Sweeping weakens 2-way Transducers even with a unary output alphabet

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Non-Classical Models of Automata and Applications
Porto 2015
1-way automaton over $\Sigma$

$A$

$(Q, q, F, \delta)$

transition set: $Q \times \Sigma \times Q$

```plaintext
Automaton
```

```plaintext
READ
```
2-way automaton over $\Sigma$

$A$

$(Q, q, F, \delta) \leftarrow$

transition set: $Q \times \Sigma_{\triangleright, \triangleleft} \times \{-1, 0, 1\} \times Q$

left endmarker

right endmarker

Automaton

left endmarker

right endmarker
2-way transducer over $\Sigma, \Gamma$

$$(A, \phi)$$

$$(Q, q_-, F, \delta) \quad \text{production function: } \delta \rightarrow \text{Rat}(\Gamma^*)$$

transition set: $Q \times \Sigma_{\triangleright, \triangleleft} \times \{-1, 0, 1\} \times Q$

**Automaton**

```
▷ the input word ◁
```

```
READ
```

```
WRITE
```

```
t the e  o u t p u t
```

```
▷ the output word ◁
```

```
t the e  o u t p u t
```

```
```
A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$
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- ▶ copy the input word
- ▶ rewind the input tape
- ▶ append a copy of the input word
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Another example: \( UnaryMult = \{ (a^n, a^{kn}) \mid k, n \in \mathbb{N} \} \)
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- Copy the input word
- Rewind the input tape
- Accept and halt with nondeterminism
Rational operations

- Union

- Componentwise concatenation

\[ R_1 \cdot R_2 = \{ (u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2 \} \]

- Kleene star

\[ R^* = \{ (u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall \, i \, (u_i, v_i) \in R \} \]
Rational operations

- **Union**

- **Componentwise concatenation**

\[ R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\}\]

- **Kleene star**

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**Definition (\(Rat(\Sigma^* \times \Gamma^*)\))**

The class of **rational relations** is the smallest class:

- that contains finite relations
- and which is closed under rational operations
Rational operations

- Union
  \[ R_1 \cup R_2 \]

- Componentwise concatenation
  \[ R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\} \]

- Kleene star
  \[ R^* = \{(u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R\} \]

Definition (\( \text{Rat}(\Sigma^* \times \Gamma^*) \))

The class of rational relations is the smallest class:
- that contains finite relations
- and which is closed under rational operations

Theorem (Elgot, Mezei - 1965)

\( 1\text{-way transducers} \equiv \text{the class of rational relations.} \)
Hadamard operations

- H-product

\[ R_1 \otimes R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2\} \]
Hadamard operations

- H-product

\[ R_1 \oplus R_2 = \{ (u, \nu_1 \nu_2) \mid (u, \nu_1) \in R_1 \text{ and } (u, \nu_2) \in R_2 \} \]

Example: \( SQUARE = \{ (w, \text{w} \text{w}) \mid w \in \sum^* \} = \text{Identity} \oplus \text{Identity} \)

- copy the input word
- rewind the input tape
- append a copy of the input word
Hadamard operations

- **H-product**
  \[ R_1 \oplus R_2 = \{(u, \nu_1 \nu_2) \mid (u, \nu_1) \in R_1 \text{ and } (u, \nu_2) \in R_2\} \]

- **H-star**
  \[ R^{H*} = \{(u, \nu_1 \nu_2 \cdots \nu_k) \mid \forall i \ (u, \nu_i) \in R\} \]
Hadamard operations

- **H-product**
  \[ R_1 \oplus R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2 \} \]

- **H-star**
  \[ R^{H \star} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \ (u, v_i) \in R \} \]

**Example:** \( UnaryMult = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N} \} = Identity^{H \star} \)
Hadamard operations

- H-product
  \[ R_1 \oplus R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2\} \]

- H-star
  \[ R^{H*} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \ (u, v_i) \in R\} \]

Property

two-way transducers are closed under H-operations.
**H-Rat relations**

**Definition**

A relation $R$ is in $H$-$\text{Rat}(\Sigma^* \times \Gamma^*)$ if

$$R = \bigcup_{0 \leq i \leq n} A_i \oplus B_i^{\text{H*}}$$

where for each $i$, $A_i$ and $B_i$ are rational relations.
**H-Rat relations**

**Definition**
A relation $R$ is in $H$-$Rat(\Sigma^* \times \Gamma^*)$ if

$$R = \bigcup_{0 \leq i \leq n} A_i \oplus B_i^{H*}$$

where for each $i$, $A_i$ and $B_i$ are rational relations.

**Theorem (Choffrut, G. - 2014)**
When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers $\iff$ H-Rat relations
Main result

Theorem (Choffrut, G. - 2014)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

\[
\text{2-way transducers} \not\subseteq H-\text{Rat}
\]
Main result

Theorem (Choffrut, G. - 2014)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

- 2-way transducers $\overset{?}{=} H$-Rat $\overset{\equiv}{=} \text{sweeping transducers}$
Main result

Theorem (Choffrut, G. - 2014)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers $\neq$ H-Rat $\subseteq$ sweeping transducers

This talk

8/16
Main result

Theorem (Choffrut, G. - 2014)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

- 2-way transducers $\not= H$-Rat $\equiv$ sweeping transducers
- $H$-Rat $\subsetneq$ 2-way transducers

This talk
Known results on 2-way transducers

- Functional $\equiv$ Deterministic $\equiv$ MSO definable functions
- General incomparable MSO definable relations

[Engelfriet, Hoogeboom – 2001]
Known results on 2-way transducers

- functional $\iff$ deterministic $\iff$ MSO definable functions
- general incomparable MSO definable relations
  [Engelfriet, Hoogeboom - 2001]

- general uniformizable by deterministic
  [de Souza - 2013]
Known results on 2-way transducers

- Functional ▀ deterministic ▀ MSO definable functions
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- 1-way simulation of 2-way functional transducer:
  decidable and constructible
  [Filiot et al. - 2013]
Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:
Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:

- unambiguous $\rightarrow$ 1-way
  
  [Anselmo - 1990]

- unambiguous $\Rightarrow$ deterministic
  
  [Carnino, Lombardy - 2014]
Known results on 2-way transducers with unary output

When \( \Gamma = \{a\} \):

- unambiguous \(\rightarrow\) 1-way

  [Anselmo - 1990]

- unambiguous \(\equiv\) deterministic

  [Carnino, Lombardy - 2014]

- general uniformizable by 1-way

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Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:

- unambiguous $\rightarrow$ 1-way
  
  [Anselmo - 1990]

- unambiguous $=$ deterministic
  
  [Carnino, Lombardy - 2014]

- general uniformizable by 1-way
  
  [Choffrut, G. - 2014]

- tropical $=$ 1-way
  
  [Carnino, Lombardy - 2014]

production function $\Phi : \delta \rightarrow \{a^n a^* | n \in \mathbb{N}\}$

rational of period 1
Sketch of the proof

Theorem

When $\Gamma = \{a\}$.

\[
\begin{array}{c}
\text{two-way transducer} \neq H\text{-Rat} \\
(U_i A_i \oplus B_i^{\text{H*}})
\end{array}
\]
Sketch of the proof

Theorem

When $\Gamma = \{a\}$.

two-way transducer $\neq$ H-Rat

$(\bigcup_i A_i \oplus B_i^{H\star})$

- Establish a non-trivial property satisfied by rational relations;
Sketch of the proof

**Theorem**

*When \( \Gamma = \{a\} \).*

\[
\text{two-way transducer} \neq H-\text{Rat}
\]

\[
(U_i A_i \oplus B_i^{H\ast})
\]

- Establish a non-trivial property satisfied by rational relations;
- Extend it to \( H-\text{Rat} \) relations;
Sketch of the proof

**Theorem**

When $\Gamma = \{ a \}$.

- Establish a non-trivial property satisfied by rational relations;
- Extend it to $H$-Rat relations;
- Find a relation accepted by a two-way transducer which does not satisfy the previous property.
Revisiting the family $Rat(a^*)$

The family $Rat(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$ by the canonical mapping $a^n \mapsto n$. 

\[ L = A \cup (t + M + pN) \]

where:
- $t, p \in \mathbb{N}$
- $A \subseteq J_0$
- $tJ$ and $M \subseteq J_0$
- $pJ$

$t$ is a threshold for $L$

$p$ is a period for $L$
Revisiting the family $Rat(a^*)$

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Revisiting the family \( \text{Rat}(a^*) \)

The family \( \text{Rat}(a^*) \) is isomorphic to the rational subsets of \( \mathbb{N} \) by the canonical mapping \( a^n \mapsto n \)

\[
L = A \cup (t + M + p\mathbb{N})
\]

where: \( t, p \in \mathbb{N} \), \( A \subseteq [0, t] \) and \( M \subseteq [0, p] \)

- \( t \) is a threshold for \( L \)
- \( p \) is a period for \( L \)
Periods of images

\[ R \subseteq \Sigma^* \times \Gamma^* \]. The image of \( u \in \Sigma^* \) is:

\[ R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*} \]
Periods of images

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**Theorem**

\( R \) is rational \( \Rightarrow \exists t, p \) such that \( \forall u \)

- \( t (|u| + 1) \) is a threshold and
- \( p \) is a period

of \( R(u) \).
Periods of images

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- \( t (|u| + 1) \) is a threshold and
- \( p \) is a period

of \( R(u) \).

**Theorem**

\( R \) is **H-Rat** \( \Rightarrow \exists k \) such that \( \forall u \), \( R(u) \) has a period \( p \in O\left(|u|^k\right) \).
The counter example

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R = \left\{ (u, a^{kn}) \mid k, n \in \mathbb{N}, \text{ #}a^k\# \text{ is a factor of } u \right\} \]
The counter example

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

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start \rightarrow \text{choose block}
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\[ R = \left\{ (u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u \right\} \]

\[ u = \#a^{n_1}\#a^{n_2}\# \cdots \#a^{n_r}\# \]

\[ R(u) = \bigcup_{0 < i \leq r} \{ a^{kn_i} \} \text{ has minimal period } \text{lcm}_{0 < i \leq r}(n_i) \]

\[ |u| = \sum_{0 < i \leq r} n_i + r + 1 \]
The counter example

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\[ g(n) = \max \left( \{\text{lcm}(n_i) \mid \sum n_i = n\} \right) \quad \text{(Landau's function)} \]
The counter example

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the period is super-polynomial in \(|u|\)
Example with polynomial period

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \left\{ \left(\# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n}\right) \mid n \in \mathbb{N} \right\} \]
Example with polynomial period

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\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}
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\[ u = \#aaa\#aaaaa\#aaaaaaaa\# \quad |u| = 20 \]

the period of \( R(u) \) is \( \text{lcm}(3, 5, 7) = 105 \)
Example with polynomial period

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \left\{ \left( \# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n} \right) \mid n \in \mathbb{N} \right\} \]

the period of \( R(u) \) is in \( O(|u|^r) \)
Conclusion

When $\Gamma = \{a\}$:

- two-way transducers:

<table>
<thead>
<tr>
<th>transducer</th>
<th>family</th>
</tr>
</thead>
<tbody>
<tr>
<td>deterministic</td>
<td>$= \text{rational}$</td>
</tr>
<tr>
<td>unambiguous functional</td>
<td>$= \text{rational}$</td>
</tr>
<tr>
<td>sweeping outer-nondeterm</td>
<td>$= H\text{-Rat}$</td>
</tr>
<tr>
<td>input unary</td>
<td></td>
</tr>
<tr>
<td>general</td>
<td>$\supseteq H\text{-Rat}$</td>
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Thank you for your attention.
Conclusion

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  \hline
  \end{array}
  \]

- images of \( u \):

  \[
  \begin{array}{|c|c|c|}
  \hline
  \text{family} & \text{threshold} & \text{period} \\
  \hline
  \text{rational} & \text{linear} & \text{constant} \\
  \text{H-Rat} & \text{polynomial} & \text{polynomial} \\
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- images of $u$:

<table>
<thead>
<tr>
<th>family</th>
<th>threshold</th>
<th>period</th>
</tr>
</thead>
<tbody>
<tr>
<td>rational</td>
<td>linear</td>
<td>constant</td>
</tr>
<tr>
<td>$H$-Rat</td>
<td></td>
<td>polynomial</td>
</tr>
</tbody>
</table>

Thank you for your attention.