Characterization of relations accepted by two-way transducers

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Séminaire Automate
1-way automaton over $\Sigma$

$A = (Q, q_-, F, \delta)$

transition set: $Q \times \Sigma \times Q$

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the input word
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Automaton
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READ
2-way automaton over $\Sigma$

$A$

$(Q, q_-, F, \delta) \leftarrow$

transition set: $Q \times \Sigma_{\triangleright, \triangleleft} \times \{-1, 0, 1\} \times Q$

left endmarker

right endmarker

Automaton

$A$

$(Q, q_-, F, \delta) \leftarrow$

transition set: $Q \times \Sigma_{\triangleright, \triangleleft} \times \{-1, 0, 1\} \times Q$
2-way transducer over $\Sigma$, $\Gamma$

$$(A, \phi)$$

$$(Q, q_-, F, \delta)$$

production function: $\delta \rightarrow \text{RAT}(\Gamma^*)$

transition set: $Q \times \Sigma_{\triangleright, \triangleleft} \times \{-1, 0, 1\} \times Q$
A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$
A simple example: $\text{SQUARE} = \{(w, ww) \mid w \in \Sigma^*\}$

- copy the input word
A simple example: \( \text{SQUARE} = \{(w, ww) \mid w \in \Sigma^*\} \)

- Copy the input word
- Rewind the input tape
A simple example: $SQUARE = \{(w, ww) \mid w \in \Sigma^*\}$

- copy the input word
- rewind the input tape
- append a copy of the input word
A simple example: $\text{SQUARE} = \{(w, ww) \mid w \in \Sigma^*\}$

- copy the input word
- rewind the input tape
- append a copy of the input word
A simple example: \( \text{SQUARE} = \{(w, ww) \mid w \in \Sigma^*\} \)

- copy the input word
- rewind the input tape
- append a copy of the input word
Another example: \( \text{uMULT} = \{ (a^n, a^{kn}) \mid k, n \in \mathbb{N} \} \)
Another example: $\text{UMULT} = \{(a^n, a^{kn}) | k, n \in \mathbb{N}\}$
Another example: \( \text{UMULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\} \)
Another example: $\text{uMULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$
Another example: \( \text{uMULT} = \{(a^n, a^{kn}) | k, n \in \mathbb{N}\} \)
Another example: $\mathcal{U}\text{MULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$

copy the input word $\rightarrow$ rewind the input tape
Another example: $\text{uMULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$
Another example: $\text{uMULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\}$
Rational operations

- Union

- Componentwise concatenation

\[ R_1 \cdot R_2 = \{ (u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2 \} \]

- Kleene star

\[ R^* = \{ (u_1 u_2 \cdots u_k, v_1 v_2 \cdots v_k) \mid \forall i \ (u_i, v_i) \in R \} \]
Rational operations

- Union

- Componentwise concatenation

\[ R_1 \cdot R_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in R_1 \text{ and } (u_2, v_2) \in R_2\} \]

- Kleene star

\[ R^* = \{(u_1 u_2 \ldots u_k, v_1 v_2 \ldots v_k) \mid \forall \ i \ (u_i, v_i) \in R\} \]

**Definition** \((Rat(\Sigma^* \times \Gamma^*))\)

The class of rational relations is the smallest class:

- contains finite relations
- closed under rational operations
Rational operations

- Union
  \[ R_1 \cup R_2 \]

- Componentwise concatenation
  \[ R_1 \cdot R_2 = \{(u_1 u_2, \nu_1 \nu_2) \mid (u_1, \nu_1) \in R_1 \text{ and } (u_2, \nu_2) \in R_2\} \]

- Kleene star
  \[ R^* = \{(u_1 u_2 \cdots u_k, \nu_1 \nu_2 \cdots \nu_k) \mid \forall i \ (u_i, \nu_i) \in R\} \]

Definition \((\text{Rat}(\Sigma^* \times \Gamma^*))\)

The class of rational relations is the smallest class:
- contains finite relations
- closed under rational operations

Theorem (Elgot, Mezei - 1965)

1-way transducers \(\equiv\) the class of rational relations.
Hadamard operations

- Union
  \( R_1 \cup R_2 \)

- H-product
  \[
  R_1 \boxplus R_2 = \{ (u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2 \}
  \]
Hadamard operations

- Union
- H-product

\[
R_1 \oplus R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2\}
\]

Example: \textit{SQUARE} = \{(w, ww) \mid w \in \Sigma^*\} = \text{Id} \oplus \text{Id}

- copy the input word
- rewind the input tape
- append a copy of the input word
Hadamard operations

- Union
- H-product

\[ R_1 \oplus R_2 = \{ (u, \nu_1 \nu_2) \mid (u, \nu_1) \in R_1 \text{ and } (u, \nu_2) \in R_2 \} \]

- H-star

\[ R^{H*} = \{ (u, \nu_1 \nu_2 \cdots \nu_k) \mid \forall i \ (u, \nu_i) \in R \} \]
Hadamard operations

- **Union**
  \[ R_1 \cup R_2 \]

- **H-product**
  \[ R_1 \oplus R_2 = \{(u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2\} \]

- **H-star**
  \[ R^{H*} = \{(u, v_1 v_2 \cdots v_k) \mid \forall i \ (u, v_i) \in R\} \]

**Example:** \( u\text{MULT} = \{(a^n, a^{kn}) \mid k, n \in \mathbb{N}\} = u\text{ID}^{H*} \)
Hadamard operations

- Union

\[ R_1 \cup R_2 \]

- H-product

\[ R_1 \oplus R_2 = \{ (u, v_1 v_2) \mid (u, v_1) \in R_1 \text{ and } (u, v_2) \in R_2 \} \]

- H-star

\[ R^{H*} = \{ (u, v_1 v_2 \cdots v_k) \mid \forall i (u, v_i) \in R \} \]

Definition \( (\text{HAD}(\Sigma^* \times \Gamma^*)) \)

The class of Hadamard relations is the smallest class:

- contains rational relations
- closed under Hadamard operations
Hadamard relations

Proposition

two-way transducers are closed under $H$-operations.
Hadamard relations

Proposition

two-way transducers are closed under H-operations.

Proof

- $R_1 \cup R_2$:
  - simulate $T_1$ or $T_2$
Hadamard relations

**Proposition**

*two-way transducers are closed under H-operations.*

**Proof**

- $R_1 \cup R_2$:
  - simulate $T_1$ or $T_2$

- $R_1 \otimes R_2$:
  - simulate $T_1$
  - rewind the input tape
  - simulate $T_2$
Hadamard relations

**Proposition**

Two-way transducers are **closed** under **H-operations**.

**Proof**

- \( R_1 \cup R_2 \):  
  - simulate \( T_1 \) or \( T_2 \)
- \( R_1 \oplus R_2 \):  
  - simulate \( T_1 \)
  - rewind the input tape
  - simulate \( T_2 \)
- \( R^{H*} \):  
  - repeat an arbitrary number of times:  
    - simulate \( T \)
    - rewind the input tape  
  - reach the right endmarker and accept
Hadamard relations

Proposition

two-way transducers are closed under $H$-operations.

Proposition

$\text{HAD} \equiv \text{rotating}$
Hadamard relations

Proposition

two-way transducers are closed under H-operations.

Proposition

\[ \text{HAD} = \text{rotating} \subseteq \text{two-way} \]
Hadamard relations

Proposition
two-way transducers are closed under H-operations.

Proposition
| Rat | HAD | = | rotating | ⊆ | two-way |

Example

$$\text{uMult} = \{ (a^n, a^{kn}) \mid k, n \in \mathbb{N} \} = \{ (a^n, a^n) \mid n \in \mathbb{N} \}^{H^*} = \text{uId}^{H^*}$$
Main result

Theorem (Elgot, Mezei - 1965)

1-way transducers \(\equiv\) the class of rational relations.
Main result

Theorem (This talk)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers $\equiv$ the class of HAD relations.
Main result

Theorem (Elgot, Mezei - 1965)

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers $\equiv$ the class of $\text{HAD relations}$.

This talk

Theorem

With $\Sigma = \{a, \#\}$:

$\text{HAD} \not\subset \text{two-way}$

With $\Gamma = \{a, b\}$:

$\text{HAD} \not\subset \text{two-way}$
Known results on 2-way transducers

- functional $\iff$ deterministic $\iff$ MSO definable functions
- general incomparable MSO definable relations

[Engelfriet, Hoogeboom - 2001]
Known results on 2-way transducers

- functional $\equiv$ deterministic $\equiv$ MSO definable functions
- general incomparable MSO definable relations

[Engelfriet, Hoogeboom - 2001]

- 1-way simulation of 2-way functional transducer:
  decidable and constructible

[Filiot et al. - 2013]
Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:
Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:

- Simulation of unambiguous by 1-way [Anselmo - 1990]
- Simulation of unambiguous by deterministic [Carnino, Lombardy - 2014]
Known results on 2-way transducers with unary output

When $\Gamma = \{a\}$:

- Simulation of unambiguous by 1-way [Anselmo - 1990]
- Simulation of unambiguous by deterministic [Carnino, Lombardy - 2014]
- Tropical $\equiv$ 1-way [Carnino, Lombardy - 2014]

Production function $\Phi : \delta \rightarrow \{a^n a^* | n \in \mathbb{N}\} \cup \emptyset$

Rational of period 1
$\Sigma = \{a\} \text{ and } \Gamma = \{a\}$
From 2-way transducers to $\mathbb{HAD}$ (unary case) [1]

Theorem

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

$$\mathbb{HAD} \iff \text{two-way transducers}$$

Proof

- $\subseteq$: done.
- $\supseteq$: to do.
From 2-way transducers to \( \text{HAD} \) (unary case) [1]

**Theorem**

When \( \Sigma = \{a\} \) and \( \Gamma = \{a\} \):

\[
\text{HAD} \iff \text{two-way transducers}
\]

**Proof**

- \( \subseteq \): done.
- \( \supseteq \): to do.

We fix a transducer \( T \).
From 2-way transducers to $\text{HAD}$ (unary case) [2]

- Consider border to border run segments;

<table>
<thead>
<tr>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$R_1 = \{(u, v_1)\}$

$R_2 = \{(u, v_2)\}$

$R_3 = \{(u, v_3)\}$

$R_1 \cap R_2 \cap R_3 = \{(u, v_1v_2v_3)\}$
From 2-way transducers to $\text{HAD}$ (unary case) [2]

- Consider border to border run segments;

\[ R_1 = \{(u, v_1)\} \]
\[ R_2 = \{(u, v_2)\} \]
\[ R_3 = \{(u, v_3)\} \]
From 2-way transducers to $\text{HAD}$ (unary case) [2]

- Consider border to border run segments;
- Compose border to border segments;

\[
R_1 = \{(u, v_1)\} \quad R_2 = \{(u, v_2)\} \quad R_3 = \{(u, v_3)\}
\]
From 2-way transducers to $\mathbb{HAD}$ (unary case) [2]

- Consider border to border run segments;
- Compose border to border segments;

\[
R_1 = \{(u, v_1)\} \\
R_2 = \{(u, v_2)\} \\
R_3 = \{(u, v_3)\}
\]

\[
R_1 \boxplus R_2 \boxplus R_3 = \{(u, v_1v_2v_3)\}
\]
From 2-way transducers to $\text{HAD}$ (unary case) [3]

\[
\begin{array}{c|c|c}
\triangleright & u & \triangleleft \\
\hline
q_1 & & q_2 \\
\end{array}
\]

define a relation $R_{b_i, b_j}$
From 2-way transducers to \(\text{HAD}\) (unary case) [3]

\[
\begin{array}{ccc}
\triangleright & u & \triangleleft \\
q_1 & \longrightarrow & q_2 \\
\end{array}
\]

define a relation \(R\)

\[
Q \times \{\triangleright, \triangleleft\}
\]

\[b_i, b_i\]
From 2-way transducers to HAD (unary case) [3]

\[
\begin{array}{c|c|c}
\triangleright & u & \triangleright \\
q_1 & \rightarrow & q_2
\end{array}
\]

define a relation \( R_{b_i, b_j} \) of \( Q \times \{\triangleright, \triangleleft\} \)

\[
\text{HIT} = \begin{pmatrix}
R_{0,0} & R_{0,1} & \cdots & R_{0,k} \\
R_{1,0} & R_{1,1} & \cdots & R_{1,k} \\
\vdots & \vdots & \ddots & \vdots \\
R_{k,0} & R_{k,1} & \cdots & R_{k,k}
\end{pmatrix}
\]

\[Q \times \{\triangleright, \triangleleft\}\]
From 2-way transducers to $\mathbb{HAD}$ (unary case) [4]

\begin{equation}
\langle \mathbb{HAD}, \cup, \oplus, \cdot \rangle \text{ is a Conway semiring}.
\end{equation}
From 2-way transducers to $\text{HAD}$ (unary case) [4]

$\langle \text{HAD}, \cup, \oplus, \ast \rangle$ is a Conway semiring.

Look at the successive power of the matrix $\text{HIT}$: $\text{HIT}^k$

...that is, the compositions of $k$ border to border runs...
From 2-way transducers to $\text{HAD}$ (unary case) [4]

$\langle \text{HAD}, \cup, \oplus, \ast \rangle$ is a **Conway** semiring.

Look at the star of the matrix $\text{HIT}$: $\text{HIT}^\ast$

...that is, the behavior of $\mathcal{T}$.

**Remark**

*The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^\ast$.*
From 2-way transducers to $\text{HAD}$ (unary case) [4]

$\langle \text{HAD}, \cup, \oplus, \cdot^* \rangle$ is a Conway semiring.

Look at the star of the matrix $\text{HIT}$: $\text{HIT}^{\cdot^*}$

...that is, the behavior of $\mathcal{T}$.

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^{\cdot^*}$.

entries of $\text{HIT} \in \text{HAD} \quad \rightarrow \quad \text{entries of } \text{HIT}^{\cdot^*} \in \text{HAD}$
From 2-way transducers to $\text{HAD}$ (unary case) [5]
From 2-way transducers to HAD (unary case) [5]
From 2-way transducers to \( \text{HAD} \) (unary case) [5]
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From 2-way transducers to $\text{HAD}$ (unary case) [5]
From 2-way transducers to $\text{HAD}$ (unary case) [6]

$\langle \text{HAD}, \cup, \oplus, \cdot \rangle$ is a Conway semiring.

Look at the star of the matrix $\text{HIT}$: $\text{HIT}^{H^*}$

\[ \text{HIT}^{H^*} \]

\[ \text{entries of } \text{HIT} \in \text{HAD} \iff \text{entries of } \text{HIT}^{H^*} \in \text{HAD} \]

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^{H^*}$. 

\[ \text{entries of } \text{HIT} \in \text{HAD} \iff \text{entries of } \text{HIT}^{H^*} \in \text{HAD} \]
From 2-way transducers to $\text{HAD}$ (unary case) [6]

$\langle \text{HAD}, \cup, \oplus, \cdot^* \rangle$ is a Conway semiring.

Look at the star of the matrix $\text{HIT}: \text{HIT}^{\cdot^*}$

...that is, the behavior of $\mathcal{T}$.

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^{\cdot^*}$.

entries of $\text{HIT} \in \text{HAD} \iff$ entries of $\text{HIT}^{\cdot^*} \in \text{HAD}$

Proposition

unary 2-way transducers $\subseteq \text{HAD}$
From 2-way transducers to $\text{HAD}$ (unary case) \[6\]

\[
\langle \text{HAD}, \cup, \oplus, \text{H}^* \rangle \text{ is a Conway semiring.}
\]

Look at the star of the matrix $\text{HIT}$: $\text{HIT}^{\text{H}^*}$

\[
\text{... that is, the behavior of } \mathcal{T}.
\]

Remark

*The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^{\text{H}^*}$.*

entries of $\text{HIT} \in \text{HAD} \iff$ entries of $\text{HIT}^{\text{H}^*} \in \text{HAD}$

Proposition

*unary 2-way transducers* $\subseteq \text{HAD}$

Proposition

*with $\Gamma = \{a\}$ only, sweeping transducer* $\subseteq \text{HAD}$
From 2-way transducers to $\text{HAD}$ (unary case) [6]

\[
\langle \text{HAD}, \cup, \oplus, \ast \rangle \text{ is a Conway semiring}.
\]

Look at the star of the matrix $\text{HIT}$: $\text{HIT}^{\ast}$

\[
\ldots \text{that is, the behavior of } \mathcal{T}.
\]

Remark

The relation accepted by $\mathcal{T}$ is a union of entries of $\text{HIT}^\ast$.

Proposition

unary 2-way transducers $\equiv$ $\text{HAD}$

Proposition

with $\Gamma = \{a\}$ only, sweeping transducer $\equiv$ $\text{HAD}$
Generalizations?

**Theorem**

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers accept exactly the HAD relations.

With only $\Gamma = \{a\}$:

sweeping transducer $= \text{HAD}$
Generalizations?

Theorem
When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers accept exactly the \textbf{HAD relations}.

With only $\Gamma = \{a\}$:

sweeping transducer $=$ \textbf{HAD}
Theorem

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers accept exactly the HAD relations.

With only $\Gamma = \{a\}$:

sweeping transducer $\equiv$ HAD

2-way transducers $\equiv$ sweeping transducers

effective
Generalizations?

**Theorem**

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

- **2-way transducers** accept exactly the **HAD relations**.

2-way transducers $\equiv$ sweeping transducers

With only $\Gamma = \{a\}$:

- **Sweeping transducer** $\equiv$ **HAD**

**Question**

Generalization to arbitrary $\Sigma$?
Generalizations?

**Theorem**

When $\Sigma = \{a\}$ and $\Gamma = \{a\}$:

2-way transducers accept exactly the HAD relations.

With only $\Gamma = \{a\}$:

sweeping transducer $\equiv$ HAD

**Question**

Generalization to arbitrary $\Sigma$? to arbitrary $\Gamma$?
\[ \Sigma = \{ a, \# \} \quad \text{and} \quad \Gamma = \{ a \} \]
On $\Sigma = \{a, \#\}$, and $\Gamma = \{a\}$: counter example

$$R = \{(u, a^{kn}) | k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u\}$$
On $\Sigma = \{a, \#\}$, and $\Gamma = \{a\}$: counter example

$$R = \{(u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u\}$$
On $\Sigma = \{a, \#\}$, and $\Gamma = \{a\}$: counter example

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On $\Sigma = \{a, \#\}$, and $\Gamma = \{a\}$: counter example

$$R = \{(u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u\}$$
Sweeping weakens two-way transducers

Proposition

With $\Sigma = \{a, \#\}$ and $\Gamma = \{a\}$, Halad $\neq$ sweeping and two-way.
Sweeping weakens two-way transducers

Proposition
With $\Sigma = \{a, \#\}$ and $\Gamma = \{a\}$,

$\text{Had} \equiv \text{sweeping \cancel{two-way}}$

Proof

- Establish a non trivial property satisfied by rational relations
Sweeping weakens two-way transducers

Proposition

With \( \Sigma = \{ a, \# \} \) and \( \Gamma = \{ a \} \),

\[
\text{HAD} = \text{sweeping} \quad \not\in \quad \text{two-way}
\]

Proof

- Establish a non trivial property satisfied by rational relations
  
  \[ R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*} \]

  ... a property on the language of images
Sweeping weakens two-way transducers

Proposition

With $\Sigma = \{a, \#\}$ and $\Gamma = \{a\}$,

\[ \text{HAD} = \text{sweeping} \neq \text{two-way} \]

Proof

- Establish a non trivial property satisfied by rational relations
  ... a property on the language of images

  \[ R(u) = \{v \mid (u, v) \in R\} \in 2^\Gamma^* \]

- Extend it to Hadamard relations
Sweeping weakens two-way transducers

**Proposition**

With $\Sigma = \{a, \#\}$ and $\Gamma = \{a\}$,

$\text{HAD} = \text{sweeping} \not\subseteq \text{two-way}$

**Proof**

- Establish a non trivial property satisfied by rational relations
  
  ...a property on the language of images

  $$R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*}$$

- Extend it to Hadamard relations

- Prove that the previous relation does not satisfy the property
Revisiting the family $\text{Rat}(a^*)$

the family $\text{Rat}(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$

by the canonical mapping $a^n \mapsto n$
Revisiting the family \( \text{Rat}(a^*) \)

The family \( \text{Rat}(a^*) \) is isomorphic to the rational subsets of \( \mathbb{N} \) by the canonical mapping \( a^n \mapsto n \).
Revisiting the family $Rat(a^*)$

The family $Rat(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$ by the canonical mapping $a^n \mapsto n$.

Diagram showing a line with dots representing elements of the family $Rat(a^*)$ and a mapping to the natural numbers $\mathbb{N}$.
Revisiting the family $Rat(a^*)$

The family $Rat(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$ by the canonical mapping $a^n \mapsto n$.
Revisiting the family $\text{Rat}(a^*)$

The family $\text{Rat}(a^*)$ is isomorphic to the rational subsets of $\mathbb{N}$ by the canonical mapping $a^n \mapsto n$

$L = A \cup (t + M + p\mathbb{N})$

where: $t, p \in \mathbb{N}$, $A \subseteq [0, t]$ and $M \subseteq [0, p]$

- $t$ is a threshold for $L$
- $p$ is a period for $L$
Periods of images

\[ R \subseteq \Sigma^* \times \Gamma^*. \text{ The image of } u \in \Sigma^* \text{ is:} \]

\[ R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*} \]
Periods of images

\[ R \subseteq \Sigma^* \times \Gamma^* \]. The image of \( u \in \Sigma^* \) is:

\[ R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*} \]

**Theorem**

\( R \) is **rational** \( \Rightarrow \exists t, p \) such that \( \forall u \)

- \( t (|u| + 1) \) is a **threshold** and of \( R(u) \).
- \( p \) is a **period**
Periods of images

\[ R \subseteq \Sigma^* \times \Gamma^* \]. The image of \( u \in \Sigma^* \) is:

\[ R(u) = \{ v \mid (u, v) \in R \} \in 2^{\Gamma^*} \]

Theorem

\( R \) is rational \( \Rightarrow \exists t, p \) such that \( \forall u \)

\[ t(|u| + 1) \text{ is a threshold and } \]
\[ p \text{ is a period} \]

Theorem

\( R \) is HAD \( \Rightarrow \exists k \) such that \( \forall u, R(u) \) has a period \( p \in O\left(|u|^k\right) \).
The counter example

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R = \{ (u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u \} \]
The counter example

\[ \Sigma = \{ \#, a \} \text{ and } \Gamma = \{ a \} \]

\[ R = \{ (u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u \} \]

\[ u = \#a^{n_1}\#a^{n_2}\# \cdots \#a^{n_r}\# \]

\[ R(u) = \bigcup_{0 < i \leq r} \{ a^{kn_i} \} = \bigcup_{0 < i \leq r} n_i \mathbb{N} \]

has minimal period \( \text{lcm}_{0 < i \leq r}(n_i) \)

\[ |u| = \sum_{0 < i \leq r} n_i + r + 1 \]
The counter example

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

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\[ |u| = \sum_{0<i\leq r} n_i + r + 1 \]

\[ g(n) = \max\left(\{|\text{lcm}(n_i)| \sum n_i = n\}\right) \text{ (Landau’s function)} \]
The counter example

\[ \Sigma = \{ \#, a \} \text{ and } \Gamma = \{ a \} \]

\[ R = \left\{ (u, a^{kn}) \mid k, n \in \mathbb{N}, \#a^k\# \text{ is a factor of } u \right\} \]

\[ u = \#a^{n_1}\#a^{n_2}\# \cdots \#a^{n_r}\# \]

\[ R(u) = \bigcup_{0<i\leq r} \{ a^{kn_i} \} = \bigcup_{0<i\leq r} n_i \mathbb{N} \quad \text{has minimal period } \operatorname{lcm}_{0<i\leq r}(n_i) \]

\[ |u| = \sum_{0<i\leq r} n_i + r + 1 \]

\[ g(n) = \max \left( \{ \operatorname{lcm}(n_i) \mid \sum n_i = n \} \right) \quad \text{(Landau’s function)} \]

the period is super-polynomial in \(|u|\)
\[ \Sigma = \{a\} \quad \text{and} \quad \Gamma = \{a, b\} \]
On $\Sigma = \{a\}$ and $\Gamma = \{a, b\}$

Proposition

$\text{HAD} \subseteq \text{two-way}$

Example

$R = \{(a^n, a^p b^p) \mid n \in \mathbb{N}, 0 \leq p < n\}$
On $\Sigma = \{a\}$ and $\Gamma = \{a, b\}$

**Proposition**  
$\text{HAD} \subseteq \text{two-way}$

**Example**  
$R = \{(a^n, a^p b^p) \mid n \in \mathbb{N}, 0 \leq p < n\}$
On $\Sigma = \{a\}$ and $\Gamma = \{a, b\}$

**Proposition**

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**Example**

$$R = \{(a^n, a^p b^p) \mid n \in \mathbb{N}, 0 \leq p < n\}$$
On $\Sigma = \{a\}$ and $\Gamma = \{a, b\}$

Proposition

$\text{Had} \subseteq \text{two-way}$

Example

$R = \{(a^n, a^p b^p) | n \in \mathbb{N}, 0 \leq p < n\}$
## Conclusion

<table>
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<th>Rotating</th>
<th>Sweeping</th>
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Deterministic (== functional) case: everything is effective...

Thank you for your attention.
### Conclusion

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everything is effective...
# Conclusion

Deterministic (= functional) case

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## Conclusion

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Deterministic (== functional) case: everything is effective...

Thank you for your attention.
Conclusion

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## Conclusion

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Thank you for your attention.
Appendix 1

On the optimality of:

Theorem

\[ R \text{ is HAD} \Rightarrow \exists k \text{ such that } \forall u, R(u) \text{ has a period } p \in \mathcal{O}\left(|u|^k\right). \]
Example of Hadamard relation with polynomial period
\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \left\{ (\# a^{k_1} \# a^{k_2} \# \cdots \# a^{k_r} \#, a^{k_i n}) \mid n \in \mathbb{N} \right\} \]
Example of Hadamard relation with polynomial period
\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \left\{ \left( \#a^{k_1} \#a^{k_2} \# \cdots \#a^{k_r} \#, a^{k_i n} \right) \mid n \in \mathbb{N} \right\} \]
Example of Hadamard relation with polynomial period

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \{ (#a_1^k #a_2^k \ldots #a_r^k \#, a_1^{kn}) | n \in \mathbb{N} \} \]
Example of Hadamard relation with polynomial period

$$\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}$$

$$R_r = \{ (#a^{k_1} #a^{k_2} \#\cdots\#a^{k_r} #, a^{k_i}n) \mid n \in \mathbb{N} \}$$
Example of Hadamard relation with polynomial period

$$\Sigma = \{\#, a\} \text{ and } \Gamma = \{a\}$$

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Example of Hadamard relation with polynomial period

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\[ \Sigma = \{ \#, a \} \text{ and } \Gamma = \{ a \} \]

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\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \left\{ (\#^{a_1} \#^{a_2} \# \cdots \#^{a_r} \#, a^{k_in}) \mid n \in \mathbb{N} \right\} \]

\[ u = \#\text{aaa}#\text{aaaaa}#\text{aaaaaaa}# \quad |u| = 20 \]

the period of \( R(u) \) is \( \text{lcm}(3, 5, 7) = 105 \)
Example of Hadamard relation with polynomial period

\[ \Sigma = \{\#, a\} \text{ and } \Gamma = \{a\} \]

\[ R_r = \{(#a^{k_1}#a^{k_2}#\cdots#a^{k_r}#, a^{k_in}) \mid n \in \mathbb{N} \} \]

the period of \( R(u) \) is in \( \mathcal{O}(|u|^r) \)
Appendix 2

On central loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the language:

$$L^\infty_q = \{ \phi(r) \mid r \text{ is a } q\text{-central loop over some input } u \}$$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

> Consider the subset of $\mathbb{N}$:

$$L_q^\infty = \{ |\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u \}$$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

$$L_q^\infty = \{|\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u\}$$

- It is a submonoid of $2^\mathbb{N}$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

$$L_q^\infty = \{ |\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u \}$$

- It is a submonoid of $2^\mathbb{N}$

- $\Rightarrow$ it is finitely generated: $\{g_1, \ldots, g_n\}$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

$$L_q^\infty = \{\phi(r) \mid r \text{ is a } q\text{-central loop over some input } u\}$$

- It is a submonoid of $2^\mathbb{N}$

- $\Rightarrow$ it is finitely generated: $\{g_1, \ldots, g_n\}$

- each generator $g_i$ is produced by a $q$-central loop $r_i$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

\[
L_q^\infty = \{ |\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u \}
\]

- It is a submonoid of $2^\mathbb{N}$
- $\Rightarrow$ it is finitely generated: $\{g_1, \ldots, g_n\}$
- each generator $g_i$ is produced by a $q$-central loop $r_i$
- each $r_i$ needs a finite space
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:

$$L^\infty_q = \{|\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u\}$$

- It is a submonoid of $2^\mathbb{N}$

$\implies$ it is finitely generated: $\{g_1, \ldots, g_n\}$

- each generator $g_i$ is produced by a $q$-central loop $r_i$

- each $r_i$ needs a finite space bounded by $N$
Center loops when $\Sigma = \{a\}$ and $\Gamma = \{a\}$

We fix $q \in Q$.

- Consider the subset of $\mathbb{N}$:
  \[ L_q^\infty = \{ |\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u \} \]

- It is a submonoid of $2^\mathbb{N}$
- $\Rightarrow$ it is finitely generated: $\{g_1, \ldots, g_n\}$
- each generator $g_i$ is produced by a $q$-central loop $r_i$
- each $r_i$ needs a finite space bounded by $N$
- if a position is at distance $> N$ of both endmarkers, then each $r_i$ may occur
We fix \( q \in Q \).

- Consider the subset of \( \mathbb{N} \):

\[
L_q^\infty = \{ |\phi(r)| \mid r \text{ is a } q\text{-central loop over some input } u \}
\]

- It is a submonoid of \( 2^\mathbb{N} \)
- \( \Rightarrow \) it is finitely generated: \( \{g_1, \ldots, g_n\} \)
- each generator \( g_i \) is produced by a \( q\)-central loop \( r_i \)
- each \( r_i \) needs a finite space bounded by \( N \)
- if a position is at distance \( > N \) of both endmarkers, then each \( r_i \) may occur
- and thus the language \( L_q^\infty \) can be produced on the output tape