Caractérisation algébrique des relations acceptées par transducteurs bidirectionnels unaires

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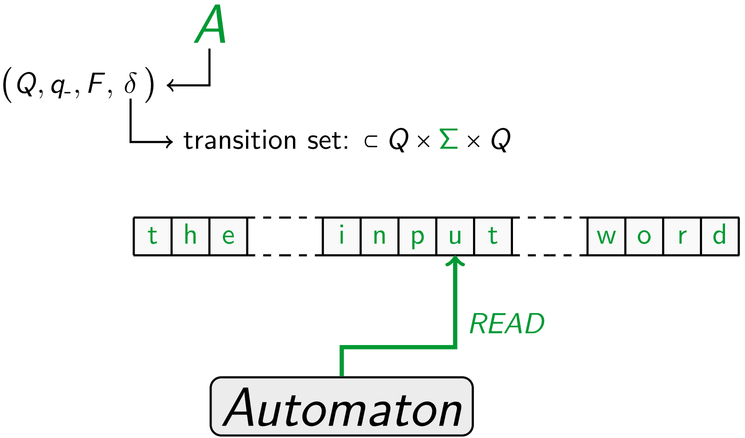
1-way automaton over Σ

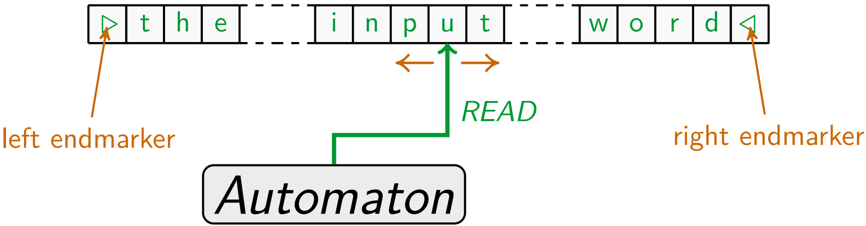


2-way automaton over Σ

A

(Q, q_{-}, F, δ)

δ: transition set: $\subset Q \times \Sigma_{\triangleright,\triangleleft} \times \{-1, 0, 1\} \times Q$

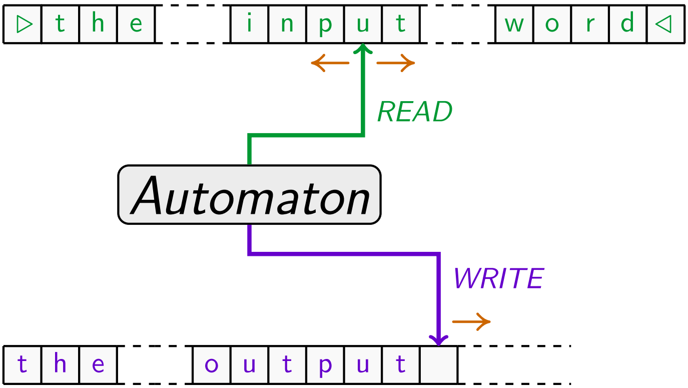


2-way transducer over Σ, Γ

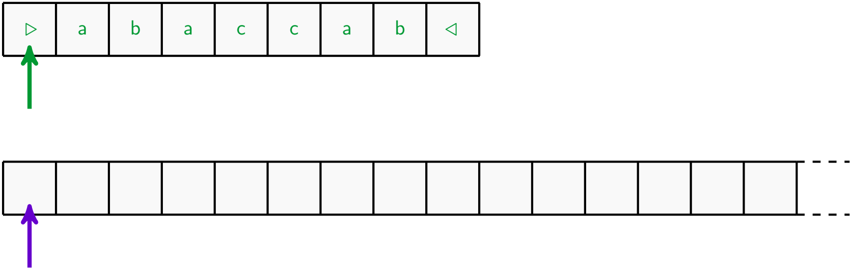
$$(A, \phi)$$

$(Q, q_-, F, \delta) \hookleftarrow A$

$\phi \hookrightarrow$ production function: $\delta \to Rat(\Gamma^*)$



A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$



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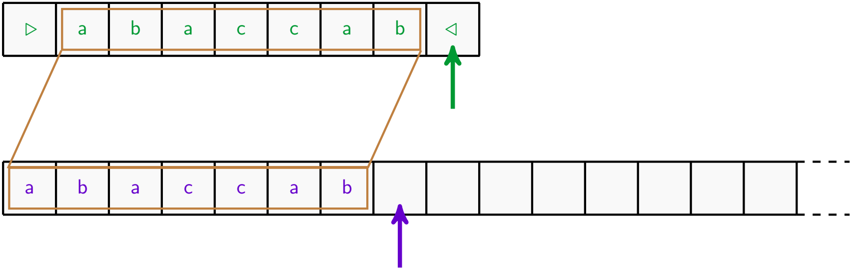

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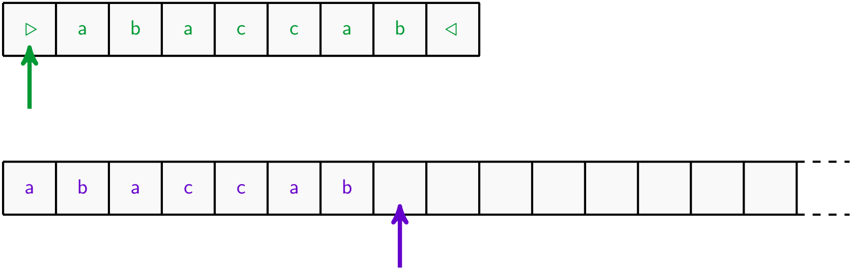

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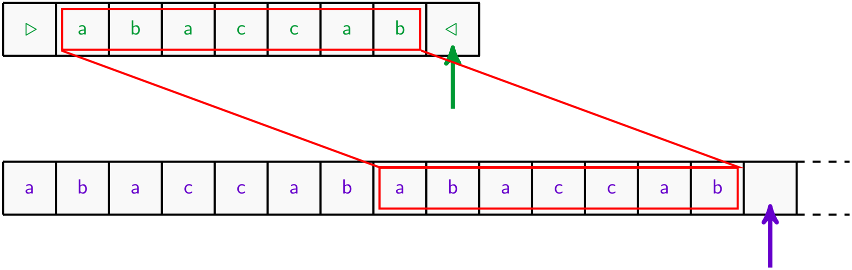

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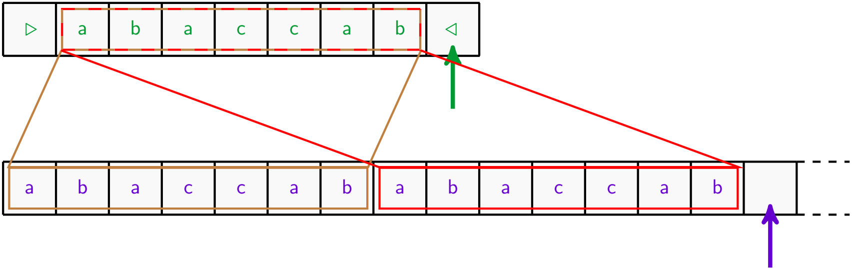

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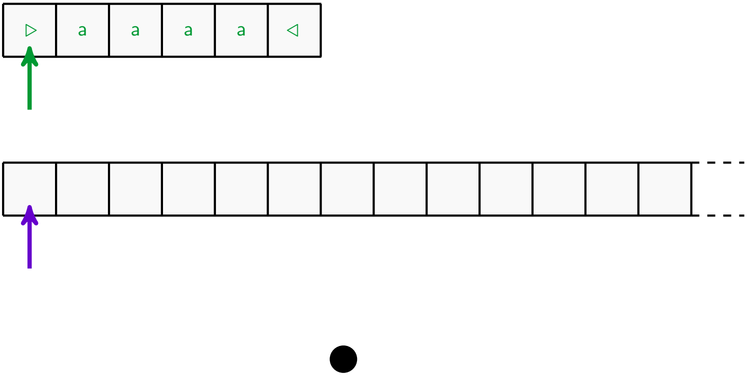
Another example: $\mathit{UnaryMult} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$



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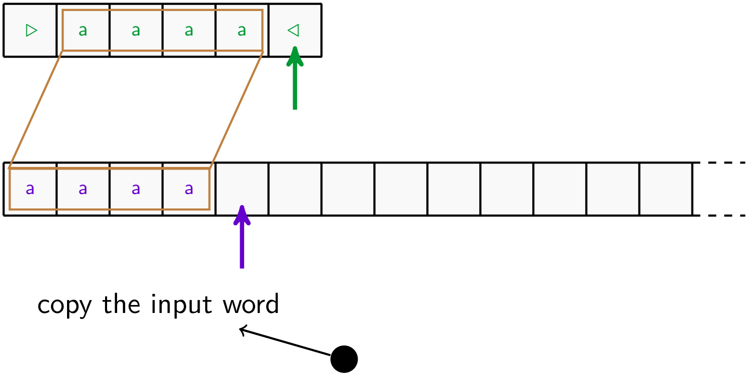

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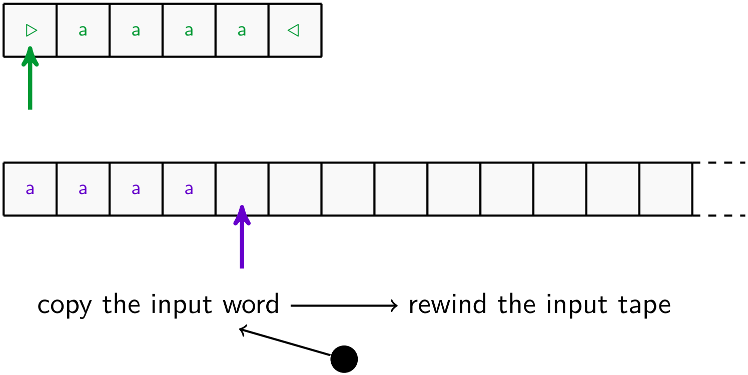

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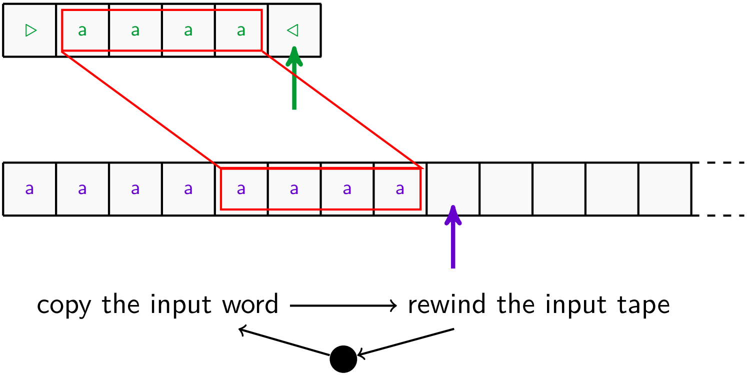

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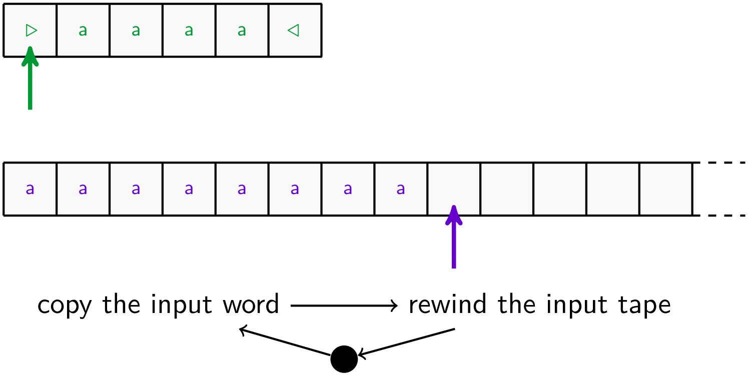

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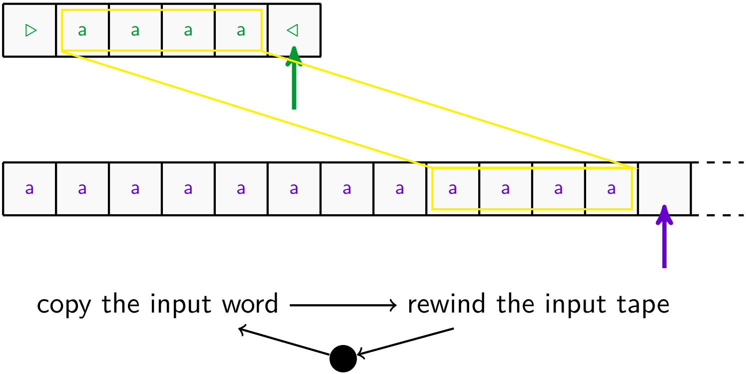

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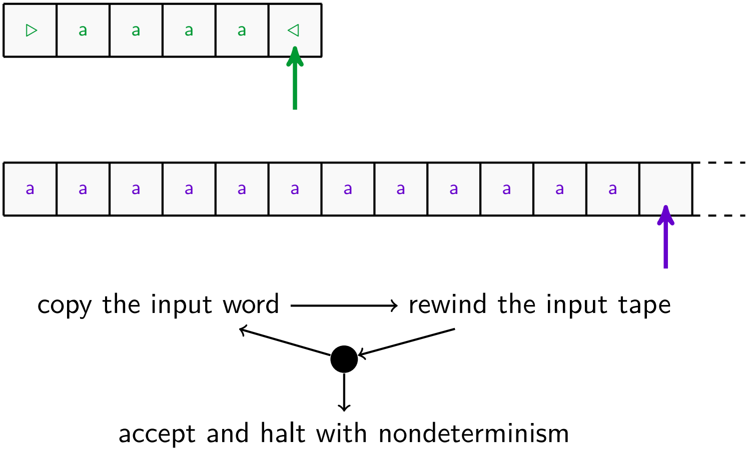

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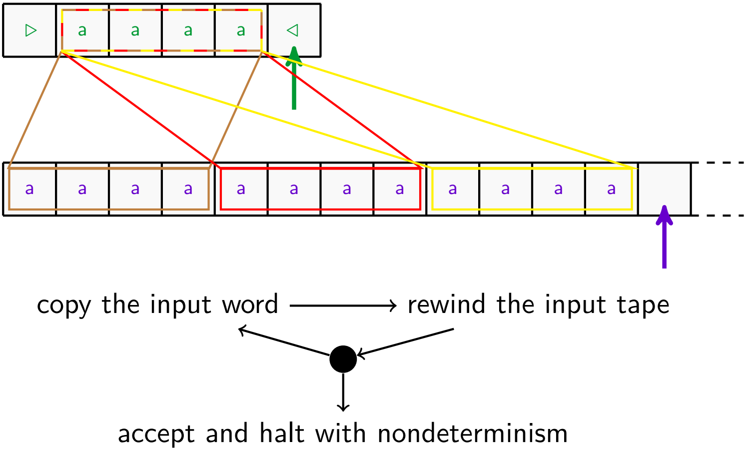

Rational operations

- Union

$$R_1 \cup R_2$$

- Componentwise concatenation

$$R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\}$$

- Kleene star

$$R^* = \{(u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R\}$$

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Definition ($Rat(\Sigma^* \times \Gamma^*)$)

The class of rational relations is the smallest class:

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Theorem (Elgot, Mezei - 1965)

1-way transducers = the class of rational relations.

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- H-product

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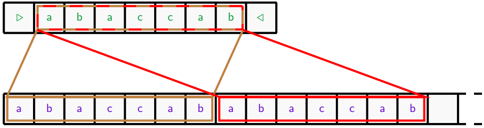
Example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\} = Identity \circledH Identity$



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$$R^{\mathrm{H}\star} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \ (u, v_i) \in R\}$$

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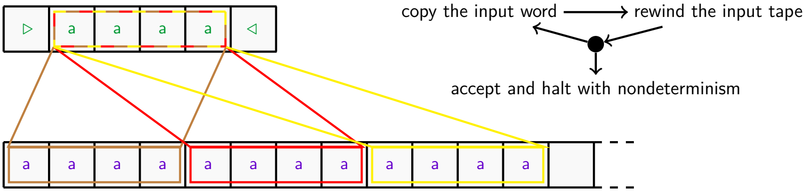

H-Rat relations

Definition

A relation R is in $H\text{-}Rat(\Sigma^* \times \Gamma^*)$ if

$$R = \bigcup_{0 \leq i \leq n} A_i \circledH B_i^{\mathsf{H}\star}$$

where for each i, A_i and B_i are rational relations.

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$$\left\{ (a^n, a^{kn}) \mid k, n \in \mathbb{N},\ k \geq 2 \right\}$$

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$$\left\{(a^n, a^{2n}) \mid n \in \mathbb{N}\right\} \circledast \left\{(a^n, a^n) \mid n \in \mathbb{N}\right\}^{\mathsf{H}\star}$$

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Main result

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

Theorem (Elgot, Mezei - 1965)

1-way transducers $=$ *the class of* *rational relations*.

Main result

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

Theorem (*This talk*)

2-way transducers = *the class of H-Rat relations*.

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When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

Theorem (This talk)

2-way transducers = *the class of H-Rat relations*.

Proof

- $\supseteq$: *easy*
- $\subseteq$: *difficult part*

Known results

- 2-way functional = MSO definable functions

  [Engelfriet, Hoogeboom - 2001]
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When $\Gamma = \{a\}$:

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When $\Gamma = \{a\}$:

- 2-way unambiguous $\longrightarrow$ 1-way

  [Anselmo - 1990]
- 2-way unambiguous = 2-way deterministic

  [Carnino, Lombardy - 2014]

From *H-Rat* to 2-way transducers

Property

The family of relations accepted by 2-way transducers is closed under $\cup$, $\textcircled{H}$ *and* $_H\star$.

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- $R_1 \cup R_2$:
  - simulate T_1 or T_2

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The family of relations accepted by 2-way transducers is closed under $\cup$, Ⓗ *and* $_H\star$.

Proof.

- $R_1 \cup R_2$:
  - simulate T_1 or T_2
- R_1 Ⓗ R_2:
  - simulate T_1
  - rewind the input tape
  - simulate T_2

□

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- $R_1 \textcircled{H} R_2$:
  - simulate T_1
  - rewind the input tape
  - simulate T_2
- $R^{H\star}$:
  - repeat an arbitrary number of times:
    - simulate T
    - rewind the input tape
  - reach the right endmarker and accept

□

From *H-Rat* to 2-way transducers

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The family of relations accepted by 2-way transducers is closed under $\cup$, $\circledH$ and ${}_{H}\star$.

Corollary

H-Rat $\subseteq$ *accepted by 2-way transducers*

$$\left(\bigcup_{0\leq i\leq n} A_i \circledH B_i^{\mathrm{H}\star}\right)$$

From *H-Rat* to 2-way transducers

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The family of relations accepted by 2-way transducers is closed under $\cup$, $Ⓗ$ *and* ^{H*}.

Corollary

H-Rat $\subseteq$ *accepted by* sweeping transducer

$$\left(\bigcup_{0\leq i\leq n} A_i \, Ⓗ \, B_i^{H\star}\right)$$

From 2-way transducers to *H-Rat* (unary case)

A first ingredient, a preliminary result:

Lemma

With arbitrary Σ *and* $\Gamma = \{a\}$*:*

H-Rat is closed under $\cup$, $\textcircled{H}$ *and* $H\star$.

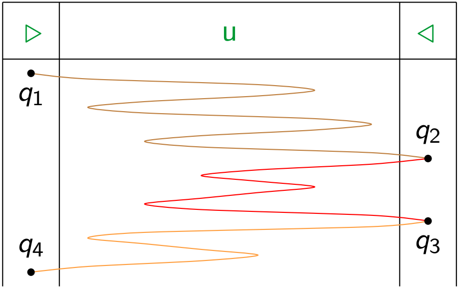
Proof.

Tedious formal computations... □

From 2-way transducers to *H-Rat* (unary case)

We fix a transducer $\mathcal{T}$.

- Consider border to border run segments;



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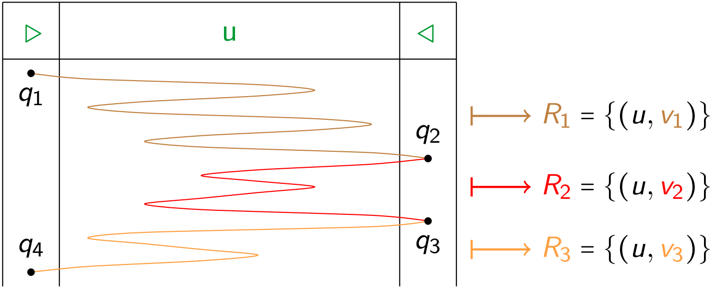

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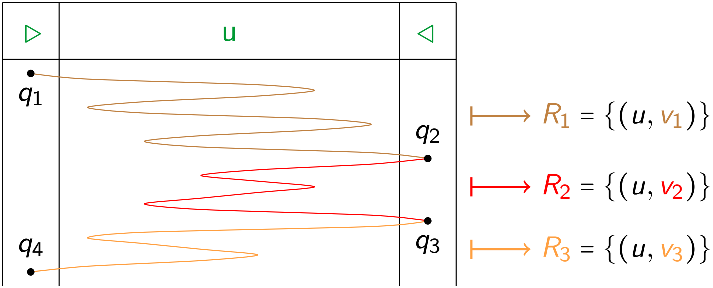

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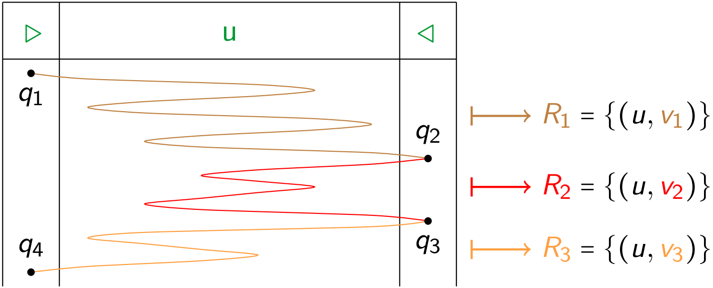
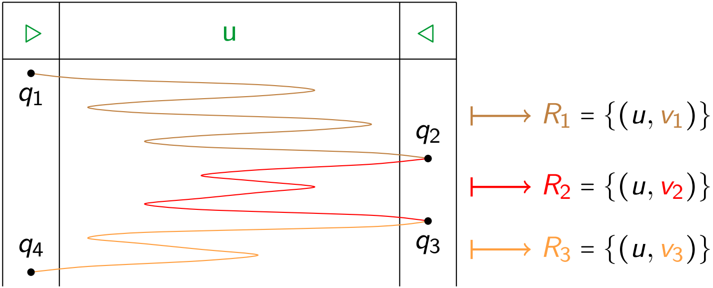


$$R_1 \circledH R_2 \circledH R_3 = \{(u, v_1 v_2 v_3)\}$$

From 2-way transducers to *H-Rat* (unary case)

We fix a transducer $\mathcal{T}$.

- Consider border to border run segments;
- Compose border to border segments;
- Conclude using the closure properties of *H-Rat*.



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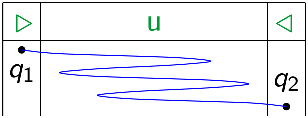

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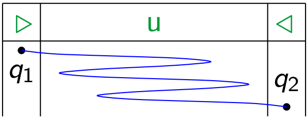

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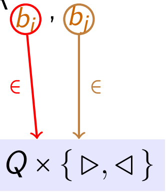

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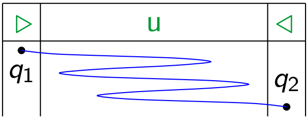

define a relation R_{b_i, b_j}

$b_i \in Q \times \{ \triangleright, \triangleleft \}$, $b_j \in Q \times \{ \triangleright, \triangleleft \}$

$$HIT = \underbrace{\overbrace{\begin{pmatrix} R_{0,0} & R_{0,1} & \cdot & \cdot & \cdot & R_{0,k} \\ R_{1,0} & R_{1,1} & \cdot & \cdot & \cdot & R_{1,k} \\ \cdot & & & & & \cdot \\ \cdot & & & R_{i,j} & & \cdot \\ \cdot & & & & & \cdot \\ R_{k,0} & R_{k,1} & \cdot & \cdot & \cdot & R_{k,k} \end{pmatrix}}^{2|Q|}}_{}\Bigg\} 2|Q|$$

From 2-way transducers to *H-Rat* (unary case)

Second ingredient:
The behavior of $\mathcal{T}$ is given by the matrix $HIT^{\mathsf{H}\star}$.

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The relation accepted by $\mathcal{T}$ is a union of entries of $HIT^{H\star}$.

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$\Gamma = \{a\}$, by closure property:

entries of $HIT \in$ *H-Rat* $\Rightarrow$ entries of $HIT^{\mathsf{H}\star} \in$ *H-Rat*

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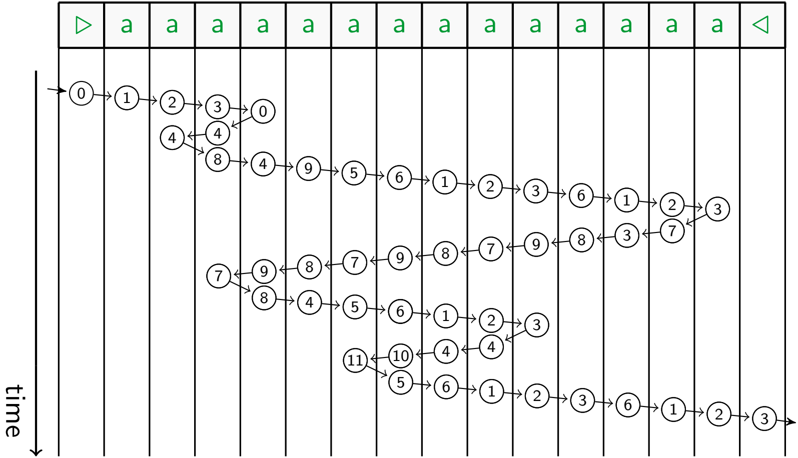

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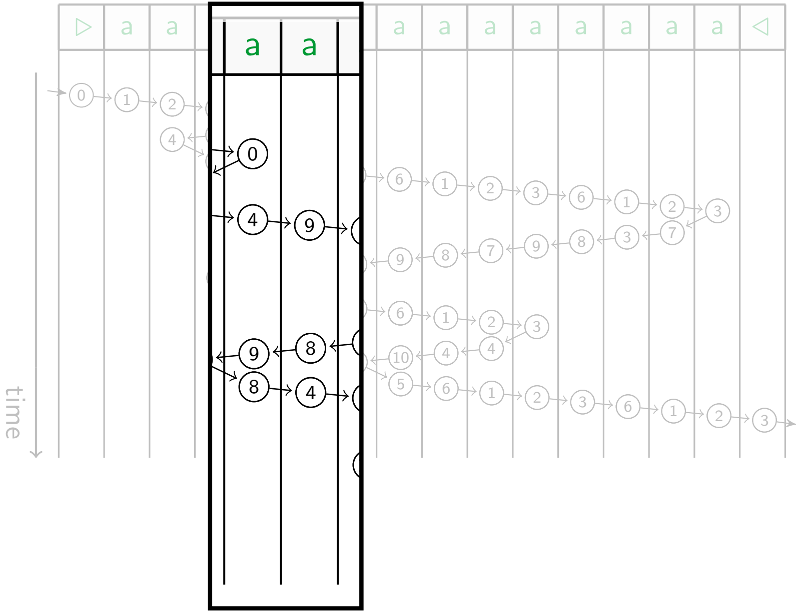

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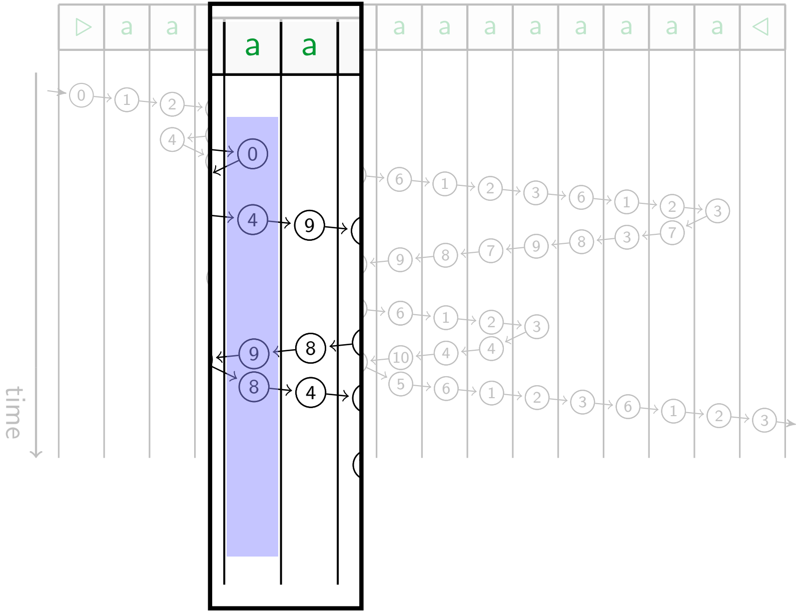

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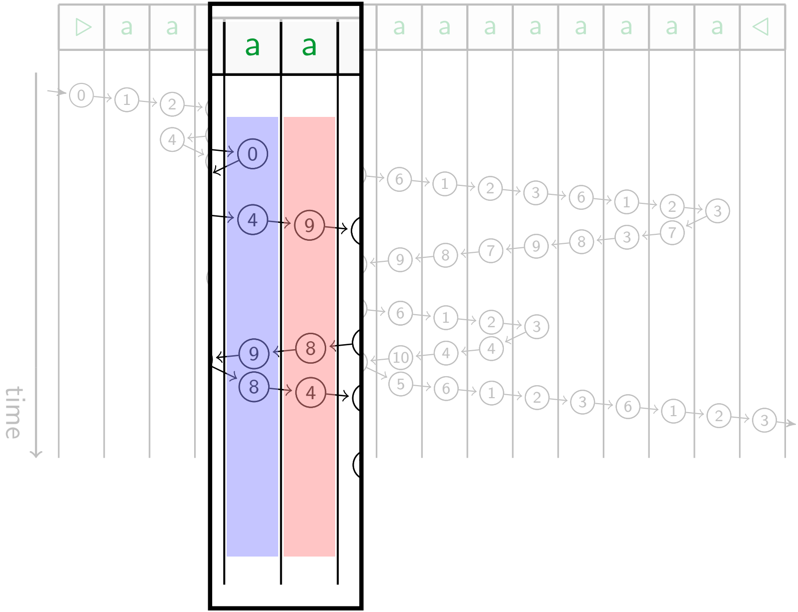

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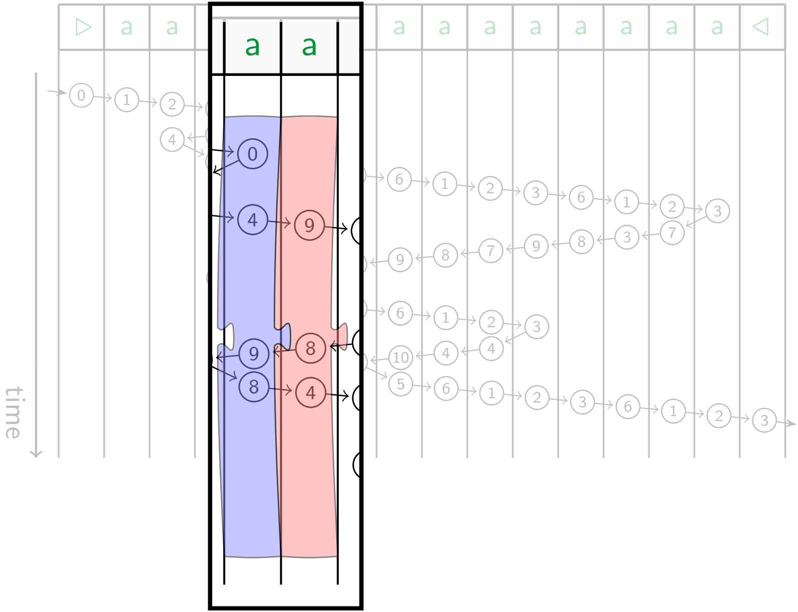

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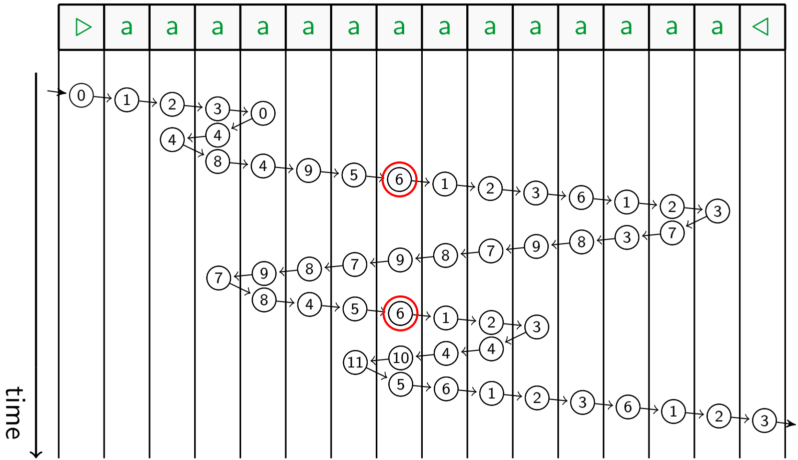

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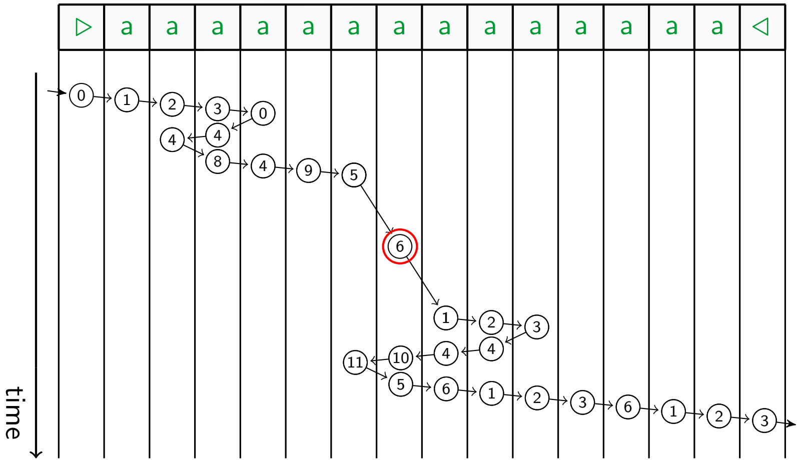

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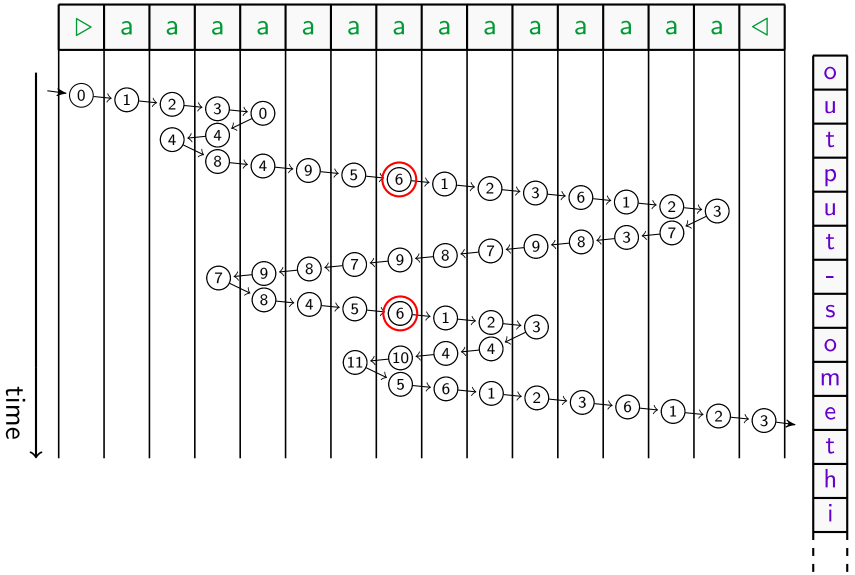

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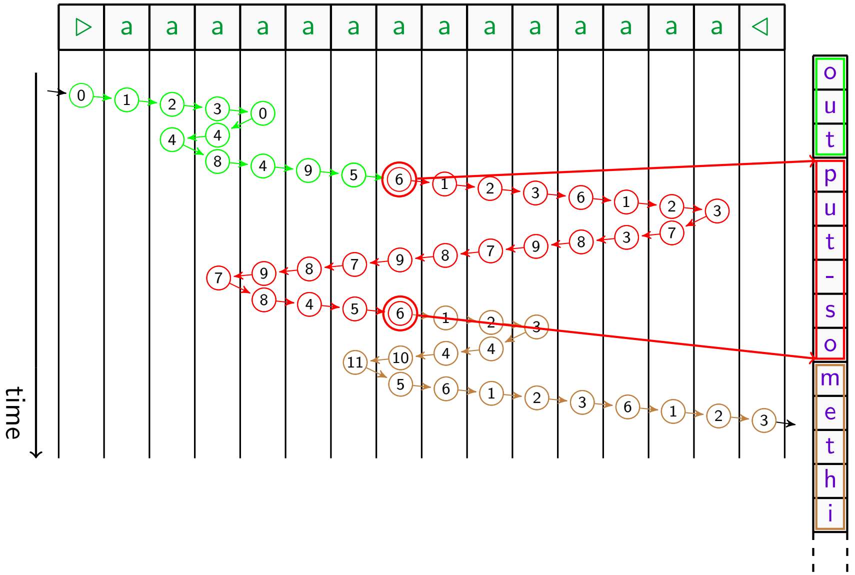

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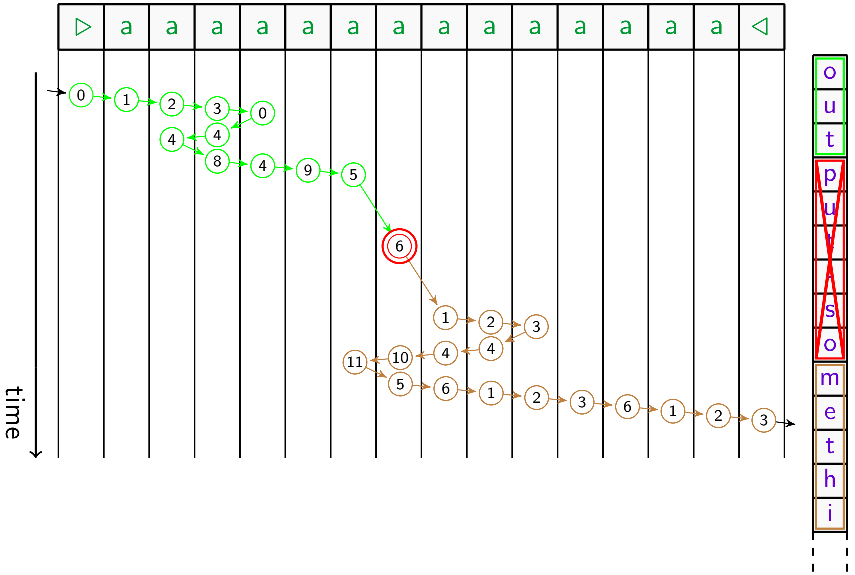

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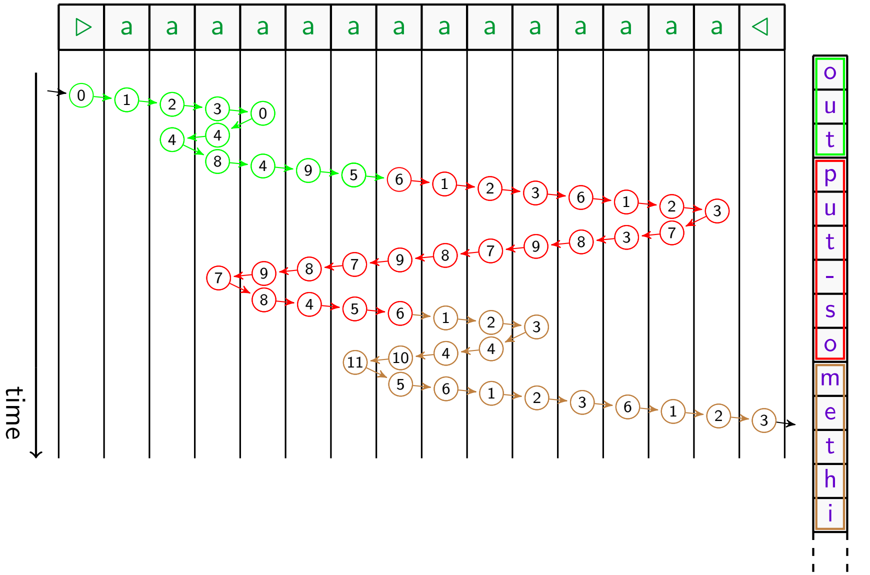

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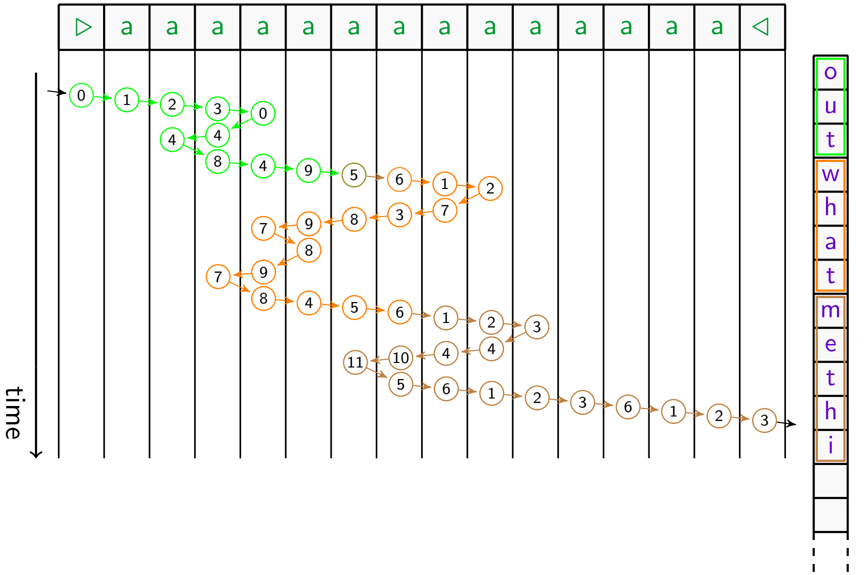

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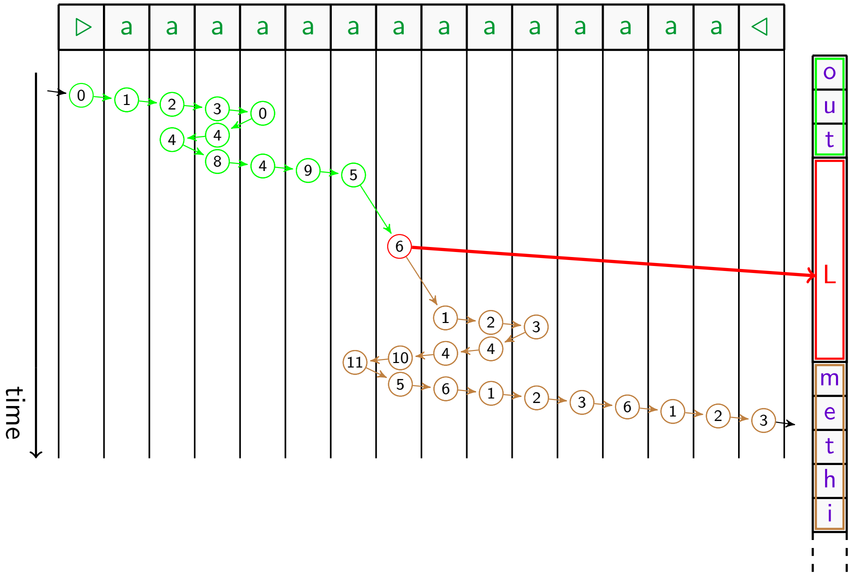

From 2-way transducers to *H-Rat* (unary case)

Second ingredient:
The behavior of $\mathcal{T}$ is given by the matrix $HIT^{\mathsf{H}\star}$.

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $HIT^{H\star}$.

$\Gamma = \{a\}$, by closure property:

entries of $HIT \in H\text{-}Rat$ $\Rightarrow$ entries of $HIT^{\mathsf{H}\star} \in H\text{-}Rat$

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Proposition
unary 2-way transducers $\subseteq$ *H-Rat*

Generalizations?

Theorem

When $\Sigma = \{a\}$ *and* $\Gamma = \{a\}$*:*

2-way transducers accept exactly the H-Rat relations.

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$$R = \left\{ \left(u, a^{kn}\right) \mid k, n \in \mathbb{N},\ \#a^k\# \text{ is a factor of } u \right\}$$

The counter example

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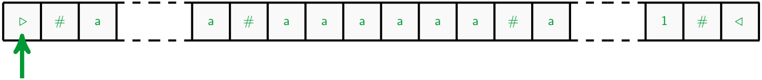

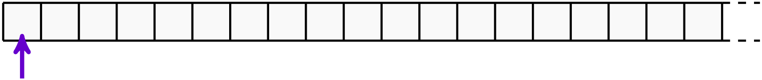

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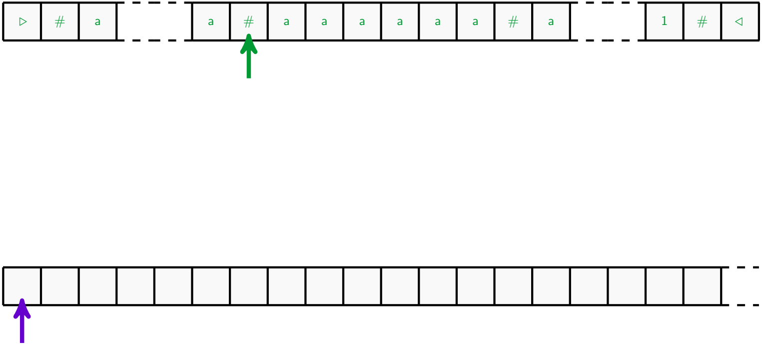

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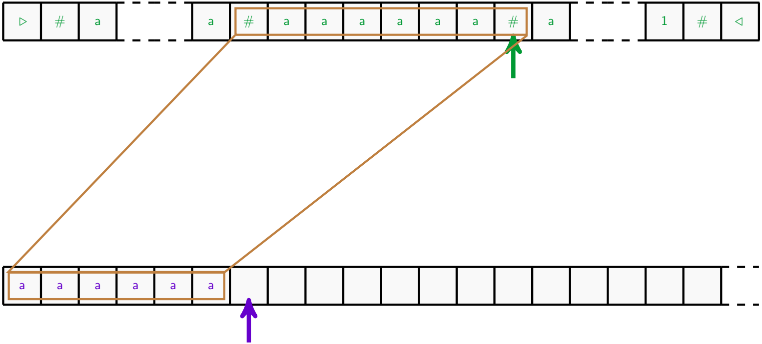

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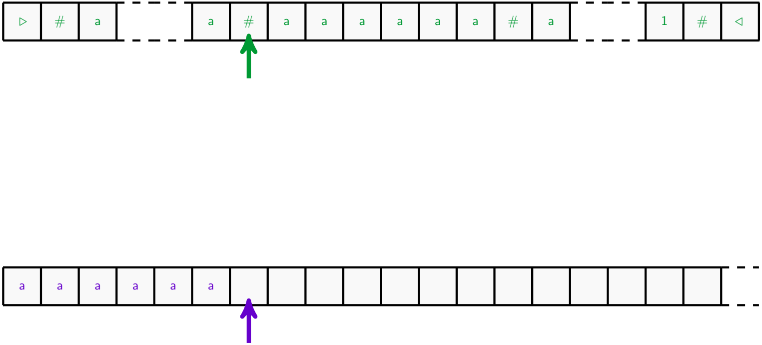

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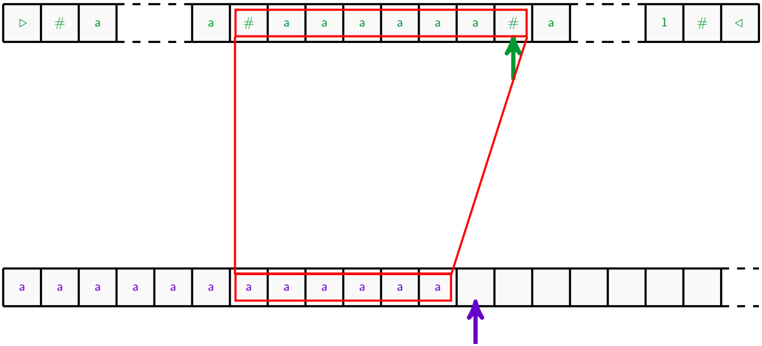

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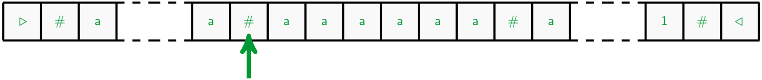

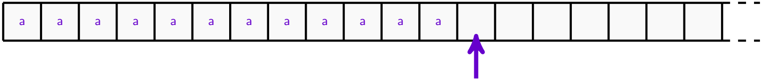

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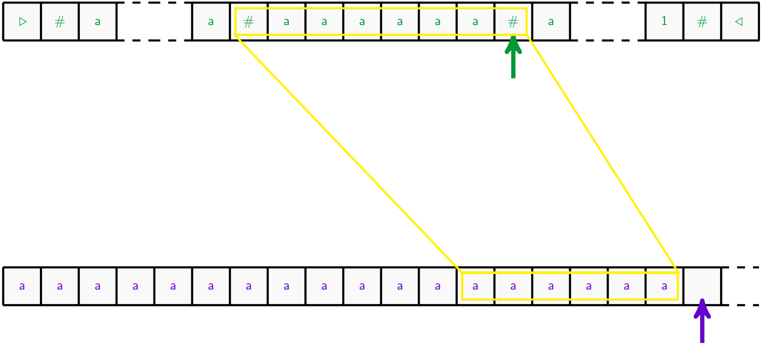

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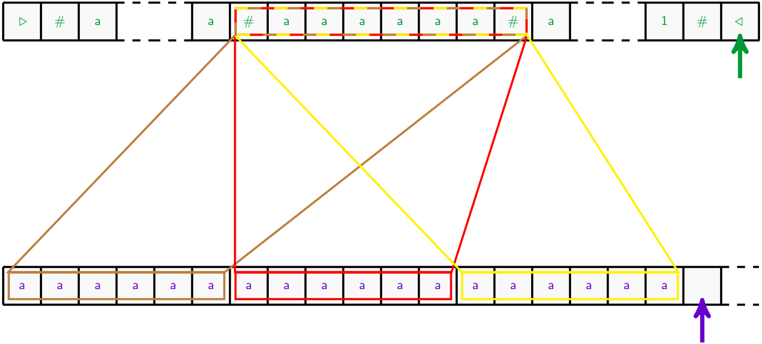

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Thank you for your attention.