Böhm's Theorem for Resource Lambda Calculus through Taylor Expansion*

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Abstract. We study the resource calculus, an extension of the λ-calculus allowing to model resource consumption. We achieve an internal separation result, in analogy with Böhm's theorem of λ-calculus. We define an equivalence relation on the terms, which we prove to be the maximal non-trivial congruence on normalizable terms respecting β-reduction. It is significant that this equivalence extends the usual η-equivalence and is related to Ehrhard's Taylor expansion – a translation mapping terms into series of finite resources.

Keywords: differential linear logic, resource lambda-calculus, separation property, Böhm-out technique.

Introduction

Böhm's theorem in the λ-calculus. Böhm's theorem [1] is a fundamental result in the untyped λ-calculus [2] stating that, given two closed distinct $\beta\eta$-normal λ-terms M and N, there exists a sequence of λ-terms $\vec{L}$, such that $M\vec{L}$ β-reduces to the first projection $\lambda xy.x$ and $N\vec{L}$ β-reduces to the second projection $\lambda xy.y$. The original issue motivating this result was the quest for solutions of systems of equations between λ-terms: given closed terms $M_1, N_1, \dots, M_n, N_n$, is there a λ-term S such that $SM_1 \equiv_\beta N_1 \;\wedge \cdots \wedge\; SM_n \equiv_\beta N_n$ holds? The answer is trivial for $n = 1$ (just take $S = \lambda z.N_1$ for a fresh variable z) and Böhm's theorem gives a positive answer for $n = 2$ and M_1, M_2 distinct $\beta\eta$-normal forms (apply the theorem to M_1, M_2 and set $S = \lambda f.f\vec{L}N_1N_2$). The result has been then generalized in [3] to treat every finite family $M_1, \dots, M_n$ of pairwise distinct $\beta\eta$-normal forms. This generalization is non-trivial since each M_i may differ from the other ones at distinct addresses of its syntactic tree.

As an important consequence of Böhm's theorem we have that the $\beta\eta$-equivalence is the maximal non-trivial congruence on normalizable terms extending the β-equivalence. The case of non-normalizable terms has been addressed by Hyland in [4]. Indeed, the $\beta\eta$-equivalence is not maximal on such terms, and one must consider the congruence $\mathcal{H}^\star$ equating two λ-terms whenever they have

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the same Böhm tree up to possibly infinite η-expansions [2, §16.2]. Then one proves that for all closed λ-terms $M \not\equiv_{\mathcal{H}^\star} N$ there is a sequence of λ-terms $\vec{L}$ such that $M\vec{L}$ β-reduces to the identity $\lambda x.x$ while $N\vec{L}$ is unsolvable (i.e., it does not interact with the environment [2, §8.3]) or *vice versa.* This property is called *semi-separation* because of the asymmetry between the two values: the identity, on the one hand, and any unsolvable term, on the other hand. In fact, non-normalizable terms represent partial functions, and one cannot hope to separate a term less defined than another one without sending the first to an unsolvable term (corresponding to the empty function). Despite the fact that Hyland's semi-separability is weaker than the full separability achieved by Böhm's theorem, it is sufficient to entail that $\mathcal{H}^\star$ is the maximal non-trivial congruence on λ-terms extending β-equivalence and equating all unsolvable terms [2, Thm. 16.2.6].

The resource λ-calculus. We study Böhm's theorem in the resource λ-calculus (Λ^r for short), which is an extension of the λ-calculus along two directions. First, Λ^r is resource sensitive. Following Girard's linear logic [5], the λ-calculus application can be written as $M(N^!)$ emphasizing the fact that the argument N is actually infinitely available for the function M, i.e. it can be erased or copied as many times as needed during the evaluation. Λ^r extends this setting by allowing also applications of the form $M(N^n)$ where N^n denotes a finite resource that must be used exactly n-times during the evaluation. If the number n does not match the needs of M then the application evaluates to the empty sum 0, expressing the absence of a result. In fact, 0 is a β-normal form giving a notion of unsolvable different from the λ-calculus one represented by looping terms[3]. The second feature of Λ^r is the non-determinism. Indeed, the argument of an application, instead of being a single term, is a bag of resources, each being either finite or infinitely available. In the evaluation several possible choices arise, corresponding to the different possibilities of distributing the resources among the occurrences of the formal parameter. The outcome is a finite formal sum of terms collecting all possible results.

Boudol has been the first to extend the λ-calculus with a resource sensitive application [8]. His resource calculus was designed to study Milner's encoding of the lazy λ-calculus into the π-calculus [9,10]. Some years later, Ehrhard and Regnier introduced the differential λ-calculus [11], drawing on insights gained from the quantitative semantics of linear logic, denoting proofs/terms as smooth (i.e. infinitely differentiable) functions. As remarked by the authors, the differential λ-calculus is quite similar to Boudol's calculus, the resource sensitive application $M(N^n)$ corresponds to applying the n-th derivative of M at 0 to N. This intuition was formalized by Tranquilli, who defined the present syntax of Λ^r and showed a Curry-Howard correspondence between this calculus and Ehrhard and Regnier's differential nets [12]. The main differences between Boudol's calculus and Λ^r are that the former is equipped with explicit substitution and lazy operational semantics, while the latter is a true extension of the regular λ-calculus. Since we cannot separate M from $M + M$ we will conveniently suppose that

[3] Denotational models of Λ^r distinguishing between 0 and the usual unsolvable terms are built in [6]. For more details on the notion of solvability in Λ^r see [7].

the sum on Λ^r is idempotent as in [13]; this amounts to say that we only check whether a term appears in a result, not how many times it appears.

A resource conscious Böhm's theorem. A notable outcome of Ehrhard and Regnier's work has been to develop the λ-calculus application as an infinite series of finite applications, $M(N^!) = \sum_{n=0}^{\infty} \frac{1}{n!} M(N^n)$, in analogy with the Taylor expansion of the entire functions. In [14], the authors relate the Böhm tree of a λ-term with its Taylor expansion, giving the intuition that the latter is a resource conscious improvement of the former. Following this intuition, we achieve the main result of this paper, namely a separation property in Λ^r that can be seen as a resource sensitive Böhm's theorem (Theorem 2). Such a result states that for all closed β-normal M, N having η-different Taylor expansion, there is a sequence $\vec{L}$, such that $M\vec{L}$ β-reduces to $\lambda x.x$ and $N\vec{L}$ β-reduces to 0, or *vice versa*.

This theorem reveals a first sharp difference between Λ^r and the λ-calculus, as our result is much similar to Hyland's semi-separation than Böhm's theorem, even if we consider the β-normal forms. This is due to the empty sum 0, the unsolvable β-normal form, outcome of the resource consciousness of Λ^r.

Taylor expansion is a semantical notion, in the sense that it is an infinite series of finite terms. It is then notable that we give a syntactic characterization of the Taylor equality introducing the τ-equivalence in Definition 3 (Proposition 1). As expected, our semi-separability is strong enough to entail that the $\eta\tau$-equivalence induces the maximal non-trivial congruence on β-normalizable terms extending the β-equivalence (Corollary 1).

A crucial ingredient in the classic proof of Böhm's theorem is the fact that it is possible to erase subterms in order to pull out of the terms their structural difference. This is not an easy task in Λ^r, since the finite resources must be consumed and cannot be erased. In this respect, our technique has some similarities with the one developed to achieve the separation for the λI-calculus (i.e., the λ-calculus without weakening, [2, §10.5]). Moreover, since the argument of an application is a bag of resources, comparing a difference between two terms may turn into comparing the differences between two multisets of terms, and this problem presents analogies with that of separating a finite set of terms [3].

Basic definitions and notations. We let $\mathbf{N}$ denote the set of natural numbers. Given a set $\mathcal{X}$, $\mathcal{M}_{\mathsf{f}}(\mathcal{X})$ is the set of all finite multisets over $\mathcal{X}$. Given a reduction $\xrightarrow{\mathsf{r}}$ we let $\xleftarrow{\mathsf{r}}$, $\xrightarrow{\mathsf{r}*}$ and $\equiv_{\mathsf{r}}$ denote its transpose, its transitive-reflexive closure and its symmetric-transitive-reflexive closure, respectively.

An operator $F(-)$ (resp. $F(-,-)$) is *extended by linearity* (resp. *bilinearity*) by setting $F\big(\Sigma_i A_i\big) = \Sigma_i F(A_i)$ (resp. $F\big(\Sigma_i A_i, \Sigma_j B_j\big) = \Sigma_{i,j} F(A_i, B_j)$).

1 Resource Calculus

Syntax. The *resource calculus* has three syntactic categories: *terms* that are in functional position, *bags* that are in argument position and represent unordered lists of resources, and *finite formal sums* that represent the possible results of

$$\begin{array}{llll}
\Lambda^r: & M, N, L & ::= x \mid \lambda x.M \mid MP & \text{terms} \\
\Lambda^b: & P, Q, R & ::= [M_1, \dots, M_n, \mathbb{M}^!] & \text{bags} \\
\Lambda^e: & A, B & ::= M \mid P & \text{expressions} \\
\multicolumn{3}{l}{\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle \quad \mathbb{P}, \mathbb{Q} \in \mathbf{2}\langle \Lambda^b \rangle \quad \mathbb{A}, \mathbb{B} \in \mathbf{2}\langle \Lambda^e \rangle := \mathbf{2}\langle \Lambda^r \rangle \cup \mathbf{2}\langle \Lambda^b \rangle} & \text{sums}
\end{array}$$

(a) Grammar of terms, resources, bags, expressions, sums.

$$\lambda x.(\textstyle\sum_i M_i) := \sum_i \lambda x.M_i \qquad \mathbb{M}(\sum_i P_i) := \sum_i \mathbb{M}P_i$$

$$(\textstyle\sum_i M_i)\mathbb{P} := \sum_i M_i\mathbb{P} \qquad [(\sum_i M_i)]\cdot\mathbb{P} := \sum_i [M_i]\cdot\mathbb{P}$$

(b) Notation on $\mathbf{2}\langle \Lambda^e \rangle$.

$$y\langle N/x \rangle := \begin{cases} N & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases} \qquad \begin{array}{l} (\lambda y.M)\langle N/x \rangle := \lambda y.(M\langle N/x \rangle), \\ (MP)\langle N/x \rangle := M\langle N/x \rangle P + M(P\langle N/x \rangle), \end{array}$$

$$[\mathbb{M}^!]\langle N/x \rangle := [\mathbb{M}\langle N/x \rangle, \mathbb{M}^!], \qquad ([M]\cdot P)\langle N/x \rangle := [M\langle N/x \rangle]\cdot P + [M]\cdot P\langle N/x \rangle,$$

(c) Linear substitution, in the abstraction case we suppose $y \notin \mathrm{FV}(N) \cup \{x\}$.

Fig. 1: Syntax, notations and linear substitution of resource calculus.

a computation. Figure 1(a) provides the grammar for generating the set Λ^r of terms and the set Λ^b of bags, together with their typical metavariables.

A *bag* $[\vec{M}, \mathbb{M}^!]$ is a compound object, consisting of a multiset of *linear resources* $[\vec{M}]$ and a set of terms $\mathbb{M}$ presented in additive notation (see the discussion on sets and sums below) representing the *reusable resources*. Roughly speaking, the linear resources in $\vec{M}$ must be used exactly once during a reduction, while the reusable ones in $\mathbb{M}$ can be used ad libitum (hence, following the linear logic notation, $\mathbb{M}$ is decorated with a ! superscript). We shall deal with bags as if they were multisets presented in multiplicative notation, defining union by $[\vec{M}, \mathbb{M}^!] \cdot [\vec{N}, \mathbb{N}^!] := [\vec{M}, \vec{N}, (\mathbb{M} + \mathbb{N})^!]$. This operation is commutative, associative and has the *empty bag* $1 := [0^!]$ as neutral element. To avoid confusion with application we will never omit the dot "$\cdot$". To lighten the notations we write $[L_1, \dots, L_k]$ for the bag $[L_1, \dots, L_k, 0^!]$, and $[M^k]$ for the bag $[M, \dots, M]$ containing k copies of M. Such a notation allows to decompose a bag in several ways, and this will be used throughout the paper. As for example: $[x, y, (x+y)^!] = [x] \cdot [y, (x+y)^!] = [x^!] \cdot [x, y, y^!] = [x, y] \cdot [(x+y)^!] = [x, x^!] \cdot [y, y^!]$.

Expressions (whose set is denoted by Λ^e) are either terms or bags and will be used to state results holding for both categories.

Let $\mathbf{2}$ be the semiring $\{0, 1\}$ with $1 + 1 = 1$ and multiplication defined in the obvious way. For any set $\mathcal{X}$, we write $\mathbf{2}\langle \mathcal{X} \rangle$ for the free $\mathbf{2}$-module generated by $\mathcal{X}$, so that $\mathbf{2}\langle \mathcal{X} \rangle$ is isomorphic to the finite powerset of $\mathcal{X}$, with addition corresponding to union, and scalar multiplication defined in the obvious way. However we prefer to keep the algebraic notations for elements of $\mathbf{2}\langle \mathcal{X} \rangle$, hence set union will be denoted by $+$ and the empty set by 0. This amounts to say that $\mathbf{2}\langle \Lambda^r \rangle$ (resp. $\mathbf{2}\langle \Lambda^b \rangle$) denotes the set of finite formal sums of terms (resp.

bags), with an idempotent sum. We also set $\mathbf{2}\langle \Lambda^e\rangle = \mathbf{2}\langle \Lambda^r\rangle \cup \mathbf{2}\langle \Lambda^b\rangle$. This is an abuse of notation, as $\mathbf{2}\langle \Lambda^e\rangle$ here does not denote the $\mathbf{2}$-module generated over $\Lambda^e = \Lambda^r \cup \Lambda^b$ but rather the union of the two $\mathbf{2}$-modules; this amounts to say that sums may be taken only in the same sort.

The *size of* $\mathbb{A} \in \mathbf{2}\langle \Lambda^e\rangle$ is defined inductively by: $\mathrm{size}(\Sigma_i A_i) = \Sigma_i \,\mathrm{size}(A_i)$, $\mathrm{size}(x) = 1$, $\mathrm{size}(\lambda x.M) = \mathrm{size}(M) + 1$, $\mathrm{size}(MP) = \mathrm{size}(M) + \mathrm{size}(P) + 1$, $\mathrm{size}([M_1, \dots, M_k, \mathbb{M}^!]) = \Sigma_{i=1}^k \mathrm{size}(M_i) + \mathrm{size}(\mathbb{M}) + 1$.

Notice that the grammar for terms and bags does not include any sums, but under the scope of a $(\cdot)^!$. However, as syntactic sugar – and *not* as actual syntax – we extend all the constructors to sums as shown in Figure 1(b). In fact all constructors except the $(\cdot)^!$ are (multi)linear, as expected. The intuition is that a reusable sum $(M + N)^!$ represents a resource that can be used several times and each time one can choose non-deterministically M or N.

Observe that in the particular case of empty sums, we get $\lambda x.0 := 0$, $M0 := 0$, $0P := 0$, $[0] := 0$ and $0 \cdot P := 0$, but $[0^!] = 1$. Thus 0 annihilates any term or bag, except when it lies under a $(\cdot)^!$. As an example of this *extended (meta-)syntax*, we may write $(x_1 + x_2)[y_1 + y_2, (z_1 + z_2)^!]$ instead of $x_1[y_1, (z_1 + z_2)^!] + x_1[y_2, (z_1 + z_2)^!] + x_2[y_1, (z_1 + z_2)^!] + x_2[y_2, (z_1 + z_2)^!]$. This kind of meta-syntactic notation is discussed thoroughly in [14].

The α-equivalence and the set $\mathrm{FV}(\mathbb{A})$ of free variables are defined as in ordinary λ-calculus. From now on expressions are considered up to α-equivalence. Concerning specific terms we set:

$$\mathbf{I} := \lambda x.x, \qquad \mathbf{X}_n := \lambda x_1 \dots \lambda x_n \lambda x.x[x_1^!] \dots [x_n^!] \text{ for } n \in \mathbf{N},$$

where $\mathbf{X}_n$ is called the *n-th Böhm permutator*.

Due to the presence of two kinds of resources, we need two different notions of substitutions: the usual λ-calculus substitution and a linear one, which is particular to differential and resource calculi (see [14,15]).

Definition 1 (Substitutions). *We define the following substitution operations.*

1. *$A\{N/x\}$ is the usual capture-free substitution of N for x in A. It is extended to sums as in $\mathbb{A}\{\mathbb{N}/x\}$ by linearity in $\mathbb{A}$, and using the notations of Figure 1(b) for $\mathbb{N}$.*
2. *$A\langle N/x\rangle$ is the* linear substitution *defined inductively in Figure 1(c). It is extended to $\mathbb{A}\langle \mathbb{N}/x\rangle$ by bilinearity in both $\mathbb{A}$ and $\mathbb{N}$.*

Intuitively, linear substitution replaces the resource to *exactly one* linear free occurrence of the variable. In presence of multiple occurrences, all possible choices are made and the result is the sum of them. E.g., $(x[x])\langle \mathbf{I}/x\rangle = \mathbf{I}[x] + x[\mathbf{I}]$.

Notice the difference between $[x, x^!]\{M + N/x\} = [(M + N), (M + N)^!] = [M, (M+N)^!] + [N, (M+N)^!]$ and $[x, x^!]\langle M+N/x\rangle = [x, x^!]\langle M/x\rangle + [x, x^!]\langle N/x\rangle = [M, x^!] + [x, M, x^!] + [N, x^!] + [x, N, x^!]$.

Linear substitution bears resemblance to differentiation, as shown clearly in Ehrhard and Regnier's differential λ-calculus [11]. For instance, it enjoys the following Schwarz lemma, whose proof is rather classic and is omitted.

Lemma 1 (Schwarz Lemma [14,11]). *Given* $\mathbb{A} \in \mathbf{2}\langle \Lambda^e \rangle$, $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle$ *and* $y \notin \mathrm{FV}(\mathbb{M}) \cup \mathrm{FV}(\mathbb{N})$ *we have* $\mathbb{A}\langle \mathbb{M}/y \rangle \langle \mathbb{N}/x \rangle = \mathbb{A}\langle \mathbb{N}/x \rangle \langle \mathbb{M}/y \rangle + \mathbb{A}\langle \mathbb{M}\langle \mathbb{N}/x \rangle /y \rangle$. *In particular, if* $x \notin \mathrm{FV}(\mathbb{M})$ *the two substitutions commute.*

$$\frac{M \;\mathsf{R}\; \mathbb{M}}{\lambda x.M \;\mathsf{R}\; \lambda x.\mathbb{M}}\;\texttt{lam} \qquad \frac{M \;\mathsf{R}\; \mathbb{M}}{MP \;\mathsf{R}\; \mathbb{M}P}\;\texttt{appl} \qquad \frac{P \;\mathsf{R}\; \mathbb{P}}{MP \;\mathsf{R}\; M\mathbb{P}}\;\texttt{appr}$$

$$\frac{M \;\mathsf{R}\; \mathbb{M}}{[M]\cdot P \;\mathsf{R}\; [\mathbb{M}]\cdot P}\;\texttt{lin} \qquad \frac{M \;\mathsf{R}\; \mathbb{M}}{[M^!]\cdot P \;\mathsf{R}\; [\mathbb{M}^!]\cdot P}\;\texttt{bng} \qquad \frac{A \;\mathsf{R}\; \mathbb{A}}{A + \mathbb{B} \;\mathsf{R}\; \mathbb{A} + \mathbb{B}}\;\texttt{sum}$$

(a) Rules defining the context closure of a relation $\mathsf{R} \subseteq \Lambda^e \times \mathbf{2}\langle \Lambda^e \rangle$.

$$\frac{\mathbb{M} \;\mathsf{R}\; \mathbb{N}}{\lambda x.\mathbb{M} \;\mathsf{R}\; \lambda x.\mathbb{N}}\;\texttt{lam} \qquad \frac{\mathbb{M} \;\mathsf{R}\; \mathbb{N} \quad \mathbb{P} \;\mathsf{R}\; \mathbb{Q}}{\mathbb{M}\mathbb{P} \;\mathsf{R}\; \mathbb{N}\mathbb{Q}}\;\texttt{app}$$

$$\frac{\mathbb{M} \;\mathsf{R}\; \mathbb{N} \quad \mathbb{P} \;\mathsf{R}\; \mathbb{Q}}{[\mathbb{M}]\cdot \mathbb{P} \;\mathsf{R}\; [\mathbb{N}]\cdot \mathbb{Q}}\;\texttt{lin} \qquad \frac{\mathbb{M} \;\mathsf{R}\; \mathbb{N}}{[\mathbb{M}^!] \;\mathsf{R}\; [\mathbb{N}^!]}\;\texttt{bng} \qquad \frac{\mathbb{A} \;\mathsf{R}\; \mathbb{B} \quad \mathbb{A}' \;\mathsf{R}\; \mathbb{B}'}{\mathbb{A} + \mathbb{A}' \;\mathsf{R}\; \mathbb{B} + \mathbb{B}'}\;\texttt{sum}$$

(b) Rules defining a compatible relation $\mathsf{R} \subseteq \mathbf{2}\langle \Lambda^e \rangle \times \mathbf{2}\langle \Lambda^e \rangle$.

Fig. 2: Definition of context closure and compatible relation.

Operational semantics. Given a relation $\mathsf{R} \subseteq \Lambda^e \times \mathbf{2}\langle \Lambda^e \rangle$ its *context closure* is the smallest relation in $\mathbf{2}\langle \Lambda^e \rangle \times \mathbf{2}\langle \Lambda^e \rangle$ containing R and respecting the rules of Figure 2(a). The main notion of reduction of resource calculus is β*-reduction*, which is defined as the context closure of the following rule:

$$(\beta) \qquad (\lambda x.M)[L_1, \dots, L_k, \mathbb{N}^!] \xrightarrow{\beta} M\langle L_1/x \rangle \cdots \langle L_k/x \rangle \{\mathbb{N}/x\}.$$

Notice that the β-rule is independent of the ordering of the linear substitutions, as shown by the Schwarz lemma above. We say that $A \in \Lambda^e$ *is in* β*-normal form* (β*-nf*, for short) if there is no $\mathbb{A}$ such that $A \xrightarrow{\beta} \mathbb{A}$. A sum $\mathbb{A}$ *is in* β*-nf* if all its summands are. Notice that 0 is a β-nf. It is easy to check that a term M is in β-nf iff $M = \lambda x_1 \dots x_n.yP_1 \dots P_k$ for $n, k \geq 0$ and, for every $1 \leq i \leq k$, all resources in P_i are in β-nf. The variable y in M is called *head-variable.*

The regular λ-calculus [2] can be embedded into the resource one by translating every application MN into $M[N^!]$. In this fragment the β-reduction defined above coincides with the usual one. Hence the resource calculus has usual looping terms like $\mathbf{\Omega} := (\lambda x.x[x^!])[(\lambda x.x[x^!])^!]$, but also terms like $\mathbf{I}1$ or $\mathbf{I}[y, y]$ reducing to 0 because there is a mismatch between the number of linear resources needed by the functional part of the application and the number it actually receives.

Theorem 1 (Confluence [15]). *The* β*-reduction is Church-Rosser on* Λ^r*.*

The resource calculus is *intensional*, indeed just like in the λ-calculus there are different programs having the same extensional behaviour. In order to achieve

$$(\sum_i A_i)^\circ := \bigcup_i A_i^\circ \qquad x^\circ := \{x\} \qquad (\lambda x.M)^\circ := \lambda x.M^\circ \qquad (MP)^\circ := M^\circ P^\circ$$
$$([\mathbb{M}^!])^\circ := \mathcal{M}_\mathrm{f}(\mathbb{M}^\circ) \qquad ([M]\cdot P)^\circ := [M^\circ]\cdot P^\circ$$

Fig. 3: Taylor expansion $\mathbb{A}^\circ$ of $\mathbb{A}$.

an internal separation, we need to consider the η-*reduction* that is defined as the contextual closure of the following rule:

$$(\eta) \qquad \lambda x.M[x^!] \xrightarrow{\eta} M, \text{ if } x \notin \mathrm{FV}(M).$$

The Taylor expansion. The *finite resource calculus* is the fragment of resource calculus having only linear resources (every bag has the set of reusable resources empty). The terms (resp. bags, expressions) of this sub-calculus are called *finite* and their set is denoted by Λ^r_f (resp. Λ^b_f, Λ^e_f). Notice that the bags of Λ^b_f are actually finite multisets. It is easy to check that the above sets are closed under β-reduction, while η-reduction cannot play any role here.

In Definition 2, we describe the Taylor expansion as a map $(\cdot)^\circ$ from $\mathbf{2}\langle\Lambda^r\rangle$ (resp. $\mathbf{2}\langle\Lambda^b\rangle$) to possibly infinite sets of finite terms (resp. finite bags). The Taylor expansion defined in [11,14], in the context of λ-calculus, is a translation developing every application as an infinite series of finite applications with rational coefficients. In our context, since the coefficients are in $\mathbf{2}$, the Taylor expansion of an expression is a (possibly infinite) set of finite expressions. Indeed, for all sets $\mathcal{X}$, the set of the infinite formal sums $\mathbf{2}\langle\mathcal{X}\rangle_\infty$ with coefficients in $\mathbf{2}$ is isomorphic to the powerset of $\mathcal{X}$. Our Taylor expansion corresponds to the support[4] of the Taylor expansion taking rational coefficients given in [11,14].

To lighten the notations, we adopt for sets of expressions the same abbreviations introduced for finite sums in Figure 1(b). E.g., given $\mathcal{M} \subseteq \Lambda^r_\mathrm{f}$ and $\mathcal{P},\mathcal{Q} \subseteq \Lambda^b_\mathrm{f}$ we have $\lambda x.\mathcal{M} = \{\lambda x.M \mid M \in \mathcal{M}\}$ and $\mathcal{P}\cdot\mathcal{Q} = \{P\cdot Q \mid P \in \mathcal{P},\ Q \in \mathcal{Q}\}$.

Definition 2. *Let* $\mathbb{A} \in \mathbf{2}\langle\Lambda^e\rangle$*. The* Taylor expansion of $\mathbb{A}$ *is the set* $\mathbb{A}^\circ \subseteq \Lambda^e_\mathrm{f}$ *which is defined (by structural induction on* $\mathbb{A}$*) in Figure 3.*

As previously announced, the Taylor expansion of an expression A can be infinite, e.g., $(\lambda x.x[x^!])^\circ = \{\lambda x.x[x^n] \mid n \in \mathbf{N}\}$. Different terms may share the same Taylor expansion: $(x[(z[y^!])^!])^\circ = (x[(z1 + z[y,y^!])^!])^\circ$. The presence of linear resources permits situations where $M^\circ \subsetneq N^\circ$, e.g. $M := x[x,x^!], N := x[x^!]$. The presence of non-determinism allows to build terms like $M_1 := x[(y+z)^!], M_2 := x[(y+h)^!]$ such that $M_1^\circ \cap M_2^\circ = \{x[y^n] \mid n \in \mathbf{N}\}$ is infinite. However the intersection can also be finite as in $N_1 := x[y,z^!], N_2 := x[z,y^!]$ where $N_1^\circ \cap N_2^\circ = \{x[y,z]\}$.

2 A Syntactic Characterization of Taylor Equality

In Λ^r, there are distinct $\beta\eta$-nf's which are inseparable, in contrast with what we have in the regular λ-calculus. In fact, all normal forms having equal Taylor

[4] I.e., the set of those finite terms appearing in the series with a non-zero coefficient.

expansions are inseparable, since the first author proved in [16] that there is a non-trivial denotational model of resource calculus equating all terms having the same Taylor expansion. For example, $x[(z[y^!])^!]$ and $x[(z1+z[y,y^!])^!]$ are distinct inseparable $\beta\eta$-nf's since they have the same Taylor expansion.

Because of its infinitary nature, the property of having the same Taylor expansion is more semantical than syntactical. In this section, we provide an alternative syntactic characterization (Definition 3, Proposition 1).

A relation $\mathsf{R} \subseteq \mathbf{2}\langle\Lambda^e\rangle\times\mathbf{2}\langle\Lambda^e\rangle$ is *compatible* if it satisfies the rules in Figure 2(b). The *congruence generated by* a relation R, denoted $\equiv_{\mathsf{R}}$, is the smallest compatible equivalence relation containing R. Given two relations $\mathsf{R},\mathsf{S} \subseteq \mathbf{2}\langle\Lambda^e\rangle \times \mathbf{2}\langle\Lambda^e\rangle$, we write $\equiv_{\mathsf{RS}}$ for the congruence generated by their union $\mathsf{R}\cup\mathsf{S}$.

Definition 3. *The* Taylor equivalence $\equiv_\tau$ *is the congruence generated by:*

$$(\tau) \qquad [M^!] \equiv_\tau 1 + [M, M^!]$$

Moreover, we set $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$ *iff* $\mathbb{A}+\mathbb{B} \equiv_\tau \mathbb{B}$.

It is not difficult to check that $\sqsubseteq_\tau$ is a compatible preorder. We now prove that it captures exactly the inclusion between Taylor expansions (Proposition 1).

E.g., $z[x,x^!] + z1 \equiv_\tau z[x^!] \sqsubseteq_\tau z[x,x^!] + z[y^!]$, while $x[x^!] \not\equiv_\tau x[y^!] \not\sqsubseteq_\tau x[y,y^!]$.

Note that all elements of A° share the same minimum structure, called here *skeleton*, obtained by taking 0 occurrences of every reusable resource.

Definition 4. *Given* $A \in \Lambda^e$*, its* skeleton $\mathfrak{s}(A) \in \Lambda^e_{\mathsf{f}}$ *is obtained by erasing all the reusable resources occurring in* A*. That is, inductively:*

$$\mathfrak{s}(x) := x, \quad \mathfrak{s}(\lambda x.M) := \lambda x.\mathfrak{s}(M), \quad \mathfrak{s}(MP) := \mathfrak{s}(M)\mathfrak{s}(P),$$
$$\mathfrak{s}([M_1,\dots,M_n,\mathbb{M}^!]) := [\mathfrak{s}(M_1),\dots,\mathfrak{s}(M_n)].$$

Obviously $\mathfrak{s}(A) \in A^\circ$. In general it is false that $\mathfrak{s}(A) \in B^\circ$ entails $A \sqsubseteq_\tau B$. Take for instance $A := x[x^!]$ and $B := x[y^!]$; indeed $\mathfrak{s}(A) = x1 \in \{x[y^n] \mid n \in \mathbf{N}\} = B^\circ$. The above implication becomes however true when A is "expanded enough".

Definition 5. *Given* $k \in \mathbf{N}$*, we say that* $A \in \Lambda^e$ *is* k-expanded *if, whenever it contains a bag that can be decomposed into* $[M^!]\cdot P$*, we have* $P = [M^k]\cdot P'$ *for some* P' k*-expanded. A sum* $\mathbb{A} \in \mathbf{2}\langle\Lambda^e\rangle$ *is* k*-expanded if all its summands are.*

E.g., x, $x1$, $x[y^4, x^3, (x+y)^!]$ are 3-expanded, but the latter is not 4-expanded.

Lemma 2. *Let* $A \in \Lambda^e$ *be* k*-expanded for some* $k \in \mathbf{N}$*. Then for every* $B \in \Lambda^e$ *such that* $\mathrm{size}(B) \leq k$*, we have that* $\mathfrak{s}(A) \in B^\circ$ *entails* $A \sqsubseteq_\tau B$*.*

Proof. By induction on A. The only significant case is when A is a bag, which splits in three subcases, depending on how such a bag can be decomposed.

Case I ($A = [M^k, M^!]\cdot P$, P k-expanded). By definition $\mathfrak{s}(A) = [\mathfrak{s}(M)^k]\cdot\mathfrak{s}(P)$. From $\mathfrak{s}(A) \in B^\circ$, we deduce that $B = Q_1\cdot Q_2\cdot Q_3$ where $Q_1 = [L_1,\dots,L_\ell]$, $Q_2 = [(N_1+\cdots+N_n)^!]$ for some $\ell, n \geq 0$ and Q_1, Q_2, Q_3 are such that $[\mathfrak{s}(M)^\ell] \in Q_1^\circ$,

$[\mathfrak{s}(M)^{k-\ell}] \in Q_2^\circ$ and $\mathfrak{s}(P) \in (Q_2 \cdot Q_3)^\circ$. From $\mathrm{size}(B) \le k$ we get $\ell < k$ and this entails $n > 0$. We then have that $\mathfrak{s}(M) \in L_i^\circ$, for every $i \le \ell$, and there is a j such that $\mathfrak{s}(M) \in N_j^\circ$. By induction hypothesis, we have $M \sqsubseteq_\tau L_i$ for every $i \le \ell$, $M \sqsubseteq_\tau N_j$ and $P \sqsubseteq_\tau Q_2 \cdot Q_3$. Hence, by compatibility of $\sqsubseteq_\tau$, we derive $[M^\ell] \sqsubseteq_\tau Q_1$ and $[M^!] \sqsubseteq_\tau [N_j^!]$, that entails $[M^k, M^!] \sqsubseteq_\tau [M^\ell, M^!] \sqsubseteq_\tau Q_1 \cdot [N_j^!]$. From this, using $[N_j^!] \cdot Q_2 \equiv_\tau Q_2$ and $P \sqsubseteq_\tau Q_2 \cdot Q_3$, we conclude $A \sqsubseteq_\tau Q_1 \cdot [N_j^!] \cdot Q_2 \cdot Q_3 \equiv_\tau B$.

Case II ($A = [M] \cdot P$, P without reusable resources). By definition $\mathfrak{s}(A) = [\mathfrak{s}(M)] \cdot \mathfrak{s}(P)$. Suppose $\mathfrak{s}(A) \in B^\circ$, then two subcases are possible.

Case $B = [N] \cdot Q$ such that $\mathfrak{s}(M) \in N^\circ$ and $\mathfrak{s}(P) \in Q^\circ$. By induction hypothesis $M \sqsubseteq_\tau N$ and $P \sqsubseteq_\tau Q$. By compatibility, $A = [M] \cdot P \sqsubseteq_\tau [N] \cdot Q = B$.

Case $B = [N^!] \cdot Q$ such that $\mathfrak{s}(M) \in N^\circ$ and $\mathfrak{s}(P) \in ([N^!] \cdot Q)^\circ$. Then by induction hypothesis, $M \sqsubseteq_\tau N$ and $P \sqsubseteq_\tau [N^!] \cdot Q$, and, always by compatibility of $\sqsubseteq_\tau$, we conclude that $A = [M] \cdot P \sqsubseteq_\tau [N, N^!] \cdot Q \sqsubseteq_\tau B$.

Case III ($A = 1$). From $\mathfrak{s}(A) = 1 \in B^\circ$, we deduce B is a bag containing only reusable resources, hence trivially $A = 1 \sqsubseteq_\tau B$. □

Lemma 3. *For all $A \in \Lambda^e$ and $k \in \mathbf{N}$ there is a k-expanded $\mathbb{A}$ such that $A \equiv_\tau \mathbb{A}$.*

Proof. By immediate structural induction on A. The crucial case is when $A = [M^!] \cdot P$. Then by induction hypothesis $M \equiv_\tau \mathbb{M}$ and $P \equiv_\tau \mathbb{P}$ for some k-expanded $\mathbb{M}, \mathbb{P}$. Let us explicit the first sum into $\mathbb{M} = M_1 + \cdots + M_m$. Then, we have:

$$[M^!] \cdot P \equiv_\tau [\mathbb{M}^!] \cdot \mathbb{P} \equiv_\tau [M_1^k, \ldots, M_m^k, \mathbb{M}^!] \cdot \mathbb{P} + \textstyle\sum_{n_1=0}^{k-1} \cdots \sum_{n_m=0}^{k-1} [M_1^{n_1}, \ldots, M_m^{n_m}] \cdot \mathbb{P}.$$

Note that all summands in the last sum are k-expanded, since so are $\mathbb{M}, \mathbb{P}$. □

Proposition 1. *For all $\mathbb{A}, \mathbb{B} \in \mathbf{2}\langle \Lambda^e \rangle$ we have that $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$ iff $\mathbb{A}^\circ \subseteq \mathbb{B}^\circ$.*

Proof. ($\Rightarrow$) By a trivial induction on a derivation of $\mathbb{A} + \mathbb{B} \equiv_\tau \mathbb{B}$, remarking that all rules defining $\equiv_\tau$ preserve the property of having equal Taylor expansion.

($\Leftarrow$) By induction on the number of terms in the sum $\mathbb{A}$. If $\mathbb{A} = 0$, then clearly $\mathbb{A} \sqsubseteq_\tau \mathbb{B}$ by the reflexivity of $\equiv_\tau$. If $\mathbb{A} = A + \mathbb{A}'$, then by induction hypothesis we have $\mathbb{A}' \sqsubseteq_\tau \mathbb{B}$. As for A, let $k \ge \max(\mathrm{size}(A), \mathrm{size}(\mathbb{B}))$, by Lemma 3, we have a k-expanded sum $\mathbb{A}'' = A_1 + \cdots + A_a \equiv_\tau A$. From $\mathbb{A}'' \equiv_\tau A$ and the already proved left-to-right direction of the proposition we get $(A'')^\circ = A^\circ \subseteq \mathbb{B}^\circ$. This means that $A_i^\circ \subseteq \mathbb{B}^\circ$ for all $i \le a$. In particular $\mathfrak{s}(A_i) \in B_{j_i}^\circ$, for a particular summand B_{j_i} of $\mathbb{B}$. Since we are supposing that A_i is k-expanded and $\mathrm{size}(B_{j_i}) \le k$, we can apply Lemma 2, so arguing $A_i \sqsubseteq_\tau B_{j_i} \sqsubseteq_\tau \mathbb{B}$. Since this holds for every $i \le a$, we get $\mathbb{A}'' \sqsubseteq_\tau \mathbb{B}$. Then we can conclude $\mathbb{A} \equiv_\tau \mathbb{A}' + \mathbb{A}'' \sqsubseteq_\tau \mathbb{B} + \mathbb{B} \equiv_\tau \mathbb{B}$. □

We conclude that $\equiv_\tau$ deserves the name of Taylor equivalence.

3 Separating a Finite Term from Infinitely Many Terms

We now know that two $\beta\eta$-nf's $\mathbb{M}, \mathbb{N}$ such that $\mathbb{M} \equiv_\tau \mathbb{N}$ are inseparable; hence, in order to achieve an internal separation, we need to consider the $\eta\tau$-difference.

One may hope that $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ is equivalent to first compute the η-nf of two β-normal $\mathbb{M}, \mathbb{N}$ and then check whether they are τ-different. Unfortunately, this is false as shown in the following counterexample (which is a counterexample to the confluence of η-reduction modulo $\equiv_\tau$ [17, §14.3]).

For all variables a, b we set $\mathbb{M}_{a,b} := v[\lambda z.a1, \lambda z.b[z^!]] + v[\lambda z.a[z, z^!], \lambda z.b[z^!]]$ and $\mathbb{N}_{a,b} := v[\lambda z.a1, b] + v[\lambda z.a[z, z^!], b]$. Note that $\mathbb{M}_{a,b} \xrightarrow{\eta*} \mathbb{N}_{a,b}$, hence:

$$\mathbb{N}_{x,y} \xleftarrow{\eta*} \mathbb{M}_{x,y} \equiv_\tau v[\lambda z.x[z^!], \lambda z.y[z^!]] \equiv_\tau \mathbb{M}_{y,x} \xrightarrow{\eta*} \mathbb{N}_{y,x}.$$

If $x \neq y$, $\mathbb{N}_{x,y}$ and $\mathbb{N}_{y,x}$ are two inseparable but distinct $\beta\eta$-nf's s.t. $\mathbb{N}_{x,y} \not\equiv_\tau \mathbb{N}_{y,x}$.

This means that, to study the $\eta\tau$-difference, we cannot simply analyze the structure of the η-nf's. Our solution will be to introduce a relation $\preceq_s$ such that $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ entails either $\mathbb{M} \not\preceq_s \mathbb{N}$ or vice versa (Definition 7). Basically, this approach corresponds to first compute the Taylor expansion of $\mathbb{M}, \mathbb{N}$ and then compute their η-nf pointwise, using the following partial η-reduction on Λ^r_{f}.

Definition 6. *The* partial η-reduction $\xrightarrow{\wp}$ *is the contextual closure of the rule* $\lambda x.M[x^n] \xrightarrow{\wp} M$ *if* $x \notin \mathrm{FV}(M)$.

We define the relation $\preceq_s$ corresponding to the *Smith extension* of $\xrightarrow{\wp*}$ and $\preceq_\tau$ corresponding to the relation "${}^{\eta}\sqsubseteq^{\eta}$" of [2, Def. 10.2.32] keeping in mind the analogy between Taylor expansions and Böhm trees discussed in [14].

Definition 7. *Given* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle\Lambda^r\rangle$ *we define:*

- $\mathbb{M} \preceq_s \mathbb{N}$ *iff* $\forall M \in \mathbb{M}^\circ, \exists N \in \mathbb{N}^\circ$ *such that* $M \xrightarrow{\wp*} N$.
- $\mathbb{M} \preceq_\tau \mathbb{N}$ *iff* $\exists \mathbb{M}' \xrightarrow{\eta*} \mathbb{M}$, $\exists \mathbb{N}' \xrightarrow{\eta*} \mathbb{N}$ *such that* $\mathbb{M}' \sqsubseteq_\tau \mathbb{N}'$.

It is easy to check that $\preceq_s$ is a preorder, and we conjecture that also $\preceq_\tau$ is (we will not prove it because unnecessary for the present work).

Remark 1. It is clear that $\mathbb{M} \xrightarrow{\eta} \mathbb{N}$ implies $\mathbb{M} \preceq_s \mathbb{N}$. More generally, by transitivity of $\preceq_s$ we have that $\mathbb{M} \xrightarrow{\eta*} \mathbb{N}$ entails $\mathbb{M} \preceq_s \mathbb{N}$.

Lemma 4. *Let* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle\Lambda^r\rangle$. *Then* $\mathbb{M} \preceq_\tau \mathbb{N}$ *and* $\mathbb{N} \preceq_\tau \mathbb{M}$ *entails* $\mathbb{M} \equiv_{\eta\tau} \mathbb{N}$

Proof. By hypothesis we have $\mathbb{M}', \mathbb{M}'' \xrightarrow{\eta*} \mathbb{M}$, $\mathbb{N}', \mathbb{N}'' \xrightarrow{\eta*} \mathbb{N}$, such that $\mathbb{M}' \sqsubseteq_\tau \mathbb{N}'$ and $\mathbb{N}'' \sqsubseteq_\tau \mathbb{M}''$. Then, $\mathbb{N}' \equiv_\tau \mathbb{M}' + \mathbb{N}' \xrightarrow{\eta*} \mathbb{M} + \mathbb{N}$, hence $\mathbb{N} \equiv_{\eta\tau} \mathbb{M} + \mathbb{N}$. Symmetrically, $\mathbb{M}'' \equiv_\tau \mathbb{M}'' + \mathbb{N}'' \xrightarrow{\eta*} \mathbb{M} + \mathbb{N}$, hence $\mathbb{M} \equiv_{\eta\tau} \mathbb{M} + \mathbb{N}$, and we conclude $\mathbb{M} \equiv_{\eta\tau} \mathbb{N}$. ☐

proof in tech. app. ←

Lemma 5. *Let* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle\Lambda^r\rangle$. *Then* $\mathbb{M} \preceq_s \mathbb{N}$ *entails* $\mathbb{M} \preceq_\tau \mathbb{N}$.

To sum up, $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ implies that, say, $\mathbb{M} \not\preceq_s \mathbb{N}$, which means $\exists M \in \mathbb{M}^\circ$ such that $\forall N \in \mathbb{N}^\circ$ we have $M \stackrel{\wp*}{\not\to} N$. Hence, what Lemma 7 below does, is basically to separate such finite term M from all $\mathbb{N}$'s satisfying the condition $\forall N \in \mathbb{N}^\circ$ $M \stackrel{\wp*}{\not\to} N$ (that are infinitely many). For technical reasons, we will need to suppose that M has a number of λ-abstractions greater than (or equal to) the number of λ-abstractions of $\mathbb{N}$, at any depth where its syntactic tree is defined.

Definition 8. *Let $M, N \in \Lambda^r$ be β-nf's of the shape $M = \lambda x_1 \dots \lambda x_a.yP_1 \dots P_p$, $N = \lambda x_1 \dots \lambda x_b.zQ_1 \dots Q_q$. We say that M* is λ-wider than N *if $a \geq b$ and each (linear or reusable) resource L in P_i is λ-wider than every (linear or reusable) resource $L' \in Q_i$, for all $i \leq q$. Given $\mathbb{M}, \mathbb{N}$ in β-nf we say that $\mathbb{M}$* is λ-wider than $\mathbb{N}$ *iff each summand of $\mathbb{M}$ is λ-wider than all summands of $\mathbb{N}$.*

Notice that empty bags in M make it λ-wider than N, independently from the corresponding bags in N. For example, $y1$ is λ-wider than zQ for any bag Q. The term $x[\lambda y.\mathbf{I}, \mathbf{I}]1$ is λ-wider than $x[\mathbf{I}][\mathbf{I}]$ but not than himself.

Remark 2. If $\mathbb{M}$ is λ-wider than $\mathbb{N}$ then every $M \in \mathbb{M}^\circ$ is λ-wider than $\mathbb{N}$.

Lemma 6. *For all $\mathbb{M}, \mathbb{N}$ in β-nf there is a β-normal $\mathbb{M}'$ such that $\mathbb{M}' \xrightarrow{\eta*} \mathbb{M}$ and $\mathbb{M}'$ is λ-wider than both $\mathbb{M}$ and $\mathbb{N}$.*

Proof. (Outline) For $m > \mathrm{size}(\mathbb{M})$, define $\mathcal{E}^h_m(M)$ by induction as follows:

$$\begin{array}{ll}
\mathcal{E}^0_m(\mathbb{A}) = \mathbb{A} & \mathcal{E}^{h+1}_m(A + \mathbb{A}) = \mathcal{E}^{h+1}_m(A) + \mathcal{E}^{h+1}_m(\mathbb{A}) \\
\multicolumn{2}{l}{\mathcal{E}^{h+1}_m(\lambda x_1 \dots \lambda x_n.y\vec{P}) = \lambda x_1 \dots \lambda x_m.y\mathcal{E}^h_m(\vec{P})\mathcal{E}^h_m([x^!_{n+1}]) \dots \mathcal{E}^h_m([x^!_m])} \\
\multicolumn{2}{l}{\mathcal{E}^{h+1}_m([M_1, \dots, M_k, \mathbb{M}^!]) = [\mathcal{E}^{h+1}_m(M_1), \dots, \mathcal{E}^{h+1}_m(M_k), (\mathcal{E}^{h+1}_m(\mathbb{M}))^!]}
\end{array}$$

Consider then $\mathbb{M}' = \mathcal{E}^k_k(\mathbb{M})$ for some $k > \max(\mathrm{size}(\mathbb{M}), \mathrm{size}(\mathbb{N}))$. ☐

The next lemma will be the key ingredient for proving the resource Böhm's theorem (Theorem 2, below).

Lemma 7. *Let $M \in \Lambda^r_{\mathrm{f}}$ be a finite β-nf and $\Gamma = \{x_1, \dots, x_d\} \supseteq \mathrm{FV}(M)$. Then, there exist a substitution σ and a sequence $\vec{R}$ of closed bags such that, for all β-normal $\mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle$ such that M is λ-wider than $\mathbb{N}$ and $\mathrm{FV}(\mathbb{N}) \subseteq \Gamma$, we have:*

$$\mathbb{N}\sigma\vec{R} \xrightarrow{\beta*} \begin{cases} \mathbf{I} & \text{if } \exists N' \in \mathbb{N}^\circ,\ M \xrightarrow{\varphi*} N', \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Proof. The proof requires an induction loading, namely the fact that $\sigma = \{\mathbf{X}_{k_1}/x_1, \dots, \mathbf{X}_{k_d}/x_d\}$ such that for all distinct $i, j \leq d$ we have $k_i, |k_i - k_j| > 2k$ for some fixed $k > \mathrm{size}(M)$. Recall that $\mathbf{X}_n$ has been defined in section 1.

The proof is carried by induction on $\mathrm{size}(M)$. Let

$$M = \lambda x_{d+1} \dots \lambda x_{d+a}.x_h P_1 \dots P_p$$

where $a, p \geq 0$, $h \leq d + a$ and, for every $i \leq p$, $P_i = [M_{i,1}, \dots, M_{i,m_i}]$ with $m_i \geq 0$. Notice that, for every $j \leq m_i$, $\mathrm{FV}(M_{i,j}) \subseteq \Gamma \cup \{x_{d+1}, \dots, x_{d+a}\}$ and $k > \mathrm{size}(M_{i,j})$. So, define $\sigma' = \sigma \cdot \{\mathbf{X}_{k_{d+1}}/x_{d+1}, \dots, \mathbf{X}_{k_{d+a}}/x_{d+a}\}$ such that $k_i, |k_i - k_j| > 2k$, for every different $i, j \leq d + a$.

By induction hypothesis on $M_{i,j}$, we have a sequence $\vec{R}_{i,j}$ of closed bags satisfying condition (1) for every $\mathbb{N}'$ such that $M_{i,j}$ is λ-wider than $\mathbb{N}'$ and $\mathrm{FV}(\mathbb{N}') \subseteq \Gamma \cup \{x_{d+1}, \dots, x_{d+a}\}$.

We now build the sequence of bags $\vec{R}$ starting from such $\vec{R}_{i,j}$'s. First, we define a closed term H as follows (setting $m = \max\{k_1, \ldots, k_{d+a}\}$):

$$H := \lambda z_1 \ldots \lambda z_p \lambda w_1 \ldots \lambda w_{m+k_h-p}.\mathbf{I}[z_1\vec{R}_{1,1}] \ldots [z_1\vec{R}_{1,m_1}] \ldots [z_p\vec{R}_{p,1}] \ldots [z_p\vec{R}_{p,m_p}].$$

Then we set: $\vec{R} := [\mathbf{X}^!_{k_{d+1}}] \ldots [\mathbf{X}^!_{k_{d+a}}] \underbrace{1 \ldots 1}_{k_h - p \text{ times}} [H^!] \underbrace{1 \ldots 1}_{m \text{ times}}$.

Notice the base of induction is when $p = 0$ or for every $i \leq p$, $m_i = 0$. In these cases H will be of the form $\lambda\vec{z}\lambda\vec{w}.\mathbf{I}$.

We prove condition (1) for any $\mathbb{N}$ satisfying the hypothesis of the lemma. Note that we can restrict to the case $\mathbb{N}$ is a single term N, the general case follows by distributing the application $\mathbb{N}\sigma\vec{R}$ on every summand of $\mathbb{N}$. So, let

$$N = \lambda x_{d+1} \ldots \lambda x_{d+b}.x_{h'}Q_1 \ldots Q_q$$

and let us prove that $N\sigma\vec{R} \xrightarrow{\beta*} \mathbf{I}$ if there is $N' \in N^\circ, M \overset{\varphi*}{\to} N'$, otherwise $N\sigma\vec{R} \xrightarrow{\beta*} 0$. Since M is λ-wider than N, we have $a \geq b$. We then get, setting $\sigma'' = \sigma \cdot \{\mathbf{X}_{k_{d+1}}/x_{d+1}, \ldots, \mathbf{X}_{k_{d+b}}/x_{d+b}\}$:

$$N\sigma\vec{R} \xrightarrow{\beta*} \mathbf{X}_{k_{h'}}Q_1 \ldots Q_q\sigma''[\mathbf{X}^!_{k_{d+b+1}}] \ldots [\mathbf{X}^!_{k_{d+a}}]\underbrace{1 \ldots 1}_{k_h-p}[H^!]\underbrace{1 \ldots 1}_{m}. \tag{2}$$

Indeed, since $a \geq b$, the variables $x_{d+b+1}, \ldots, x_{d+a}$ can be considered not occurring in $Q_1, \ldots, Q_q$, so that σ' acts on $Q_1, \ldots, Q_q$ exactly as σ''. Moreover, since $k > a$ and $k > q$, we have $k_{h'} \geq 2k > q + (a - b)$, so, setting $g = q + (a - b)$, we have (2) β-reduces to:

$$\big(\lambda y_{g+1}...\lambda y_{k_{h'}}\lambda y.yQ_1...Q_q[\mathbf{X}^!_{k_{d+b+1}}]...[\mathbf{X}^!_{k_{d+a}}][y^!_{g+1}]...[y^!_{k_{h'}}]\big)\sigma' \underbrace{1...1}_{k_h-p}[H^!]\underbrace{1...1}_{m} \tag{3}$$

We consider now three cases.

Case I ($h' \neq h$). This means M and N differ on their head-variable, in particular, for every $N' \in N^\circ$, $M \overset{\varphi*}{\not\to} N'$. We prove then (3)$\xrightarrow{\beta*} 0$. By the hypothesis on $k_h, k_{h'}$, we have either $k_h > k_{h'} + 2k$ or $k_{h'} > k_h + 2k$. In the first case, we get (by the hypothesis on k and N) $k_h > k_{h'} + p + q + a + b > k_{h'} + p - g$, so that (3)$\xrightarrow{\beta*} 0$ since the head-variable y will get 0 from an empty bag of the bunch of the $k_h - p$ empty bags. In the second case, we get $m \geq k_{h'} - g > k_h - p$, so that (3)$\xrightarrow{\beta*} 0$ since the head-variable y will get 0 from an empty bag of the bunch of the m empty bags.

Case II ($h' = h$ and $p - a \neq q - b$). This means that, for every $N' \in N^\circ$, $M \overset{\varphi*}{\not\to} N'$ (in fact, note that $(\cdot)^\circ$ preserve the length of the head prefix of abstractions and that of the head sequence of applications, while $\overset{\varphi}{\to}$ preserves the difference between the two). We prove then (3)$\xrightarrow{\beta*} 0$. As before, we have two subcases. If $p - a < q - b$, then $k_{h'} - g = k_h - g < k_h - p$ so (3)$\xrightarrow{\beta*} 0$, the head variable y

getting 0 from an empty bag of the bunch of the $k_h - p$ empty bags. Otherwise, $p-a > q-b$ implies $k_{h'} - g = k_h - g > k_h - p$ and so (3)$\xrightarrow{\beta*}$ 0, the head variable y getting 0 from an empty bag of the bunch of the m empty bags.

Case III ($h' = h$ and $p-a = q-b$). In this case we have $k_{h'} - g = k_h - g = k_h - p$, so that (3) β-reduces to $HQ_1 \dots Q_q[\mathbf{X}^!_{k_{d+b+1}}] \dots [\mathbf{X}^!_{k_{d+a}}] \underbrace{1 \dots 1}_{k_h - p} \underbrace{1 \dots 1}_{m} \sigma'$. Notice that, by the definition of the substitution σ', we can rewrite this term as

(4) $$HQ_1 \dots Q_q[x^!_{k_{d+b+1}}] \dots [x^!_{k_{d+a}}] \underbrace{1 \dots 1}_{k_h - p} \underbrace{1 \dots 1}_{m} \sigma'.$$

Since N can be in $\Lambda^r - \Lambda^r_{\mathrm{f}}$, the bags Q_i's may contain reusable resources. Hence, for every $i \le q$, let us explicit $Q_i = [N_{i,1}, \dots, N_{i,\ell_i}, (N_{i,\ell_i+1} + \dots + N_{i,n_i})^!]$, where $n_i \ge \ell_i \ge 0$. We split into three subcases. Notice that $p \ge q$, indeed $a \ge b$ (as M is λ-wider than N) and we are considering the case $p - a = q - b$. Also, recall that m_i is the number of resources in P_i.

Subcase III.a ($\exists i \le q, m_i < \ell_i$). In this case, for every $N' \in N^\circ$, we have $M \overset{\varphi*}{\not\to} N'$. In fact, any $N' \in N^\circ$ is of the form $\lambda x_{d+1} \dots \lambda x_{d+b}.x_{h'} Q'_1 \dots Q'_q$, with $Q'_j \in Q^\circ_j$ for every $j \le q$. In particular, Q'_i has at least $\ell_i > m_i$ linear resources, hence $P_i \overset{\varphi*}{\not\to} Q'_i$, and hence $M \overset{\varphi*}{\not\to} N'$.

We have (4)$\xrightarrow{\beta*}$ 0. Indeed, applying the β-reduction to (4) will eventually match the abstraction λz_i in H with the bag Q_i. The variable z_i has m_i linear occurrences in H and no reusable ones. This means there will be not enough occurrences of z_i to accommodate all the ℓ_i linear resources of Q_i, so giving (4)$\xrightarrow{\beta*}$ 0.

Subcase III.b ($\exists i \le q, m_i \ne \ell_i$ and $n_i = \ell_i$). This case means that Q_i has no reusable resources and a number of linear resources different from P_i. Hence $M \overset{\varphi*}{\not\to} N'$, for every $N' \in N^\circ$. Also, (4)$\xrightarrow{\beta*}$ 0 since the number of linear resources in Q_i does not match the number of linear occurrences of the variable z_i in H.

Subcase III.c ($\forall i \le q, m_i \ge \ell_i,$ and $m_i > \ell_i$ entails $n_i > \ell_i$). The hypothesis of the case says that $\ell_i < m_i$ entails that Q_i has some reusable resources. Let $\mathcal{F}_i$ be the set of maps $s : \{1, \dots, m_i\} \to \{1, \dots, \ell_i, \ell_i + 1, \dots, n_i\}$ such that

ℓ_i-injectivity: for every $j, h \le m_i$, if $s(j) = s(h) \le \ell_i$, then $j = h$,
ℓ_i-surjectivity: for every $h \le \ell_i$, there is $j \le m_i$, $s(j) = h$.

Intuitively, $\mathcal{F}_i$ describes the possible ways of replacing the m_i occurrences of the variable z_i in H by the n_i resources in Q_i: the two conditions say that each of the ℓ_i linear resources of Q_i must replace exactly one occurrence of z_i. Notice that, being under the hypothesis that $p - a = q - b$, we have $p - q = a - b$, hence (4) β-reduces to the following sum

$$\sum_{\substack{s_1\in\mathcal{F}_1\\ \vdots\\ s_q\in\mathcal{F}_q}} \mathbf{I}[N_{1,s_1(1)}\vec{R}_{1,1}]...[N_{1,s_1(m_1)}\vec{R}_{1,m_1}]...[N_{q,s_q(1)}\vec{R}_{q,1}]...[N_{q,s_q(m_q)}\vec{R}_{q,m_q}]$$
$$[x_{d+b+1}\vec{R}_{q+1,1}]...[x_{d+b+1}\vec{R}_{q+1,m_1}]...[x_{d+a}\vec{R}_{p,1}]...[x_{d+a}\vec{R}_{p,m_p}]\sigma' \tag{5}$$

Notice that for every $i \leq q$, $s_i \in \mathcal{F}_i$, $j \leq m_i$ the term $M_{i,j}$ is λ-wider than $N_{i,s_i(j)}$ and $\mathrm{FV}(N_{i,s_i(j)}) \subseteq \Gamma \cup \{x_{d+1}, \ldots, x_{d+a}\}$, so by the induction hypothesis

$$N_{i,s_i(j)}\sigma'\vec{R}_{i,j} \xrightarrow{\beta*} \begin{cases} \mathbf{I} & \text{if } \exists N' \in N^\circ_{i,s_i(j)}, M_{i,j} \overset{\varphi*}{\to} N', \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Also, for every $i, 1 \leq i \leq p-q = a-b$, $j \leq m_{q+i}$, we have that $M_{q+i,j}$ is λ-wider than $x_{d+b+i} \in \Gamma \cup \{x_{d+1}, \ldots, x_{d+a}\}$, so by induction hypothesis

$$x_{d+b+i}\sigma'\vec{R}_{q+i,j} \xrightarrow{\beta*} \begin{cases} \mathbf{I} & \text{if } M_{i,j} \overset{\varphi*}{\to} x_{d+b+i}, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

From (6) and (7) we deduce that (5)$\xrightarrow{\beta*}\mathbf{I}$ if, and only if,

(i) for all $i \leq q$, there exists $s_i \in \mathcal{F}_i$ such that, for all $j \leq m_i$, $\exists N' \in N^\circ_{i,s_i(j)}$ satisfying $M_{i,j} \overset{\varphi*}{\to} N'$, and

(ii) for all $i < p-q = a-b$ and for all $j \leq m_{q+i}$, we have $M_{q+i,j} \overset{\varphi*}{\to} x_{d+b+i}$.

Thanks to the conditions on the function s_i, item (i) is equivalent to say that for all $i \leq q$, $\exists Q'_i \in Q^\circ_i, P_i \overset{\varphi*}{\to} Q'_i$, while item (ii) is equivalent to say that for all $i < p-q = a-b$, $P_{q+i} \overset{\varphi*}{\to} [x^{m_{q+i}}_{d+b+i}]$. This means that (5)$\xrightarrow{\beta*}\mathbf{I}$ if, and only if,

$$\begin{aligned} M &= \lambda x_{d+1} \ldots \lambda x_{d+a}.x_h P_1 \ldots P_p \\ &\overset{\varphi*}{\to} \lambda x_{d+1} \ldots \lambda x_{d+b} \lambda x_{d+b+1} \ldots \lambda x_{d+a}.x_h Q'_1 \ldots Q'_q [x^{m_{q+1}}_{d+b+1}] \ldots [x^{m_p}_{d+a}] \\ &\overset{\varphi*}{\to} \lambda x_{d+1} \ldots \lambda x_{d+b}.x_h Q'_1 \ldots Q'_q \end{aligned}$$

where the last term is in N°. To sum up, $\mathbb{N}\sigma\vec{R}$ β-reduces to $\mathbf{I}$ if $\exists N' \in N^\circ$ such that $M \overset{\varphi*}{\to} N'$ and to 0, otherwise. We conclude that condition (1) holds. □

4 A Resource Conscious Böhm's Theorem

In this section we will prove the main result of our paper, namely Böhm's theorem for the resource calculus. We first need the following technical lemma.

Lemma 8. *Let $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle\Lambda^r\rangle$. If $\mathbb{M} \not\preceq_\tau \mathbb{N}$ then $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ for all $\mathbb{M}' \xrightarrow{\eta*} \mathbb{M}$.*

Proof. Suppose, by the way of contradiction, that there is an $\mathbb{M}' \xrightarrow{\eta*} \mathbb{M}$ such that $\mathbb{M}' \preceq_\tau \mathbb{N}$. Then, there are $\mathbb{M}'' \xrightarrow{\eta*} \mathbb{M}'$ and $\mathbb{N}' \xrightarrow{\eta*} \mathbb{N}$, such that $\mathbb{M}'' \sqsubseteq_\tau \mathbb{N}'$. By transitivity of $\xrightarrow{\eta*}$, $\mathbb{M}'' \xrightarrow{\eta*} \mathbb{M}$ holds, so we get $\mathbb{M} \preceq_\tau \mathbb{N}$ which is impossible. □

We are now able to prove the main result of this paper.

Theorem 2 (Resource Böhm's Theorem). *Let* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle$ *be closed sums in* β*-nf. If* $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ *then there is a sequence* $\vec{P}$ *of closed bags such that either* $\mathbb{M}\vec{P} \xrightarrow{\beta*} \mathbf{I}$ *and* $\mathbb{N}\vec{P} \xrightarrow{\beta*} 0$*, or* vice versa.

Proof. Let $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$, then $\mathbb{M} \not\preceq_\tau \mathbb{N}$ or vice versa (Lemma 4): say $\mathbb{M} \not\preceq_\tau \mathbb{N}$. Applying Lemma 8 we have $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ for all $\mathbb{M}' \overset{\eta*}{\leftarrow} \mathbb{M}$; in particular, by Lemma 6, $\mathbb{M}' \not\preceq_\tau \mathbb{N}$ holds for an $\mathbb{M}'$ λ-wider than both $\mathbb{M}$ and $\mathbb{N}$. By Lemma 5 there is $M' \in (\mathbb{M}')^\circ$ such that for all $N \in \mathbb{N}^\circ$ we have $M' \overset{\varphi*}{\not\to} N$; such a term M' is in β-nf since $\mathbb{M}'$ is in β-nf and is λ-wider than both $\mathbb{M}$ and $\mathbb{N}$ by Remark 2.

From Lemma 7, recalling that $M', \mathbb{M}, \mathbb{N}$ are closed, there is a sequence $\vec{P}$ of closed bags such that: (i) $\mathbb{N}\vec{P} \overset{\beta*}{\to} 0$, since for all $N \in \mathbb{N}^\circ$ we have $M' \overset{\varphi*}{\not\to} N$, and (ii) $\mathbb{M}\vec{P} \overset{\beta*}{\to} \mathbf{I}$, since $\mathbb{M}' \overset{\eta*}{\to} \mathbb{M}$ and hence by Remark 1 there is an $M \in \mathbb{M}^\circ$ such that $M' \overset{\varphi*}{\to} M$. This concludes the proof of our main result. ◻

Corollary 1. *Let* $\sim$ *be a congruence on* $\mathbf{2}\langle \Lambda^e \rangle$ *extending* β*-equivalence. If there are two closed* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle$ *in* β*-nf such that* $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ *but* $\mathbb{M} \sim \mathbb{N}$*, then* $\sim$ *is trivial, i.e. for all sums* $\mathbb{L} \in \mathbf{2}\langle \Lambda^r \rangle$*,* $\mathbb{L} \sim 0$*.*

Proof. Suppose $\mathbb{M} \not\equiv_{\eta\tau} \mathbb{N}$ but $\mathbb{M} \sim \mathbb{N}$. From Theorem 2 there is $\vec{P}$ such that $\mathbb{M}\vec{P} \xrightarrow{\beta*} \mathbf{I}$ and $\mathbb{N}\vec{P} \xrightarrow{\beta*} 0$, or vice versa. By the congruence of $\sim$, we have $\mathbb{M}\vec{P} \sim \mathbb{N}\vec{P}$. By the hypothesis that $\sim$ extends β-equivalence, we get $\mathbf{I} \sim 0$. Now, take any term $\mathbb{L} \in \mathbf{2}\langle \Lambda^r \rangle$, we have $\mathbb{L} \equiv_\beta \mathbf{I}\mathbb{L} \sim 0\mathbb{L} = 0$. ◻

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A Technical Appendix

This technical appendix is devoted to provide the full proof of Lemma 5. To prove this result we will need some preliminary definitions and technical lemmas.

Definition 9. *The* sliced size *of an expression $A \in \Lambda^e$ is the number* $\mathrm{size}^{\mathrm{sl}}(A)$ *defined by structural induction on A as follows:*

- $\mathrm{size}^{\mathrm{sl}}(x) := 1$,
- $\mathrm{size}^{\mathrm{sl}}(MP) := \mathrm{size}^{\mathrm{sl}}(M) + \mathrm{size}^{\mathrm{sl}}(P)$,
- $\mathrm{size}^{\mathrm{sl}}(\lambda x.M) := \mathrm{size}^{\mathrm{sl}}(M) + 1$,
- $\mathrm{size}^{\mathrm{sl}}([M_1, \dots, M_m, (\Sigma_{i=m+1}^{m+k} M_i)^!]) := \max_{i\in\{1,\dots,m+k\}}(\mathrm{size}^{\mathrm{sl}}(M_i)) + 1$.

Lemma 9. *For $\mathbb{A}, \mathbb{B} \in \mathbf{2}\langle\Lambda^e\rangle$ and $A, B \in \Lambda^e_{\mathrm{f}}$, we have:*

(i) $\forall A' \in \mathbb{A}^\circ$, $\mathrm{size}^{\mathrm{sl}}(A') \le \mathrm{size}(\mathbb{A})$*;*
(ii) *if $A \xrightarrow{\varphi} B$, then* $\mathrm{size}^{\mathrm{sl}}(A) \ge \mathrm{size}^{\mathrm{sl}}(B)$*;*
(iii) *if $\mathbb{A} \xrightarrow{\eta} \mathbb{B}$, then $\forall A \in \mathbb{A}^\circ, \exists B \in \mathbb{B}^\circ$ such that* $\mathrm{size}^{\mathrm{sl}}(A) \ge \mathrm{size}^{\mathrm{sl}}(B)$ *and* vice versa*, $\forall B \in \mathbb{B}^\circ, \exists A \in \mathbb{A}^\circ$ such that* $\mathrm{size}^{\mathrm{sl}}(A) \ge \mathrm{size}^{\mathrm{sl}}(B)$*.*

Proof. (i) By a straightforward induction on $\mathbb{A}$.

(ii) By a straightforward inspection of the rules defining $A \xrightarrow{\varphi} B$.

(iii) By an easy inspection of the rules defining $\mathbb{A} \xrightarrow{\eta} \mathbb{B}$, one gets $\forall A \in \mathbb{A}^\circ, \exists B \in \mathbb{B}^\circ$ such that $A \xrightarrow{\varphi} B$ (hence by (ii), $\mathrm{size}^{\mathrm{sl}}(A) \ge \mathrm{size}^{\mathrm{sl}}(B)$) and *vice versa* $\forall B \in \mathbb{B}^\circ, \exists A \in \mathbb{A}^\circ$ such that $A \xrightarrow{\varphi} B$, hence $\mathrm{size}^{\mathrm{sl}}(A) \ge \mathrm{size}^{\mathrm{sl}}(B)$. ☐

Lemma 10. *Let $A, B \in \Lambda^e_{\mathrm{f}}$ such that $A \xrightarrow{\varphi*} B$, then:*

(i) *if $B = x$ then either $A = x$ or $A \xrightarrow{\varphi*} \lambda y.x[y^k]$, for some $k \in \mathbf{N}$;*
(ii) *if $B = MP$ then there are M', P' such that $M' \xrightarrow{\varphi*} M$, $P' \xrightarrow{\varphi*} P$ and either $A = M'P'$ or $A \xrightarrow{\varphi*} \lambda y.M'P'[y^k]$ for some $y \notin \mathrm{FV}(M'P')$ and $k \in \mathbf{N}$;*
(iii) *if $B = \lambda x.M$, then there exists M' such that $M' \xrightarrow{\varphi*} M$ and either $A = \lambda x.M'$ or $A \xrightarrow{\varphi*} \lambda y.(\lambda x.M')[y^k]$ for some $y \notin \mathrm{FV}(\lambda x.M')$ and $k \in \mathbf{N}$;*
(iv) *if $B = [N_1, \dots, N_n]$ then $A = [M_1, \dots, M_n]$ with $M_i \xrightarrow{\varphi*} N_i$ for every $i \le n$.*

Proof. By induction on the length of the reduction chain $A \xrightarrow{\varphi*} B$. ☐

Lemma 5 is a direct consequence of the following result. To ease the formulation of its statement we define $\preceq_s$ on sets $\mathcal{A}, \mathcal{B} \subseteq \Lambda^e_{\mathrm{f}}$ by setting $\mathcal{A} \preceq_s \mathcal{B}$ iff $\forall A \in \mathcal{A}$, $\forall B \in \mathcal{B}$ we have $A \xrightarrow{\varphi*} B$.

Lemma 11. *Let $\mathcal{A} \subseteq \Lambda^e_{\mathrm{f}}$ be such that $\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$ is finite and let $\mathbb{B} \in \mathbf{2}\langle\Lambda^e\rangle$. Then $\mathcal{A} \preceq_s \mathbb{B}^\circ$ implies there is $\mathbb{B}' \xrightarrow{\eta*} \mathbb{B}$ such that $\mathcal{A} \subseteq (\mathbb{B}')^\circ$.*

Proof. The proof is performed by induction on the triplet

$$\left(\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big), \sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) - \inf_{B\in\mathbb{B}^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big), \mathrm{size}(\mathbb{B})\right),$$

lexicographically ordered. We split in several cases, depending on $\mathbb{B}$.

Case I ($\mathbb{B} = 0$). Then, by the hypothesis $\mathcal{A} \preceq_s \mathbb{B}^\circ$, we deduce $\mathcal{A} = \emptyset$. Clearly, $0 \xrightarrow{\eta *} 0$ and $\emptyset \subseteq 0^\circ$.

Case II ($\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2$, both non-empty). Then, for $i = 1, 2$, let $\mathcal{A}_i := \{A \in \mathcal{A} \mid \exists B \in \mathbb{B}_i^\circ \text{ s.t. } A \xrightarrow{\varphi *} B\}$. Notice that $\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) \geq \sup_{A\in\mathcal{A}_i}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$, $\inf_{B\in\mathbb{B}^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big) \leq \inf_{B\in\mathbb{B}_i^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big)$ and $\mathrm{size}(\mathbb{B}) > \mathrm{size}(\mathbb{B}_i)$. So, we can apply the induction hypothesis, getting $\mathbb{B}'_i \xrightarrow{\eta *} \mathbb{B}_i$ such that $\mathcal{A}_i \subseteq \mathbb{B}_i^\circ$. We conclude by setting $\mathbb{B}' := \mathbb{B}'_1 + \mathbb{B}'_2$.

Case III ($\mathbb{B} = x$). Let $\mathcal{A}_\eta := \{M \in \mathcal{A} \mid M \xrightarrow{\varphi *} \lambda y.x[y^k], \text{ for } k \in \mathbf{N}\}$ and notice that, by Lemma 10(i), $\mathcal{A} \subseteq \{x\} \cup \mathcal{A}_\eta$. We have that $\mathcal{A}_\eta \preceq_s (\lambda y.x[y^!])^\circ$, and $\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) \geq \sup_{A\in\mathcal{A}_\eta}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$, while $\inf_{B\in x^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big) = \mathrm{size}^{\mathrm{sl}}(x) < \inf_{B\in(\lambda y.x[y^!])^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big)$. So, we can apply the induction hypothesis, getting $\mathbb{M}_\eta \xrightarrow{\eta *} \lambda y.x[y^!]$ such that $\mathcal{A}_\eta \subseteq \mathbb{M}_\eta^\circ$. We conclude by defining $\mathbb{B}' := x + \mathbb{M}_\eta$. In fact, $\mathbb{B}' \xrightarrow{\eta *} x + \lambda y.x[y^!] \xrightarrow{\eta} x + x = x$, by sum idempotency.

Case IV ($\mathbb{B} = NP$). Then, by Lemma 10(ii), $\mathcal{A}$ is a set of terms M such that, for each of them, there exist $N' \xrightarrow{\varphi *} N''$, and $P' \xrightarrow{\varphi *} P''$, with $N'' \in N^\circ$, $P'' \in P^\circ$ and either $M = N'P'$ or $M \xrightarrow{\varphi *} \lambda y.N'P'[y^k] \xrightarrow{\varphi} N'P'$. Hence, let us decompose $\mathcal{A}$ into the set $\mathcal{A}_@$ of those M of the form $N'P'$ and the set $\mathcal{A}_\eta$ of those M reducing to $\lambda y.N'P'[y^k]$, and let us define $\mathcal{N}_@$, $\mathcal{P}_@$ (resp. $\mathcal{N}_\eta$, $\mathcal{P}_\eta$) as the set of the terms N' and the set of the bags P' associated with the M's in $\mathcal{A}_@$ (resp. in $\mathcal{A}_\eta$).

Let us consider $\mathcal{N}_\eta$. By definition, we have $\mathcal{N}_\eta \preceq_s N^\circ$. Moreover, notice that $\sup_{N'\in\mathcal{N}_\eta}\big(\mathrm{size}^{\mathrm{sl}}(N')\big) < \sup_{M\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(M)\big)$, in fact let N' be the term in $\mathcal{N}_\eta$ having maximum $\mathrm{size}^{\mathrm{sl}}$, and let P' be the bag such that there is $M \in \mathcal{A}$, $M \xrightarrow{\varphi *} \lambda y.N'P'[y^k]$: by Lemma 9(ii), $\mathrm{size}^{\mathrm{sl}}(M) \geq \mathrm{size}^{\mathrm{sl}}(\lambda y.N'P'[y^k]) > \mathrm{size}^{\mathrm{sl}}(N')$. Hence, we can apply the induction hypothesis to $\mathcal{N}_\eta \preceq_s N^\circ$, getting $\mathbb{N}_\eta \xrightarrow{\eta *} N$ such that $\mathcal{N}_\eta \subseteq \mathbb{N}_\eta^\circ$.

We can do a similar reasoning on $\mathcal{P}_\eta \preceq_s P^\circ$, getting $\mathbb{P}_\eta \xrightarrow{\eta *} P$ such that $\mathcal{P}_\eta \subseteq \mathbb{P}_\eta^\circ$. That means $\mathbb{N}_\eta\mathbb{P}_\eta \xrightarrow{\eta *} NP$ as well as $\mathcal{N}_\eta\mathcal{P}_\eta \subseteq (\mathbb{N}_\eta\mathbb{P}_\eta)^\circ$. Notice that we have, by construction, $\mathcal{A}_\eta \preceq_s \bigcup_{k=0}^{\infty} \lambda y.\mathcal{N}_\eta\mathcal{P}_\eta[y^k] \subseteq (\lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!])^\circ$, hence $\mathcal{A}_\eta \preceq_s (\lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!])^\circ$. Notice also that $\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) \geq \sup_{A\in\mathcal{A}_\eta}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$, while $\inf_{B\in(NP)^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big) \leq \inf_{B\in(\mathbb{N}_\eta\mathbb{P}_\eta)^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big)$, because of $\mathbb{N}_\eta\mathbb{P}_\eta \xrightarrow{\eta *} NP$ and Lemma 9(iii). Then, $\inf_{B\in(\mathbb{N}_\eta\mathbb{P}_\eta)^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big) < \inf_{B\in(\lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!])^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big)$.

We conclude:

$$\sup_{A\in\mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) - \inf_{B\in\mathbb{B}^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big) > \sup_{A\in\mathcal{A}_\eta}\big(\mathrm{size}^{\mathrm{sl}}(A)\big) - \inf_{B\in(\lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!])^\circ}\big(\mathrm{size}^{\mathrm{sl}}(B)\big)$$

and so we can apply the induction hypothesis to $\mathcal{A}_\eta \preceq_s (\lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!])^\circ$, getting $\mathbb{M}_\eta \xrightarrow{\eta *} \lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!]$ such that $\mathcal{A}_\eta \subseteq \mathbb{M}_\eta^\circ$. Notice that we have $\mathbb{M}_\eta \xrightarrow{\eta *} \lambda y.\mathbb{N}_\eta\mathbb{P}_\eta[y^!] \xrightarrow{\eta *} \mathbb{N}_\eta\mathbb{P}_\eta \xrightarrow{\eta *} NP$.

By an easier variant of the above reasoning one gets $\mathbb{M}_@ \xrightarrow{\eta *} NP$ such that $\mathcal{A}_@ \subseteq \mathbb{M}_@^\circ$. We conclude by defining $\mathbb{B}' := \mathbb{M}_\eta + \mathbb{M}_@$.

Case V ($\mathbb{B} = \lambda x.M$). It is an easier variant of the case $\mathbb{B}$ is an application, using Lemma 10(iii) and Lemma 9.

Case VI ($\mathbb{B} = [N_1, \dots, N_n, \mathbb{N}^!]$). We explicit the sum $\mathbb{N} = N_{n+1} + \cdots + N_{n+m}$. Then, by definition of $\mathbb{B}^\circ$ and Lemma 10(iv), we have that each element of $\mathcal{A}$ is a bag $[L_1, \dots, L_n, M_{1,1}, \dots, M_{1,k_1}, \dots, M_{m,1}, \dots, M_{m,k_m}]$, with $m \geq 0$, for every $i \leq m$, $k_i \geq 0$ and

1. for every $\ell \leq n$ there is $N' \in N_\ell^\circ$ such that $L_\ell \overset{\varphi *}{\to} N'$
2. for every $i \leq m, j \leq k_m$ there is $N'' \in N_{n+i}^\circ$ such that $M_{i,j} \overset{\varphi *}{\to} N''$

For every $\ell \leq n$, we define $\mathcal{L}_\ell$ as the set of the L_ℓ's resources in each bag of $\mathcal{A}$ associated with the linear resource N_ℓ in B. Note that $\mathcal{L}_\ell \preceq_s N_\ell^\circ$ and $\sup_{M \in \mathcal{L}_\ell}\big(\mathrm{size}^{\mathrm{sl}}(M)\big) < \sup_{A \in \mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$. By induction hypothesis we get $\mathbb{L}_\ell \overset{\eta *}{\to} N_\ell$ such that $\mathcal{L}_\ell \subseteq \mathbb{L}_\ell^\circ$.

Similarly, for all $i \leq m$, we define $\mathcal{N}_i$ as the union of the sets $\{M_{i,1}, \dots, M_{i,k_i}\}$ of the $M_{i,j}$'s resources in each bag of $\mathcal{A}$ associated with the exponential resource N_{n+i} of B. Notice that we have $\mathcal{N}_i \preceq_s N_{n+i}^\circ$ and that $\sup_{M \in \mathcal{N}_i}\big(\mathrm{size}^{\mathrm{sl}}(M)\big) < \sup_{A \in \mathcal{A}}\big(\mathrm{size}^{\mathrm{sl}}(A)\big)$, then by induction hypothesis we get $\mathbb{N}_i \overset{\eta *}{\to} N_{n+i}$ such that $\mathcal{N}_i \subseteq \mathbb{N}_i^\circ$.

We conclude by setting $\mathbb{B}' = [\mathbb{L}_1, \dots, \mathbb{L}_n, (\mathbb{N}_1 + \cdots + \mathbb{N}_m)^!]$. $\square$

Lemma 5 *Let* $\mathbb{M}, \mathbb{N} \in \mathbf{2}\langle \Lambda^r \rangle$*. Then* $\mathbb{M} \preceq_s \mathbb{N}$ *entails* $\mathbb{M} \preceq_\tau \mathbb{N}$*.*

Proof. Suppose $\mathbb{M} \preceq_s \mathbb{N}$. By Lemma 9(i) we have that $\mathbb{M}^\circ$ is a set of terms having $\sup_{M \in \mathbb{M}^\circ} \mathrm{size}^{\mathrm{sl}}(M) \leq \mathrm{size}(\mathbb{M})$. So Lemma 11 yields $\mathbb{N}' \overset{\eta *}{\to} \mathbb{N}$ such that $\mathbb{M}^\circ \subseteq (\mathbb{N}')^\circ$. This latter entails $\mathbb{M} \sqsubseteq_\tau \mathbb{N}'$ by Proposition 1. Hence $\mathbb{M} \sqsubseteq_\tau \mathbb{N}' \overset{\eta *}{\to} \mathbb{N}$ holds for some $\mathbb{N}'$, and this entails $\mathbb{M} \preceq_\tau \mathbb{N}$ by definition. $\square$