An algebraic characterization of relations accepted by two-way unary transducers

Christian Choffrut[1], Bruno Guillon[1,2], Giovanni Pighizzini[2]

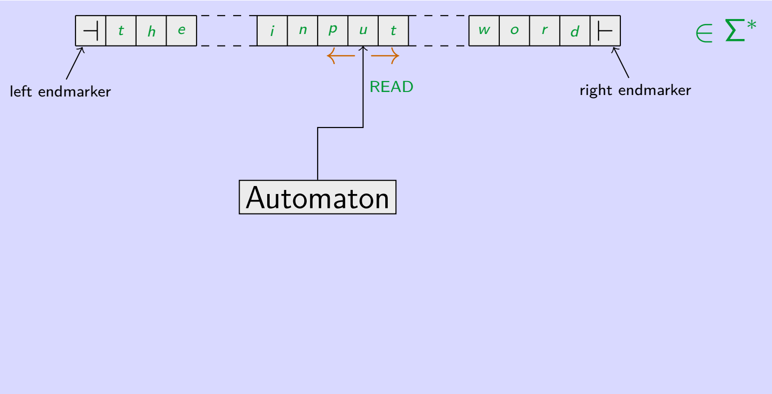
[1]LIAFA, Université Paris Diderot, Paris 7
[2]Dipartimento di Informatica, Università degli Studi di Milano

November 28, 2013

Two-way finite transducers

A

$(Q, \Sigma, I, F, \delta)$

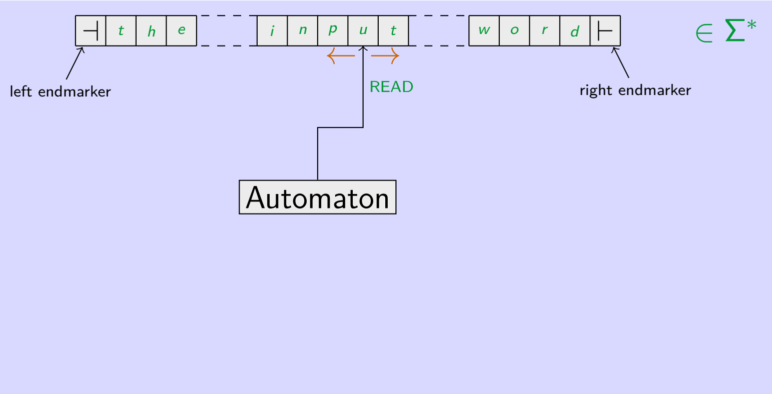


Two-way finite transducers

A

$(Q, \Sigma, I, F, \delta)$

transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$

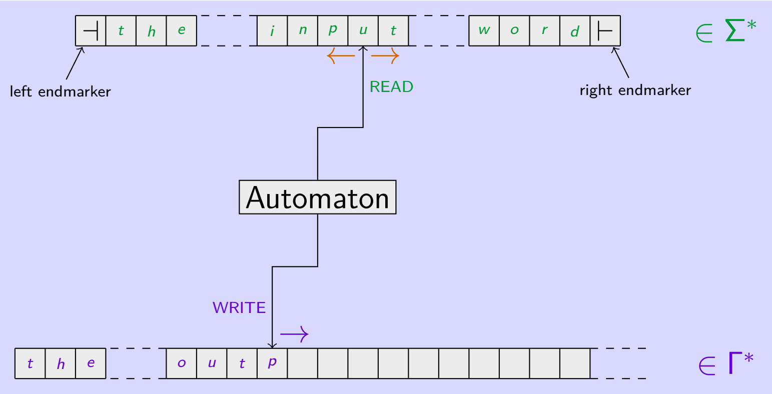


Two-way finite transducers

(A, ϕ)

$(Q, \Sigma, I, F, \delta)$

transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$

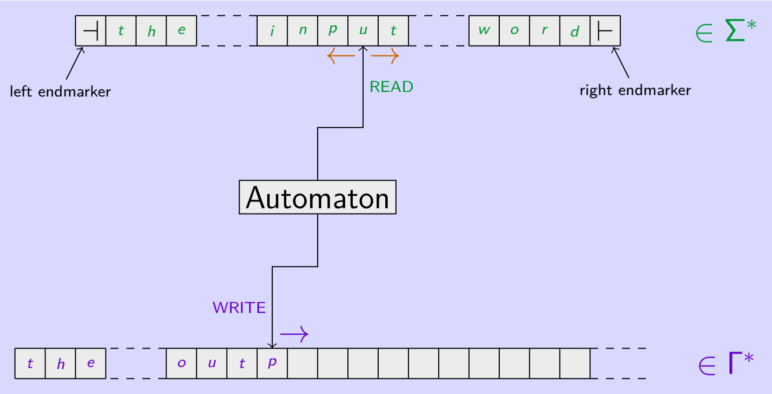


Two-way finite transducers

$$(A, \phi)$$

$(Q, \Sigma, I, F, \delta)$ ⟵ A; ϕ ⟶ production function: $\delta \to \Gamma \cup \{\epsilon\}$

δ ⟶ transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$



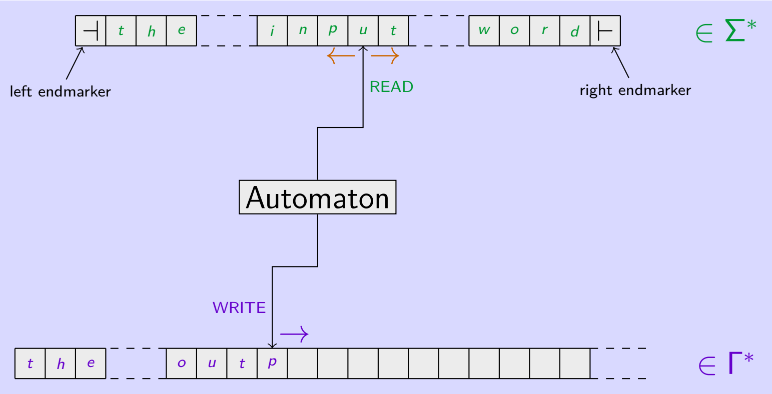
Two-way finite transducers

$$(A, \phi)$$

$(Q, \Sigma, I, F, \delta)$ ⟵ A

ϕ ⟶ production function: $\delta \to \Gamma^*$

δ ⟶ transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$

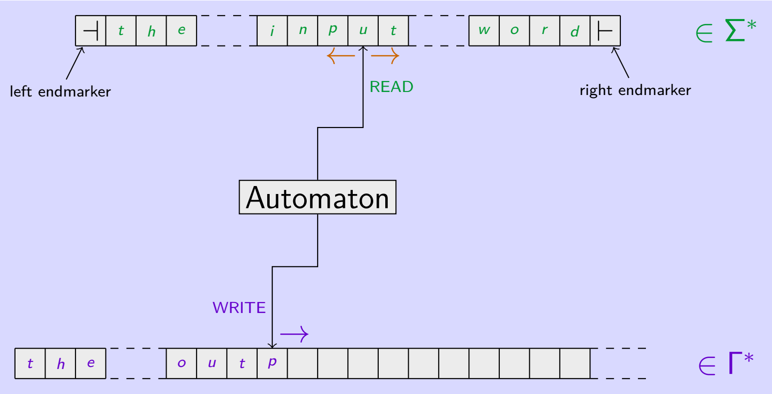


Two-way finite transducers

(A, ϕ)

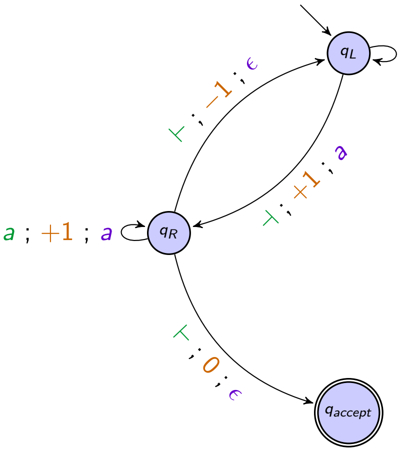
$(Q, \Sigma, I, F, \delta)$ ⟵ A; ϕ ⟶ production function: $\delta \to Rat(\Gamma^*)$

transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$



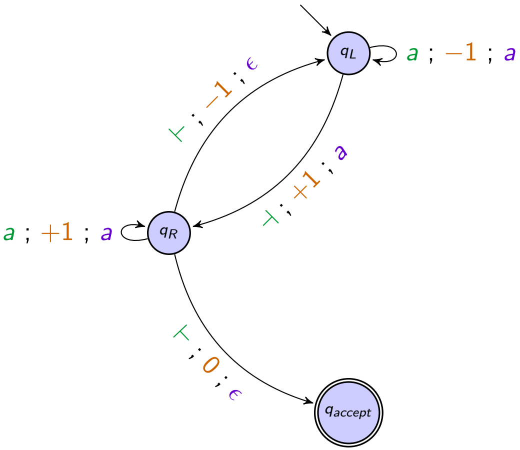
Example

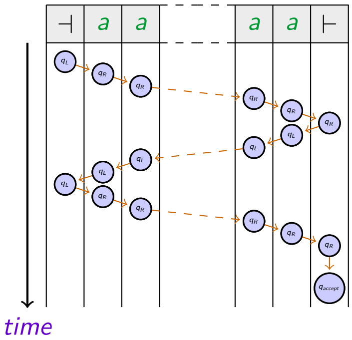
$$\Sigma = \Gamma = \{a\}$$



Example

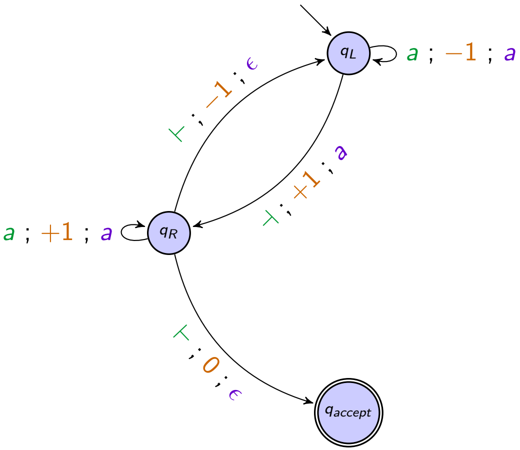
$$\Sigma = \Gamma = \{a\}$$

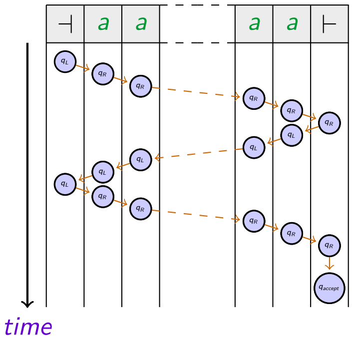




Example

$$\Sigma = \Gamma = \{a\}$$





Non-rational accepted relation: $\mathcal{R} = \left\{ \left(a^n, a^{(2k+1)n}\right),\ n, k \in \mathbb{N} \right\}$.

Formal series

Two-way transducers define binary relations (subsets of $\Sigma^* \times \Gamma^*$).

Formal series

Two-way transducers define binary relations (subsets of $\Sigma^* \times \Gamma^*$).

Given such a relation $\mathcal{R}$, we represent it as a formal serie:

$$\tau = \sum_{w \in \Sigma^*} \alpha_w \cdot w \qquad \tau(w) = \alpha_w = \{v \in \Gamma^* \mid (w, v) \in \mathcal{R}\}$$

Formal series

Two-way transducers define binary relations (subsets of $\Sigma^* \times \Gamma^*$).

Given such a relation $\mathcal{R}$, we represent it as a formal serie:

$$\tau = \sum_{w \in \Sigma^*} \alpha_w \cdot w \qquad \tau(w) = \alpha_w = \{v \in \Gamma^* \mid (w, v) \in \mathcal{R}\}$$

$$\mathcal{R} = \left\{(a^n, a^{(2k+1)n}),\ n, k \in \mathbb{N}\right\} \ \rightarrow \ \tau_{\mathcal{R}}(a^n) = \left\langle a^n \left(a^{2n}\right)^* \right\rangle$$

Rational series

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$$\boxed{2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle}$$

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle$

- contains polynomial,

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle$

- contains polynomial,
- closed under sum,

Sum

$$(\sigma + \tau)(w) = \sigma(w) + \tau(w)$$

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle$

- contains polynomial,
- closed under sum,
- Cauchy product

Cauchy Product

$$(\sigma \times \tau)(w) = \sum_{w = w_1 \cdot w_2} \sigma(w_1) \cdot \tau(w_2)$$

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle$

- contains polynomial,
- closed under sum,
- Cauchy product
- and Kleene star.

Kleene Star

$$(\sigma^*)(w) = \sum_{w = w_1 \cdot w_2 \cdots w_r} \sigma(w_1)\sigma(w_2)\ldots\sigma(w_r)$$

Rational series

$$\sum_{w \in \Sigma^*} \alpha(w) \cdot w$$

Rational series of $\mathbb{K}\langle\langle M \rangle\rangle$:

$2^{\Gamma^*}\langle\langle \Sigma^* \rangle\rangle$

- contains polynomial,
- closed under sum,
- Cauchy product
- and Kleene star.

Theorem

One-way transducers accept exactly $RAT\,(\Gamma^*)\,\langle\langle \Sigma^* \rangle\rangle$.

Two-way Transducers: known results

Theorem (Engelfriet, Hoogeboom, 2001)

- ***deterministic case****: two-way transducers accept exactly the class of MSO-definable functions.*

Two-way Transducers: known results

Theorem (Engelfriet, Hoogeboom, 2001)

- ***deterministic case****: two-way transducers accept exactly the class of MSO-definable functions.*

$$\boxed{\mathcal{T} = \{(w, w \cdot w) \mid w \in \Sigma^*\}}$$

Two-way Transducers: known results

Theorem (Engelfriet, Hoogeboom, 2001)

- ***deterministic case****: two-way transducers accept exactly the class of MSO-definable functions.*
- ***nondeterministic case****: the class of MSO-definable transductions and the class of relations accepted by two-way transducers are incomparable.*

Two-way transducers: known results

Theorem (Filiot, Gauwin, Reynier, Servais, 2013)

*It is **decidable** whether some function accepted by two-way transducer is accepted by some one-way transducer.*

→ *construction of equivalent one-way transducer, whenever one exists.*

Unary case - our result

$$\Sigma = \Gamma = \{a\}$$

Unary case - our result

$$\Sigma = \Gamma = \{a\}$$

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

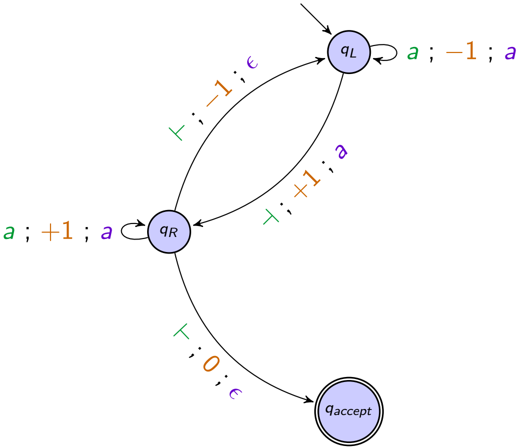
if and only if

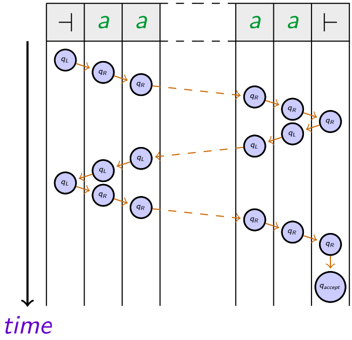
there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \; = \; \bigcup_i \; (\alpha_i(a^n) \; \cdot \; \beta_i(a^n)^*)$$

Example

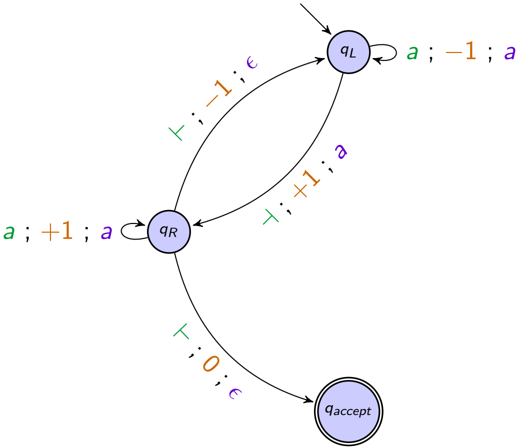
$$\mathcal{R} = \left\{ \left(a^n, a^{(2k+1)n}\right),\ n \in \mathbb{N} \right\}$$

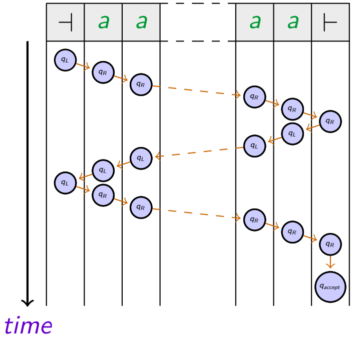




$$\mathcal{R} = \left\{ \left(a^n, a^{(2k+1)n} \right),\ n \in \mathbb{N} \right\}$$

Example

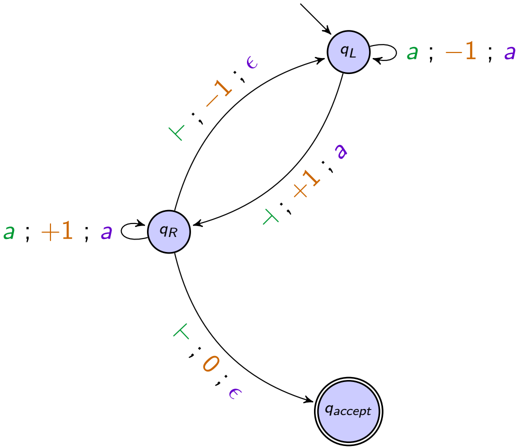


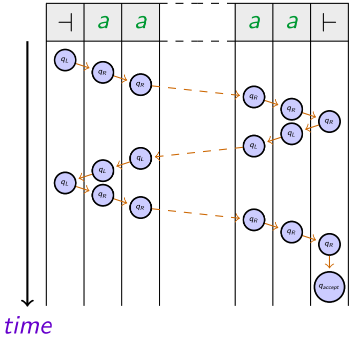


$$\tau_{\mathcal{R}}(a^n) = \langle a^n \cdot (a^{2n})^* \rangle$$

Example

$$\mathcal{R} = \left\{ \left(a^n, a^{(2k+1)n} \right), \; n \in \mathbb{N} \right\}$$



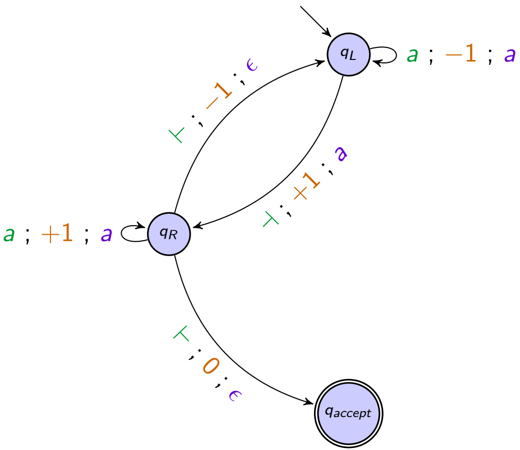


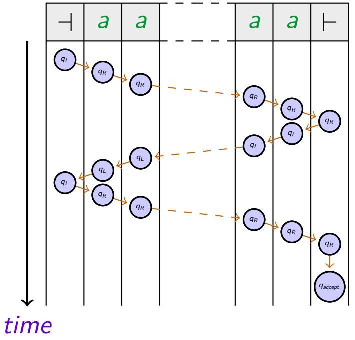
α β

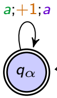
$$\tau_{\mathcal{R}}(a^n) = \langle a^n \cdot (a^{2n})^* \rangle$$

Example

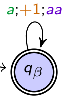
$$\mathcal{R} = \left\{ \left(a^n, a^{(2k+1)n} \right),\ n \in \mathbb{N} \right\}$$







α β



$$\tau_{\mathcal{R}}(a^n) \;=\; \langle a^n \;\cdot\; (a^{2n})^* \rangle$$

Unary case - our result

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \;\; (\alpha_i(a^n) \;\cdot\; \beta_i(a^n)^*)$$

Unary case - our result

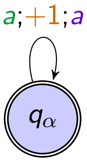
Theorem

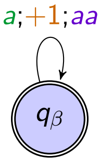
$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \; = \; \bigcup_i \; (\alpha_i(a^n) \; \cdot \; \beta_i(a^n)^*)$$





Unary case - our result

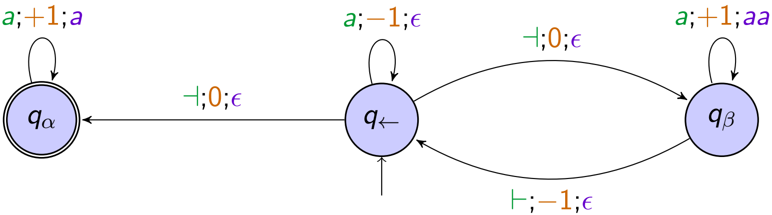
Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \quad (\alpha_i(a^n) \;\cdot\; \beta_i(a^n)^*)$$



Unary case - our result

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \; (\alpha_i(a^n) \;\cdot\; \beta_i(a^n)^*)$$

Unary case - our result

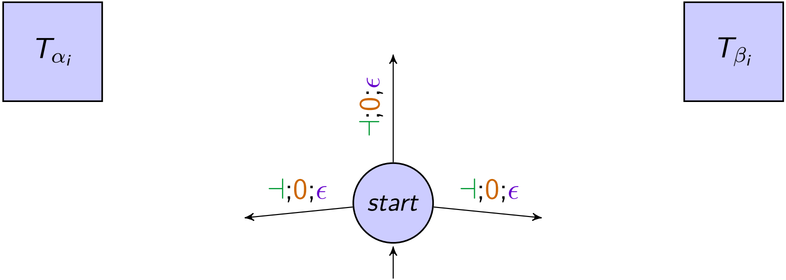
Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \; = \; \bigcup_i \; (\alpha_i(a^n) \; \cdot \; \beta_i(a^n)^*)$$



Unary case - our result

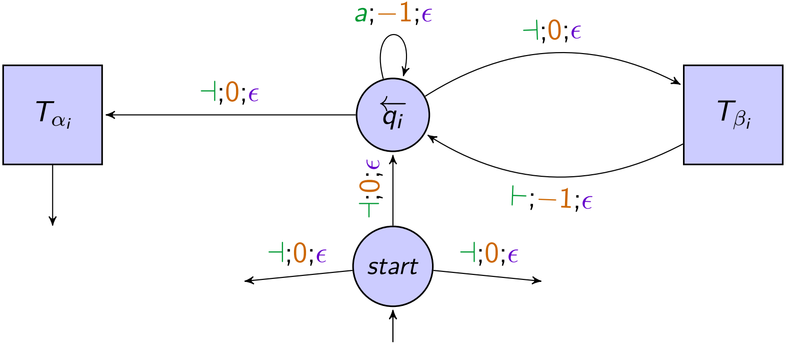
Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \;\; (\alpha_i(a^n) \;\cdot\; \beta_i(a^n)^*)$$



Unary case - our result

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \;\; (\alpha_i(a^n) \;\cdot\; \beta_i(a^n)^*)$$

Unary case - our result

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\forall n \quad \tau(a^n) \;=\; \bigcup_i \;\; (\alpha_i(a^n) \;\; \cdot \;\; \beta_i(a^n)^*)$$

Definition

- Hadamard Product: $\forall n \ (\alpha \textcircled{H} \beta)(w) = \alpha(w) \cdot \beta(w)$

Unary case - our result

Theorem

$$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}} \textit{ is accepted by a two-way transducer}$$

if and only if

there exists finitely many rational series α_i and β_i such that

Definition

- Hadamard Product: $\forall n \ (\alpha \textcircled{\scriptsize H} \beta)(w) = \alpha(w) \cdot \beta(w)$
- Hadamard Star: $\forall n \ (\alpha^{H\star})(w) = (\alpha(w))^*$

Unary case - our result

Theorem

$\tau : \Sigma^{\mathbb{N}} \to 2^{\Gamma^{\mathbb{N}}}$ is accepted by a two-way transducer

if and only if

there exists finitely many rational series α_i and β_i such that

$$\tau \;=\; \sum_i \;(\alpha_i \;\circledcirc\; \beta_i^{H\star})$$

Definition

- Hadamard Product: $\forall n \; (\alpha \circledcirc \beta)(w) = \alpha(w) \cdot \beta(w)$
- Hadamard Star: $\forall n \; (\alpha^{H\star})(w) = (\alpha(w))^*$

Analogy with Probabilistic Automata

Theorem (Anselmo,Bertoni,1994)

Acceptation probability of two-way finite automata is of the form:

$$\tau(w) = \alpha(w) \times \frac{1}{\beta(w)}$$

where α and β are rational series of $\mathbb{Q}\langle\langle\Sigma^*\rangle\rangle$.

HRAT relations

Definition

A relation is *HRAT* if and only if its serie is equal to

$$\sum_i \alpha_i \textcircled{H} \beta_i^{H\star}$$

for some finite family of rational series α_i and β_i.

Properties

- *RAT is include in HRAT.*
- *HRAT is closed under sum.*

HRAT relations

Definition

A relation is *HRAT* if and only if its serie is equal to

$$\sum_i \alpha_i \textcircled{H} \beta_i^{H\star}$$

for some finite family of rational series α_i and β_i.

Properties

- *RAT is include in HRAT.*
- *HRAT is closed under sum.*
- *In unary case, RAT is closed under H-product.*

HRAT relations

Definition

A relation is *HRAT* if and only if its serie is equal to

$$\sum_i \alpha_i \circledH \beta_i^{H\star}$$

for some finite family of rational series α_i and β_i.

Properties

- *RAT is include in HRAT.*
- *HRAT is closed under sum.*
- *In unary case, RAT is closed under H-product.*
- *In unary case, HRAT is closed under H-product and H-star.*

Sketch of the proof

Sketch of the proof

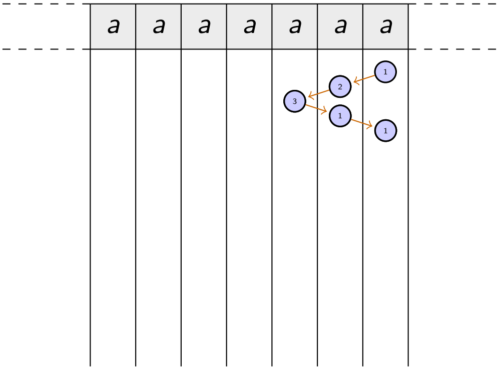
- decompose computation into traversals

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops

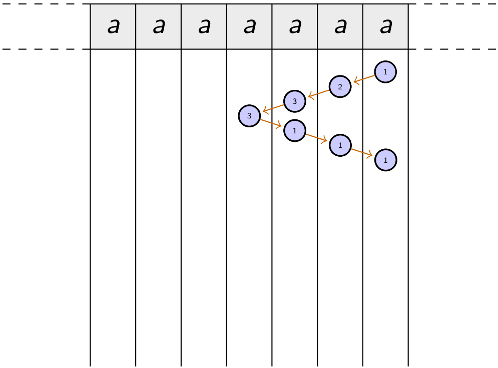
Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



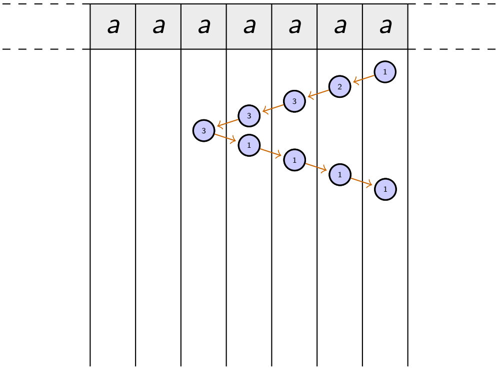
Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



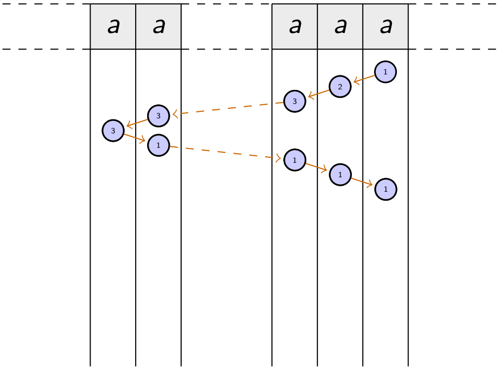
Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



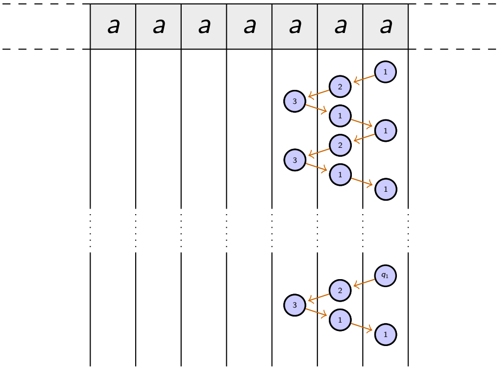
Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



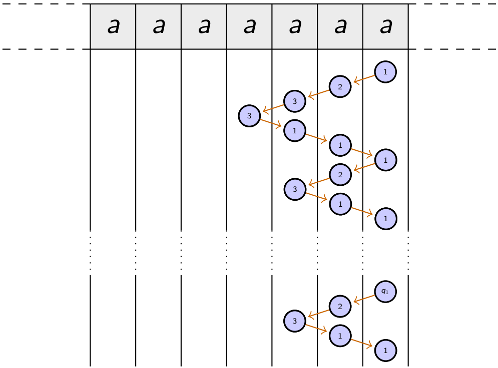
Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops



Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal
  - one traversal $TRAV_{(q_1,b_1),(q_2,b_2)}$: rational relation

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal
  - one traversal $TRAV_{(q_1,b_1),(q_2,b_2)}$: rational relation
- composition of traversals

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal
  - one traversal $TRAV_{(q_1,b_1),(q_2,b_2)}$: rational relation
- composition of traversals

Matrix $TRAV$ of size $(|Q| \times 2)^2$: $TRAV_{(q_1,b_1),(q_2,b_2)}$.

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal
  - one traversal $TRAV_{(q_1,b_1),(q_2,b_2)}$: rational relation
- composition of traversals

Matrix $TRAV$ of size $(|Q| \times 2)^2$: $TRAV_{(q_1,b_1),(q_2,b_2)}$.
$TRAV^\star$ is the composition of traversals.

Sketch of the proof

- decompose computation into traversals
- elimination of central nondeterministic loops
- one-way simulation of each traversal
  - one traversal $TRAV_{(q_1,b_1),(q_2,b_2)}$: rational relation
- composition of traversals

Matrix $TRAV$ of size $(|Q| \times 2)^2$: $TRAV_{(q_1,b_1),(q_2,b_2)}$.

$TRAV^\star$ is the composition of traversals.

$TRAV^\star \in HRAT_{2|Q| \times 2|Q|}$.

Conclusion

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i \circledH \beta_i^{H\star}$$

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i Ⓗ \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i \circledH \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way
- two-way $\rightarrow$ sweeping

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i \textcircled{H} \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way
- two-way $\rightarrow$ sweeping

- non-unary transducers?

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i \circledH \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way
- two-way $\rightarrow$ sweeping

- non-unary transducers?
- Is *HRAT* closed under Cauchy-product (unary case)?

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i \textcircled{H} \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way
- two-way $\rightarrow$ sweeping

- non-unary transducers?
- Is *HRAT* closed under Cauchy-product (unary case)?
- application to communicating automata systems?

Conclusion

- formal series accepted by two-way nondeterministic unary transducers are not rational
- characterization:

$$\tau = \sum_i \alpha_i Ⓗ \beta_i^{H\star}$$

- Deterministic case: two-way $\Leftrightarrow$ one-way
- two-way $\rightarrow$ sweeping

- non-unary transducers?
- Is *HRAT* closed under Cauchy-product (unary case)?
- application to communicating automata systems?

Do you have any questions?