, under the additional hypothesis that $\mathscr{U}$ is extensional, that the theory of $\mathscr{U}$ is $\mathcal{H}^*$.

At the end of the paper, we show that our relational model $\mathscr{D}$ of [6] fulfils these conditions, thus its equational theory is $\mathcal{H}^*$.

## 1 Preliminaries

To keep this article self-contained, we summarize some definitions and results. With regard to the $\lambda$-calculus we follow the notation and terminology of [2].

**Multisets and sequences.** Let $S$ be a set. A *multiset* $m$ over $S$ can be defined as an unordered list $m = [a_1, a_2, \ldots]$ with repetitions such that $a_i \in S$ for all $i$. A multiset $m$ is called *finite* if it is a finite list, we denote by $[]$ the empty multiset. We will write $\mathcal{M}_\mathrm{f}(S)$ for the set of all finite multisets over $S$. Given two multisets $m_1 = [a_1, a_2, \ldots]$ and $m_2 = [b_1, b_2, \ldots]$ the *multiset union* of $m_1, m_2$ is defined by $m_1 \uplus m_2 = [a_1, b_1, a_2, b_2, \ldots]$.

A $\mathbb{N}$-indexed sequence $\sigma = (m_1, m_2, \ldots)$ of multisets is *quasi-finite* if $m_i = []$ holds for all but a finite number of indices $i$. If $S$ is a set, we denote by $\mathcal{M}_\mathrm{f}(S)^{(\omega)}$ the set of all quasi-finite $\mathbb{N}$-indexed sequences of multisets over $S$.

**Cartesian closed categories.** Let $\mathbf{C}$ be a *cartesian closed category* (ccc, for short). We denote by $A \times B$ the *product* of $A$ and $B$, by $[A \Rightarrow B]$ the *exponential object* and by $ev \in \mathbf{C}([A \Rightarrow B] \times A, B)$ the *evaluation morphism*. For any $C$ and $f \in \mathbf{C}(C \times A, B)$, $\Lambda(f) \in \mathbf{C}(C, [A \Rightarrow B])$ stands for the (unique) morphism such that $ev_{AB} \circ (\Lambda(f) \times \mathrm{Id}_A) = f$. Finally, $\mathbb{1}$ denotes the terminal object and $!_A$ the only morphism in $\mathbf{C}(A, \mathbb{1})$. We recall that in a ccc the following equalities hold:

$$\text{(pair)} \quad \langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle \qquad \Lambda(f) \circ g = \Lambda(f \circ (g \times \mathrm{Id})) \quad \text{(Curry)}$$
$$\text{(beta)} \quad ev \circ \langle \Lambda(f), g\rangle = f \circ \langle \mathrm{Id}, g\rangle \qquad \Lambda(ev) = \mathrm{Id} \quad \text{(Id-Curry)}$$

We say that $\mathbf{C}$ is *well-pointed* if, for all $f, g \in \mathbf{C}(A, B)$, whenever $f \neq g$, there exists a morphism $h \in \mathbf{C}(\mathbb{1}, A)$ such that $f \circ h \neq g \circ h$.

The ccc $\mathbf{C}$ is *cpo-enriched* if every homset is a cpo $(\mathbf{C}(A,B), \sqsubseteq_{(A,B)}, \bot_{(A,B)})$, composition is continuous, pairing and currying are monotonic, and the following strictness conditions hold: (l-strict) $\bot \circ f = \bot$, (ev-strict) $ev \circ \langle \bot, f\rangle = \bot$.

**MRel: a relational semantics.** We shortly present the category $\mathbf{MRel}$. The objects of $\mathbf{MRel}$ are all the sets. A morphism from $S$ to $T$ is a relation from $\mathcal{M}_{\mathrm{f}}(S)$ to $T$, in other words, $\mathbf{MRel}(S,T) = \mathscr{P}(\mathcal{M}_{\mathrm{f}}(S) \times T)$. The identity of $S$ is the relation $\mathrm{Id}_S = \{([a], a) : a \in S\} \in \mathbf{MRel}(S,S)$. The composition of $s \in \mathbf{MRel}(S,T)$ and $t \in \mathbf{MRel}(T,U)$ is defined by:

$$\begin{aligned} t \circ s = \{(m,c) : \exists (m_1,b_1),\ldots,(m_k,b_k) \in s \text{ such that} \\ m = m_1 \uplus \ldots \uplus m_k \text{ and } ([b_1,\ldots,b_k],c) \in t\}. \end{aligned}$$

The categorical product $S \times T$ of two sets $S$ and $T$ is their disjoint union. The terminal object $\mathbb{1}$ is the empty set, and $!_S$ is the empty relation.
$\mathbf{MRel}$ is cartesian closed, non-well-pointed and has countable products [6, Sec. 4].

**The $\lambda$-calculus.** Let Var be a countably infinite set of variables. The set $\Lambda$ of $\lambda$-*terms* is inductively defined as usual: $x \in \Lambda$, for each $x \in \mathrm{Var}$; if $M, N \in \Lambda$ then $MN \in \Lambda$; if $M \in \Lambda$ then $\lambda x.M \in \Lambda$, for each $x \in \mathrm{Var}$.

Concerning specific $\lambda$-terms, we set $\mathbf{I} \equiv \lambda x.x$ and $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$.

Given a reduction rule $\to_R$ we write $\twoheadrightarrow_R$ ($=_R$) for its transitive and reflexive (and symmetric) closure. A $\lambda$-term $M$ is *solvable* if $M \twoheadrightarrow_\beta \lambda x_1 \ldots x_n.yN_1 \cdots N_k$ for some $x_1,\ldots,x_n \in \mathrm{Var}$, $N_1,\ldots,N_k \in \Lambda$ ($n,k \geq 0$); otherwise $M$ is *unsolvable*.

A $\lambda$-*theory* is any congruence on $\Lambda$, containing $=_\beta$. A $\lambda$-theory is: *consistent* if it does not equate all $\lambda$-terms; *extensional* if it contains $=_\eta$; *sensible* if it equates all unsolvable $\lambda$-terms. The set of all $\lambda$-theories, ordered by inclusion, forms a complete lattice. We denote by $\mathcal{H}^*$ the greatest consistent sensible $\lambda$-theory.

The *Böhm tree* $\mathrm{BT}(M)$ of a $\lambda$-term $M$ is defined as follows: if $M$ is unsolvable, then $\mathrm{BT}(M) = \bot$, that is, $\mathrm{BT}(M)$ is a tree with a unique node labelled by $\bot$; if $M$ is solvable and $\lambda x_1 \ldots x_n.yM_1 \cdots M_k$ is its principal hnf [2, Def. 8.3.10], then:

$$\mathrm{BT}(M) = \begin{array}{ccc} & \lambda x_1 \ldots x_n.y & \\ \swarrow & & \searrow \\ \mathrm{BT}(M_1) & \cdots & \mathrm{BT}(M_k) \end{array}$$

We call $\mathcal{B}$ the $\lambda$-theory equating all $\lambda$-terms having the same Böhm tree. Given two Böhm trees $t, t'$ we define $t \subseteq_{BT} t'$ if, and only if, $t$ results from $t'$ by replacing some subtrees with $\bot$. The relation $\subseteq_{BT}$ is transferred on $\lambda$-terms by setting $M \sqsubseteq_{BT} N$ if, and only if, $\mathrm{BT}(M) \subseteq_{BT} \mathrm{BT}(N)$.

We write $M \sqsubseteq_{\eta,\infty} N$ if $\mathrm{BT}(N)$ is a (possibly infinite) $\eta$-expansion of $\mathrm{BT}(M)$ (see [2, Def. 10.2.10]). For example, let us consider $J \equiv \Theta(\lambda jxy.x(jy))$, where $\Theta$ is Turing's fixpoint combinator [2, Def. 6.1.4]. Then $x \sqsubseteq_{\eta,\infty} Jx$, since

$$\begin{aligned} Jx &=_\beta \lambda z_0.x(Jz_0) =_\beta \lambda z_0.x(\lambda z_1.z_0(Jz_1)) \\ &=_\beta \lambda z_0.x(\lambda z_1.z_0(\lambda z_2.z_1(Jz_2))) =_\beta \ldots \end{aligned}$$

Using $\sqsubseteq_{\eta,\infty}$, we can define another relation on $\lambda$-terms which will be useful in Subsec. 2.6. For all $M, N \in \Lambda$ we set $M \precsim_\eta N$ if there exist $M', N'$ such that $M \sqsubseteq_{\eta,\infty} M' \sqsubseteq_{BT} N' \sqsupseteq_{\eta,\infty} N$. Let us provide an example of this situation:

$$
\begin{array}{ccc}
 & \lambda x.x & \\
x & \bot & \lambda z.x \\
 & & z
\end{array}
\;\sqsubseteq_{\eta,\infty}\;
\begin{array}{ccc}
 & \lambda x.x & \\
\lambda z_0.x & \bot & \lambda z.x \\
\lambda z_1.z_0 & & z \\
\lambda z_2.z_1 & & \\
\vdots & &
\end{array}
\;\sqsubseteq_{BT}\;
\begin{array}{ccc}
 & \lambda x.x & \\
\lambda z_0.x & y & \lambda z.x \\
\lambda z_1.z_0 & & z \\
\lambda z_2.z_1 & & \\
\vdots & &
\end{array}
\;\sqsupseteq_{\eta,\infty}\;
\begin{array}{ccc}
 & \lambda x.x & \\
\lambda z_0.x & y & x \\
\lambda z_1.z_0 & & \\
\lambda z_2.z_1 & & \\
\vdots & &
\end{array}
$$

Finally, we write $M \simeq_\eta N$ for $M \precsim_\eta N \precsim_\eta M$.

## 2 Well Stratifiable Categorical Models

The $\lambda$-theory $\mathcal{H}^*$ was first introduced by Hyland [10] and Wadsworth [16], who proved (independently) that the theory of $\mathscr{D}_\infty$ is $\mathcal{H}^*$. This proof has been extended by Gouy in [9] with the aim of showing that also the stable analogue of $\mathscr{D}_\infty$ had $\mathcal{H}^*$ as equational theory. Actually, his result is more powerful and covers many suitably stratifiable models living in "regular" ccc's. However, all regular ccc's have (particular) cpo's as objects and (particular) continuous functions as morphisms, hence only concrete categories can be regular. Concerning models in non-well-pointed categories, Di Gianantonio et al. provided in [7] a similar proof, but it only works for non-concrete *categories of games*.

In this section we provide sufficient conditions for models living in (possibly non-well-pointed) cpo-enriched ccc's to have $\mathcal{H}^*$ as equational theory. All syntactic notions and results we will use were already present in the literature, whilst the semantic results are our own contribution.

### 2.1 A Uniform Interpretation of $\lambda$-Terms

A *model of $\lambda$-calculus* $\mathscr{U}$ is a reflexive object in a ccc $\mathbf{C}$, i.e., a triple $(U, \mathrm{Ap}, \lambda)$ such that $U$ is an object of $\mathbf{C}$, and $\lambda \in \mathbf{C}([U \Rightarrow U], U)$ and $\mathrm{Ap} \in \mathbf{C}(U, [U \Rightarrow U])$ satisfy $\mathrm{Ap} \circ \lambda = \mathrm{Id}_{[U \Rightarrow U]}$. $\mathscr{U}$ is called *extensional* when moreover $\lambda \circ \mathrm{Ap} = \mathrm{Id}_U$.

A $\lambda$-term $M$ is usually interpreted as morphisms $|M|_I \in \mathbf{C}(U^I, U)$ for some finite subset $I \subset \mathrm{Var}$ containing the free variables of $M$. The arbitrary choice of $I$ is tedious to treat when dealing with the equalities induced by a model. Fortunately, when the underlying category has countable products, we are able to interpret all $\lambda$-terms in the homset $\mathbf{C}(U^{\mathrm{Var}}, U)$ just slightly modifying the usual definition of interpretation (see [2, Def. 5.5.3(vii)]). Indeed, given $M \in \Lambda$, we can define $|M|_{\mathrm{Var}} \in \mathbf{C}(U^{\mathrm{Var}}, U)$ by structural induction on $M$, as follows:

- $|x|_{\mathrm{Var}} = \pi_x^{\mathrm{Var}}$,
- $|NP|_{\mathrm{Var}} = |N|_{\mathrm{Var}} \bullet |P|_{\mathrm{Var}}$, where $\bullet = ev \circ (\mathrm{Ap} \times \mathrm{Id})$,
- $|\lambda x.N|_{\mathrm{Var}} = \lambda \circ \Lambda(|N|_{\mathrm{Var}} \circ \eta_x)$, where $\eta_x = \Pi^{\mathrm{Var}}_{\mathrm{Var}-\{x\}} \times \mathrm{Id} \in \mathbf{C}(U^{\mathrm{Var}} \times U, U^{\mathrm{Var}})$.

Hence, for the sake of simplicity, we will work in ccc's having countable products. This is just a simplification: all the work done in this section could be adapted to cover also categorical models living in ccc's without countable products but the statements and the proofs would be significantly more technical.

We set $\mathrm{Th}(\mathscr{U}) = \{(M,N) : |M|_{\mathrm{Var}} = |N|_{\mathrm{Var}}\}$. $\mathrm{Th}(\mathscr{U})$ is called the *$\lambda$-theory induced by $\mathscr{U}$* (or just *the (equational) theory of $\mathscr{U}$*). It is easy to check that if $\mathscr{U}$ is an extensional model then $\mathrm{Th}(\mathscr{U})$ is an extensional $\lambda$-theory.

### 2.2 Stratifiable Models in Cpo-Enriched Ccc's

The classic methods for proving that the theory of a categorical model is $\mathcal{H}^*$ require that the $\lambda$-terms are interpreted as elements of a cpo and that the morphisms involved in the definition of the interpretation are continuous functions. Thus, working *possibly outside* well-pointed categories, it becomes natural to consider categorical models living in cpo-enriched ccc's.

From now on, and until the end of the section, we consider a fixed (non-trivial) categorical model $\mathscr{U} = (U, \mathrm{Ap}, \lambda)$ living in a cpo-enriched ccc $\mathbf{C}$ having countable products.

Since in a cpo-enriched ccc pairing and currying are monotonic we get the following corollary.

**Corollary 1.** *The operations $\bullet$ and $\lambda \circ \Lambda(- \circ \eta_x)$ are continuous.*

To lighten the notation we write $\sqsubseteq$ and $\perp$ respectively for $\sqsubseteq_{(U^{\mathrm{Var}},U)}$ and $\perp_{(U^{\mathrm{Var}},U)}$.

**Definition 1.** *The model $\mathscr{U}$ is a* $\perp$-model *if the following two conditions hold:*
*(i)* $\perp \bullet a = \perp$ *for all* $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$,
*(ii)* $\lambda \circ \Lambda(\perp_{(U^{\mathrm{Var}} \times U, U)}) = \perp$.

Stratifications of models are done by using special morphisms, acting at the level of $\mathbf{C}(U,U)$ and called *projections*.

**Definition 2.** *Given an object $U$ of a category $\mathbf{C}$, a morphism $p \in \mathbf{C}(U,U)$ is a* projection from $U$ to $U$ *if* $p \sqsubseteq_{(U,U)} \mathrm{Id}_U$ *and* $p \circ p = p$.

From now on, we also fix a family $(p_k)_{k\in\mathbb{N}}$ of projections from $U$ to $U$ such that $(p_k)_{k\in\mathbb{N}}$ is increasing with respect to $\sqsubseteq_{(U,U)}$ and $\sqcup_{k\in\mathbb{N}} p_k = \mathrm{Id}_U$.

**Notation 1.** *Given a morphism $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$ we write $a_k$ for $p_k \circ a$.*

**Remark 1.** *Since the $p_k$'s are increasing, $\sqcup_{k\in\mathbb{N}} p_k = \mathrm{Id}_U$, and composition is continuous, we have for every morphism $a \in \mathbf{C}(U^{\mathrm{Var}}, U)$:*
*(i)* $a_k \sqsubseteq a$,
*(ii)* $a = \sqcup_{k\in\mathbb{N}} a_k$.

**Definition 3.** *The model $\mathscr{U}$ is called:*
*(i)* stratified *(by $(p_k)_{k\in\mathbb{N}}$) if $a_{k+1} \bullet b = (a \bullet b_k)_k$;*
*(ii)* well stratified *(by $(p_k)_{k\in\mathbb{N}}$) if, moreover, $a_0 \bullet b = (a \bullet \perp)_0$.*

Of course, the fact that $\mathscr{U}$ is a (well) stratified model depends on the family $(p_k)_{k\in\mathbb{N}}$ we are considering. Hence, it is natural and convenient to introduce the notion of (well) stratifiable model.

**Definition 4.** *The model $\mathscr{U}$ is* stratifiable *(*well stratifiable*) if there exists a family $(p_k)_{k\in\mathbb{N}}$ making $\mathscr{U}$ stratified (well stratified).*

The aim of this section is in fact to prove that every extensional well stratifiable $\perp$-model has $\mathcal{H}^*$ as equational theory.

### 2.3 Modelling the Labelled $\lambda\perp$-Calculus in $\mathscr{U}$

We recall now the definition of the *labelled* $\lambda\perp$*-calculus* (see [16] or [2, Sec. 14.1]). We consider a set $C = \{c_k : k \in \mathbb{N}\}$ of constants called *labels*, together with a constant $\underline{\perp}$ to indicate lack of information.

The set $\Lambda_{\underline{\perp}}^{lab}$ of *labelled* $\lambda\perp$*-terms* is inductively defined as follows: $\underline{\perp} \in \Lambda_{\underline{\perp}}^{lab}$; $x \in \Lambda_{\underline{\perp}}^{lab}$, for every $x \in \mathrm{Var}$; if $M, N \in \Lambda_{\underline{\perp}}^{lab}$ then $MN \in \Lambda_{\underline{\perp}}^{lab}$; if $M \in \Lambda_{\underline{\perp}}^{lab}$ then $\lambda x.M \in \Lambda_{\underline{\perp}}^{lab}$, for every $x \in \mathrm{Var}$; if $M \in \Lambda_{\underline{\perp}}^{lab}$ then $c_k M \in \Lambda_{\underline{\perp}}^{lab}$, for every $c_k \in C$.

We will denote by $\Lambda_{\underline{\perp}}$ the subset of $\Lambda_{\underline{\perp}}^{lab}$ consisting of those terms that do not contain any label; note that $\Lambda \subsetneq \Lambda_{\underline{\perp}} \subsetneq \Lambda_{\underline{\perp}}^{lab}$.

The labelled $\lambda\perp$-terms can be interpreted in $\mathscr{U}$; the intuitive meaning of $c_k M$ is the $k$-th projection applied to the meaning of $M$. Hence, we define the interpretation function as the unique extension of the interpretation function of $\lambda$-terms such that:

- $|\underline{\perp}|_{\mathrm{Var}} = \perp$,
- $|c_k M|_{\mathrm{Var}} = p_k \circ |M|_{\mathrm{Var}} = (|M|_{\mathrm{Var}})_k$, for all $M \in \Lambda_{\underline{\perp}}^{lab}$ and $k \in \mathbb{N}$.

Since the ccc $\mathbf{C}$ is cpo-enriched, all labelled $\lambda\perp$-terms are interpreted in the cpo $(\mathbf{C}(U^{\mathrm{Var}}, U), \sqsubseteq, \perp)$. Hence we can transfer this ordering, and the corresponding equality, on $\Lambda_{\underline{\perp}}^{lab}$ as follows.

**Definition 5.** *For all $M, N \in \Lambda_{\underline{\perp}}^{lab}$ we set $M \sqsubseteq_{\mathscr{U}} N$ iff $|M|_{\mathrm{Var}} \sqsubseteq |N|_{\mathrm{Var}}$, Moreover, we write $M =_{\mathscr{U}} N$ iff $M \sqsubseteq_{\mathscr{U}} N$ and $N \sqsubseteq_{\mathscr{U}} M$.*

It is straightforward to check that both $\sqsubseteq_{\mathscr{U}}$ and $=_{\mathscr{U}}$ are contextual.

The notion of substitution can be extended to $\Lambda_{\underline{\perp}}^{lab}$ by setting: $\underline{\perp}[M/x] = \underline{\perp}$ and $(c_k M)[N/x] = c_k(M[N/x])$ for all $M, N \in \Lambda_{\underline{\perp}}^{lab}$. We now show that $\mathscr{U}$ is sound for the $\beta$-conversion extended to $\Lambda_{\underline{\perp}}^{lab}$.

**Lemma 1.** *For all $M, N \in \Lambda_{\underline{\perp}}^{lab}$ we have $(\lambda x.M)N =_{\mathscr{U}} M[N/x]$.*

*Proof.* By [2, Lemma 5.5.5] we know that $(\lambda x.M)N =_{\mathscr{U}} M[N/x]$ still holds for $\lambda$-calculi extended with constants $c$, if $|c|_{\mathrm{Var}} = u \circ !_{U^{\mathrm{Var}}}$ for some $u \in \mathbf{C}(\mathbb{1}, U)$. Hence, this lemma holds since the interpretation defined above is equal to that obtained by setting: $|c_k|_{\mathrm{Var}} = \lambda \circ \Lambda(p_k) \circ !_{U^{\mathrm{Var}}}$ and $|\underline{\perp}|_{\mathrm{Var}} = \perp_{(\mathbb{1},U)} \circ !_{U^{\mathrm{Var}}}$.

We now introduce the reduction rules on labelled $\lambda\perp$-terms which generate the labelled $\lambda\perp$-calculus.

**Definition 6.**

*The* $\omega$-reduction *is defined by:*
$\underline{\perp} M \to_\omega \underline{\perp}$
$\lambda x.\underline{\perp} \to_\omega \underline{\perp}$

*The* $\gamma$-reduction *is defined by:*
$c_0(\lambda x.M)N \to_\gamma c_0(M[\underline{\perp}/x])$
$c_{k+1}(\lambda x.M)N \to_\gamma c_k(M[c_k N/x])$.

*The* $\epsilon$-reduction *is defined by:*
$c_k \underline{\perp} \to_\epsilon \underline{\perp}$,
$c_k(c_n M) \to_\epsilon c_{\min(k,m)} M$.

The calculus on $\Lambda_{\underline{\perp}}^{lab}$ generated by the $\omega$-, $\gamma$-, $\epsilon$-reductions is called *labelled $\lambda\perp$-calculus*. Note that the $\beta$-reduction is not considered here.

**Theorem 1.** *[2, Thm. 14.1.12 and 14.2.3] The labelled $\lambda\perp$-calculus is strongly normalizable and Church Rosser.*

We now show that the interpretation of a labelled $\lambda\perp$-term, in a well stratified $\perp$-model, is invariant along its $\omega$-, $\epsilon$-, $\gamma$-reduction paths.

**Proposition 1.** *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M, N \in \Lambda_{\underline{\perp}}^{lab}$:*

| | |
|---|---|
| $(i)$ $\underline{\perp} M =_{\mathscr{U}} \underline{\perp}$, | $(iv)$ $c_n(c_m M) =_{\mathscr{U}} c_{\min(n,m)} M$, |
| $(ii)$ $\lambda x.\underline{\perp} =_{\mathscr{U}} \underline{\perp}$, | $(v)$ $(c_0 \lambda x.M)N =_{\mathscr{U}} c_0(M[\underline{\perp}/x])$, |
| $(iii)$ $c_k\underline{\perp} =_{\mathscr{U}} \underline{\perp}$, | $(vi)$ $(c_{k+1}\lambda x.M)N =_{\mathscr{U}} c_k(M[c_k N/x])$. |

*Proof.* $(i)$ $|\underline{\perp} M|_{\mathrm{Var}} = |\underline{\perp}|_{\mathrm{Var}} \bullet |M|_{\mathrm{Var}} = \perp \bullet |M|_{\mathrm{Var}}$, which is $\perp$ by Def. 1(i).
$(ii)$ $|\lambda x.\underline{\perp}|_{\mathrm{Var}} = \lambda \circ \Lambda(|\underline{\perp}|_{\mathrm{Var}} \circ \eta_x) = \lambda \circ \Lambda(\perp \circ \eta_x)$. Using (l-strict) this is equal to $\lambda \circ \Lambda(\perp_{(U^{\mathrm{Var}} \times U, U)})$, which is $\perp$ by Def. 1(ii). On the other side, $|\underline{\perp}|_{\mathrm{Var}} = \perp$.
$(iii)$ $|c_k\underline{\perp}|_{\mathrm{Var}} = \perp_k$, hence by Rem. 1 we obtain $\perp_k \sqsubseteq \sqcup_{k\in\mathbb{N}} \perp_k = \perp$. The other inequality is clear.
$(iv)$ $|c_n(c_m M)|_{\mathrm{Var}} = p_n \circ p_m \circ |M|_{\mathrm{Var}}$. By continuity of $\circ$, and since the sequence $(p_k)_{k\in\mathbb{N}}$ is increasing and every $p_k \sqsubseteq_{(U,U)} \mathrm{Id}_U$ we obtain $p_n \circ p_m = p_{\min(n,m)}$.
$(v)$
$$\begin{aligned} |(c_0\lambda x.M)N|_{\mathrm{Var}} &= (|\lambda x.M|_{\mathrm{Var}})_0 \bullet |N|_{\mathrm{Var}} && \text{by def. of } |-|_{\mathrm{Var}} \\ &= (|\lambda x.M|_{\mathrm{Var}} \bullet \perp)_0 && \text{by Def. 3(ii)} \\ &= |c_0((\lambda x.M)\underline{\perp})|_{\mathrm{Var}} && \text{by def. of } |-|_{\mathrm{Var}} \\ &= |c_0(M[\underline{\perp}/x])|_{\mathrm{Var}} && \text{by Lemma 1.} \end{aligned}$$
$(vi)$
$$\begin{aligned} |(c_{k+1}\lambda x.M)N|_{\mathrm{Var}} &= (|\lambda x.M|_{\mathrm{Var}})_{k+1} \bullet |N|_{\mathrm{Var}} && \text{by def. of } |-|_{\mathrm{Var}} \\ &= (|\lambda x.M|_{\mathrm{Var}} \bullet (|N|_{\mathrm{Var}})_k)_k && \text{by Def. 3(i)} \\ &= |c_k((\lambda x.M)(c_k N))|_{\mathrm{Var}} && \text{by def. of } |-|_{\mathrm{Var}} \\ &= |c_k(M[c_k N/x])|_{\mathrm{Var}} && \text{by Lemma 1.} \end{aligned}$$

**Corollary 2.** *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M, N \in \Lambda_{\underline{\perp}}^{lab}$, $M =_{\omega\gamma\epsilon} N$ implies $M =_{\mathscr{U}} N$.*

Thus, every well stratifiable $\perp$-model is a model of the labelled $\lambda\perp$-calculus.

### 2.4 Completely Labelled $\lambda\perp$-Terms

We now study the properties of those labelled $\lambda\perp$-terms $M$ which are completely labelled. This means that every subterm of $M$ "has" a label.

**Definition 7.** *The set of* completely labelled $\lambda\perp$*-terms is defined by induction: $c_k\underline{\perp}$ is a completely labelled $\lambda\perp$-term, for every $k$; $c_k x$ is a completely labelled $\lambda\perp$-term, for every $x$ and $k$; if $M, N \in \Lambda_{\underline{\perp}}^{lab}$ are completely labelled then also $c_k(MN)$ and $c_k(\lambda x.M)$ are completely labelled for every $x$ and $k$.*

Note that every completely labelled $\lambda\perp$-term is $\beta$-normal, since every lambda abstraction is "blocked" by a $c_k$.

**Definition 8.** *A* complete labelling *$L$ of a term $M \in \Lambda_{\underline{\perp}}$ is a map which assigns to each subterm of $M$ a natural number.*

**Notation 2.** *Given a term $M \in \Lambda_{\underline{\perp}}$ and a complete labelling $L$ of $M$, we denote by $M^L$ the resulting completely labelled $\lambda\perp$-term.*

It is easy to check that the set of all complete labellings of $M$ is directed with respect to the following partial ordering: $L_1 \sqsubseteq_{lab} L_2$ iff for each subterm $N$ of $M$ we have $L_1(N) \leq L_2(N)$. By structural induction on the subterms of $M$ one proves that $L_1 \sqsubseteq_{lab} L_2$ implies $M^{L_1} \sqsubseteq_{\mathscr{U}} M^{L_2}$. Therefore, the set of $M^L$ such that $L$ is a complete labelling of $M$, is also directed with respect to $\sqsubseteq_{\mathscr{U}}$.

**Lemma 2.** *If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M \in \Lambda_{\underline{\perp}}$ we have $|M|_{\mathrm{Var}} = \sqcup_L |M^L|_{\mathrm{Var}}$.*

*Proof.* By straightforward induction on $M$, using $a = \sqcup_{k\in\mathbb{N}} a_k$ and Cor. 1.

### 2.5 The Approximation Theorem and Applications

Approximation theorems are an important tool in the analysis of the $\lambda$-theories induced by the models of $\lambda$-calculus. In this section we provide an Approximation Theorem for the class of well stratified $\perp$-models: we show that the interpretation of a $\lambda$-term in a well stratified $\perp$-model $\mathscr{U}$ is the least upper bound of the interpretations of its direct approximants. From this it follows first that $\mathrm{Th}(\mathscr{U})$ is sensible, and second that $\mathcal{B} \subseteq \mathrm{Th}(\mathscr{U})$.

**Definition 9.** *Let $M, N \in \Lambda_{\underline{\perp}}$, then:*

1. *$N$ is an* approximant *of $M$ if there is a context $C[-_1, \ldots, -_k]$ over $\Lambda_{\underline{\perp}}$, with $k \geq 0$, and $M_1, \ldots, M_k \in \Lambda_{\underline{\perp}}$ such that $N \equiv C[\underline{\perp}, \ldots, \underline{\perp}]$ and $M \equiv C[M_1, \ldots, M_k]$;*
2. *$N$ is an* approximate normal form *(*app-nf*, for short) of $M$ if, furthermore, it is $\beta\omega$-normal.*

Given $M \in \Lambda$, we define the set $\mathcal{A}(M)$ of *all direct approximants* of $M$ as follows: $\mathcal{A}(M) = \{W \in \Lambda_{\underline{\perp}} : \exists N, (M \twoheadrightarrow_\beta N) \text{ and } W \text{ is an app-nf of } N\}$.

It is easy to check that if $M$ is unsolvable then $\mathcal{A}(M) = \{\underline{\perp}\}$.

The proof of the following lemma is straightforward once recalled that, if $N \in \mathcal{A}(M)$, then $M$ results (up to $\beta$-conversion) from $N$ by replacing some $\underline{\perp}$ in $N$ by other terms.

**Lemma 3.** *If $\mathscr{U}$ is a well stratifiable $\perp$-model and $M \in \Lambda$, then for all $N \in \mathcal{A}(M)$ we have $N \sqsubseteq_{\mathscr{U}} M$.*

Given $M \in \Lambda_{\underline{\perp}}^{lab}$ we will denote by $\overline{M} \in \Lambda_{\underline{\perp}}$ the term obtained from $M$ by erasing all labels.

**Lemma 4.** *For all $M \in \Lambda_{\underline{\perp}}^{lab}$, we have that $M \sqsubseteq_{\mathscr{U}} \overline{M}$.*

*Proof.* By Rem. 1(i) we have $(|M|_{\mathrm{Var}})_k \sqsubseteq |M|_{\mathrm{Var}}$, and this implies $c_k M \sqsubseteq_{\mathscr{U}} M$. We conclude the proof since $\sqsubseteq_{\mathscr{U}}$ is contextual.

The following syntactic property is a consequence of the results in [2, Sec. 14.3].

**Proposition 2.** *Let $M \in \Lambda$ and $L$ be a complete labelling of $M$. If $\mathrm{nf}(M^L)$ is the $\omega\gamma\epsilon$-normal form of $M^L$, then $\overline{\mathrm{nf}(M^L)} \in \mathcal{A}(M)$.*

**Theorem 2.** *(Approximation Theorem) If $\mathscr{U}$ is a well stratified $\perp$-model, then for all $M \in \Lambda$:*

$$|M|_{\mathrm{Var}} = \bigsqcup \mathcal{A}(M),$$

*where $\bigsqcup \mathcal{A}(M) = \bigsqcup \{|W|_{\mathrm{Var}} : W \in \mathcal{A}(M)\}$.*

*Proof.* Let $L$ be a complete labelling for $M$. From Thm. 1 there is a unique $\omega\epsilon\gamma$-normal form of $M^L$. We denote this normal form by $\mathrm{nf}(M^L)$. Since $M^L \twoheadrightarrow_{\epsilon\gamma\omega} \mathrm{nf}(M^L)$, and $\mathscr{U}$ is a model of the labelled $\lambda\perp$-calculus (Prop. 2), we have $M^L =_{\mathscr{U}} \mathrm{nf}(M^L)$. Moreover, Prop. 2 implies that $\overline{\mathrm{nf}(M^L)} \in \mathcal{A}(M)$ and hence $\mathrm{nf}(M^L) \sqsubseteq_{\mathscr{U}} \overline{\mathrm{nf}(M^L)}$ by Lemma 4. This implies that $|\mathrm{nf}(M^L)|_{\mathrm{Var}} \sqsubseteq \bigsqcup \mathcal{A}(M)$. Since $L$ is an arbitrary complete labelling for $M$, we have: $|M|_{\mathrm{Var}} = \sqcup_L |M^L|_{\mathrm{Var}}$, by Lemma 2 this is equal to $\sqcup_L |\mathrm{nf}(M^L)|_{\mathrm{Var}} \sqsubseteq \bigsqcup \mathcal{A}(M)$. The opposite inequality is clear.

**Corollary 3.** *$M \in \Lambda$ is unsolvable iff $M =_{\mathscr{U}} \underline{\perp}$.*

*Proof.* ($\Rightarrow$) If $M$ is unsolvable, then $\mathcal{A}(M) = \{\underline{\perp}\}$. Hence, $M =_{\mathscr{U}} \underline{\perp}$ by Thm. 2. ($\Leftarrow$) If $M$ is solvable, then by [2, Thm. 8.3.14] there exist $N_1, \ldots, N_k \in \Lambda$, with $k \geq 0$, such that $MN_1 \cdots N_k =_{\mathscr{U}} \mathbf{I}$. Since $\mathscr{U}$ is a $\perp$-model, $M =_{\mathscr{U}} \underline{\perp}$ would imply $\mathbf{I} =_{\mathscr{U}} \underline{\perp}$ (by Def. 1(i)) and $\mathscr{U}$ would be trivial. Contradiction.

**Corollary 4.** *If $\mathscr{U}$ is a well stratifiable $\perp$-model, then $\mathrm{Th}(\mathscr{U})$ is sensible.*

We show that the notion of Böhm tree can be also generalized to terms in $\Lambda_{\underline{\perp}}$.

**Definition 10.** *For all $M \in \Lambda_{\underline{\perp}}$ we write $\mathrm{BT}(M)$ for the Böhm tree of the $\lambda$-term obtained by substituting $\Omega$ for all occurrences of $\underline{\perp}$ in $M$.* Vice versa*, for all $M \in \Lambda$ we denote by $M^{[k]} \in \Lambda_{\underline{\perp}}$ the (unique) $\beta\omega$-normal form such that $\mathrm{BT}(M^{[k]}) = \mathrm{BT}^k(M)$ (where $\mathrm{BT}^k(M)$ is the Böhm tree of $M$ pruned at level $k$).*

It is straightforward to check that, for every $\lambda$-term $M$, $M^{[k]} \in \mathcal{A}(M)$. *Vice versa*, the following proposition is a consequence of the Approximation Theorem.

**Proposition 3.** *If $\mathscr{U}$ is a well stratifiable $\perp$-model then, for all $M \in \Lambda$, $|M|_{\mathrm{Var}} = \sqcup_{k \in \mathbb{N}} |M^{[k]}|_{\mathrm{Var}}$.*

*Proof.* For all $W \in \mathcal{A}(M)$, there exists a $k \in \mathbb{N}$ such that all the nodes in $\mathrm{BT}(W)$ have depth less than $k$. Thus $W \sqsubseteq_{BT} M^{[k]}$ and $W \sqsubseteq_{\mathscr{U}} M^{[k]}$ by Thm. 2.

**Corollary 5.** *If $N \sqsubseteq_{BT} M$ then $N \sqsubseteq_{\mathscr{U}} M$.*

*Proof.* If $N \sqsubseteq_{BT} M$ then for all $k \in \mathbb{N}$ we have $N^{[k]} \sqsubseteq_{BT} M$. By Lemma 3 $N^{[k]} \sqsubseteq_{\mathscr{U}} M$. Thus $|N|_{\mathrm{Var}} = \sqcup_{k \in \mathbb{N}} |N^{[k]}|_{\mathrm{Var}} \sqsubseteq |M|_{\mathrm{Var}}$ by Prop. 3.

As a direct consequence we get the following result.

**Theorem 3.** *If $\mathscr{U}$ is a well stratifiable $\perp$-model, then $\mathcal{B} \subseteq \mathrm{Th}(\mathscr{U})$.*

### 2.6 A General Class of Models of $\mathcal{H}^*$

We recall that the $\lambda$-theory $\mathcal{H}^*$ can be defined in terms of Böhm trees as follows: $M =_{\mathcal{H}^*} N$ if, and only if, $M \simeq_\eta N$ (see [2, Thm. 16.2.7]).

The definition of $\simeq_\eta$ has been recalled in Sec. 1, together with those of $\sqsubseteq_{BT}$, $\sqsubseteq_{\eta,\infty}$, $\precsim_\eta$. However, for proving that $\mathrm{Th}(\mathscr{U}) = \mathcal{H}^*$, the following alternative characterization of $\sqsubseteq_{\eta,\infty}$ will be useful.

**Theorem 4.** *[2, Lemma 10.2.26] The following conditions are equivalent:*
- $M \sqsubseteq_{\eta,\infty} N$,
- *for all* $k \in \mathbb{N}$ *there exists* $P_k \in \Lambda$ *such that* $P_k \twoheadrightarrow_\eta M$, *and* $P_k^{[k]} = N^{[k]}$.

**Lemma 5.** *If* $\mathscr{U}$ *is an extensional well stratified* $\perp$*-model then, for all* $M \in \Lambda_{\underline{\perp}}$ *and* $x \in \mathrm{Var}$, $x \sqsubseteq_{\eta,\infty} M$ *implies* $c_n x \sqsubseteq_{\mathscr{U}} M$ *for all* $n \in \mathbb{N}$.

*Proof.* From [2, Def. 10.2.10], we can assume that $M \equiv \lambda y_1 \dots y_m.xM_1 \cdots M_m$ with $y_i \sqsubseteq_{\eta,\infty} M_i$. The proof is done by induction on $n$. If $n = 0$, then:

$$\begin{array}{lll}
c_0 x & =_{\mathscr{U}} \lambda y_1, \dots y_m.c_0 x y_1 \cdots y_m & \text{since } \mathscr{U} \text{ is extensional,}\\
& =_{\mathscr{U}} \lambda y_1, \dots y_m.c_0 (x\underline{\perp}) y_2 \cdots y_m & \text{since } \mathscr{U} \text{ is well stratified (Def. 3(ii)),}\\
& \vdots \qquad\qquad \vdots & \vdots\\
& =_{\mathscr{U}} \lambda y_1, \dots y_m.c_0 (x\underline{\perp} \cdots \underline{\perp}) & \text{since } \mathscr{U} \text{ is well stratified (Def. 3(ii)),}\\
& \sqsubseteq_{\mathscr{U}} \lambda y_1, \dots y_m.x\underline{\perp} \cdots \underline{\perp} & \text{by Lemma 4,}\\
& \sqsubseteq_{\mathscr{U}} \lambda y_1 \dots y_m.xM_1 \cdots M_m & \text{by } \underline{\perp} \sqsubseteq_{\mathscr{U}} M_i.
\end{array}$$

If $n > 0$, then:

$$\begin{array}{lll}
c_n x & =_{\mathscr{U}} \lambda y_1, \dots y_m.c_n x y_1 \cdots y_m & \text{since } \mathscr{U} \text{ is extensional,}\\
& =_{\mathscr{U}} \lambda y_1 \dots y_m.c_{n-1}(x(c_{n-1}y_1))y_2 \dots y_m & \text{since } \mathscr{U} \text{ is stratified (Def. 3(i)),}\\
& \vdots \qquad\qquad \vdots & \vdots\\
& =_{\mathscr{U}} \lambda y_1 \dots y_m.c_{n-m}(x(c_{n-1}y_1) \cdots (c_{n-m}y_m)) & \text{since } \mathscr{U} \text{ is stratified (Def. 3(i)).}
\end{array}$$

Recalling that $y_i \sqsubseteq_{\eta,\infty} M_i$, we have:

$$\begin{array}{lll}
& \lambda y_1 \dots y_m.c_{n-m}(x(c_{n-1}y_1) \cdots (c_{n-m}y_m)) & \\
\sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.c_{n-m}(xM_1 \cdots M_m) & \text{since } c_{n-i}y_i \sqsubseteq_{\mathscr{U}} M_i \text{ by I.H.,}\\
\sqsubseteq_{\mathscr{U}} & \lambda y_1 \dots y_m.xM_1 \cdots M_m & \text{by Lemma 4.}
\end{array}$$

**Lemma 6.** *Let* $\mathscr{U}$ *be an extensional well stratified* $\perp$*-model and* $M, N, W \in \Lambda_{\underline{\perp}}$. *If* $W$ *is a* $\beta\omega$*-normal form such that* $W \sqsubseteq_{BT} M$ *and* $M \sqsubseteq_{\eta,\infty} N$, *then* $W \sqsubseteq_{\mathscr{U}} N$.

*Proof.* The proof is done by induction on the structure of $W$.
If $W \equiv \underline{\perp}$, then it is trivial.
If $W \equiv x$ then $M \equiv x$ and we conclude by Lemma 5 since $|x|_{\mathrm{Var}} = \sqcup_{n\in\mathbb{N}}(|x|_{\mathrm{Var}})_n$.
If $W \equiv \lambda x_1 \dots x_m.yW_1 \cdots W_r$, then $M =_\beta \lambda x_1 \dots x_m.yM_1 \cdots M_r$ and every $W_i$ is a $\beta\omega$-normal form such that $W_i \sqsubseteq_{BT} M_i$ (for $i \leq r$). By $M \sqsubseteq_{\eta,\infty} N$, we can assume that $N =_{\beta\eta} \lambda x_1 \dots x_{m+s}.yN_1 \cdots N_{r+s}$, with $x_{m+k} \sqsubseteq_{\eta,\infty} N_{r+k}$ (for $1 \leq k \leq s$) and $M_i \sqsubseteq_{\eta,\infty} N_i$ (for $i \leq r$). From $x_{m+k} \sqsubseteq_{\eta,\infty} N_{r+k}$ we obtain, using the previous lemma, that $x_{m+k} \sqsubseteq_{\mathscr{U}} N_{r+k}$. Moreover, since $W_i \sqsubseteq_{BT} M_i \sqsubseteq_{\eta,\infty} N_i$, the induction hypothesis implies $W_i \sqsubseteq_{\mathscr{U}} N_i$. Hence, $W \sqsubseteq_{\mathscr{U}} N$.

**Lemma 7.** *If* $\mathscr{U}$ *is an extensional well stratifiable* $\perp$*-model then for all* $M, N \in \Lambda$:

*(i)* $M \sqsubseteq_{\eta,\infty} N$ *implies* $M =_{\mathscr{U}} N$*,*
*(ii)* $M \precsim_\eta N$ *implies* $M \sqsubseteq_{\mathscr{U}} N$*.*

*Proof.* $(i)$ Suppose that $M \sqsubseteq_{\eta,\infty} N$. Since all $W \in \mathcal{A}(M)$ are $\beta\omega$-normal forms such that $W \sqsubseteq_{BT} M$, the Approximation Theorem and Lemma 6 imply that $M \sqsubseteq_{\mathscr{U}} N$. We prove now that also $N \sqsubseteq_{\mathscr{U}} M$ holds. By the characterization of $\sqsubseteq_{\eta,\infty}$ given in Thm. 4 we know that for all $k \in \mathbb{N}$ there exists a $\lambda$-term $P_k$ such that $P_k \twoheadrightarrow_\eta M$ and $P_k^{[k]} = N^{[k]}$. Since every $P_k^{[k]} \in \mathcal{A}(P_k)$, we have $P_k^{[k]} \sqsubseteq_{\mathscr{U}} P_k$; also, from the extensionality of $\mathscr{U}$, $P_k =_{\mathscr{U}} M$. Thus, by Prop. 3, we have $|N|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}} |N^{[k]}|_{\mathrm{Var}} = \sqcup_{k\in\mathbb{N}} |P_k^{[k]}|_{\mathrm{Var}} \sqsubseteq |M|_{\mathrm{Var}}$. This implies $N \sqsubseteq_{\mathscr{U}} M$.

$(ii)$ Suppose now that $M \precsim_\eta N$. By definition, there exist two $\lambda$-terms $M'$ and $N'$ such that $M \sqsubseteq_{\eta,\infty} M' \sqsubseteq_{BT} N' \sqsupseteq_{\eta,\infty} N$. We conclude as follows: $M =_{\mathscr{U}} M'$ by $(i)$, $M' \sqsubseteq_{\mathscr{U}} N'$ by Thm. 3, and $N' =_{\mathscr{U}} N$, again by $(i)$.

**Theorem 5.** *If $\mathscr{U}$ is a well stratifiable extensional $\bot$-model living in a cpo-enriched ccc (having countable products), then* $\mathrm{Th}(\mathscr{U}) = \mathcal{H}^*$*.*

*Proof.* By Lemma 7(ii) we have that $M \simeq_\eta N$ implies $M =_{\mathscr{U}} N$. Thus, $\mathcal{H}^* \subseteq \mathrm{Th}(\mathscr{U})$. We conclude since $\mathcal{H}^*$ is the maximal sensible consistent $\lambda$-theory.

## 3 An Extensional Relational Model of $\lambda$-Calculus

In this section we recall the definition of our model $\mathscr{D}$ of [6], which is extensional by construction. Finally, we prove that $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^*$ by applying Thm. 5.

### 3.1 A Relational Analogue of $\mathscr{D}_\infty$

We build a family of sets $(D_n)_{n\in\mathbb{N}}$ as follows: $D_0 = \emptyset$, $D_{n+1} = \mathcal{M}_{\mathrm{f}}(D_n)^{(\omega)}$. Since the operation $S \mapsto \mathcal{M}_{\mathrm{f}}(S)^{(\omega)}$ is monotonic on sets, and since $D_0 \subseteq D_1$, we have $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$. Finally, we set $D = \bigcup_{n\in\mathbb{N}} D_n$.

So we have $D_0 = \emptyset$ and $D_1 = \{([], [], \dots)\}$. The elements of $D_2$ are quasi-finite sequences of multisets over a singleton, and so on.

To define an isomorphism in **MRel** between $D$ and $[D \Rightarrow D] (= \mathcal{M}_{\mathrm{f}}(D) \times D)$ just remark that every element $\sigma = (\sigma_0, \sigma_1, \sigma_2, \dots) \in D$ stands for the pair $(\sigma_0, (\sigma_1, \sigma_2, \dots))$ and *vice versa*. Given $\sigma \in D$ and $m \in \mathcal{M}_{\mathrm{f}}(D)$, we write $m \cdot \sigma$ for the element $\tau \in D$ such that $\tau_1 = m$ and $\tau_{i+1} = \sigma_i$. This defines a bijection between $\mathcal{M}_{\mathrm{f}}(D) \times D$ and $D$, and hence an isomorphism in **MRel** as follows:

**Proposition 4.** *The triple $\mathcal{D} = (D, \mathrm{Ap}, \lambda)$ where:*
- $\lambda = \{([(m, \sigma)], m \cdot \sigma) : m \in \mathcal{M}_{\mathrm{f}}(D), \sigma \in D\} \in \mathbf{MRel}([D \Rightarrow D], D)$*,*
- $\mathrm{Ap} = \{([m \cdot \sigma], (m, \sigma)) : m \in \mathcal{M}_{\mathrm{f}}(D), \sigma \in D\} \in \mathbf{MRel}(D, [D \Rightarrow D])$*,*

*is an extensional categorical model of $\lambda$-calculus.*

*Proof.* It is easy to check that $\lambda \circ \mathrm{Ap} = \mathrm{Id}_D$ and $\mathrm{Ap} \circ \lambda = \mathrm{Id}_{[D\Rightarrow D]}$.

We now prove that $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^*$. From Thm. 5 it is enough to check that **MRel** is cpo-enriched and $\mathscr{D}$ is a well stratifiable $\bot$-model.

**Theorem 6.** *The ccc* **MRel** *is cpo-enriched.*

*Proof.* It is clear that, for all sets $S, T$, the homset $(\mathbf{MRel}(S,T), \subseteq, \emptyset)$ is a cpo, that composition is continuous, and pairing and currying are monotonic. Finally, it is easy to check that the strictness conditions hold.

**Theorem 7.** *$\mathscr{D}$ is a well stratifiable $\perp$-model, thus* $\mathrm{Th}(\mathscr{D}) = \mathcal{H}^*$.

*Proof.* By definition of Ap and $\lambda$ it is straigthforward to check that $\emptyset \bullet a = \emptyset$, for all $a \in \mathbf{MRel}(D^{\mathrm{Var}}, D)$, and that $\lambda \circ \Lambda(\emptyset) = \emptyset$, hence $\mathscr{D}$ is a $\perp$-model. Let now $p_n = \{([\sigma], \sigma) : \sigma \in D_n\}$, where $(D_n)_{n\in\mathbb{N}}$ is the family of sets which has been used to build $D$. Since $(D_n)_{n\in\mathbb{N}}$ is increasing also $(p_n)_{n\in\mathbb{N}}$ is, and furthermore $\sqcup_{n\in\mathbb{N}} p_n = \{([\sigma], \sigma) : \sigma \in D\} = \mathrm{Id}_D$. Then, easy calculations show that $\mathscr{D}$ enjoys conditions (i) and (ii) of Def. 3.

## References


1. S. Abramsky, R. Jagadeesan and P. Malacaria. Full abstraction for PCF. *Information and Computation*, 163(2): 409-470, 2000.
2. H.P. Barendregt. The lambda calculus: Its syntax and semantics. *North-Holland Publishing Co.*, Amsterdam, 1984.
3. G. Berry. Stable models of typed lambda-calculi. In *Proc. of the Fifth Colloquium on Automata, Languages and Programming, LNCS 62*, Berlin, 1978. Springer-Verlag.
4. A. Bucciarelli and T. Ehrhard. On phase semantics and denotational semantics: the exponentials. *Ann. Pure Appl. Logic* 109(3): 205-241, 2001.
5. A. Bucciarelli and T. Ehrhard. Sequentiality and strong stability. In LICS'91, pp 138–145, 1991.
6. A. Bucciarelli, T. Ehrhard, and G. Manzonetto. Not enough points is enough. In *Proc. CSL'07*, *LNCS* 4646, pp 298–312, 2007.
7. P. Di Gianantonio, G. Franco and F. Honsell. Game semantics for untyped lambda calculus. *Proc. of TLCA*, LNCS 1591, pp 114-128, 1999.
8. J.-Y. Girard. Normal functors, power series and the $\lambda$-calculus. *Annals of pure and applied logic*, 37:129-177, 1988.
9. X. Gouy. Etude des théories équationnelles et des propriétés algébriques des modèles stables du $\lambda$-calcul, 1995. Thèse, Université de Paris 7.
10. J.M.E. Hyland. A syntactic characterization of the equality in some models for the lambda calculus. *J. London Math. Soc. (2)*, 12(3):361–370, 1975/76.
11. M. Hyland, M. Nagayama, J. Power and G. Rosolini. A category theoretic formulation for Engeler-style models of the untyped $\lambda$-calculus. *Electronic notes in theor. comp. sci.* 161: 43-57, 2006.
12. J.M.E. Hyland and C.-H.L. Ong. On full abstraction for PCF: I, II, and III. *Inf. and Comp.*, 163(2):285–408, 2000.
13. A.D. Ker, H. Nickau and C.-H.L. Ong. A universal innocent game model for the Boehm tree lambda theory. *In Proc. of CSL'99*, LNCS 1683, pp. 405-419, 1999.
14. A. Salibra. A continuum of theories of lambda calculus without semantics. In *Proc. LICS'01*, pp. 334-343, 2001.
15. D.S. Scott. Continuous lattices. In *Toposes, algebraic geometry and logic*, Berlin, 1972. Springer-Verlag.
16. C.P. Wadsworth. The relation between computational and denotational properties for Scott's $D_\infty$-models of the lambda-calculus. *SIAM J. Comp.*, 5(3):488–521, 1976.