An algebraic characterization of unary two-way transducers

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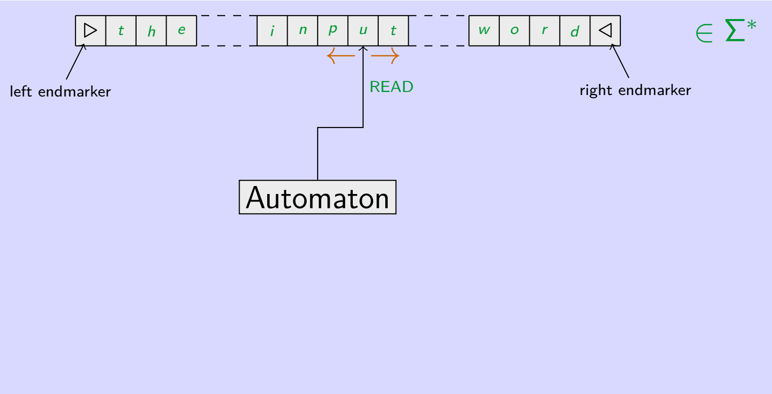
July 2, 2014

Two-way automaton over Σ

$$A \rightarrow (Q, \Sigma, I, F, \delta)$$

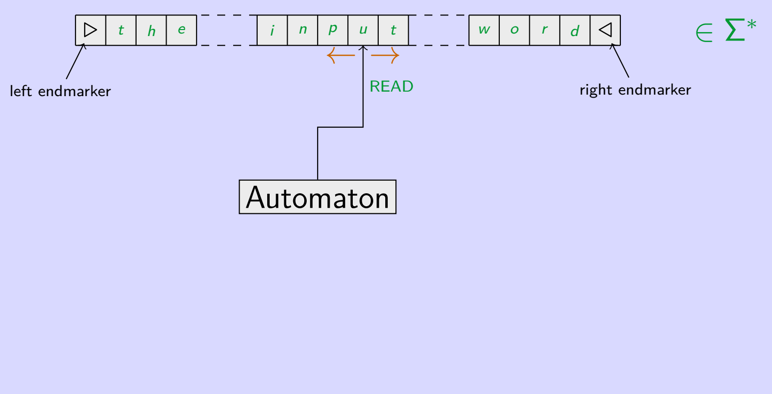


Two-way automaton over Σ

$$A$$

$$(Q, \Sigma, I, F, \delta)$$

transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$



Two-way transducer over Σ, Γ

$$(A, \phi)$$

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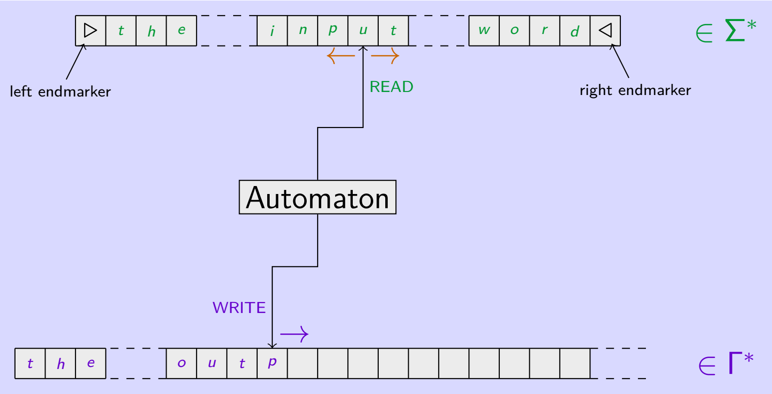
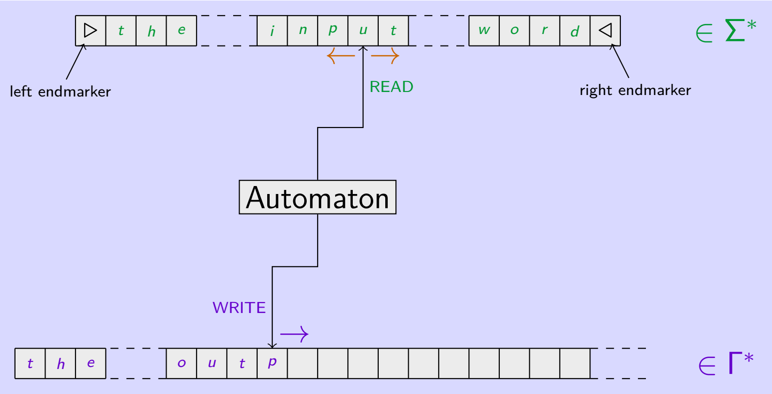


Two-way transducer over Σ, Γ

$$(A, \phi)$$

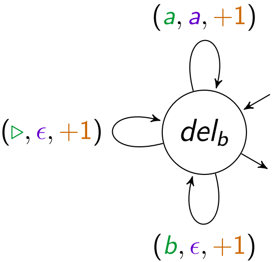
$(Q, \Sigma, I, F, \delta)$ — production function: $\delta \to \Gamma^*$

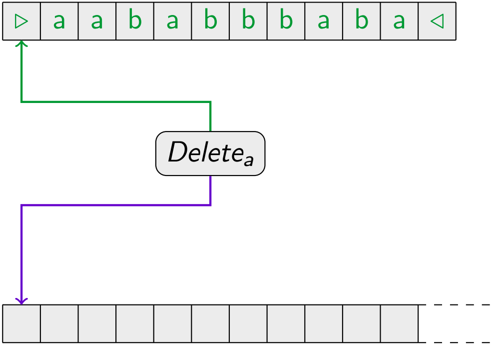
transition set: $\subseteq Q \times \overline{\Sigma} \times \{-1, 0, +1\} \times Q$



One-way simple example

$\Sigma = \Gamma = \{a, b\}$





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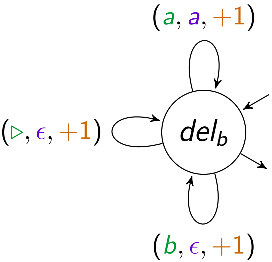

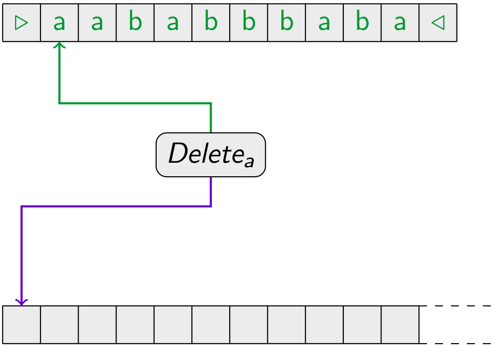

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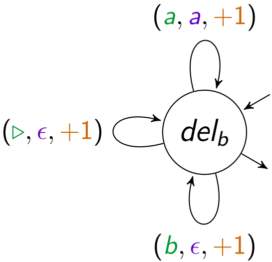

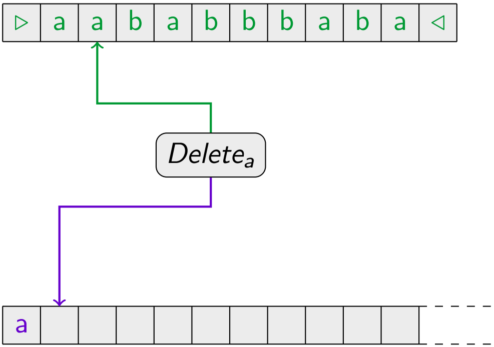

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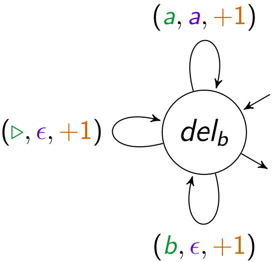

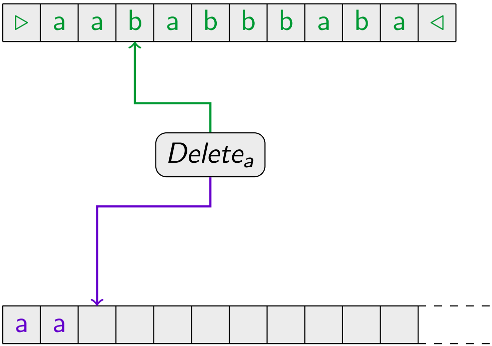

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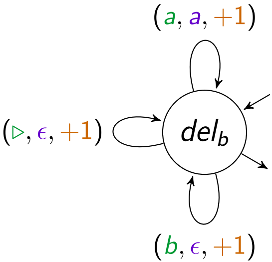

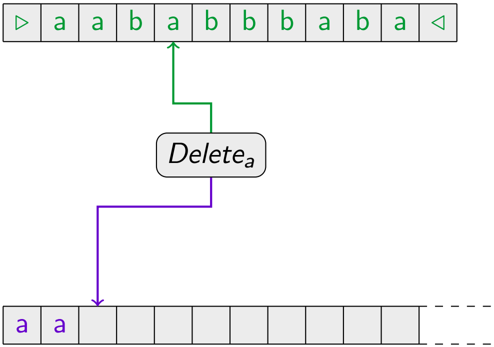

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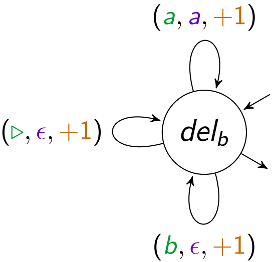

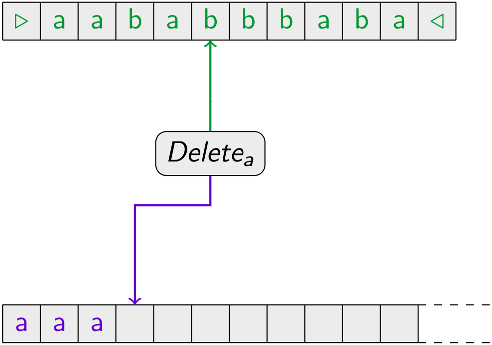

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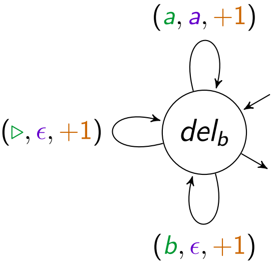

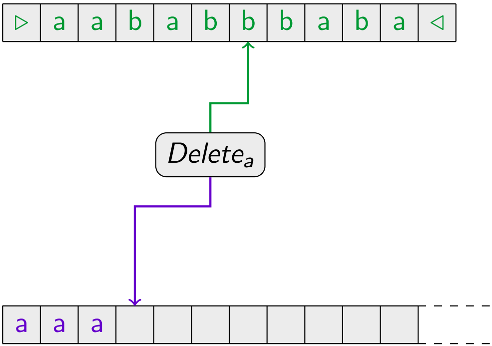

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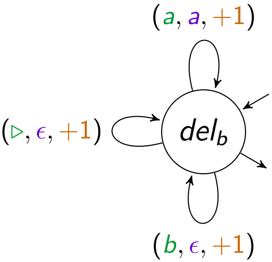

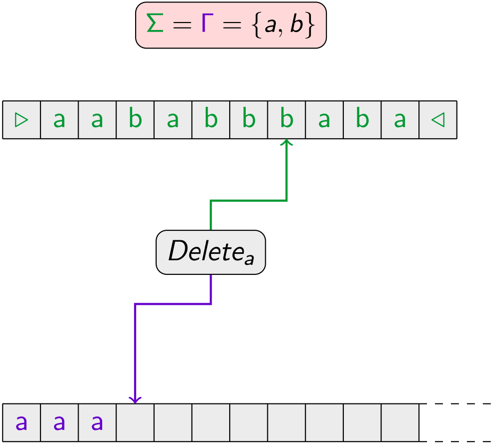

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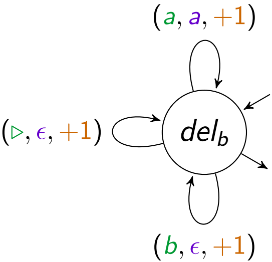

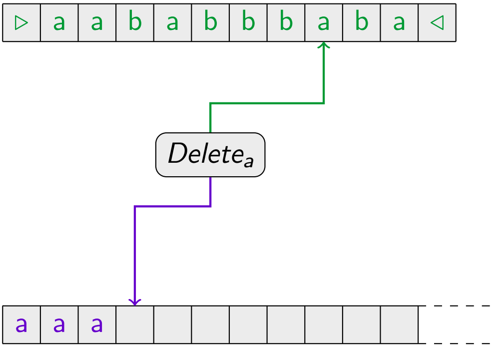

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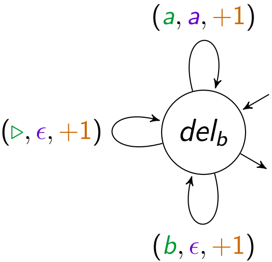

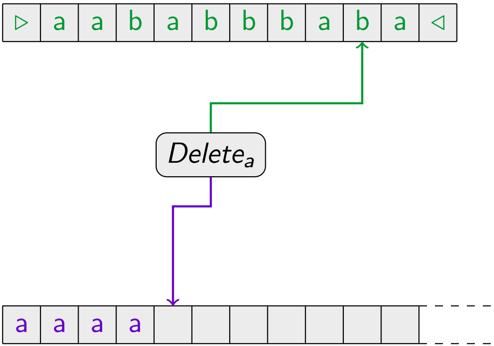

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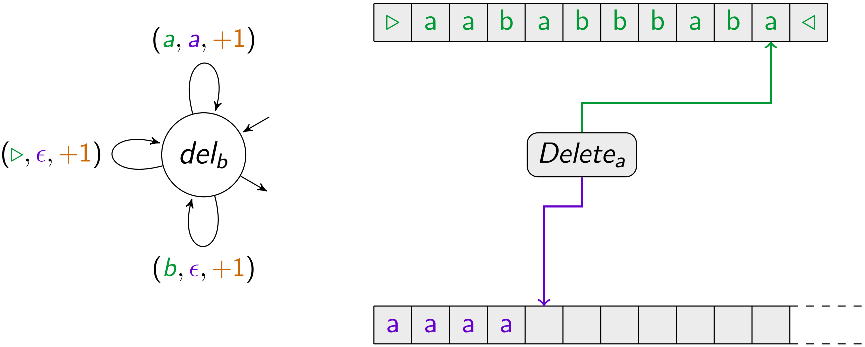

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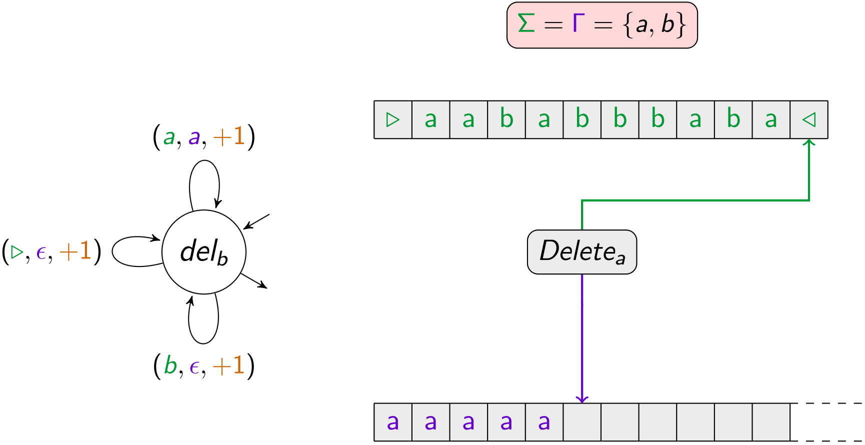

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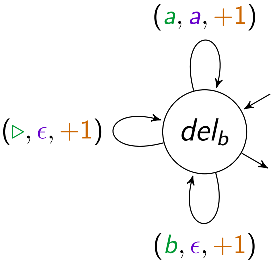
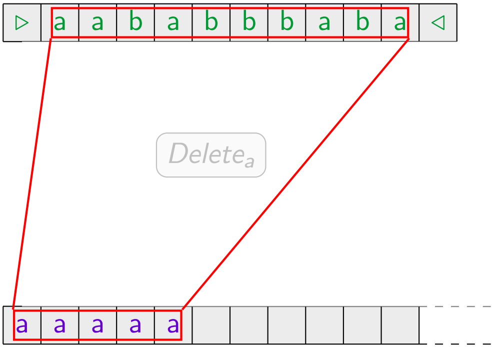
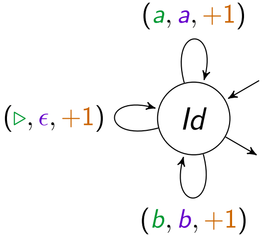




accepts: $\left\{(u, v) \mid v = a^{|u|_a}\right\}$

Simple Examples



$\boxed{\Sigma = \Gamma = \{a, b\}}$

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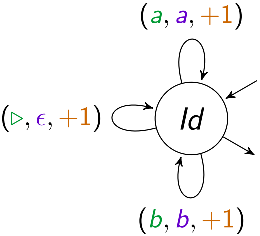

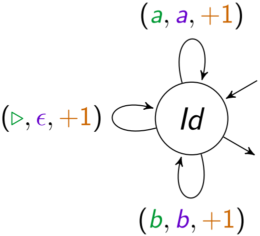

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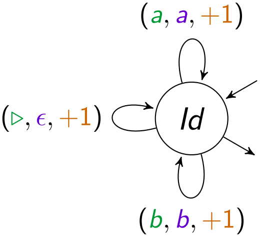

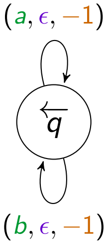

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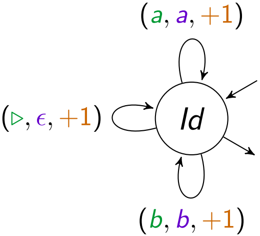

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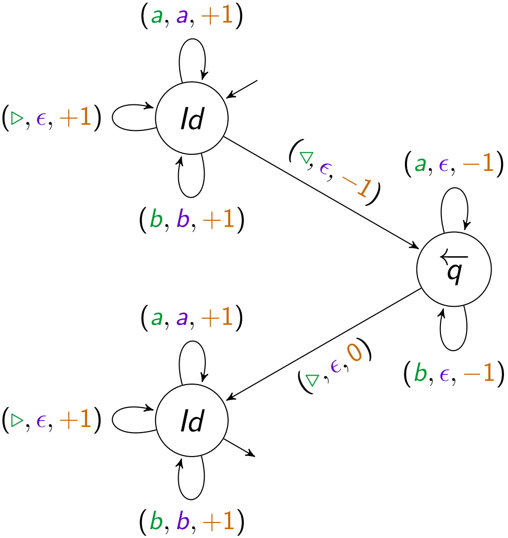

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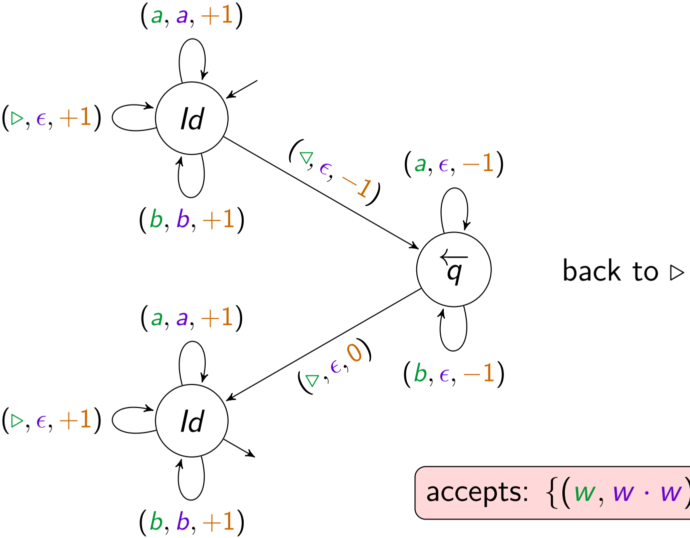
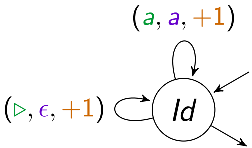


accepts: $\{(w, w \cdot w) \mid w \in \Sigma^*\}$

Simple Unary Examples

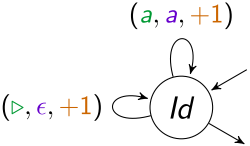
$\Sigma = \Gamma = \{a\}$

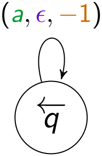


accepts: $\{(a^n, a^n) \mid n \in \mathbb{N}\}$

Simple Unary Examples

$\Sigma = \Gamma = \{a\}$





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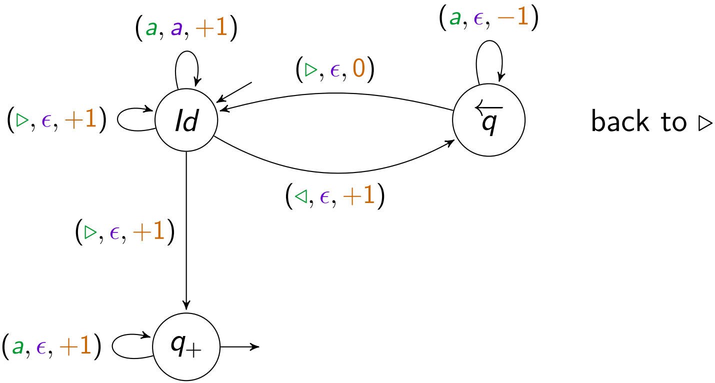

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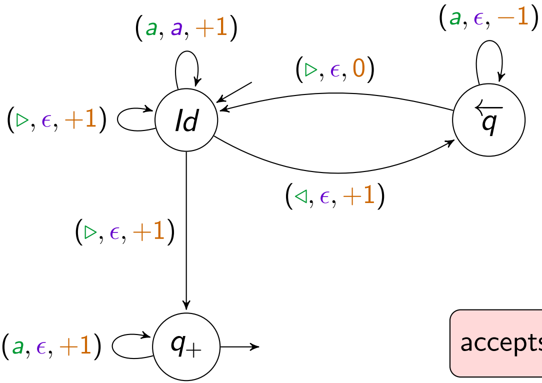

back to $\triangleright$

accepts: $\left\{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\right\}$

Relations are Series

$R \subseteq \Sigma^* \times \Gamma^*$

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the *image of* $u \in \Sigma^*$ is
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$s \in \mathcal{P}(\Gamma^*)\langle\langle \Sigma^* \rangle\rangle$

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the coefficient of $u \in \Sigma^*$ is
$\langle s, u \rangle = R(u)$

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Relations are Series

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Examples

$$\Sigma = \Gamma = \{a, b\}$$

$$R = \left\{\left(w, a^{|w|_a}\right)\right\}$$

$$s = \sum_{w \in \Sigma^*} \left\{a^{|w|_a}\right\} w$$

$$R = \{(w, ww) \mid w \in \Sigma^*\}$$

$$s = \sum_{w \in \Sigma^*} \{ww\} w$$

$$\Sigma = \Gamma = \{a\}$$

$$R = \left\{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\right\}$$

$$s = \sum_{a^n \in \Sigma^*} \left\{a^{kn} \mid k \in \mathbb{N}\right\} a^n$$

Rational operations...

- Sum: $$s + t = \sum_{u \in \Sigma^*} \left(\langle s, u \rangle \cup \langle t, u \rangle \right) u$$

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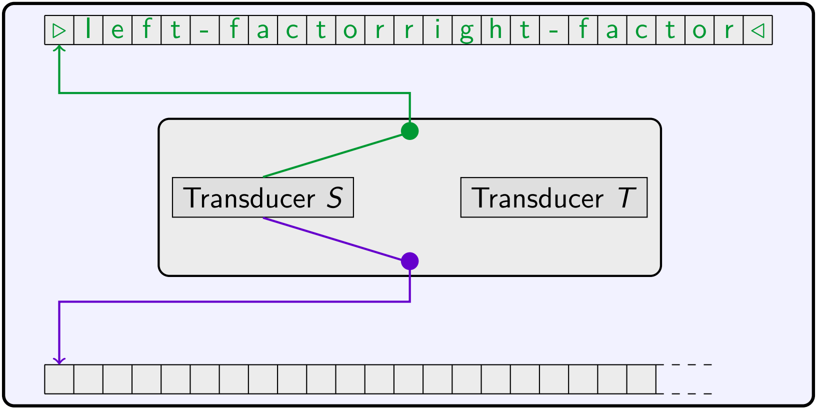
- Cauchy product: $$s \cdot t = \sum_{u \in \Sigma^*} \sum_{u_1 u_2 = u} \langle s, u_1 \rangle \, \langle t, u_2 \rangle u$$
- Kleene star: $$s^* = \sum_{u \in \Sigma^*} \sum_{u_1 u_2 \cdots u_n = u} \langle s, u_1 \rangle \, \langle s, u_2 \rangle \cdots \langle s, u_n \rangle u$$

Rational operations are one-way natural operations



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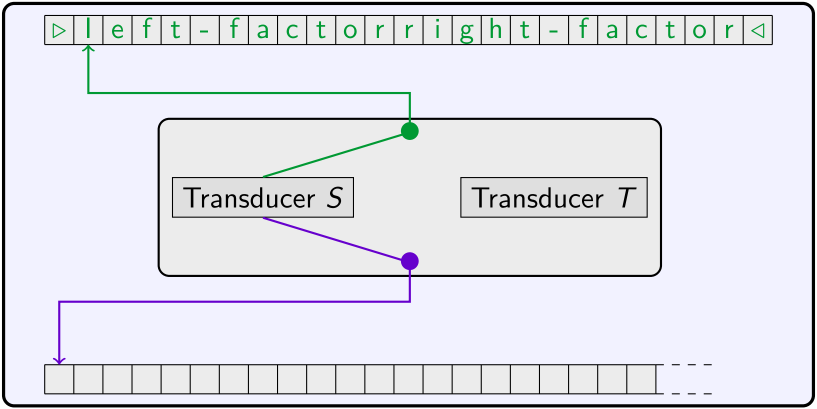

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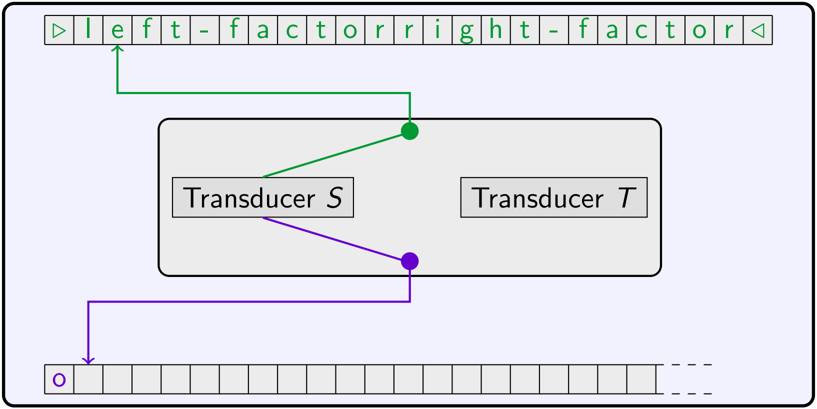

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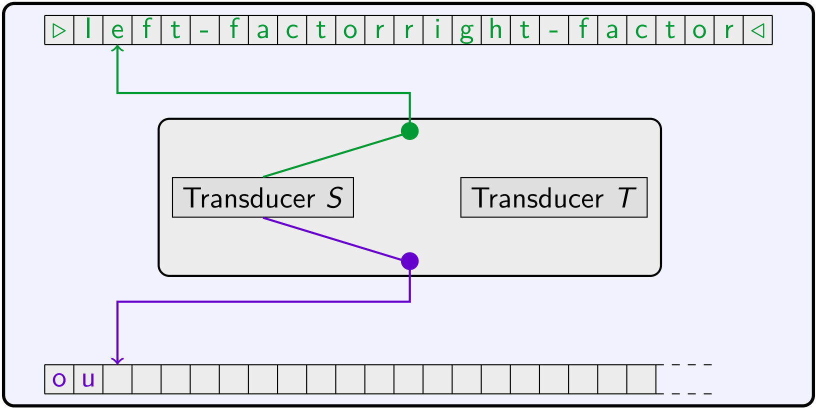

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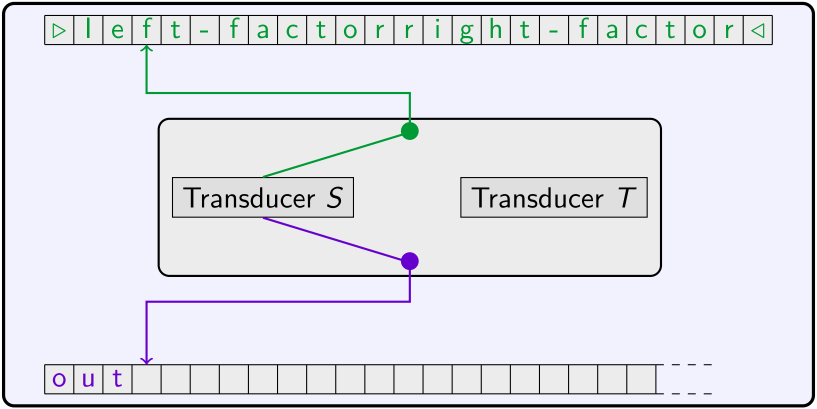

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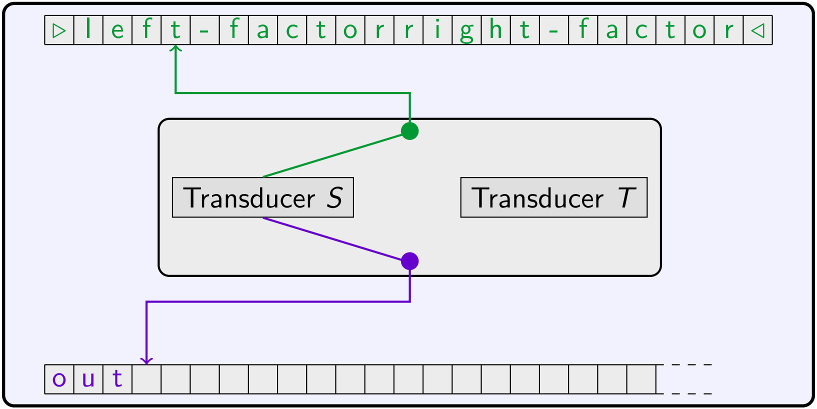

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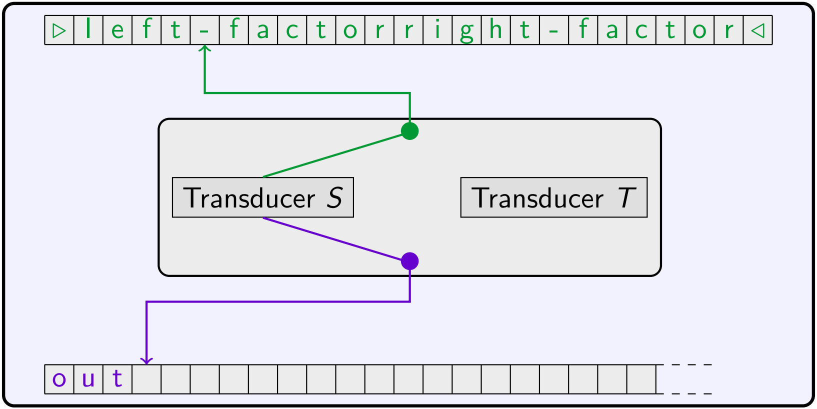

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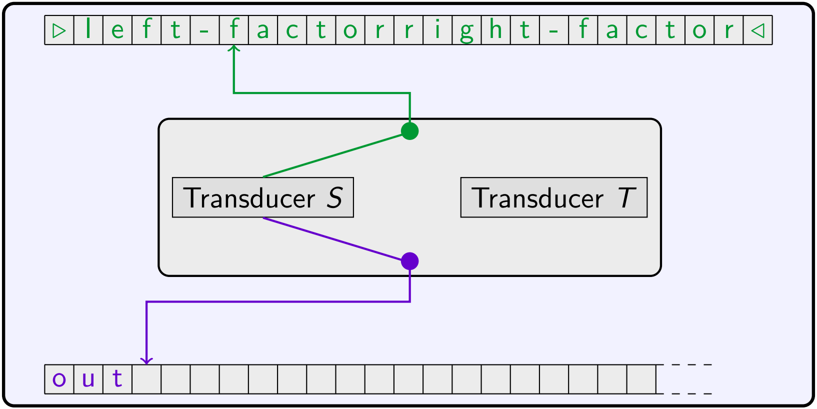

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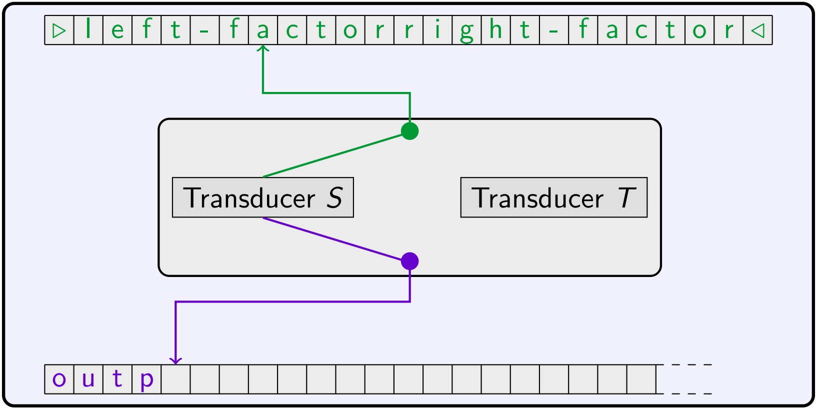

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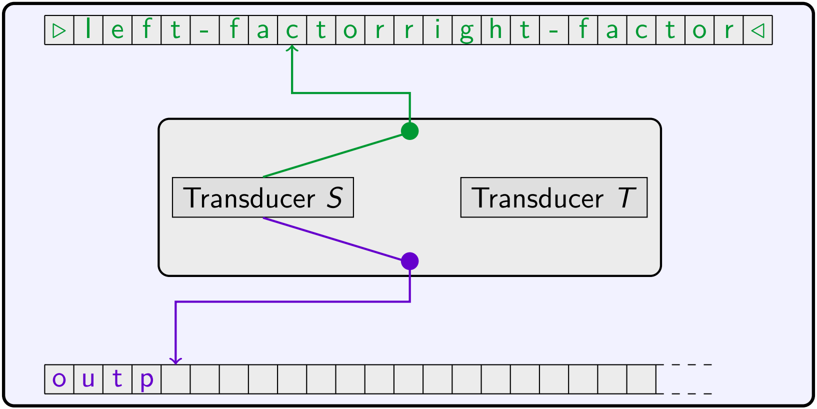

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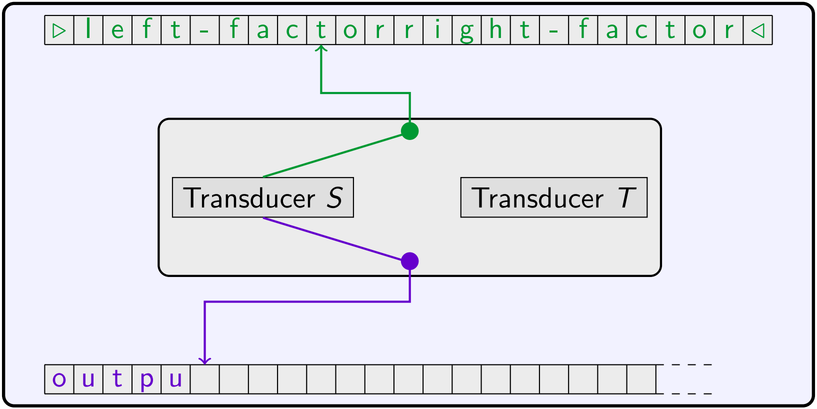

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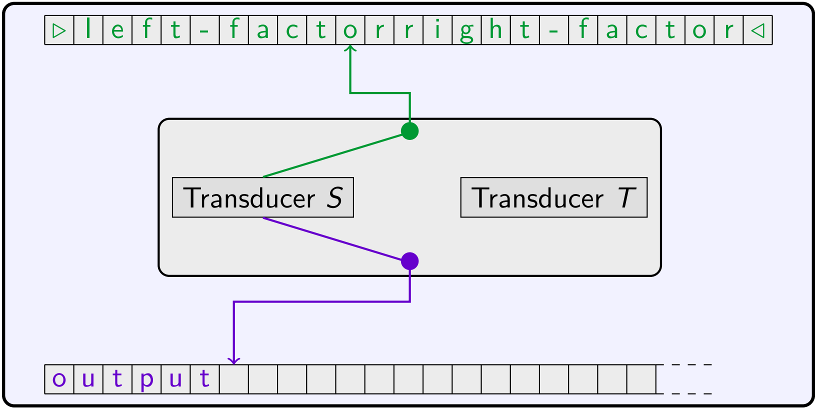

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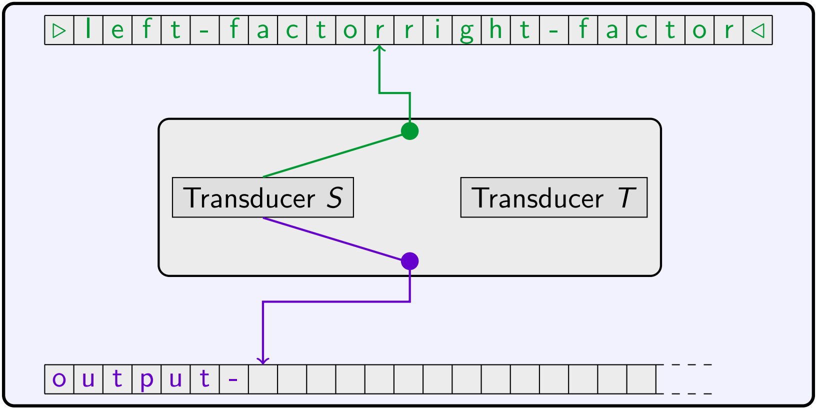

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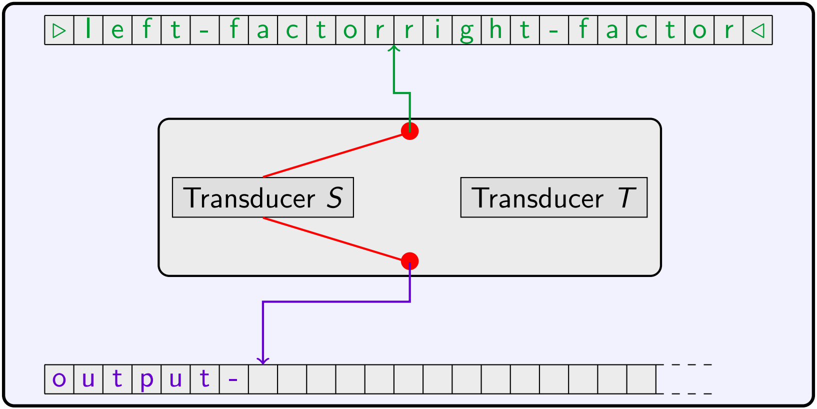

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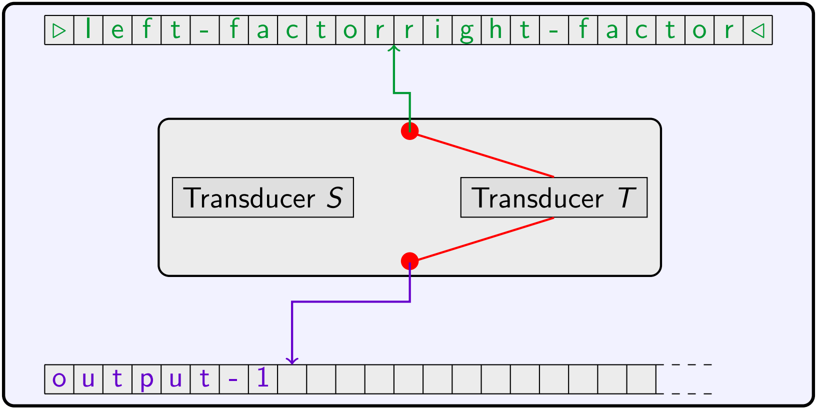

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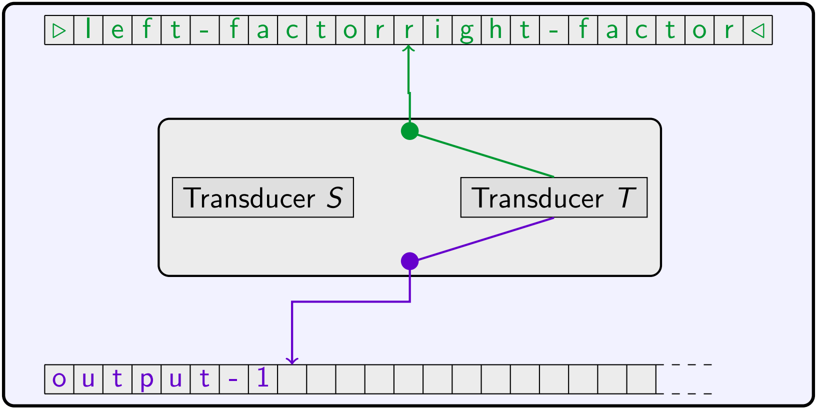

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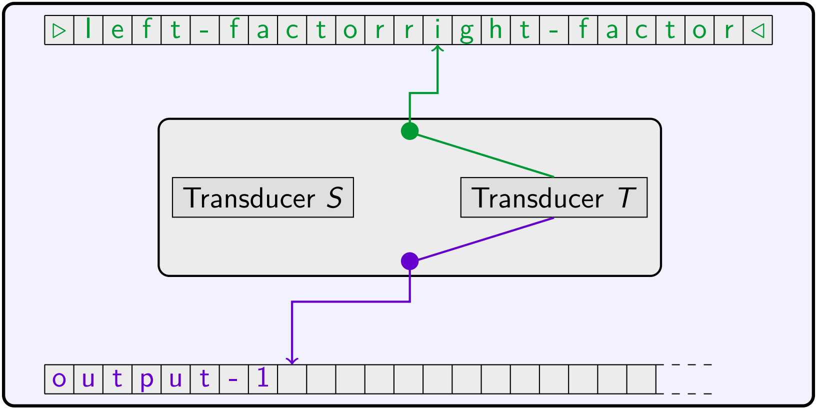

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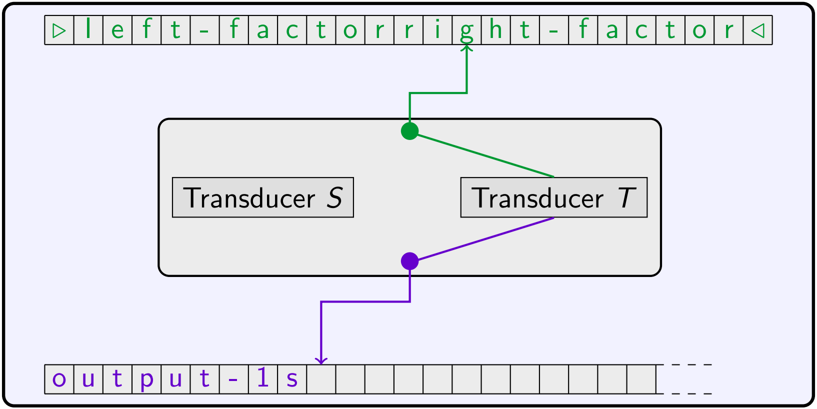

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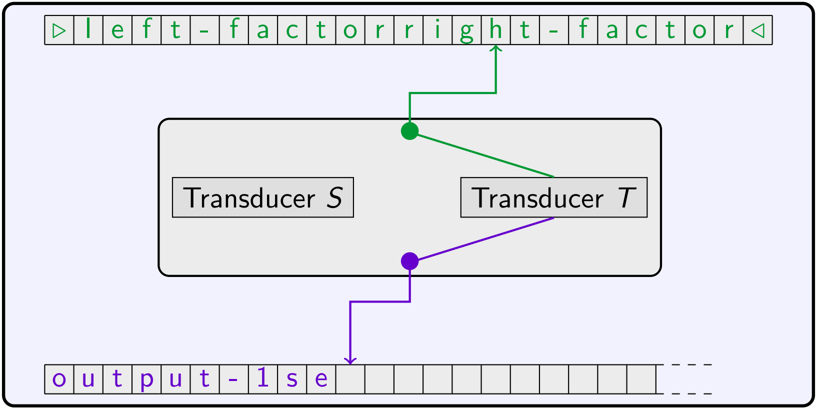

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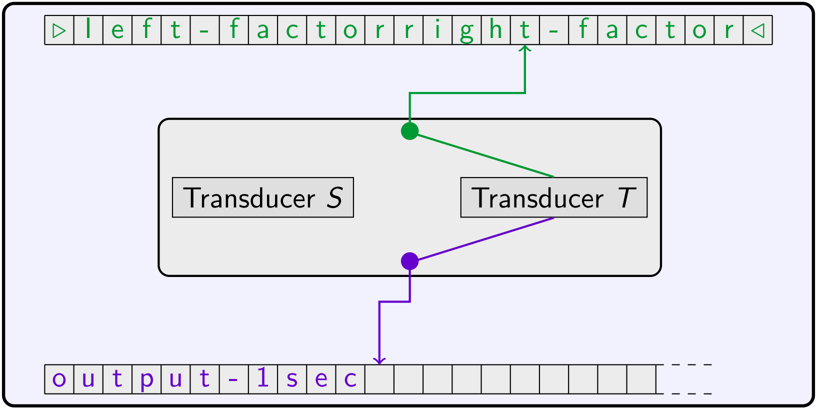

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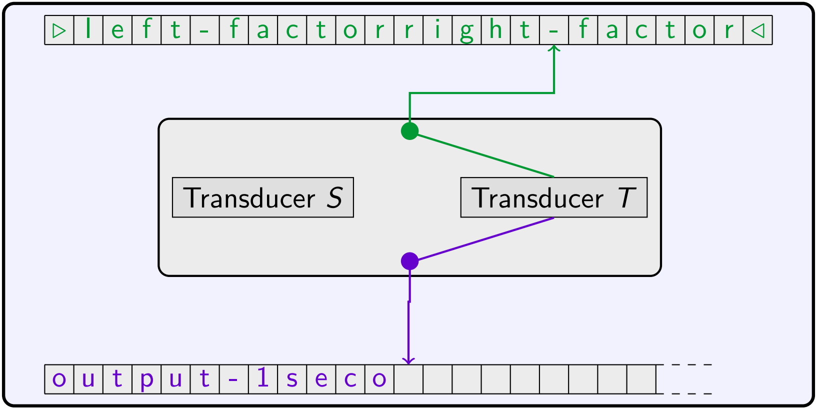

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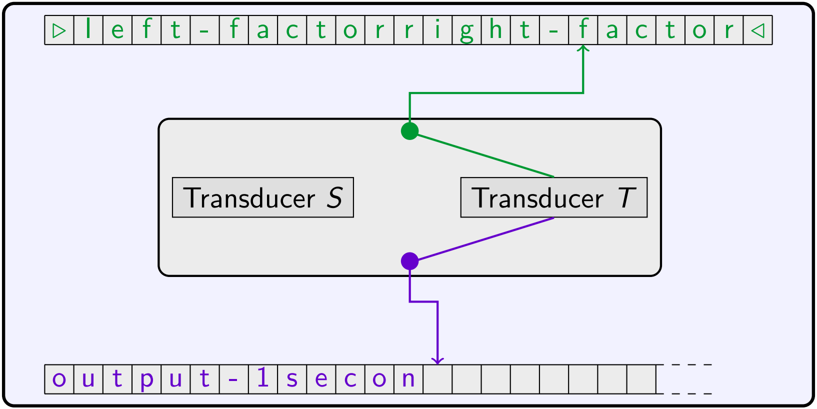

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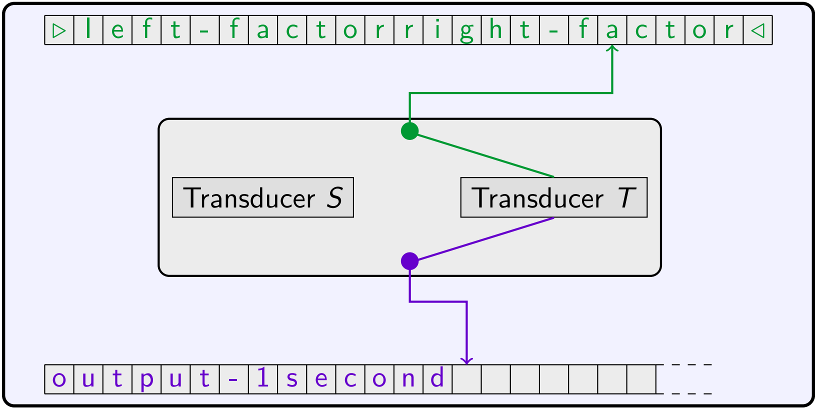

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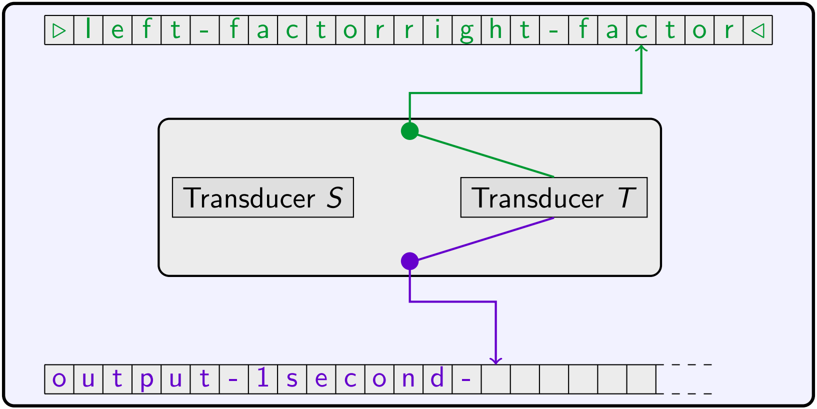

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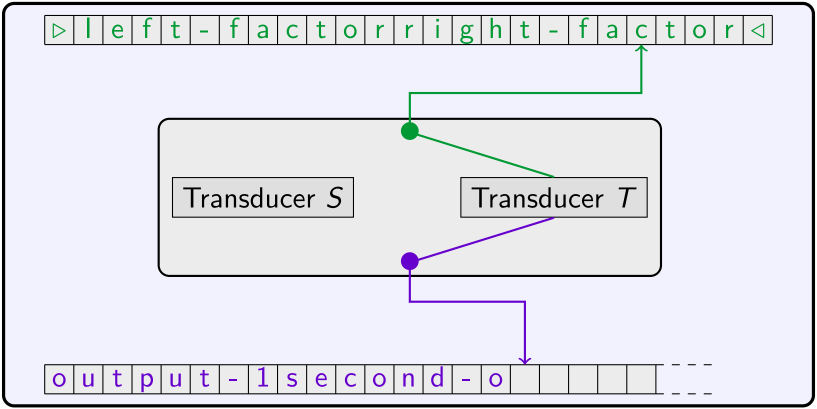

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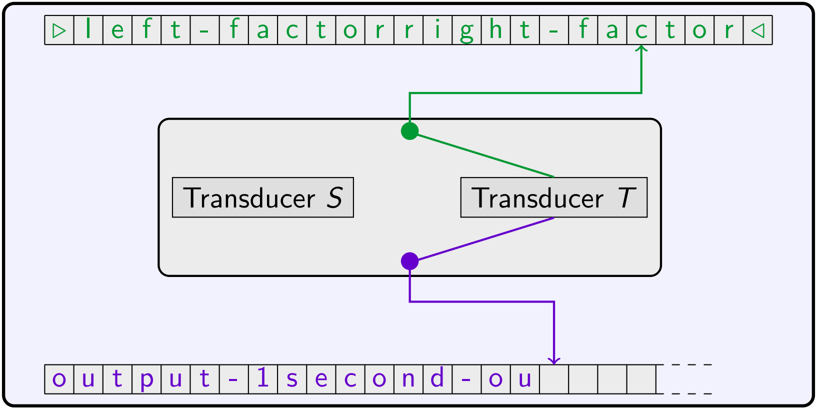

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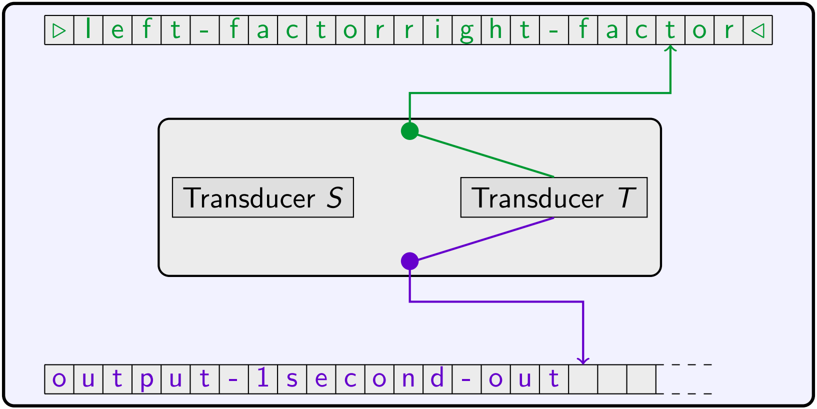

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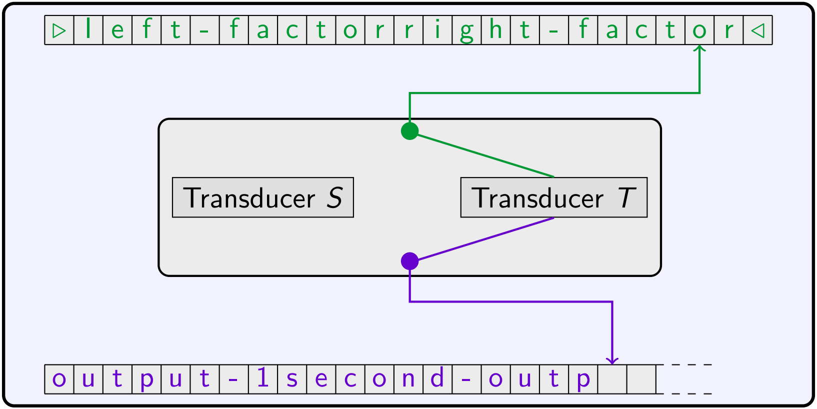

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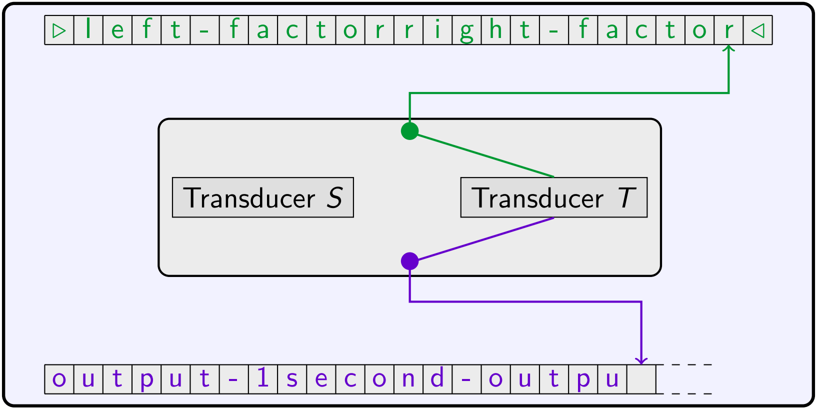

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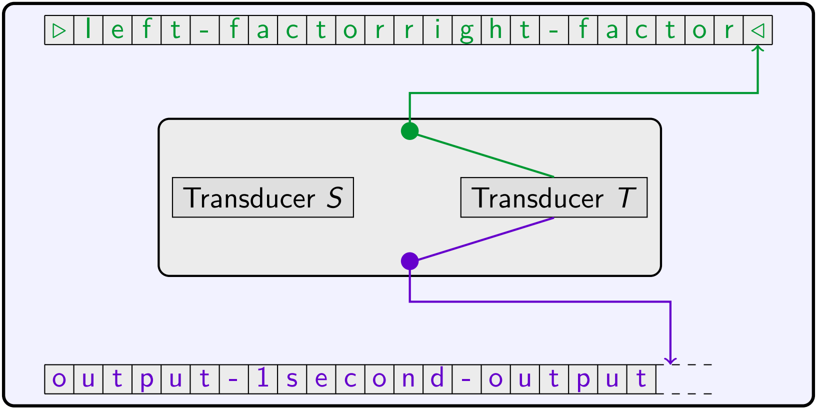

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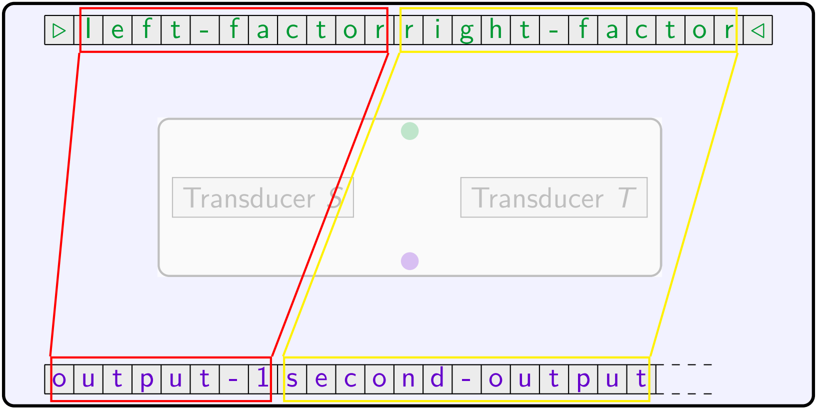

Rational operations are one-way natural operations

Definition

The class of Rational Series is the smallest class of series

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Theorem

One-way transducers accepts exactly the class of rational series.

Hadamard operations...

- Hadamard product:

$$s \circledH t = \sum_{u \in \Sigma^*} \langle s, u \rangle \cdot \langle t, u \rangle u$$

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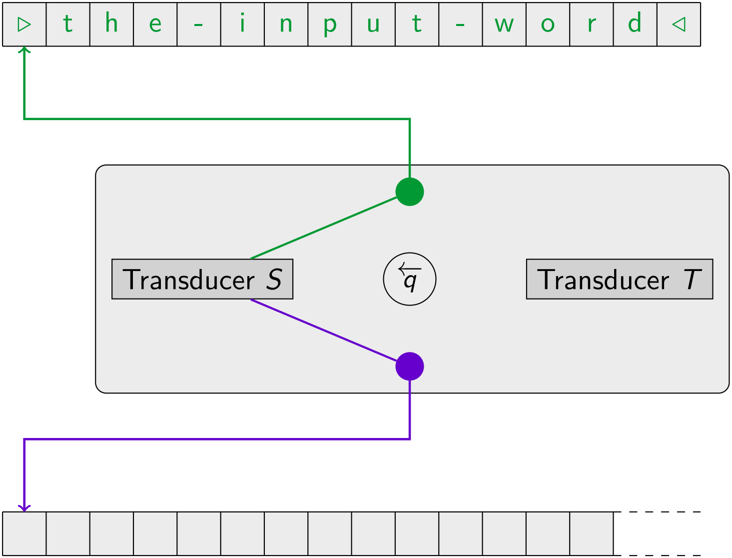
$$s \circledH t = \sum_{u \in \Sigma^*} \langle s, u \rangle \cdot \langle t, u \rangle u$$

- Hadamard star:

$$s^{\mathrm{H}\star} = \sum_{n \in \mathbb{N}} \underbrace{s \circledH s \circledH \ldots \circledH s}_{n \text{ times}} = \sum_{u \in \Sigma^*} \langle s, u \rangle^* u$$

Hadamard operations are natural for two-way transducers

Hadamard product: $s \circledH t$



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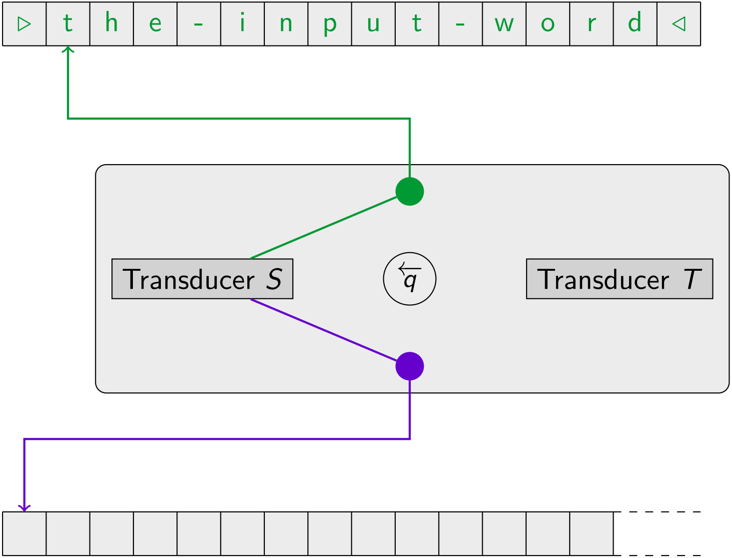

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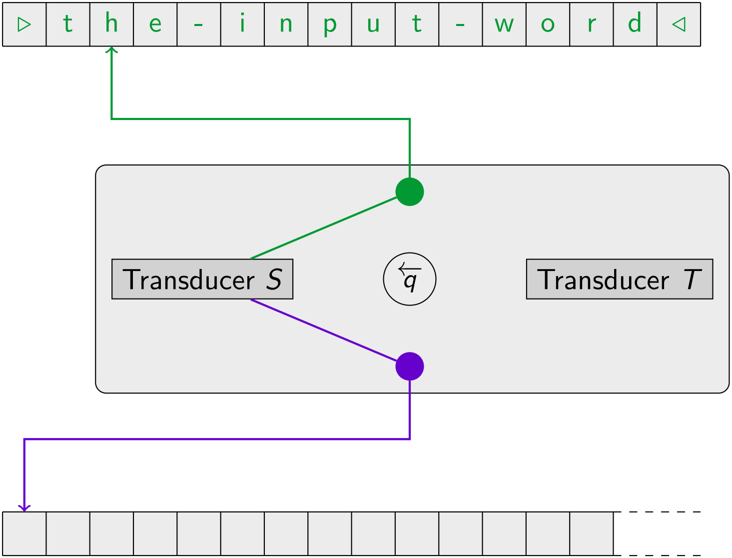

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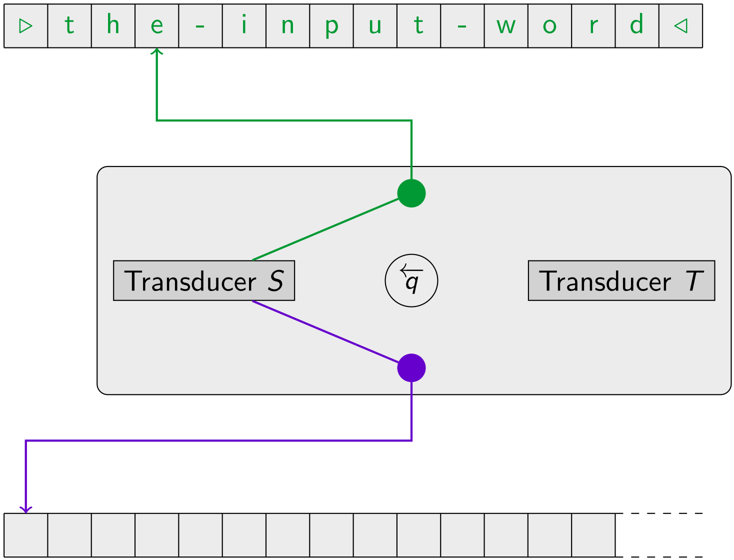

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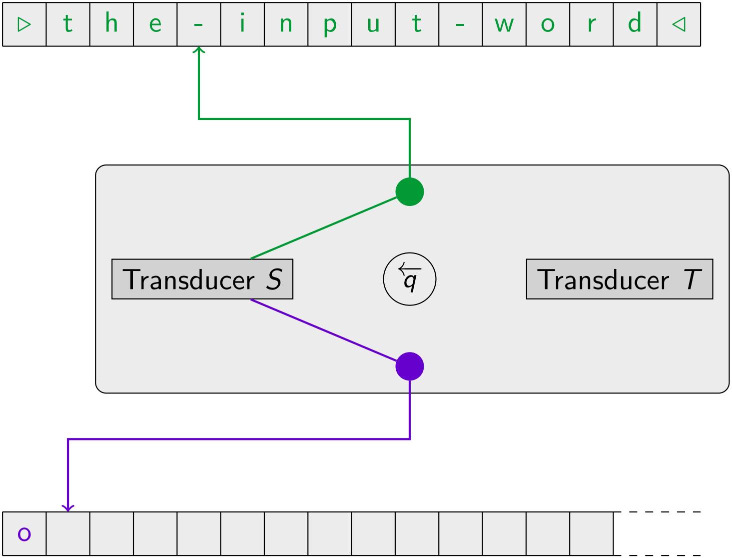

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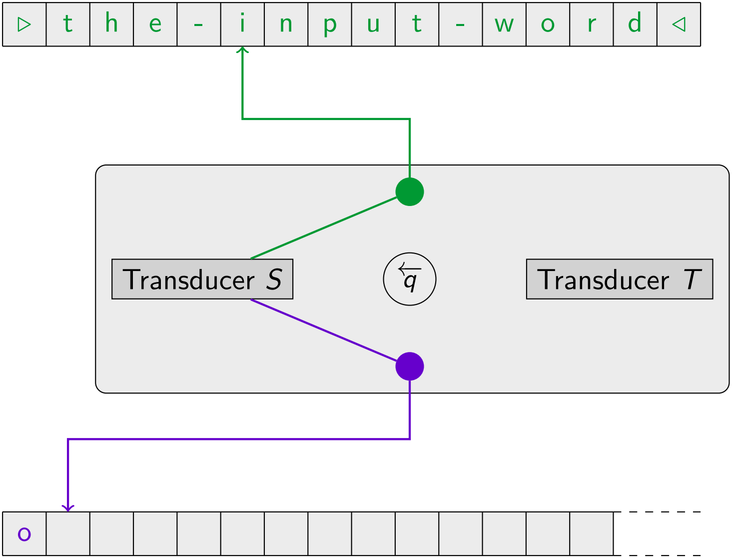

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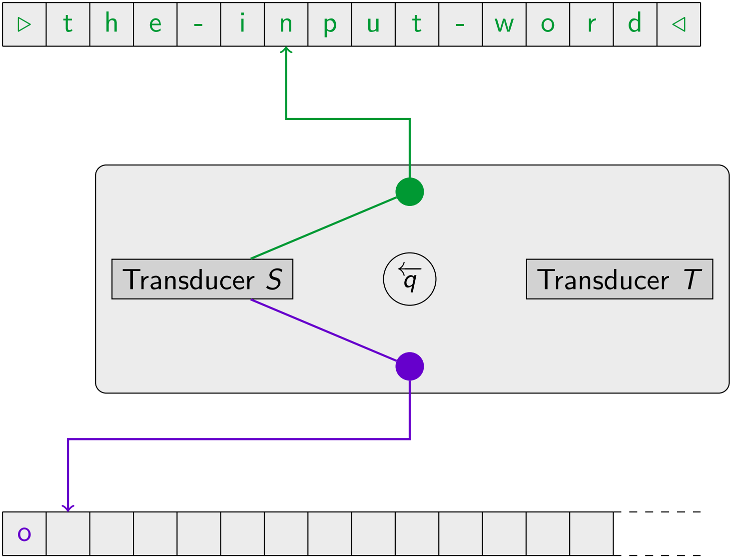

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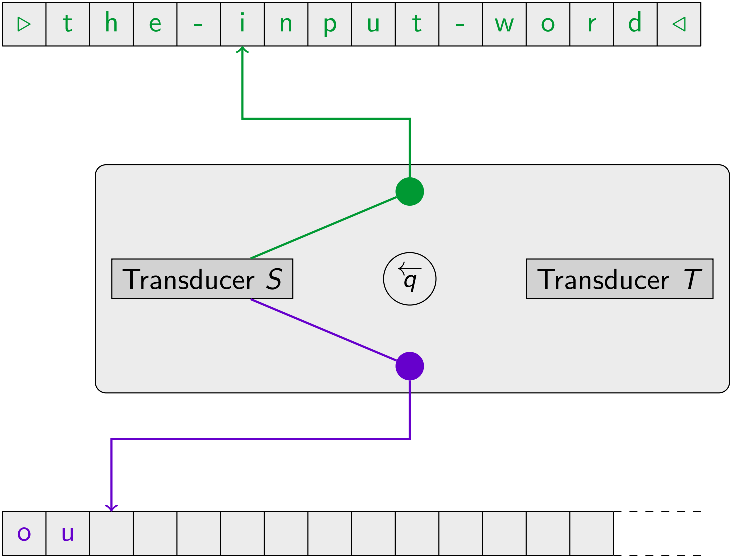

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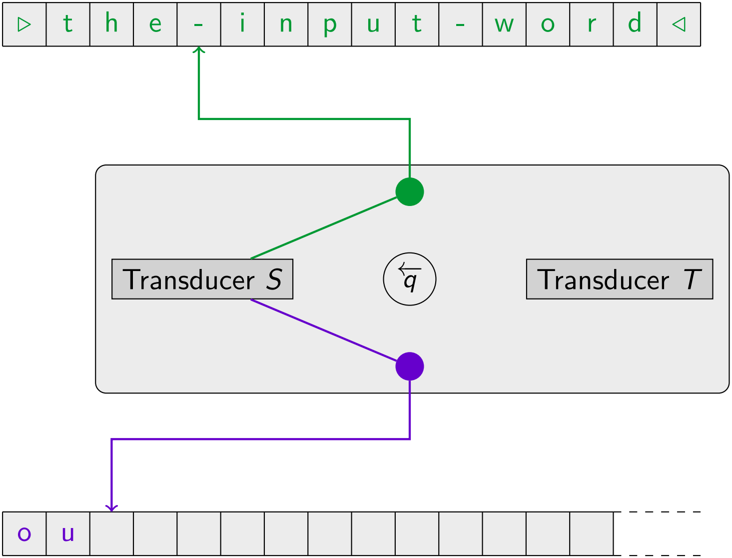

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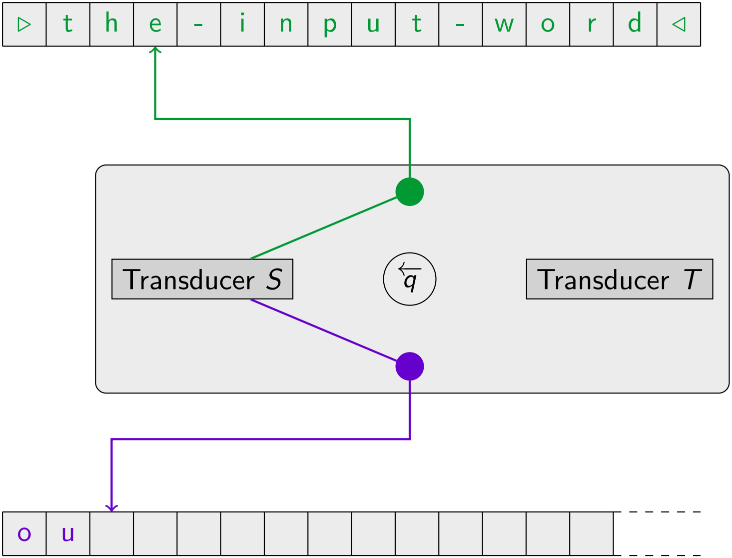

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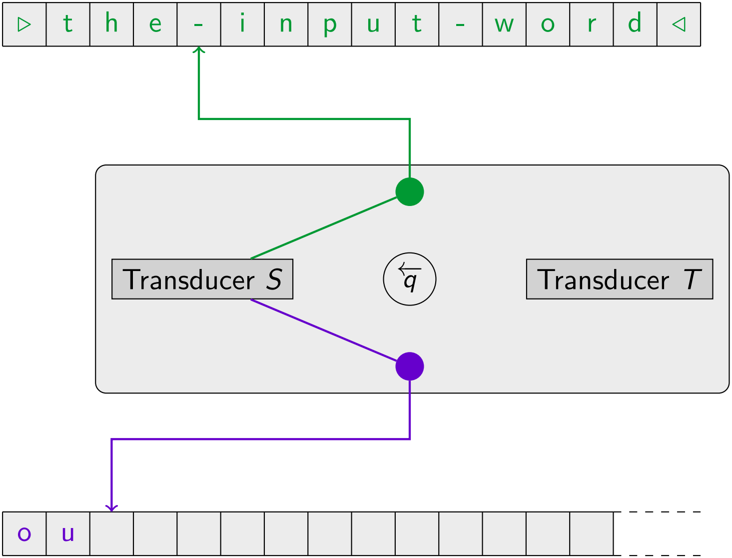

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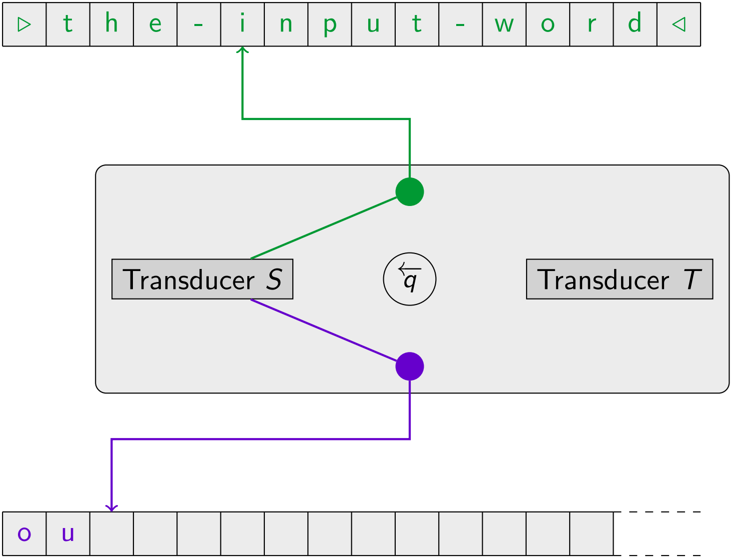

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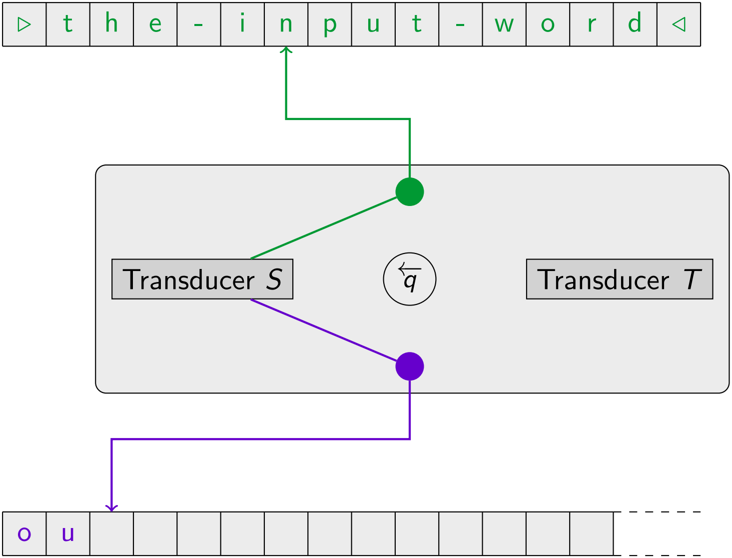

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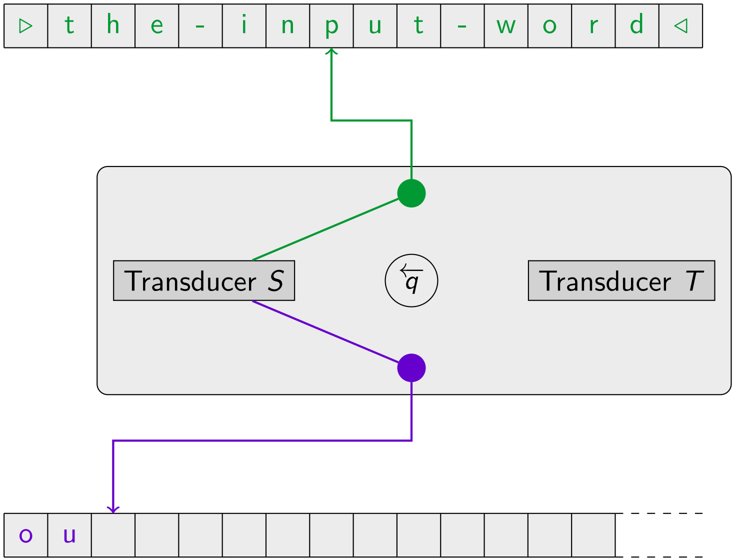

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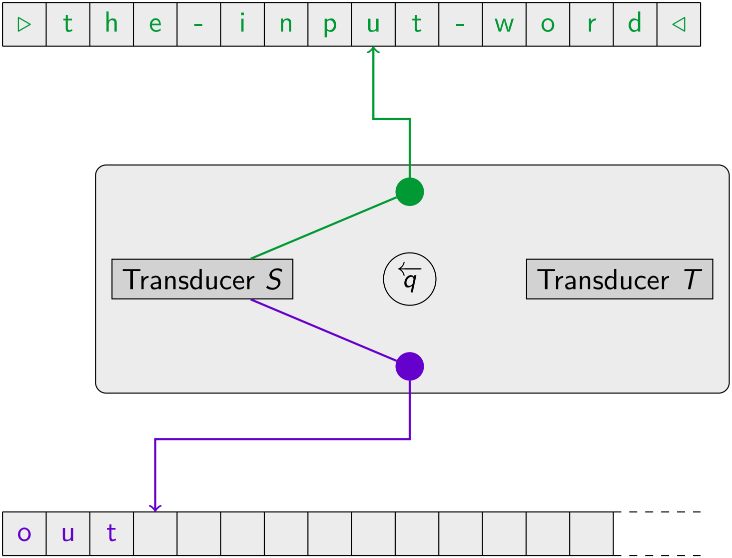

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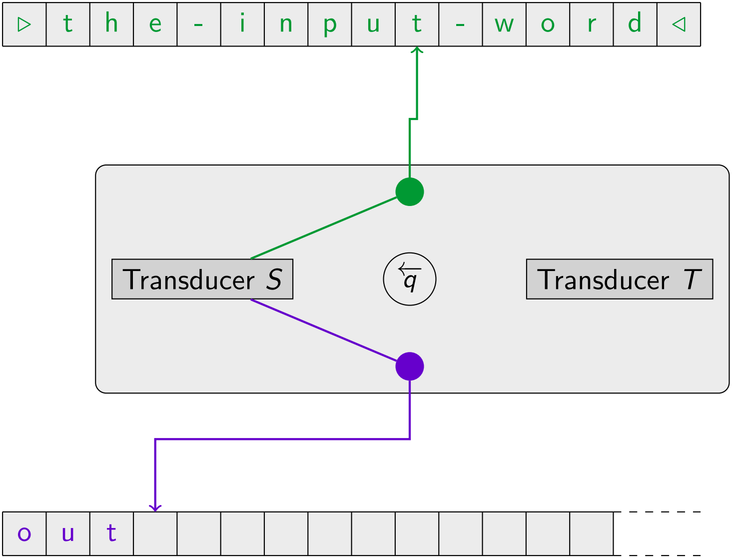

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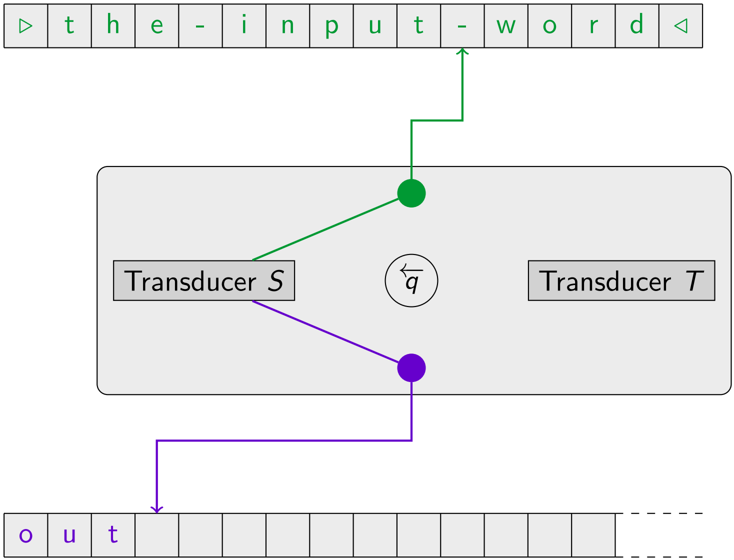

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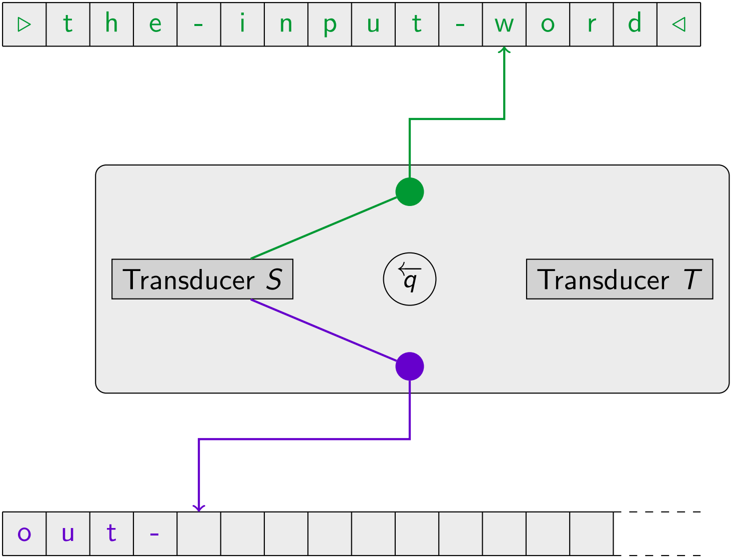

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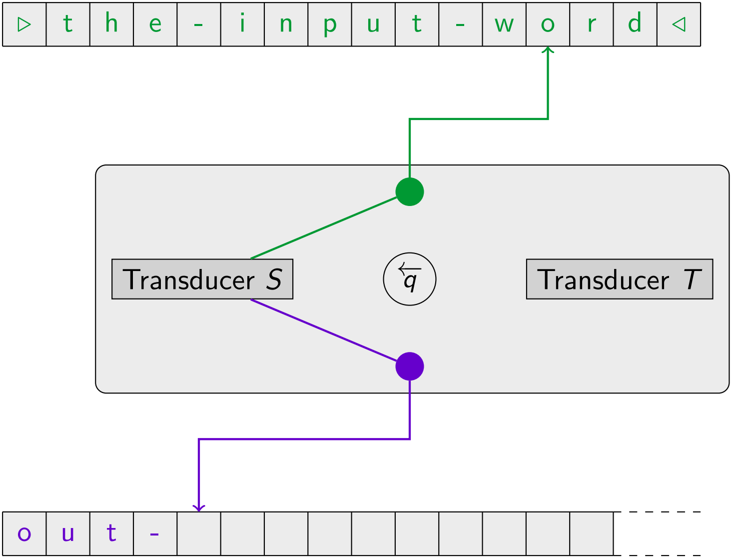

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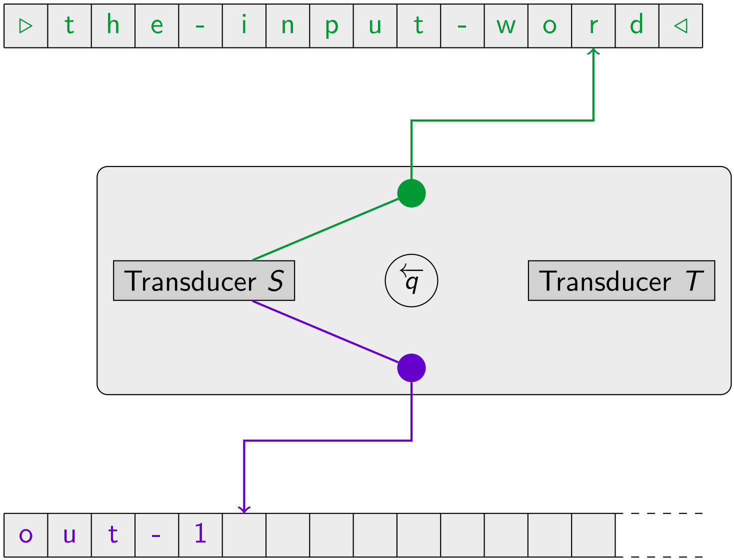

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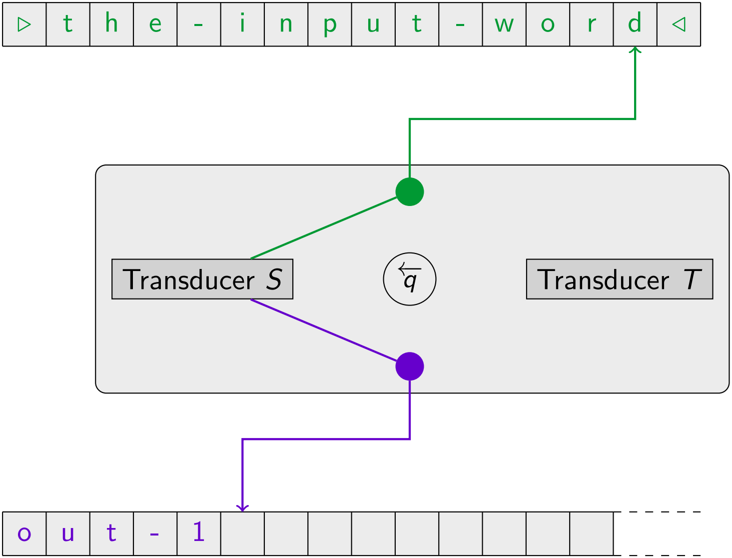

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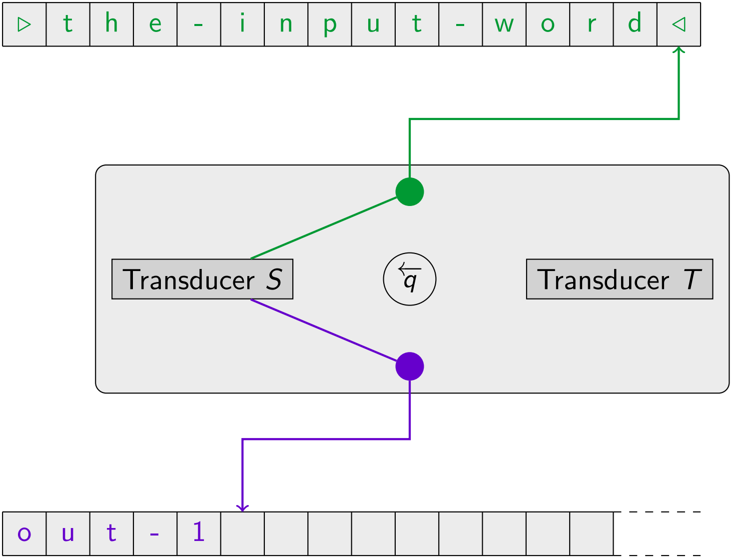

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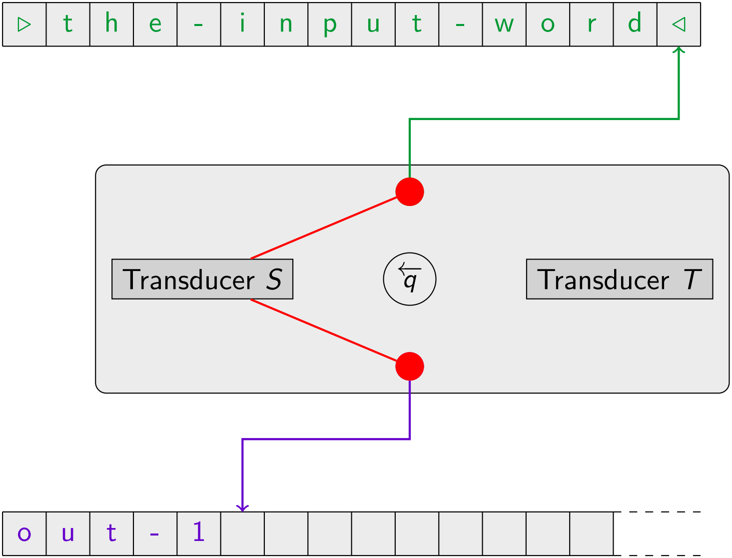

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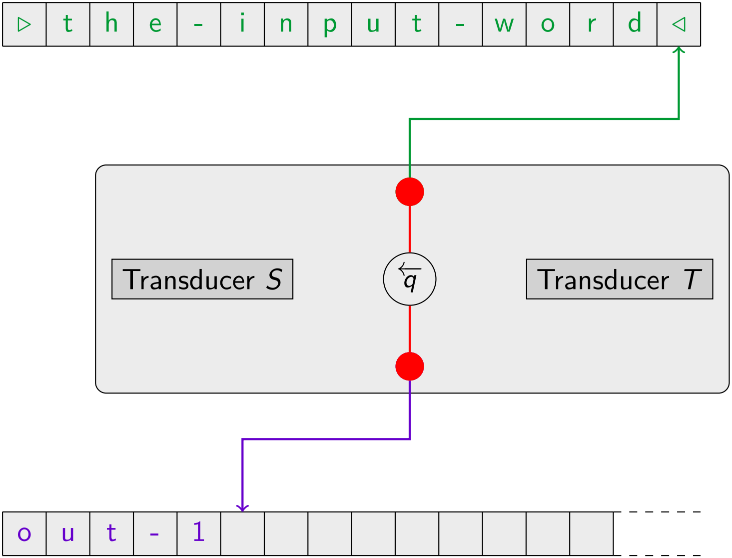

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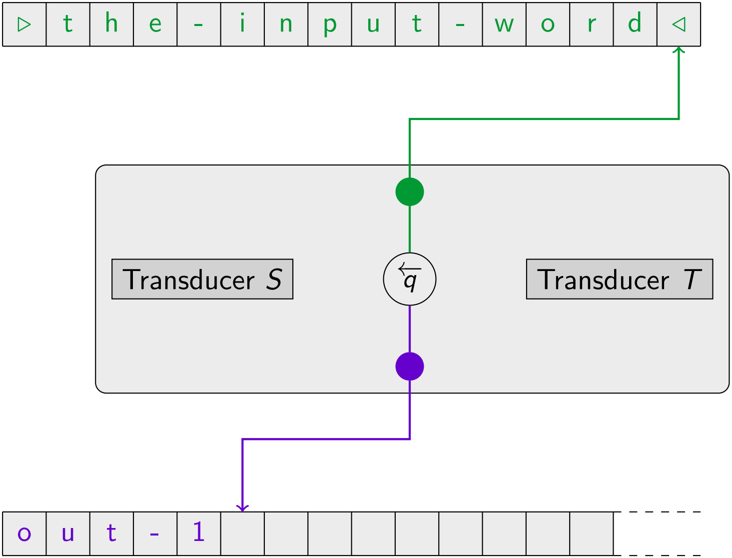

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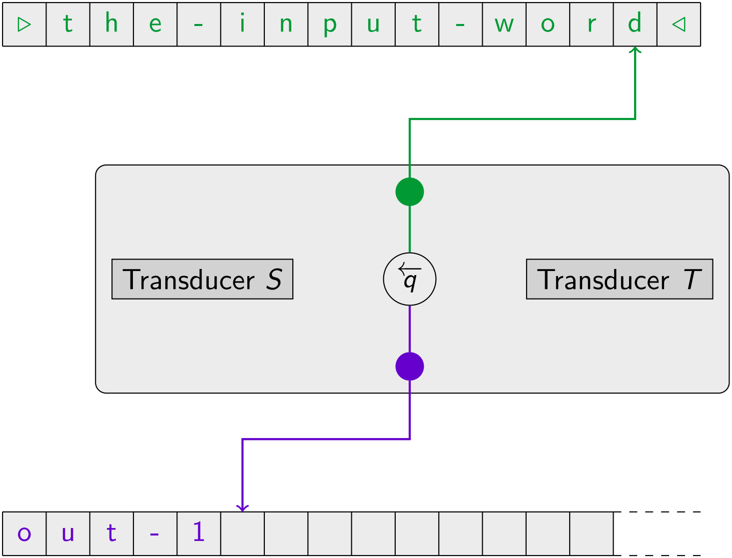

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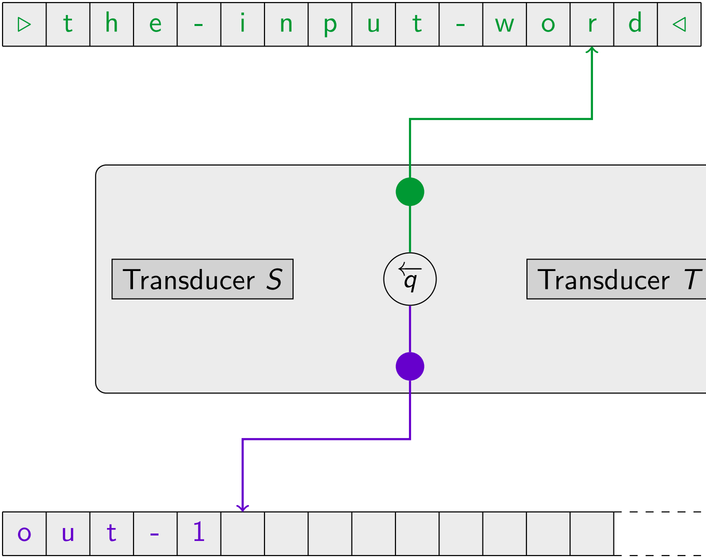

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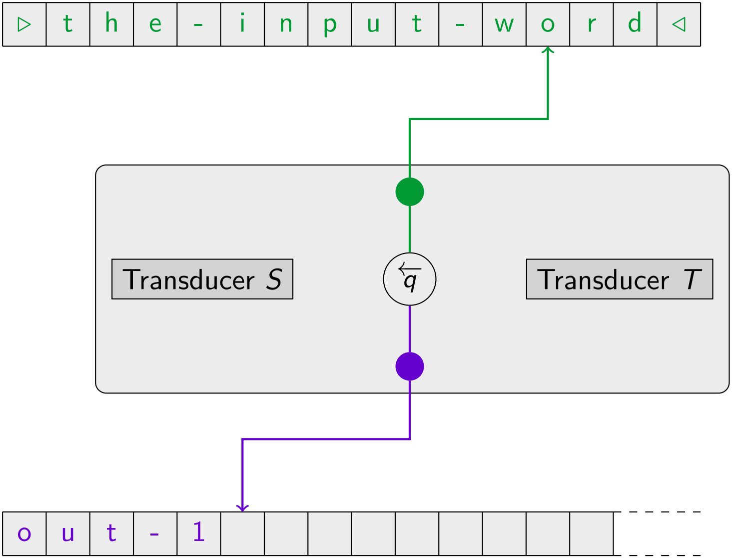

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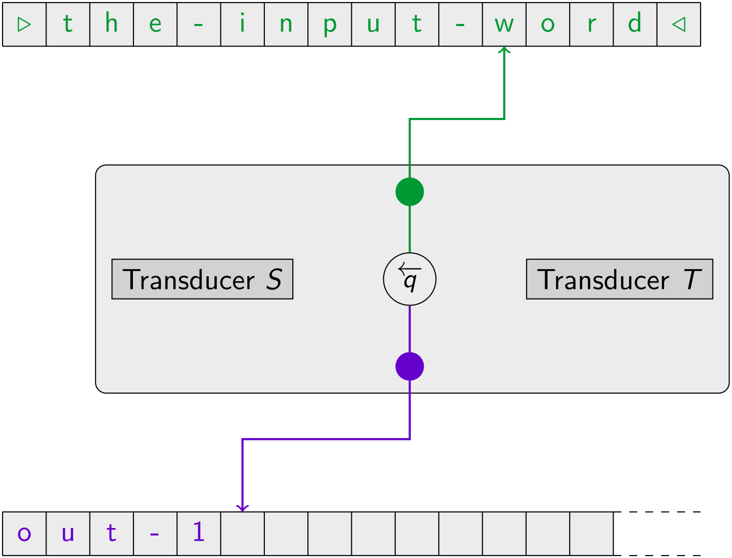

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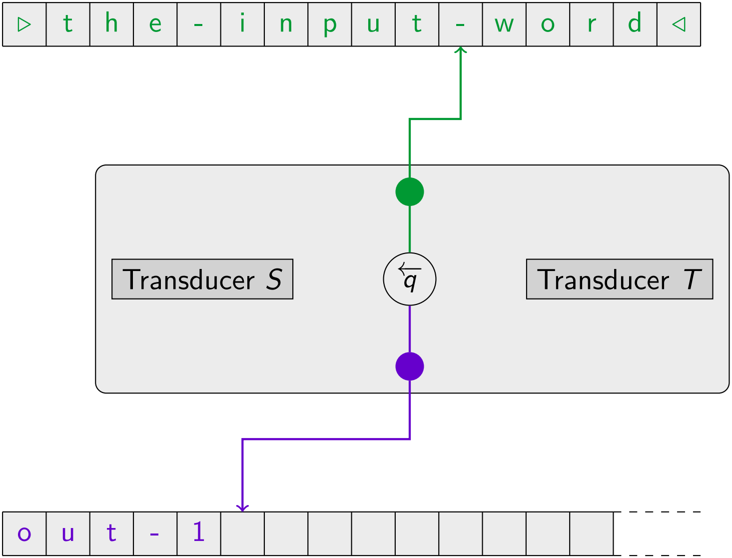

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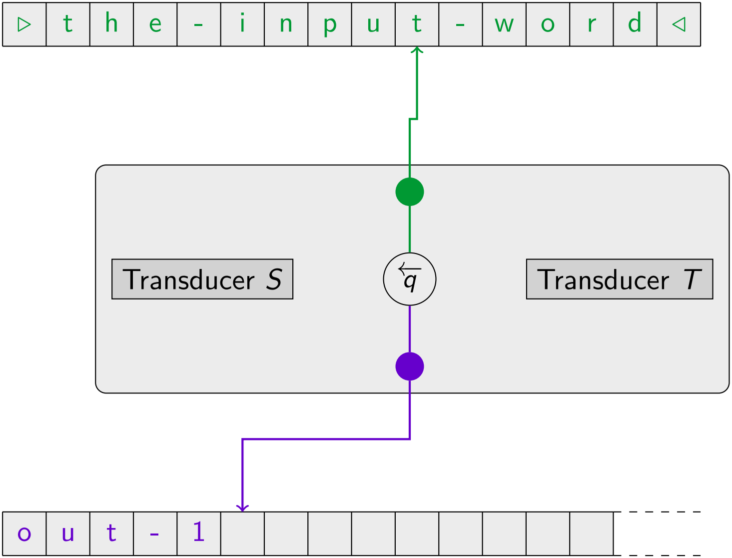

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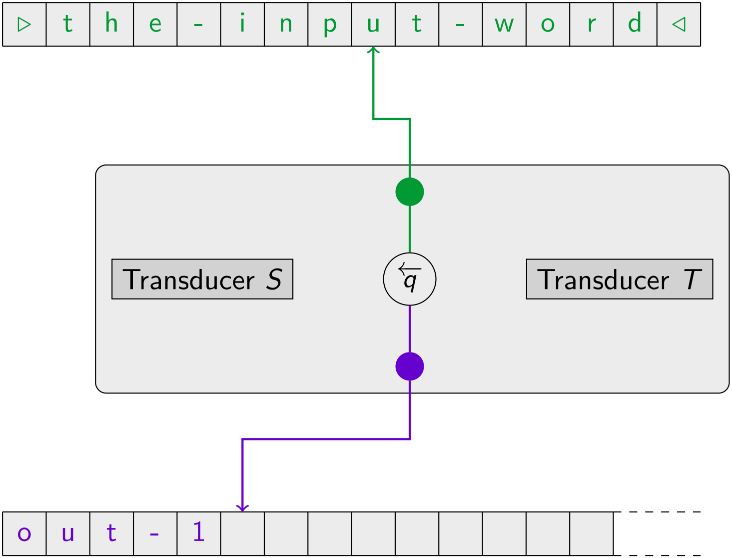

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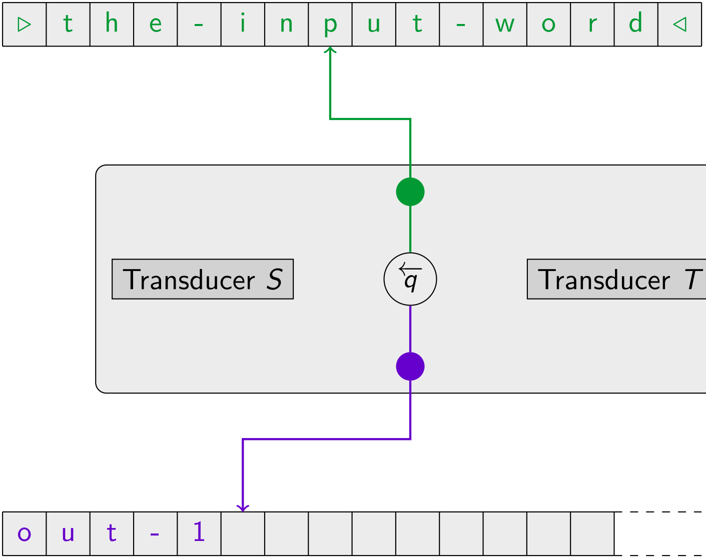

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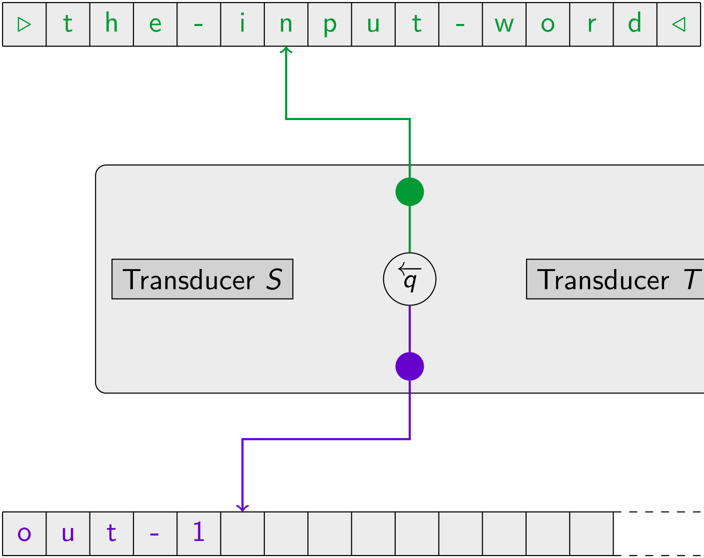

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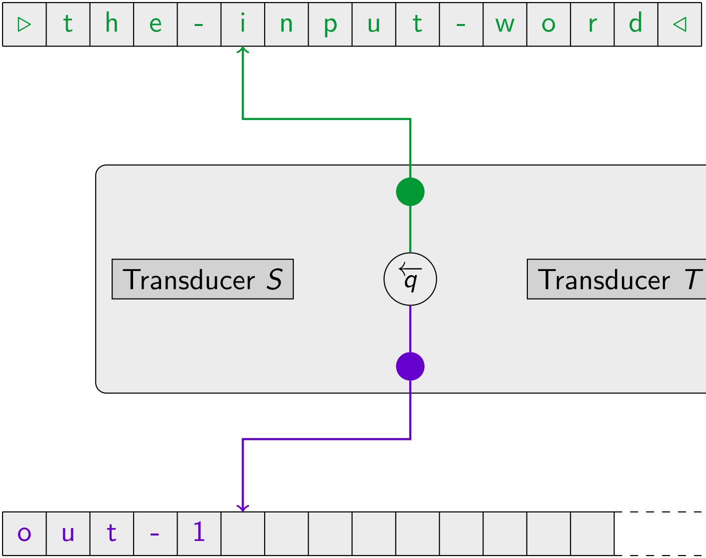

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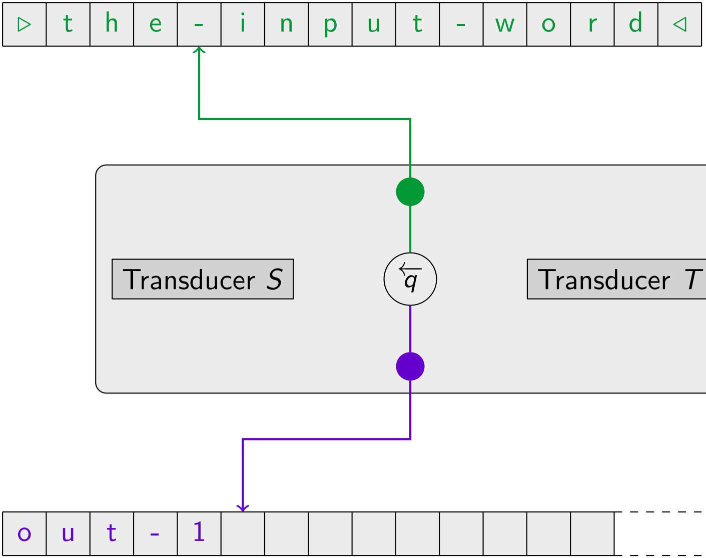

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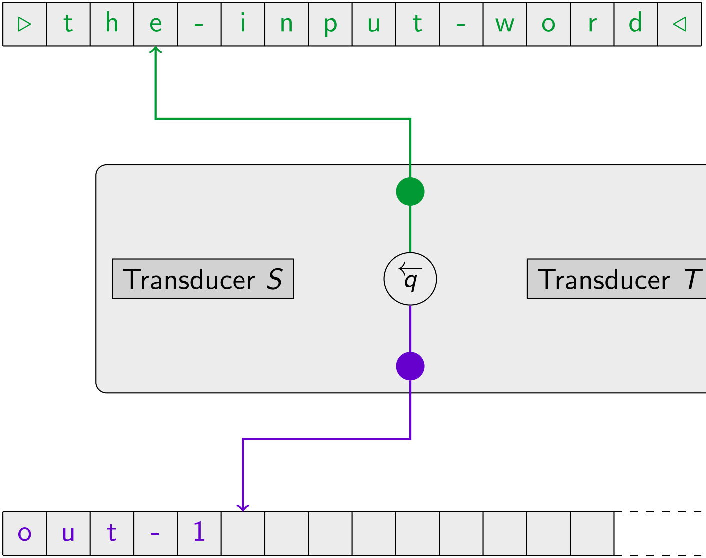

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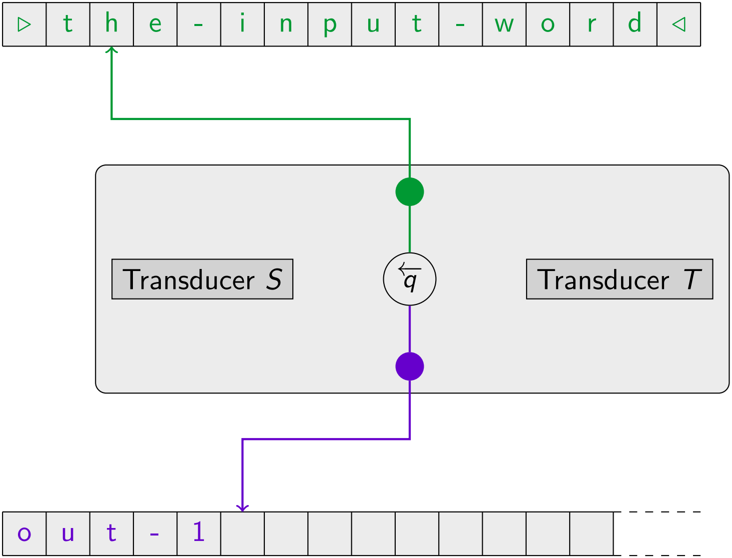

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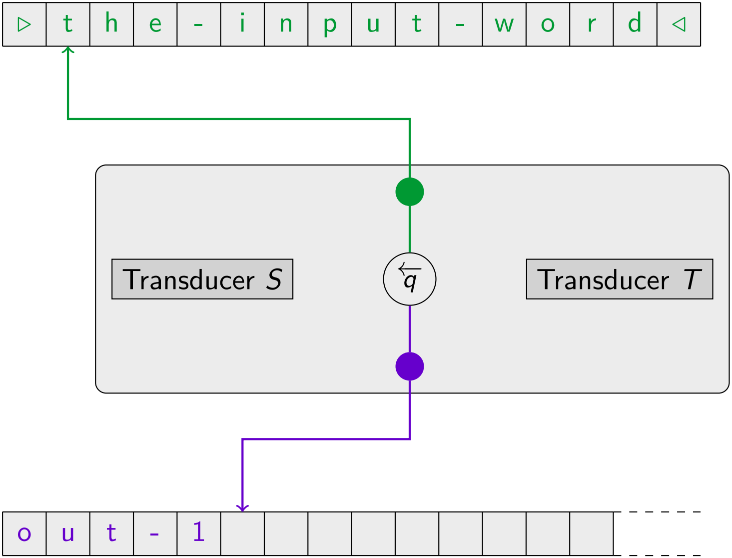

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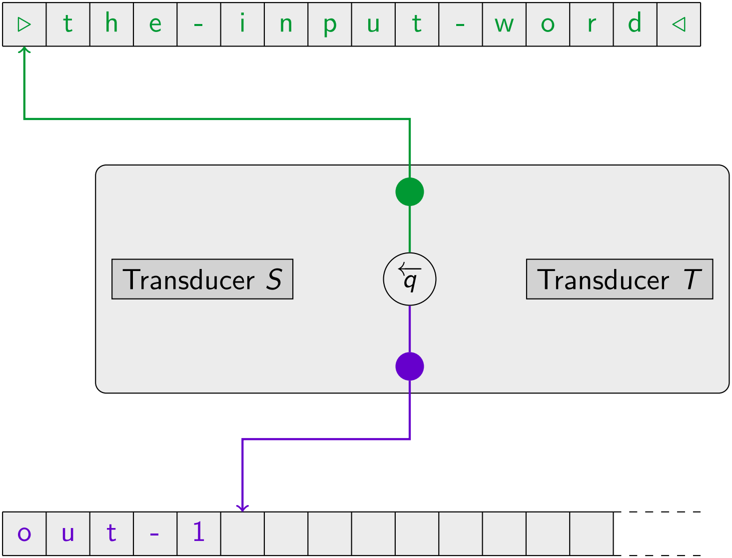

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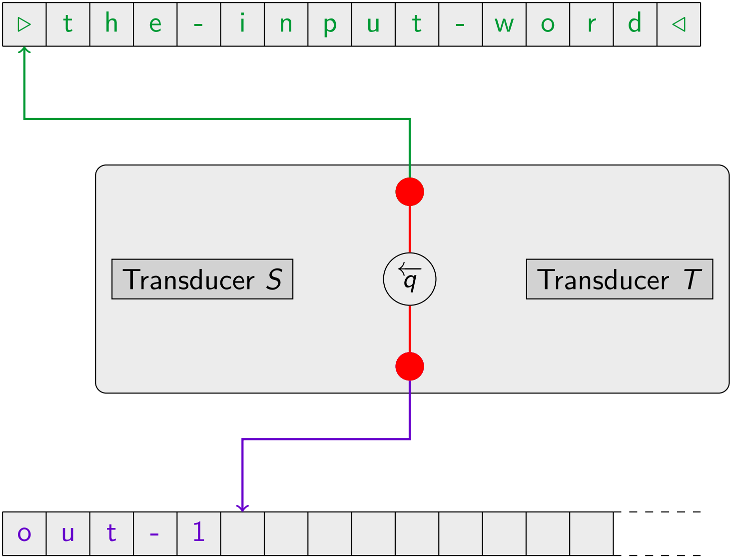

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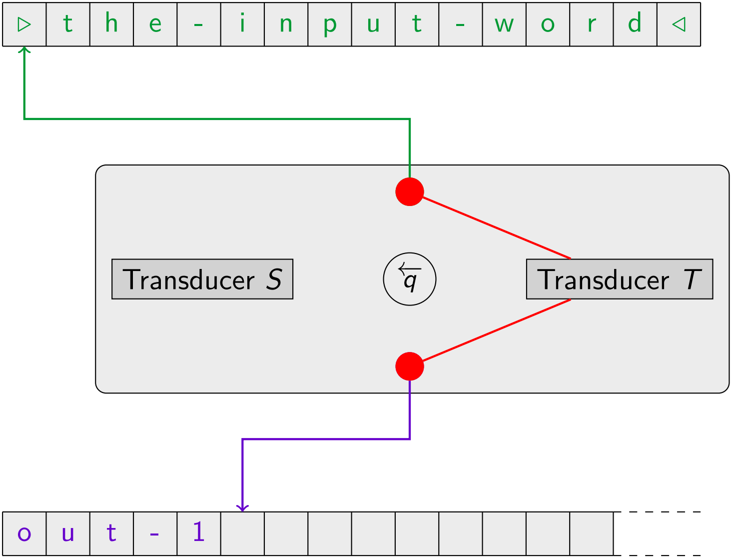

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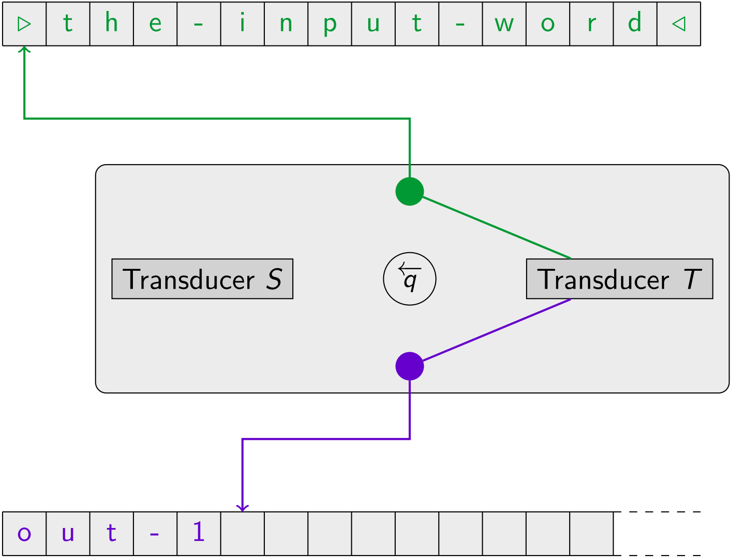

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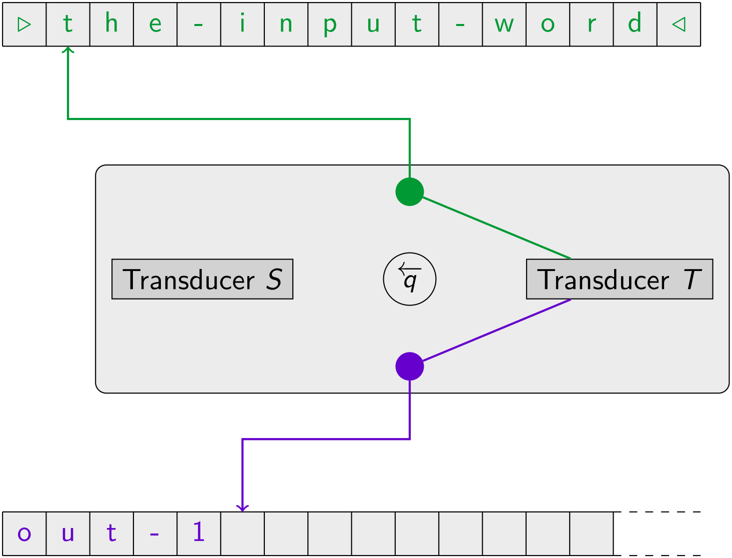

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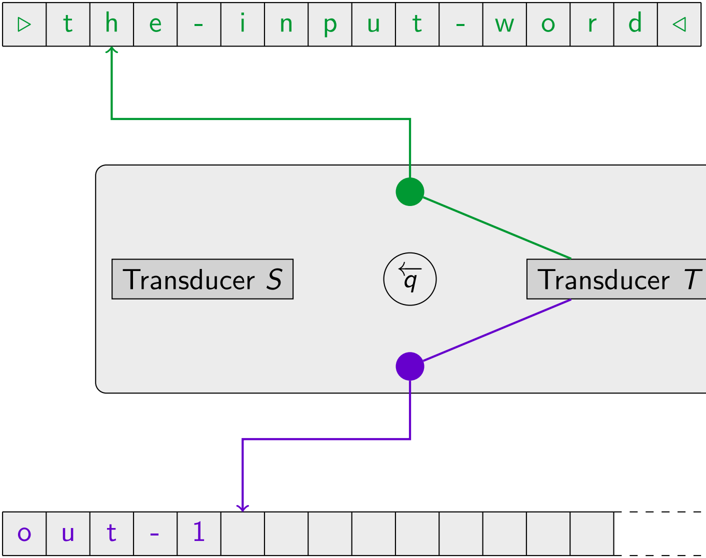

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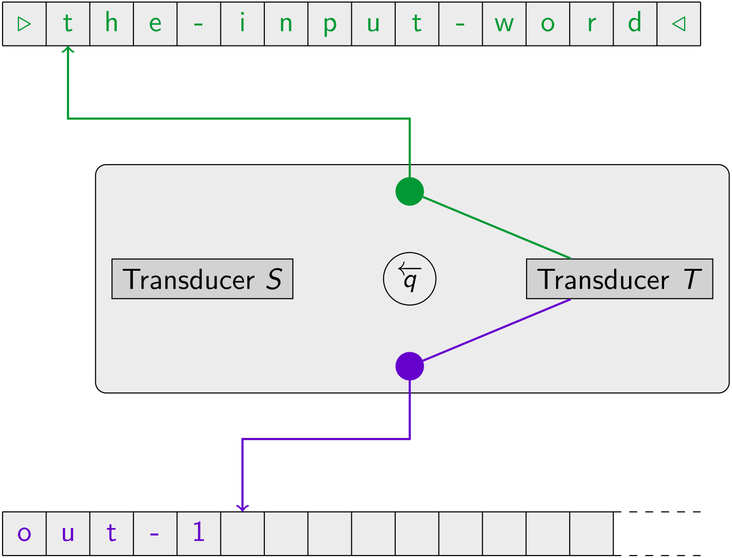

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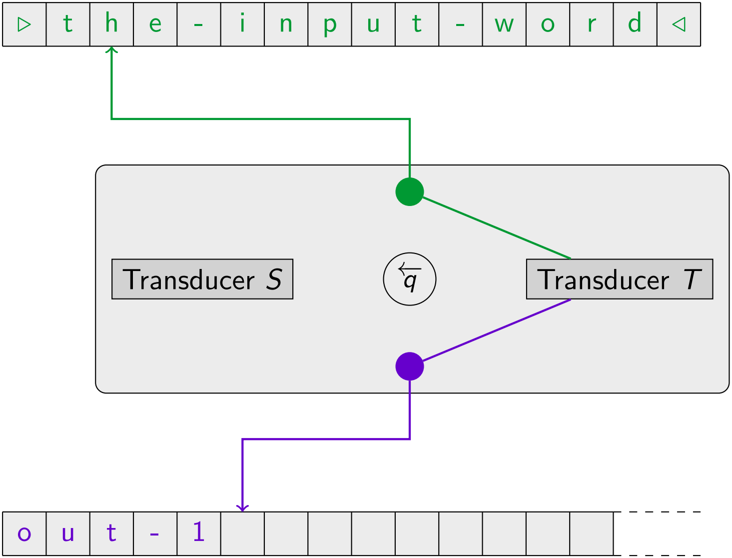

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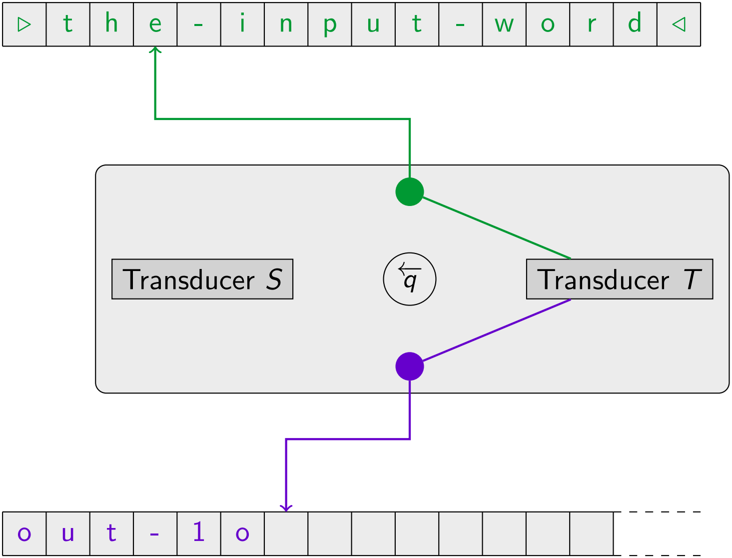

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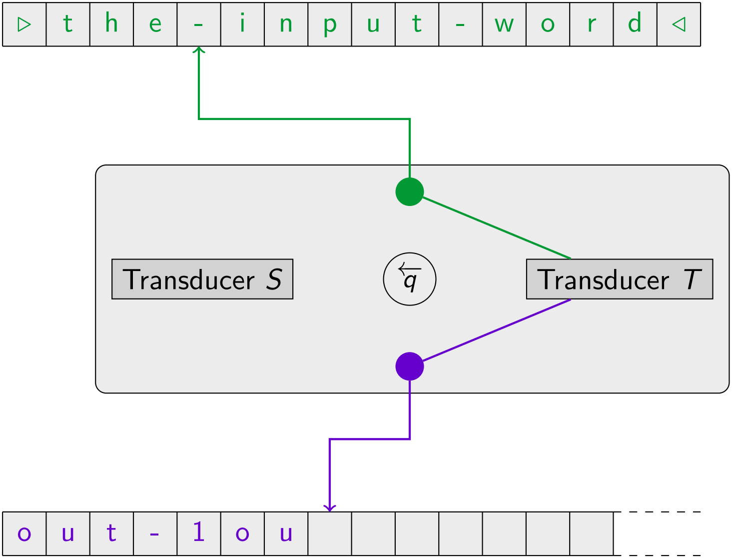

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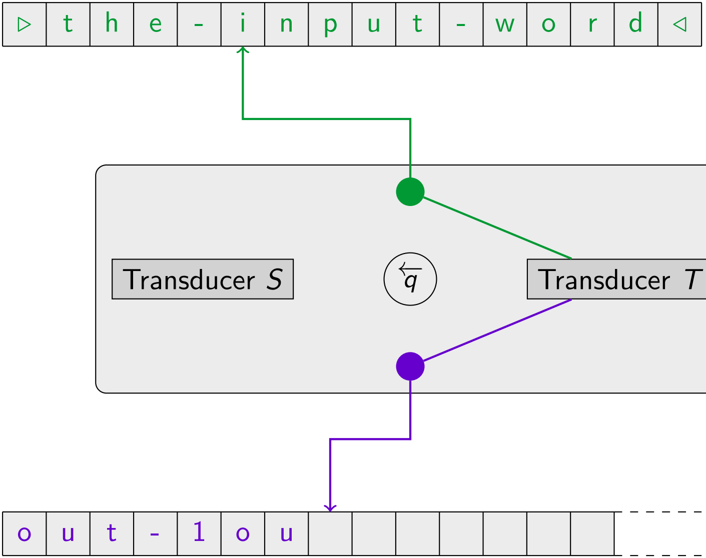

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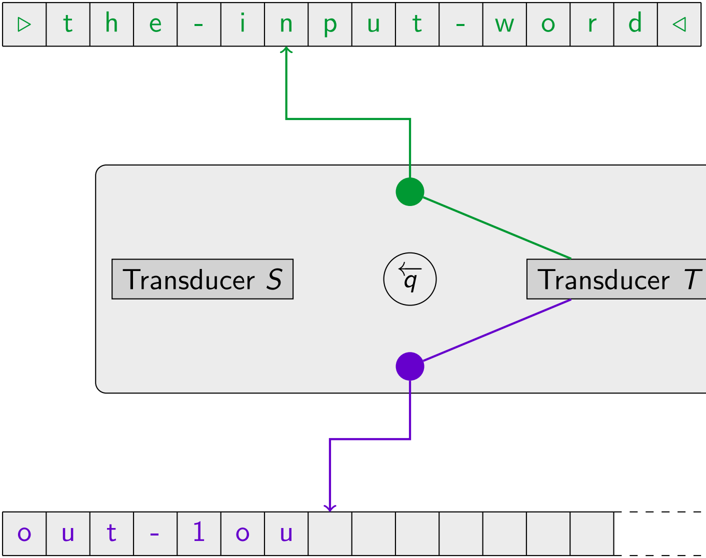

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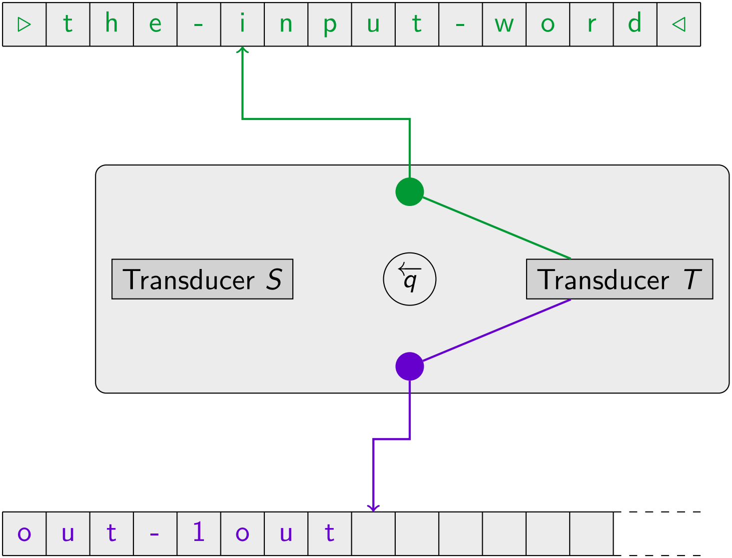

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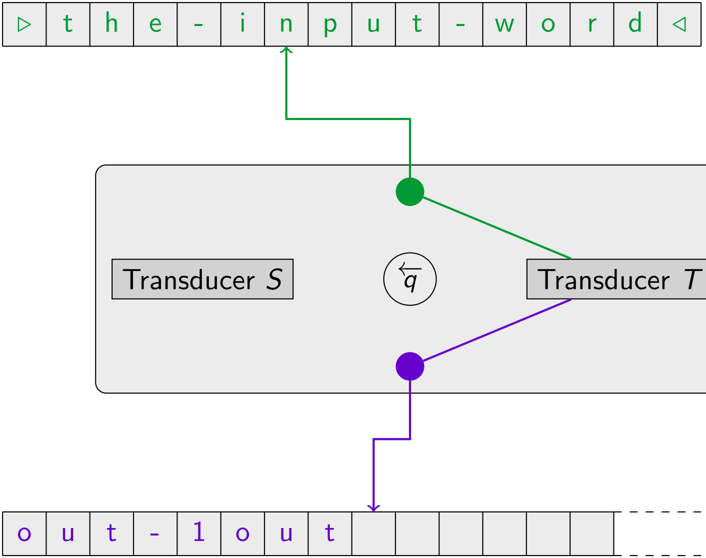

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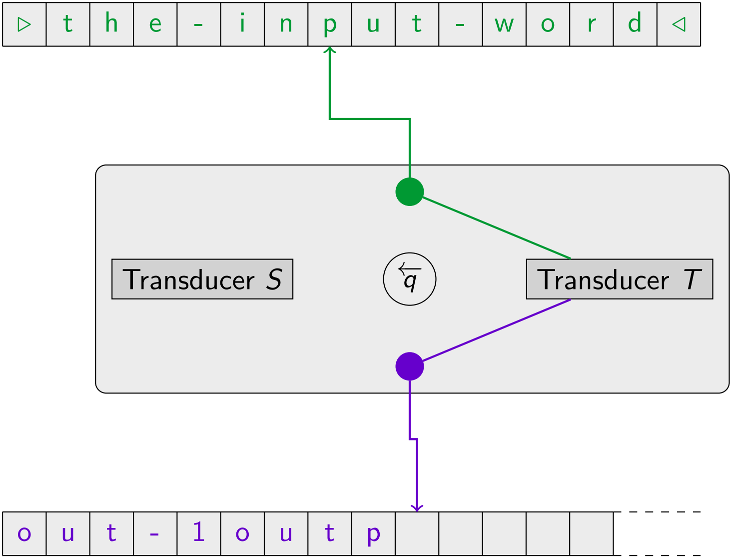

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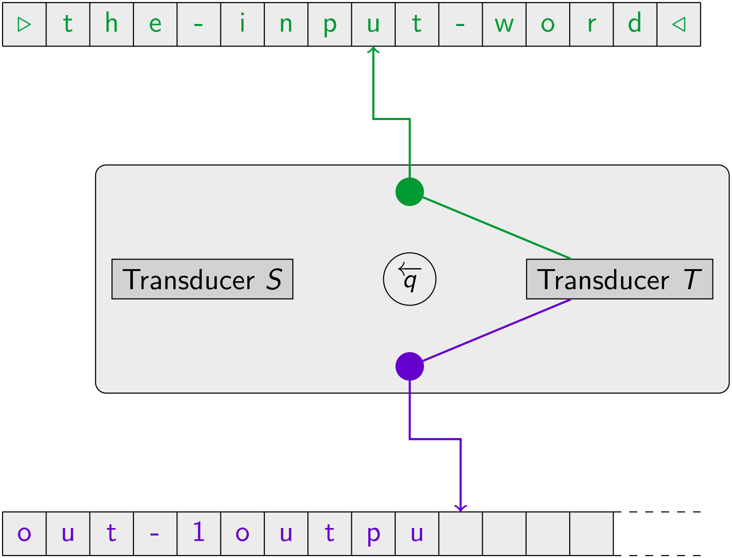

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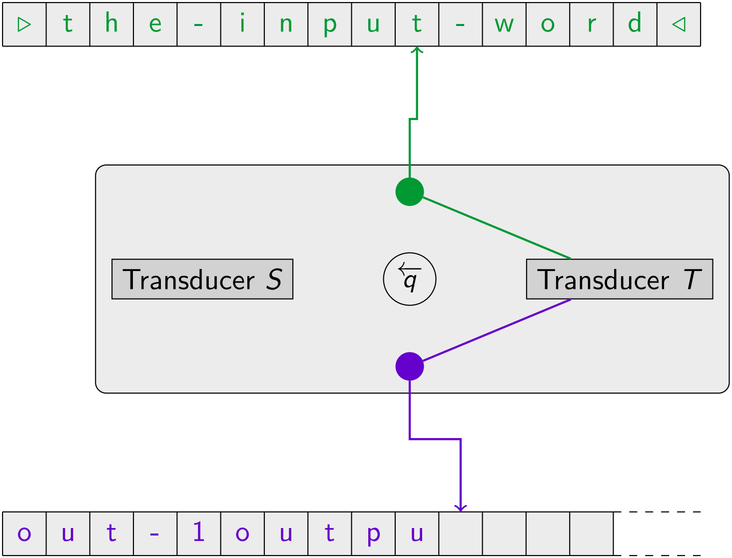

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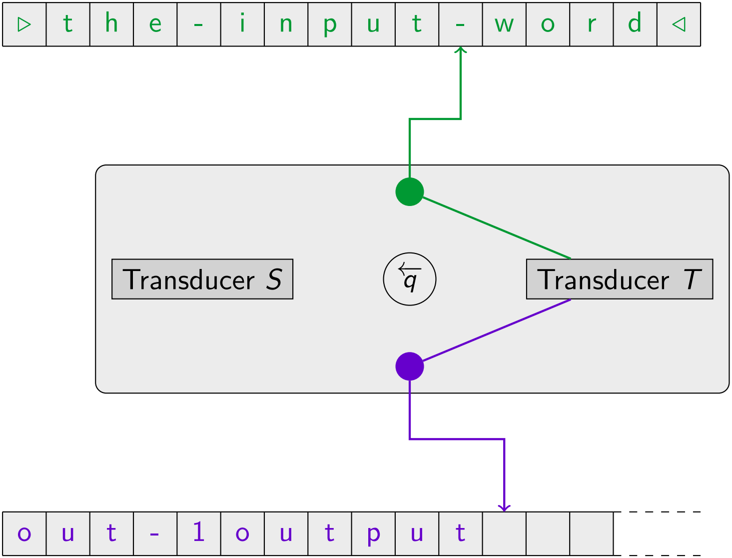

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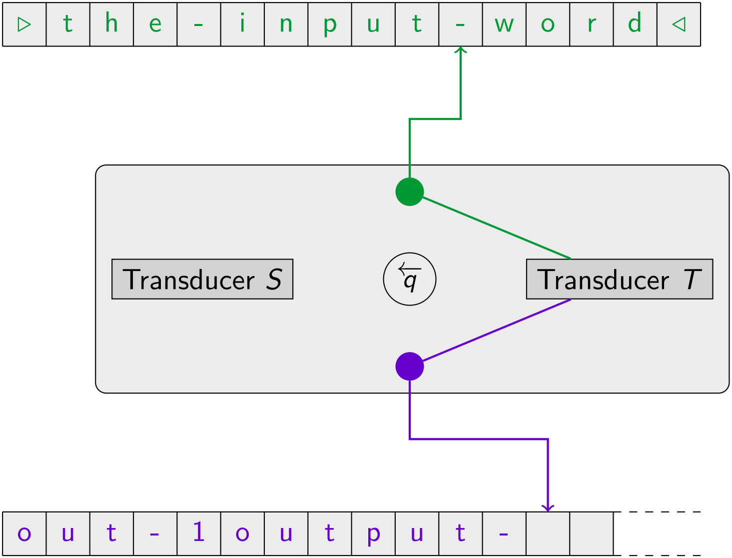

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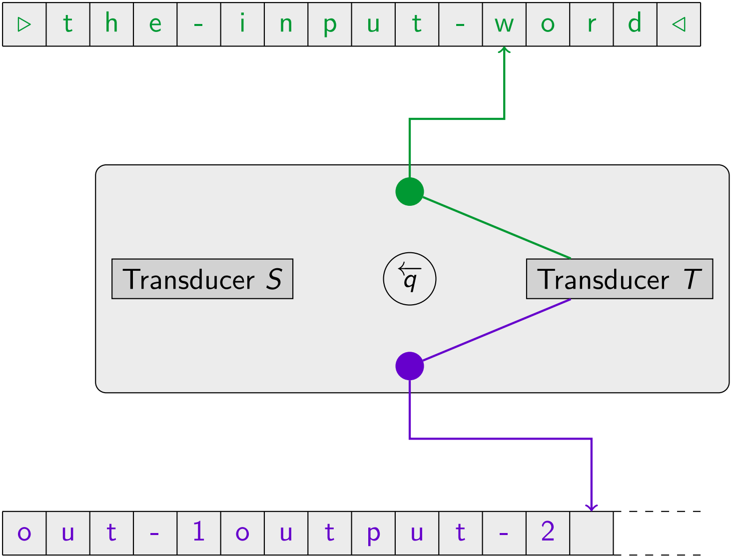

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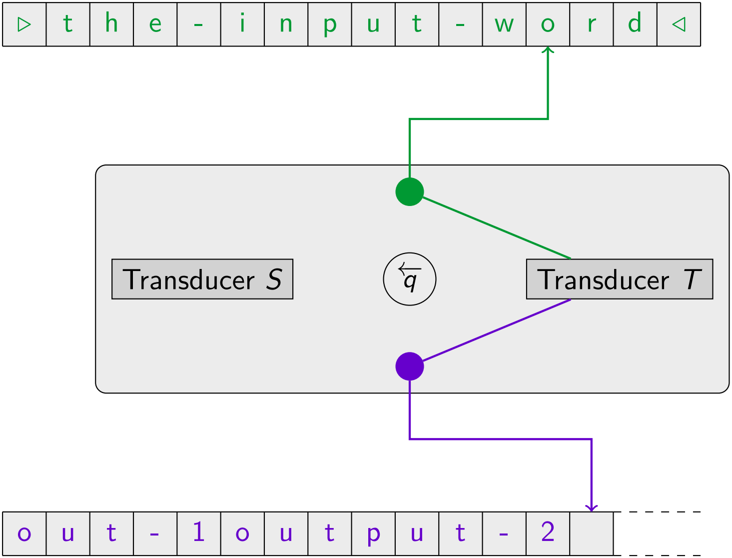

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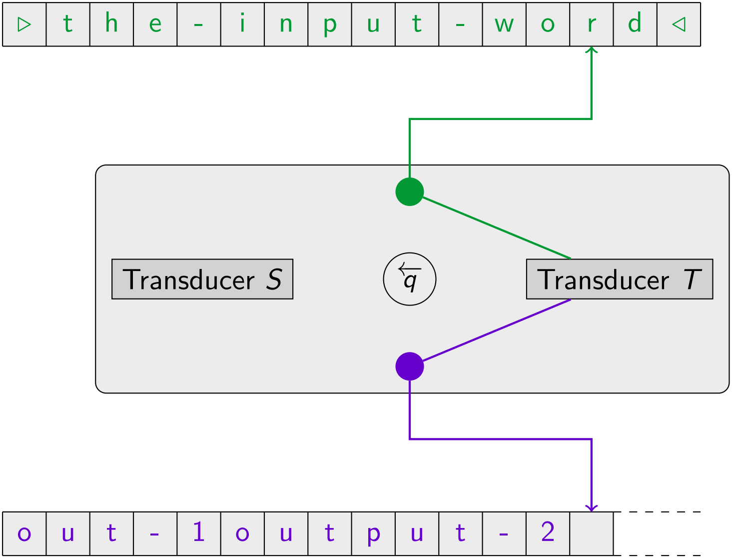

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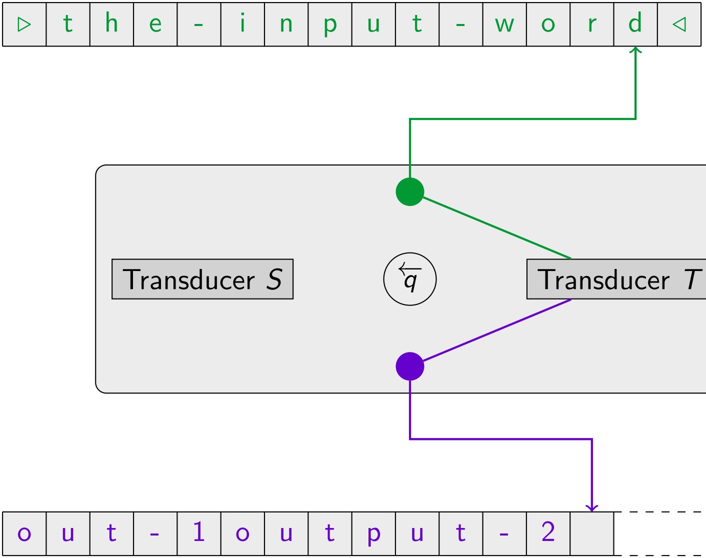

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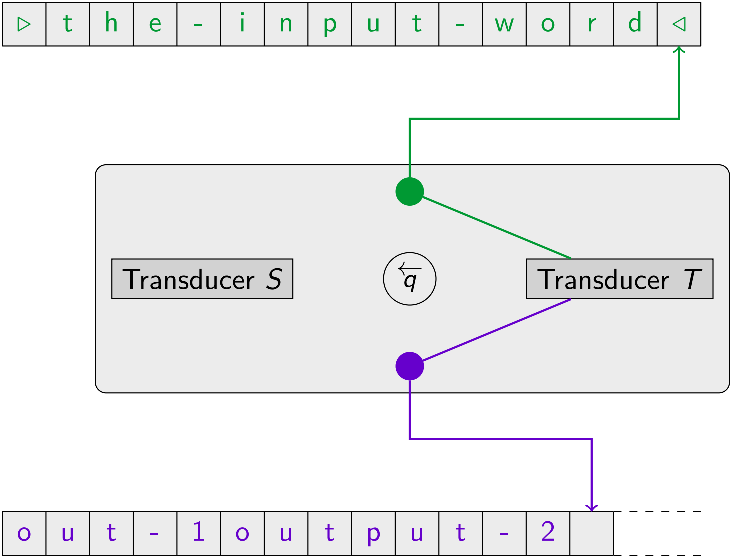

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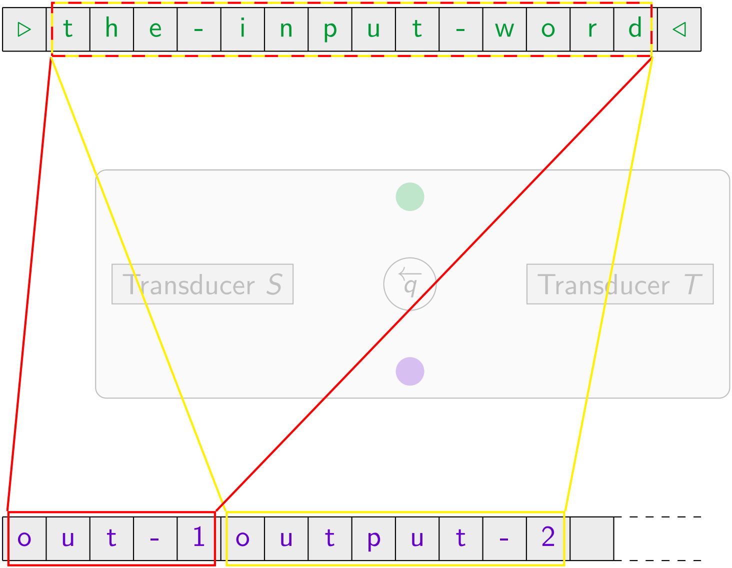

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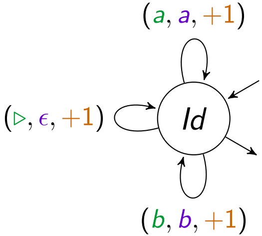

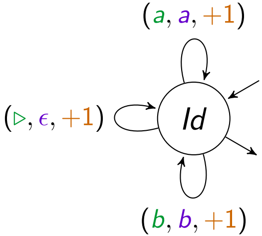

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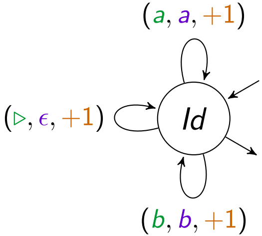
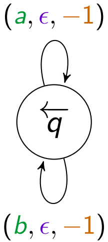
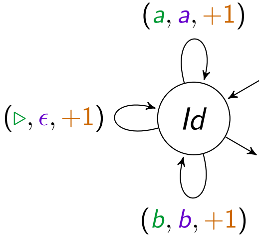




back to $\triangleright$



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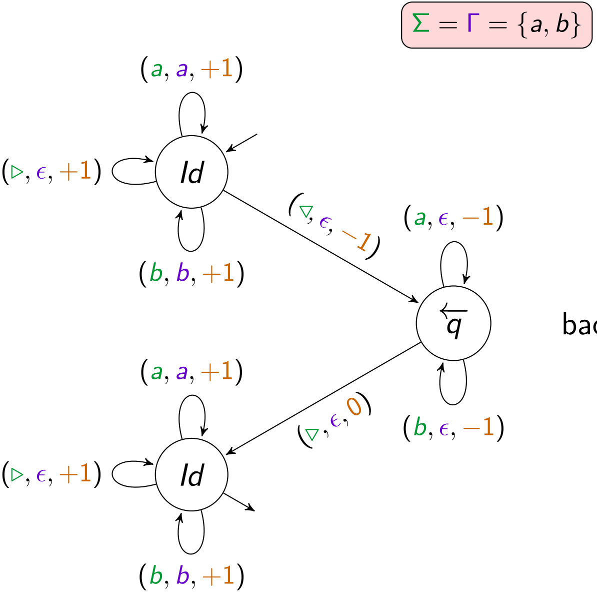

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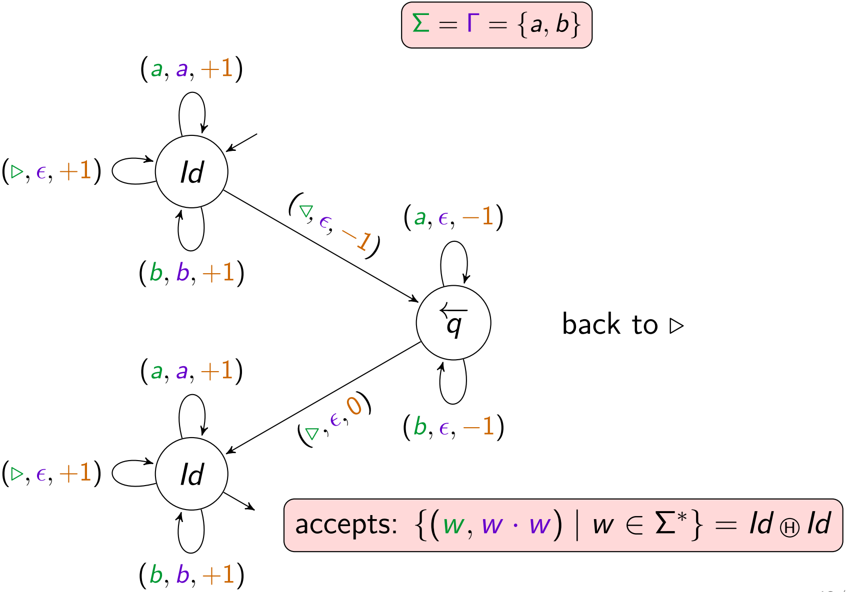

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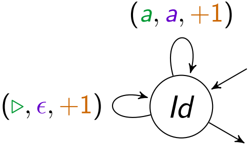

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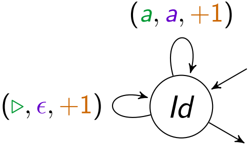

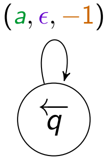

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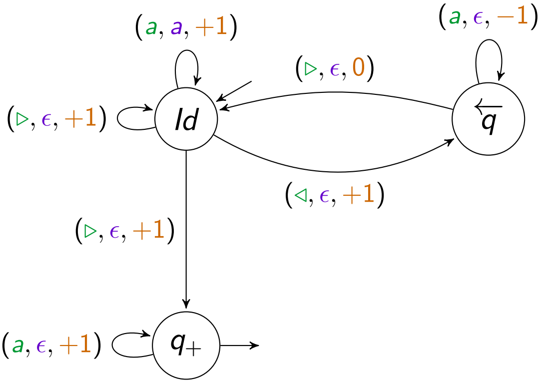

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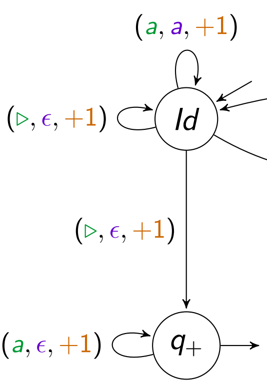

back to $\triangleright$

accepts: $\left\{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\right\} = Id^{\mathsf{H}\star}$

Hadamard-rational

Definition

$s \in \mathcal{P}(\Gamma^*)\langle\langle\Sigma^*\rangle\rangle$ is H-Rat if

$$s = \sum_i \alpha_i \circledH \beta_i^{\mathrm{H}\star}$$

where the sum is finite and α_is and β_is are rational.

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Lemma

If Γ^* is commutative,
then H-Rat is closed under finite sum, H-product and H-star.

Main Result

Definition

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Theorem

Unary two-way transducers accepts exactly H-Rat series.

Analogy with Probabilistic Automata

Theorem (Anselmo,Bertoni,1994)

Acceptation probability of two-way finite automata is of the form:

$$\tau(w) = \alpha(w) \times \frac{1}{\beta(w)}$$

where α *and* β *are rational series of* $\mathbb{Q}\langle\langle \Sigma^* \rangle\rangle$.

Known results

Theorem (Engelfriet, Hoogeboom)

Two-way transducers	***versus***	***MSO logic***

Known results

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unary alphabets?

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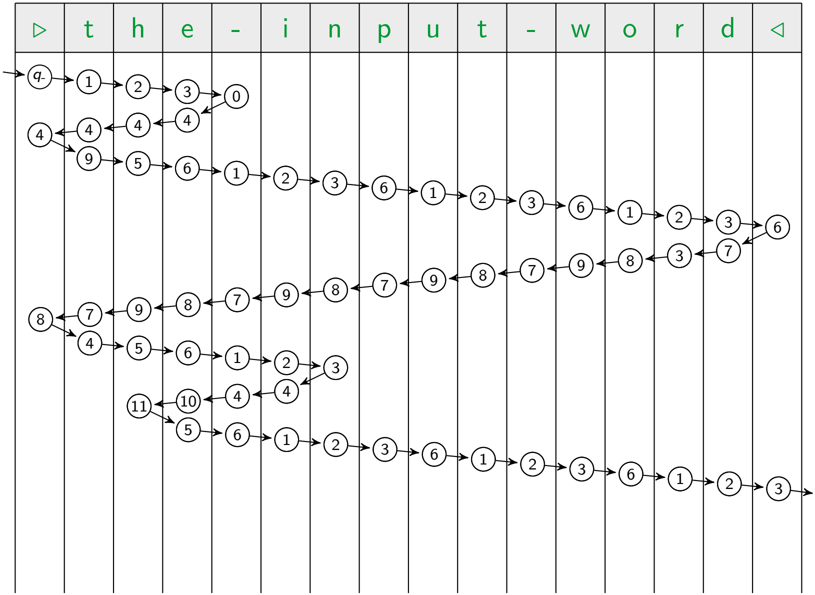
Crossing sequences...



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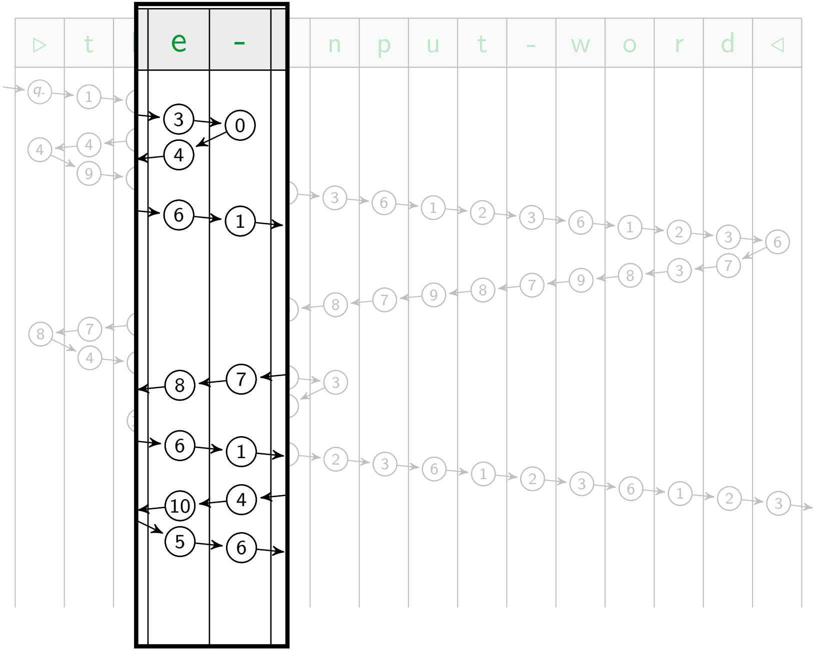

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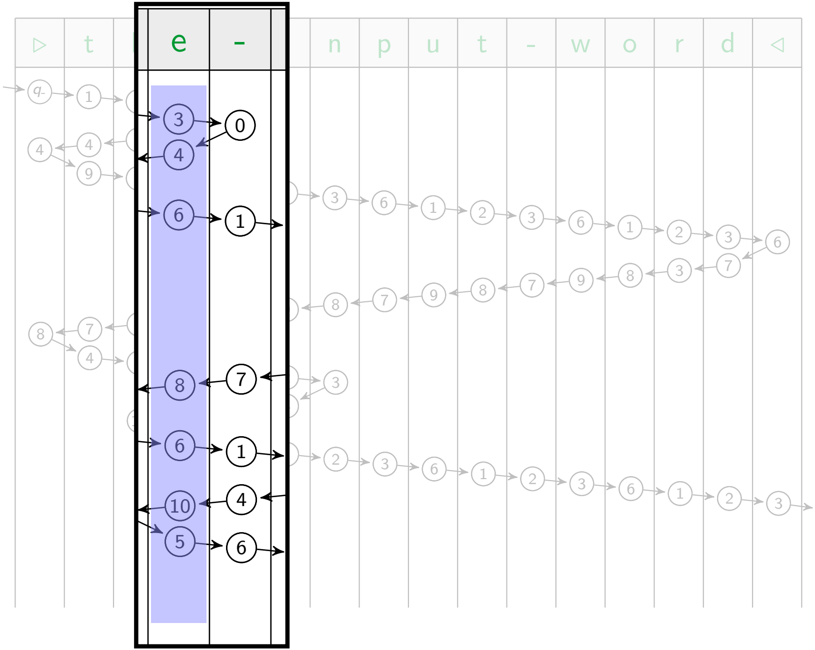

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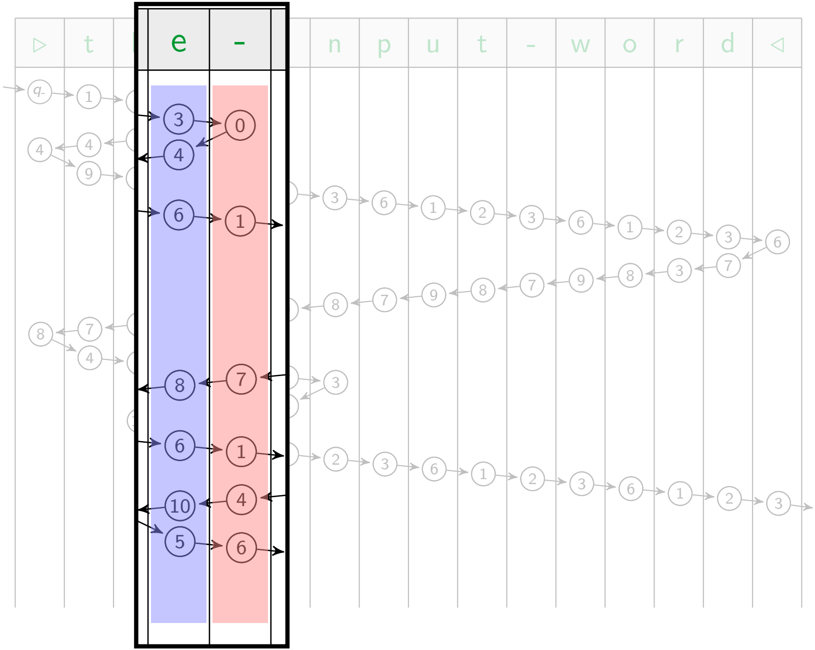

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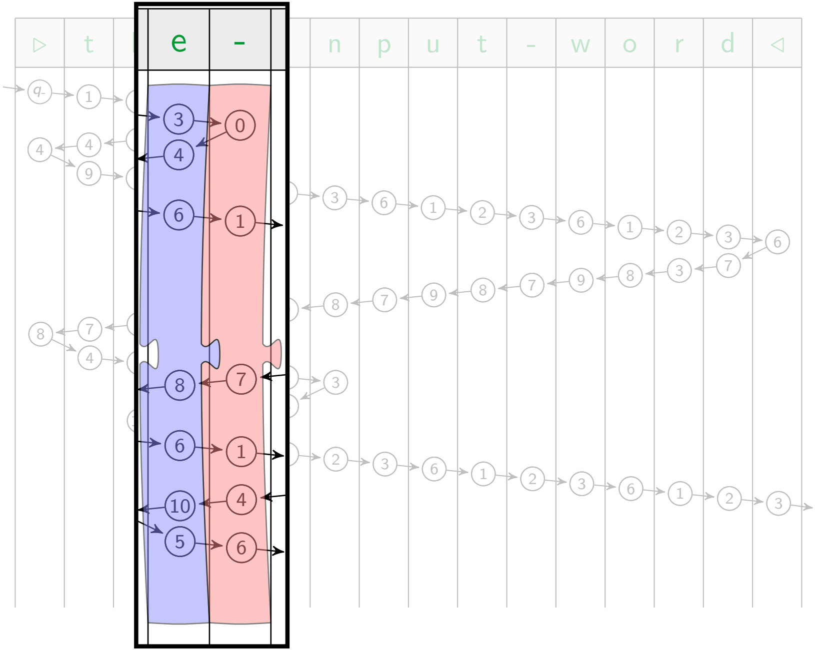

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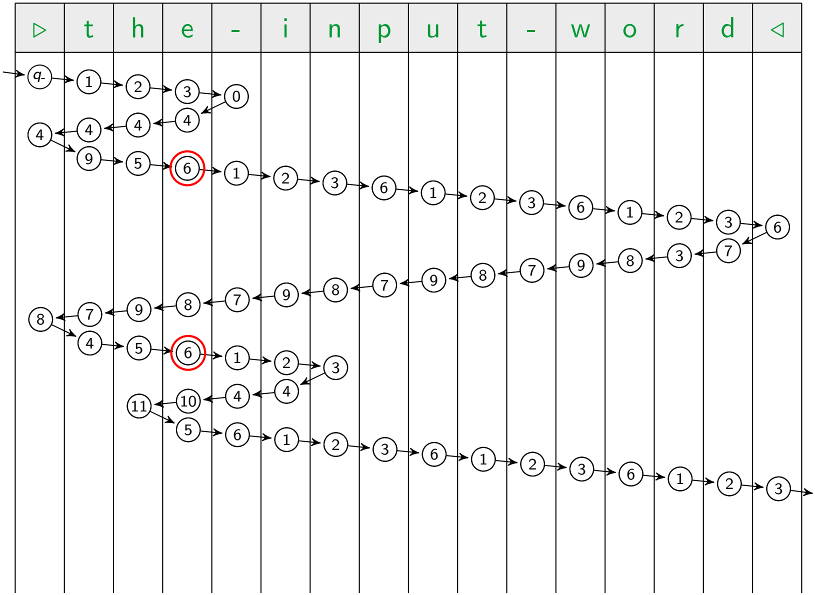

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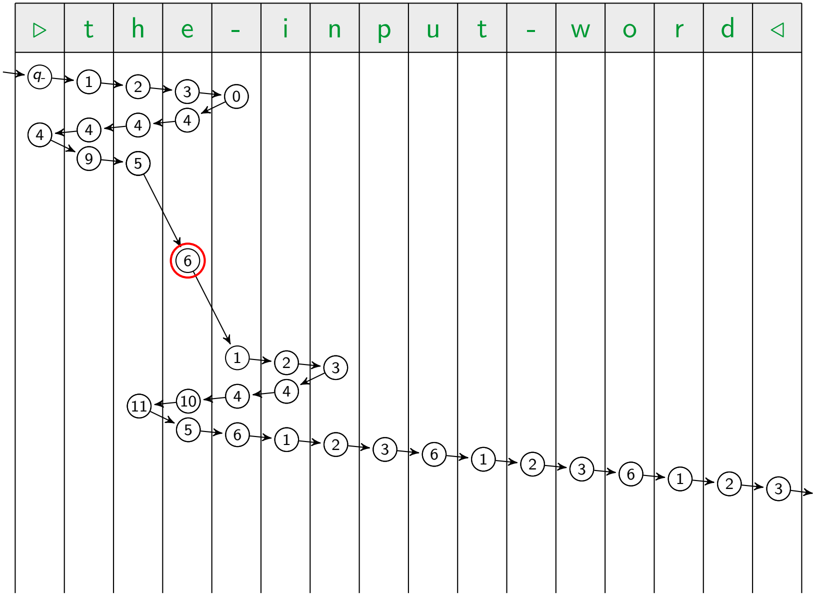

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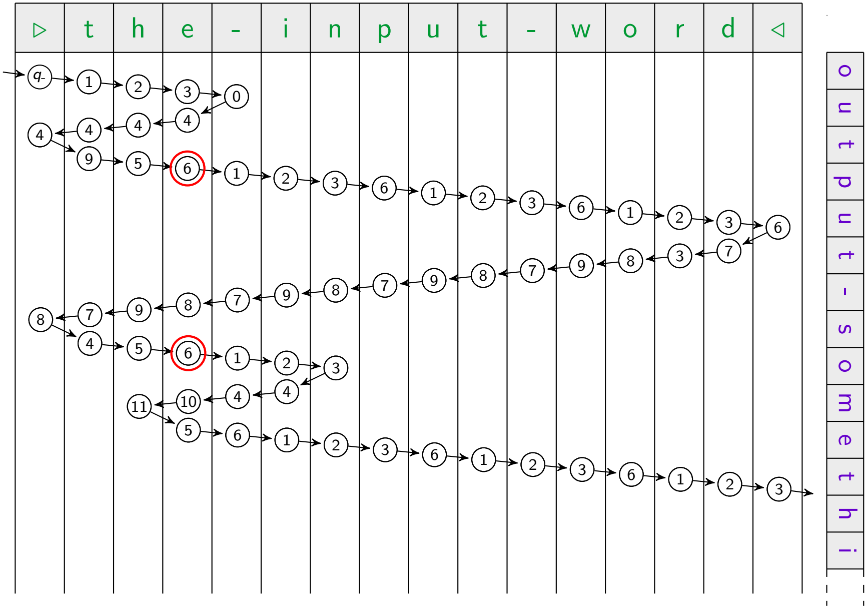

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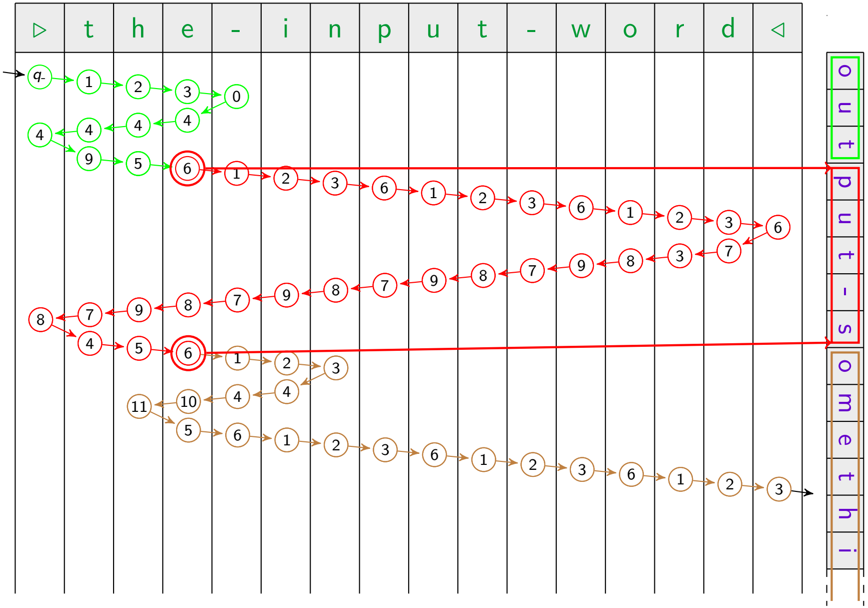

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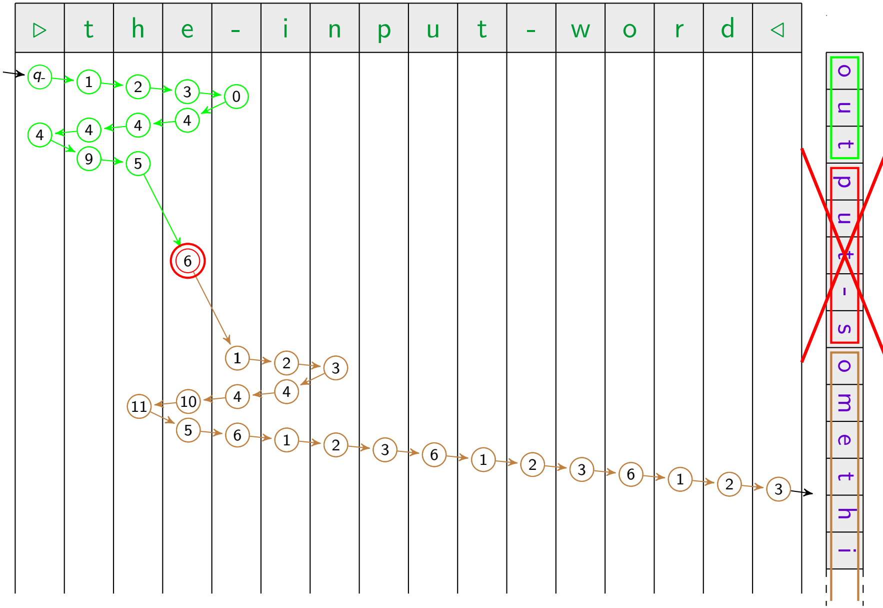

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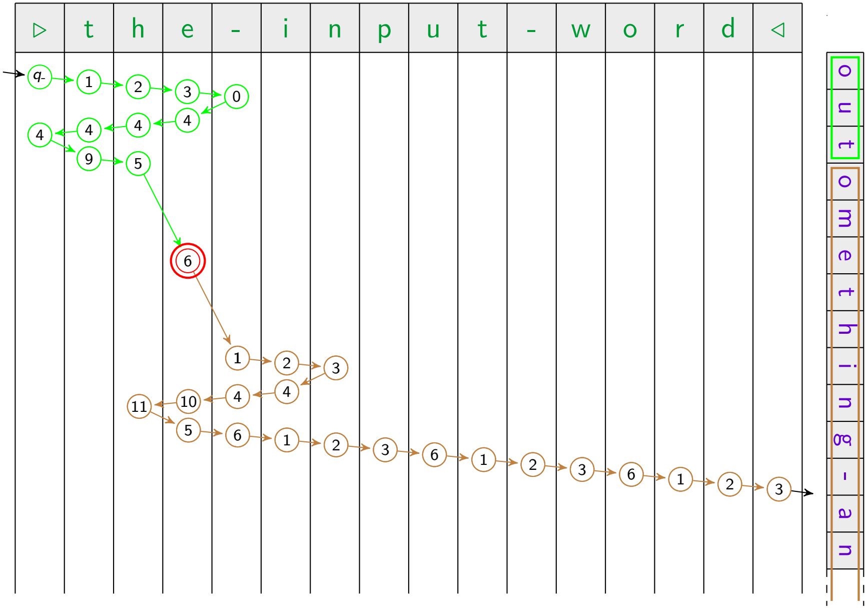

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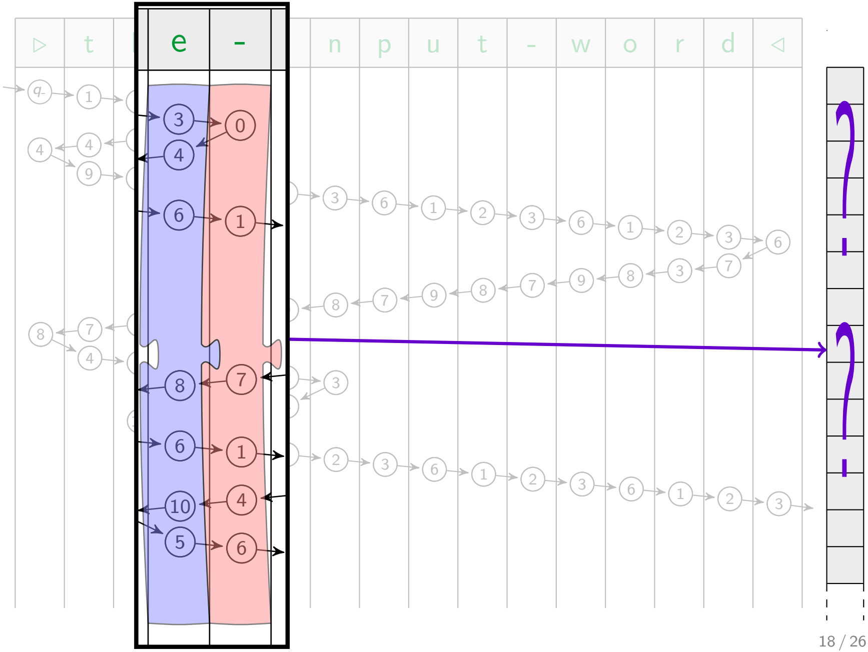

Get around the problems. . .

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Get around the problems...

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- Consider the case $\Gamma = \{a\}$, or parikh-equivalence.

Particular transducers

Γ^* is commutative

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For any transducer accepting a relation R,

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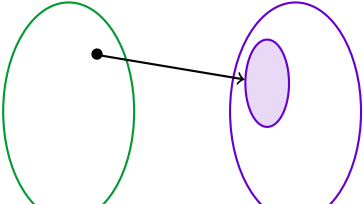

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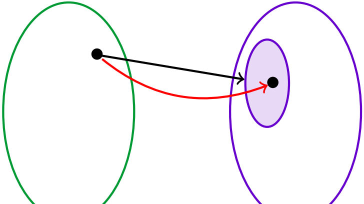

Input unary case

$\Sigma = \{a\}$

Lemma

Central loops of a two-way transducer produce finitely many rational output languages.

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Central loops of a two-way transducer produce finitely many rational output languages.

We can take into account central loops in one-way simulation.

Get around the problems...

$\boxed{\Sigma = \{a\}}$

- Consider only loop-free runs,
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Get around the problems...

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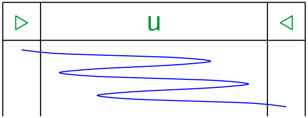
- Consider particular parts of run: hits (from border to border)
- Consider the case $\Gamma = \{a\}$, or parikh-equivalence.

One-way simulation of loop-free hits

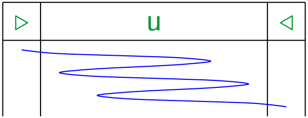
Hit: a border to border run



One-way simulation of loop-free hits

Hit: a border to border run

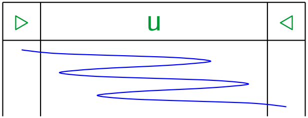
- reading u



One-way simulation of loop-free hits

Hit: a border to border run

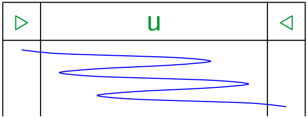
- reading u
- outputing v



One-way simulation of loop-free hits

Hit: a border to border run

- reading u
- outputing v
- no visit to endmarkers



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define a relation R_{b_i, b_j}

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Hit: a border to border run

- reading u
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$$\begin{tikzcd} b_i \arrow[dr, "\in"'] & & b_j \arrow[dl, "\in"] \\ & \boxed{Q \times \{\triangleright, \triangleleft\}} & \end{tikzcd}$$

One-way simulation of loop-free hits

Hit: a border to border run

- reading u
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define a relation R_{b_i, b_j}

$b_i \in Q \times \{\triangleright, \triangleleft\}$, $b_j \in Q \times \{\triangleright, \triangleleft\}$

$$HIT = \left.\overbrace{\begin{pmatrix} R_{0,0} & R_{0,1} & \cdot & \cdot & \cdot & R_{0,2} \\ R_{1,0} & R_{1,1} & \cdot & \cdot & \cdot & R_{1,2} \\ \cdot & & & & & \cdot \\ \cdot & & & R_{i,j} & & \cdot \\ \cdot & & & & & \cdot \\ R_{k,0} & R_{k,1} & \cdot & \cdot & \cdot & R_{k,k} \end{pmatrix}}^{2|Q|}\right\} 2|Q|$$

One-way simulation of loop-free hits

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define a relation R_{b_i,b_j}

Rational

$$HIT = \left.\overbrace{\begin{pmatrix} R_{0,0} & R_{0,1} & \cdot & \cdot & \cdot & R_{0,2} \\ R_{1,0} & R_{1,1} & \cdot & \cdot & \cdot & R_{1,2} \\ \cdot & & & & & \cdot \\ \cdot & & & R_{i,j} & & \cdot \\ \cdot & & & & & \cdot \\ R_{k,0} & R_{k,1} & \cdot & \cdot & \cdot & R_{k,k} \end{pmatrix}}^{2|Q|}\right\} 2|Q|$$

Composition of hits

Given:

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we may compose them into a b_0 to b_1 run over u producing $v_0 v_1$.

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double-hit relations are:

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triple-hit relations are:

$$R^{(3)}_{b_0,b_1} = \bigcup_{b_{x_1},b_{x_2} \in Q \times \{\triangleright,\triangleleft\}} R_{b_0,b_{x_1}} Ⓗ R_{b_{x_1},b_{x_2}} Ⓗ R_{b_{x_2},b_1}$$

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multi-hit relations are:

$$R^{(\text{H}\star)}_{b_0,b_1} = \bigcup_{n\in\mathbb{N}} \quad \bigcup_{b_{x_1},\dots,b_{x_n}} R_{b_0,b_{x_1}} \circledH \dots, R_{b_{x_n},b_1}$$

coefficient (b_0, b_1) of $HIT^{\text{H}\star}$.

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Theorem

R is in H-Rat. *(by closure properties of H-Rat)*

Conclusion

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Grazie infinite.