Lattices of Equational Theories as Church Algebras

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Abstract. We introduce the class of *Church algebras*, which is general enough to compass all Boolean algebras, Heyting algebras and rings with unit. Using a new equational characterization of central elements, we prove that Church algebras satisfy a Stone representation theorem. We show that every lattice of equational theories is isomorphic to the congruence lattice of a suitable Church algebra, and we use this property to prove a meta-Stone representation theorem which is applicable to all varieties of algebras.

We say Σ is *an equational theory* iff Σ is a set of identities closed under the rules of the equational calculus. The set $L(\Sigma) = \{T : \Sigma \subseteq T, T$ is an equational theory$\}$ forms a lattice under inclusion. L is a *lattice of equational theories* (*et-lattice*, for short) iff L is isomorphic to the lattice $L(\Sigma)$ for some equational theory Σ (or dually isomorphic to the lattice of all subvarieties of some variety of algebras). In 1966 A.I. Malcev [3] posed the following question: which lattices can be represented as et-lattices?

$L(\Sigma)$ is an algebraic and coatomic lattice, possessing a compact top element; but no stronger property was known before Lampe's discovery [2] that any et-lattice satisfies the Zipper condition: if $\vee\{a_i : i \in I\} = 1$ and $a \wedge c = z$ then $c = z$. The proof uses the following representation of the et-lattices, which is due to R. McKenzie [5]: if L is an et-lattice, then L is isomorphic to some congruence lattice of groupoids with right unit and right zero. The problem of whether any congruence lattice of groupoids with right unit and right zero is isomorphic to an et-lattice, is still open (see [2]). The representation of the et-lattices by congruence lattices of monoids with one additional unary operation was found by N. Newrly [6]. A.M. Nurakunov [7] has recently shown that a lattice L is an et-lattice iff L is isomorphic to the congruence lattice of some et-monoid.

In this paper we introduce the class of Church algebras, which are algebras modelling the "if-then-else" instruction of programming. We note that every et-lattice is isomorphic to the congruence lattice of a suitable Church algebra. Using central elements, which play here the role of idempotent elements in rings, we show that Church algebras satisfy a Stone representation theorem stating that every Church algebra can be decomposed as a weak Boolean product of directly indecomposable algebras. Finally, we prove that every variety of algebras can de decomposed as a weak Boolean product of indecomposable subvarieties. This can be seen as a meta-Stone representation theorem since it can be applied to *all* varieties and not just to a suitable class of algebras of a certain kind.

In [4] we have applied these results in the context of theoretical computer science to study the properties of the lattice of equational theories of λ-calculus.

Let $X = \{x_i : i \in \mathbb{N}\}$ be a countable set of variables and τ be an algebraic similarity type. We ambiguously denote the set of operation symbols of type τ by the same letter τ. TX is the set of all terms over X with operations symbols from τ and (TX, τ) is the term algebra. We denote by End the set of endomorphisms of the term algebra.

The lattice $L(\tau)$ of all equational theories of the given type τ can be described as $\mathrm{Con}(TX, \tau \cup \mathrm{End})$. Newrly [6] has shown that

$$\mathrm{Con}(TX, \tau \cup \mathrm{End}) = \mathrm{Con}(TX, +, 0, \phi),$$

where $(TX, +, 0)$ is a monoid and ϕ is unary. The operations are defined as follows: $0 \equiv x_0$, $s+t \equiv t[s/x_0]$ (where $t[s/x_0]$ is the term obtained by substituting s for the variable x_0 within t), $\phi(x_i) \equiv x_{i-1}$ and $\phi(x_0) \equiv x_0$.

We modify Newrly's algebra $(TX, +, 0, \phi)$ (without changing its congruence lattice). We consider the algebra $\mathbf{C}(\tau) = (TX, \mathrm{if}, \phi, 0, 1)$, where $0 \equiv x_0$, $1 \equiv x_1$, ϕ is defined as in Newrly's algebra and the ternary operation "if" is defined as follows: $\mathrm{if}(t, s, u) \equiv t[u/x_0, s/x_1]$, where $t[u/x_0, s/x_1]$ is the term obtained by substituting the term u for variable x_0 and the term s for variable x_1 within t. Newrly's algebra is a reduct of $\mathbf{C}(\tau)$ because $s + t = \mathrm{if}(t, 1, s)$. In conclusion, we have for the lattice $L(\tau)$ of all equational theories of type τ:

$$L(\tau) = \mathrm{Con}(\mathbf{C}(\tau)).$$

The algebra $\mathbf{C}(\tau)$ satisfies the identities

$$\mathrm{if}(1, s, u) = s; \quad \mathrm{if}(0, s, u) = u. \tag{1}$$

The above identities formalize the "if-then-else", which is a basic construct of programming languages; perhaps surprisingly, the variety of algebras axiomatized by these two identities has never been studied in the literature.

Our key observation is that many other algebraic structures, such as combinatory algebras, Boolean algebras, rings with unit etc., have a term operation satisfying identities (1). These identities imply strong algebraic properties, that we will apply to the study of et-lattices.

Definition 1. [4, Def. 4] *An algebra* $\mathbf{A}$ *is called* a Church algebra *if there are two constants* $0, 1 \in A$ *and a ternary term* $\mathrm{if}(e, x, y)$ *such that* $\mathrm{if}(1, x, y) = x$ *and* $\mathrm{if}(0, x, y) = y$. *A variety* $\mathcal{V}$ *is called a* Church variety *if every algebra in* $\mathcal{V}$ *is a Church algebra with respect to the same term* $\mathrm{if}(e, x, y)$ *and constants* $0, 1$.

Let τ be a similarity type. We call $\mathbf{C}(\tau)$ the *Church algebra of type* τ. Given an equational theory Σ of type τ we define the *Σ-Church algebra of type* τ as $\mathbf{C}_\Sigma(\tau) = \mathbf{C}(\tau)/\Sigma$.

The following, besides the algebras $\mathbf{C}(\tau)$ and $\mathbf{C}_\Sigma(\tau)$, are easily checked to be Church algebras:

1. Combinatory algebras: $\mathrm{if}(e, x, y) \equiv (e \cdot x) \cdot y$; $1 \equiv \mathbf{k}$; $0 \equiv \mathbf{sk}$, where $\mathbf{k}$ and $\mathbf{s}$ are the basic combinators of combinatory logic.

2. Boolean algebras: $\mathrm{if}(e, x, y) \equiv (e \vee y) \wedge (e^{-} \vee x)$.
3. Heyting algebras: $\mathrm{if}(e, x, y) \equiv (e \vee y) \wedge ((e \to 0) \vee x)$.
4. Rings with unit: $\mathrm{if}(e, x, y) \equiv (y + e - ey)(1 - e + ex)$.

Every idempotent element e of a commutative ring with unit induces a pair of complementary factor congruences (cfc-pair, for short) $(\theta(1, e), \theta(e, 0))$, where $\theta(1, e)$ is the least congruence including the pair $(1, e)$ and similarly for $\theta(e, 0)$. In other words, a ring $\mathbf{A}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\theta(1, e) \times \mathbf{A}/\theta(e, 0)$. $\mathbf{A}$ is directly indecomposable iff 0 and 1 are the unique idempotent elements. Vaggione [9] generalized the notion of idempotent to any universal algebra whose top congruence ∇ is compact, and called them *central elements*. Central elements were used to investigate the closure of varieties of algebras under Boolean products. Here we give a new equational characterization. Hereafter, we set $\theta_e \equiv \theta(1, e)$ and $\overline{\theta}_e \equiv \theta(e, 0)$.

Definition 2. *An element e of a Church algebra $\mathbf{A}$ is* central*, written $e \in \mathrm{Ce}(\mathbf{A})$, if $(\theta_e, \overline{\theta}_e)$ is a cfc-pair. A central element e is* non-trivial *if $e \neq 0, 1$.*

We now show that, in a Church algebra, factor congruences can be internally represented by central elements.

Proposition 1. *Let $\mathbf{A}$ be a Church algebra of type τ and $e \in A$. Then the following conditions are equivalent:*

(*i*) *e is central;*
(*ii*) *$\theta_e \cap \overline{\theta}_e = \Delta$;*
(*iii*) *For all x and y, $\mathrm{if}(e, x, y)$ is the unique element such that $x \theta_e \,\mathrm{if}(e, x, y)\, \overline{\theta}_e y$;*
(*iv*) *e satisfies the following identities:*
1. $\mathrm{if}(e, x, x) = x$.
2. $\mathrm{if}(e, \mathrm{if}(e, x, y), z) = \mathrm{if}(e, x, z) = \mathrm{if}(e, x, \mathrm{if}(e, y, z))$.
3. $\mathrm{if}(e, f(\overline{x}), f(\overline{y})) = f(\mathrm{if}(e, x_1, y_1), \dots, \mathrm{if}(e, x_n, y_n))$, *for every $f \in \tau$.*
4. $e = \mathrm{if}(e, 1, 0)$.

(*v*) *e is the unique element such that $0 \theta e \overline{\theta} 1$ for a suitable cfc-pair $(\theta, \overline{\theta})$.*
(*vi*) *The function f_e defined by $f_e(x, y) = \mathrm{if}(e, x, y)$ is a decomposition operator such that $f_e(1, 0) = e$.*

In the following proposition we characterize the central elements of $\mathbf{C}_\Sigma(\tau)$.

Proposition 2. *Let Σ be an equational theory and $\mathcal{V}$ be the variety of τ-algebras axiomatized by Σ. Then the following conditions are equivalent, for every $e \in \mathbf{C}_\Sigma(\tau)$ and term $t(x_1, x_0) \in e$:*

(*i*) *e is a central element.*
(*ii*) *Σ contains the identities $t(x, x) = x$; $\quad t(x, t(y, z)) = t(x, z) = t(t(x, y), z)$ and $t(f(\overline{x}), f(\overline{y})) = f(t(x_1, y_1), \dots, t(x_n, y_n))$, for $f \in \tau$.*
(*iii*) *For every $\mathbf{A} \in \mathcal{V}$, the function $t^{\mathbf{A}} : A \times A \to A$ is a decomposition operator.*
(*iv*) *$\Sigma = \Sigma_1 \cap \Sigma_0$, where Σ_i is the theory axiomatized (over Σ) by $t(x_1, x_0) = x_i$ $(i = 0, 1)$.*

Definition 3. *Let $\mathcal{V}$ be a variety and $\mathcal{V}_0, \mathcal{V}_1$ be two subvarieties of $\mathcal{V}$. $\mathcal{V}$ is* decomposable as a product of *$\mathcal{V}_0, \mathcal{V}_1$ if for every algebra $\mathbf{A} \in \mathcal{V}$ there are two non-trivial algebras $\mathbf{A}_i \in \mathcal{V}_i$ $(i = 0, 1)$ such that $\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1$. $\mathcal{V}$ is* indecomposable *if it is not decomposable as a product of any of its subvarieties.*

Notice that, if e and t satisfy the conditions of Prop. 2 and e is nontrivial as central element, then by Prop. 2(iii)-(iv) every algebra $\mathbf{A} \in \mathcal{V}$ can be decomposed as $\mathbf{A} \cong \mathbf{A}/\phi \times \mathbf{A}/\overline{\phi}$, where $(\phi, \overline{\phi})$ is the cfc-pair associated with the decomposition operator $t^{\mathbf{A}}$; moreover, the algebras $\mathbf{A}/\phi$ and $\mathbf{A}/\overline{\phi}$ satisfy respectively the equational theories Σ_1 and Σ_0. Thus, the variety $\mathcal{V}$ is decomposable as the product of the two subvarieties axiomatized respectively by Σ_1 and Σ_0.

The partial ordering on $\mathrm{Ce}(\mathbf{A})$ defined by $e \leq d$ iff $\overline{\theta}_e \subseteq \overline{\theta}_d$ is a Boolean ordering. 0 and 1 are respectively the bottom and top element of this ordering.

Theorem 1. *Let* $\mathbf{A}$ *be a Church algebra. The algebra* $(\mathrm{Ce}(\mathbf{A}), \wedge, \vee, ^-, 0, 1)$*, with operations defined by* $e \wedge d = \mathrm{if}(e, d, 0)$, $e \vee d = \mathrm{if}(e, 1, d)$, $e^- = \mathrm{if}(e, 0, 1)$, *is a* BA*, which is isomorphic to the* BA *of factor congruences of* $\mathbf{A}$.

Next we turn to the Stone representation theorem for Church algebras. It is a corollary of Thm. 1 and of theorems by Comer [1] and by Vaggione [9].

Let $\mathbf{A}$ be a Church algebra. If I is an ideal of the Boolean algebra $\mathrm{Ce}(\mathbf{A})$, then $\phi_I = \cup_{e \in I} \overline{\theta}_e$ is a congruence. In the next theorem $\mathcal{S}$ is the Boolean space of maximal ideals of $\mathrm{Ce}(\mathbf{A})$.

Theorem 2. (The Stone Representation Theorem) *Let* $\mathbf{A}$ *be a Church algebra. Then, for all* $I \in \mathcal{S}$ *the quotient algebra* $\mathbf{A}/\phi_I$ *is directly indecomposable and the map* $f : A \to \Pi_{I \in \mathcal{S}}(A/\phi_I)$*, defined by* $f(x) = ([x]_{\phi_I} : I \in \mathcal{S})$*, gives a weak Boolean product representation of* $\mathbf{A}$.

Note that, in general, Thm. 2 does not give a (non-weak) Boolean product representation (see [4] for more details).

The set of all factor congruences of an algebra $\mathbf{A}$ does not constitute in general a sublattice of the congruence lattice of $\mathbf{A}$. We now show that in every algebra there is a subset of factor congruences which always constitutes a Boolean sublattice of the congruence lattice.

We denote by $t_e^{\mathbf{A}}$ the decomposition operator associated with the central element e by Prop. 2(iii).

Lemma 1. *Let* Σ *be an equational theory and* $\mathcal{V}$ *be the variety of* τ*-algebras axiomatized by* Σ*. For every algebra* $\mathbf{A} \in \mathcal{V}$*, the function* $h : \mathrm{Ce}(\mathbf{C}_\Sigma(\tau)) \to \mathrm{Con}(\mathbf{A})$*, defined by* $h(e) = \{(x, y) : t_e^{\mathbf{A}}(x, y) = x\}$*, is a lattice homomorphism from the* BA *of central elements of* $\mathbf{C}_\Sigma(\tau)$ *into the set of factor congruences of* $\mathbf{A}$ *such that* $(h(e), h(e^-))$ *is a cfc-pair for all* $e \in \mathrm{Ce}(\mathbf{C}_\Sigma(\tau))$*. The range of* h *constitutes a Boolean sublattice of* $\mathrm{Con}(\mathbf{A})$.

Proof. (Outline) We only show that h is a homomorphism with respect to the join operator. Recall from Thm. 1 that $e \vee d = q(e, 1, d)$ and that in $\mathbf{C}_\Sigma(\tau)$ the term q is the substitution operator. Then we obtain $t_{e \vee d}(x, y) = t_e(x, t_d(x, y)) = t_d(x, t_e(x, y))$. We have $(x, y) \in h(e \vee d) \Leftrightarrow t_{e \vee d}(x, y) = x \Leftrightarrow t_e(x, t_d(x, y)) = x \Leftrightarrow (x, t_d(x, y)) \in h(e) \Leftrightarrow (x, y) \in h(d) \circ h(e)$, because $(t_d(x, y), y) \in h(d)$ holds from property $t_d(t_d(x, y), y) = t_d(x, y)$ of decomposition operators. We get $h(e \vee d) \subseteq h(e) \vee h(d)$. For the opposite it is sufficient to check $h(e), h(d) \subseteq h(e \vee d)$. Let $(x, y) \in h(e)$, i.e., $t_e(x, y) = x$. Then $t_d(x, t_e(x, y)) = x$, so that $t_{e \vee d}(x, y) = x$. A similar reasoning works for $h(d)$.

We say that a variety $\mathcal{V}$ is *decomposable as a weak Boolean product of directly indecomposable subvarieties* if there exists a family $\langle \mathcal{V}_i : i \in X \rangle$ of indecomposable subvarieties $\mathcal{V}_i$ of $\mathcal{V}$ such that every algebra $\mathbf{A} \in \mathcal{V}$ is isomorphic to a weak Boolean product $\Pi_{i\in X}\mathbf{B}_i$ of algebras $\mathbf{B}_i \in \mathcal{V}_i$.

Theorem 3. (Meta-Representation Theorem) *Every variety $\mathcal{V}$ of τ-algebras is decomposable as a weak Boolean product of directly indecomposable subvarieties.*

Proof. (Outline) Let Σ be the equational theory of $\mathcal{V}$. By Thm. 2 we can represent $\mathbf{C}_\Sigma(\tau)$ as a weak Boolean product $f : \mathbf{C}_\Sigma(\tau) \to \Pi_{I\in X}(\mathbf{C}_\Sigma(\tau)/\phi_I)$, where $\mathcal{X}$ is the Stone space of the Boolean algebra $\mathrm{Ce}(\mathbf{C}_\Sigma(\tau))$ of central elements of $\mathbf{C}_\Sigma(\tau)$, $I \in X$ ranges over the maximal ideals of $\mathrm{Ce}(\mathbf{C}_\Sigma(\tau))$, $\phi_I = \cup_{e\in I}\overline{\theta}_e$, and $\overline{\theta}_e$ is the factor congruence associated with the central element $e \in I$. Since the lattice $L(\Sigma)$ of the equational theories extending Σ is isomorphic to the congruence lattice of $\mathbf{C}_\Sigma(\tau)$, the congruence ϕ_I corresponds to an equational theory, say Σ_I. The Σ_I-Church algebra of type τ is isomorphic to $\mathbf{C}_\Sigma(\tau)/\phi_I$, so that it is directly indecomposable. Then by Prop. 2 the variety $\mathcal{V}_I$ axiomatized by Σ_I is directly indecomposable.

Let $\mathbf{A} \in \mathcal{V}$ and $h : \mathrm{Ce}(\mathbf{C}_\Sigma(\tau)) \to \mathrm{Con}(\mathbf{A})$ be the lattice homomorphism defined in Lemma 1. For every maximal ideal I of $\mathrm{Ce}(\mathbf{C}_\Sigma(\tau))$, consider the congruence $\phi_I^{\mathbf{A}} = \cup_{e\in I}h(e)$. The map $f : \mathbf{A} \to \Pi_{I\in X}(\mathbf{A}/\phi_I^{\mathbf{A}})$ defined by $f(x) = ([x]_{\phi_I^{\mathbf{A}}} : I \in X)$, determines a weak Boolean representation of $\mathbf{A}$, where $\mathbf{A}/\phi_I^{\mathbf{A}} \in \mathcal{V}_I$. The algebra $\mathbf{A}/\phi_I^{\mathbf{A}}$ may be directly decomposable also if it belongs to the directly indecomposable variety $\mathcal{V}_I$.

We remark that, if an algebra $\mathbf{A} \in \mathcal{V}$ has Boolean factor congruences, then $\mathbf{A}$ can be represented as a weak Boolean product in two different way, by using either the Comer representation theorem [1] or the meta-representation theorem. The meta representation is in general weaker than Comer's representation.

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