Higher categories with finite derivation type

ATMCS 2008

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I. Two-dimensional homotopy and String Rewriting

II. Higher dimensional rewriting

III. Polygraph with Finite Derivation Type

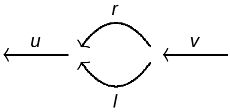
IV. An example ... and a counterexample

I. Two-dimensional Homotopy and String Rewriting

Two-dimensional homotopy and string rewriting

String Rewriting System : X a set , $R \subseteq X^* \times X^*$

$ulv \rightarrow_R urv$



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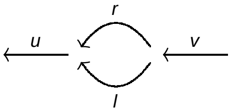

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$$w_0 \to_R w_1 \to_R \cdots \to_R w_n \to_R \cdots$$

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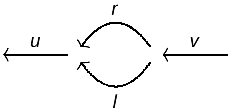

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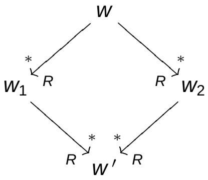
Confluent



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$$w, w' \in X^*, \text{ is } w = w' \text{ in } X^*/\leftrightarrow_R^* \quad ?$$

$\leftrightarrow_R^*$: derivation

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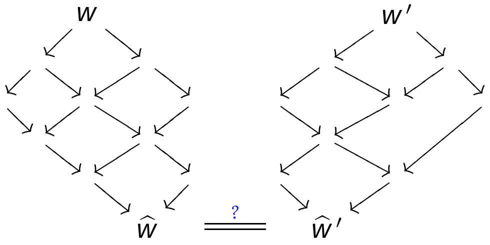

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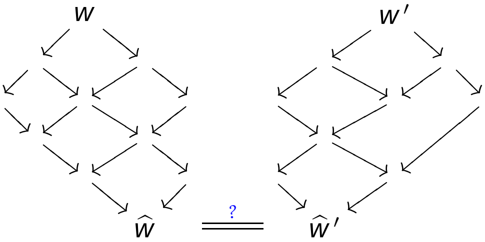

- Monoids having a finite convergent presentation are decidable.

First Squier theorem

- Rewriting is not universal to decide the word problem in finite type (f.t.) monoids.

Theorem. (Squier ’87)
There are f.t.decidable monoids which do not have a finite convergent presentation.

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Proof.

- A monoid $\mathbf{M}$ having a finite convergent presentation (X, R) is of homological type FP_3.

$$\mathrm{Ker}\, J \to \mathbb{Z}\mathbf{M}[R] \xrightarrow{J} \mathbb{Z}\mathbf{M}[X] \to \mathbb{Z}\mathbf{M} \to \mathbb{Z}$$

i.e. module of homological 3-syzygies is generated by critical branchings.

- There are f.t. decidable monoids which are not of type FP_3.

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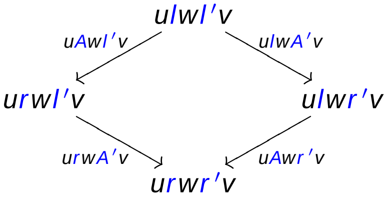
0-cells : words on X 1-cells : derivations $\leftrightarrow_R^*$
2-cells : Peiffer elements



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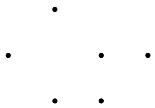
II. Higher Rewriting

Structural dimension of rewriting

dim = 0. No rewriting, no transformations : sets



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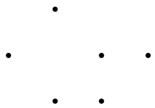
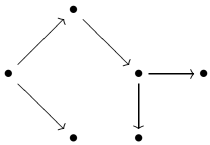


dim = 1. object rewriting : abstract rewriting systems



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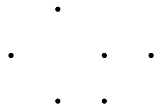

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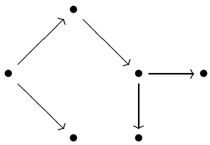
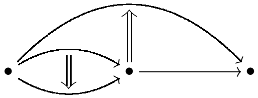


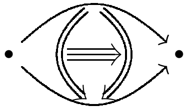
dim = 2. 1-cell rewriting : string rewriting systems



presentation of groups, monoids, 1-categories, ...

Structural dimension of rewriting

dim = 3. 2-cell rewriting : term rewriting systems



presentation of 2-categories, Lawvere theories (Burroni ’93), ...

Structural dimension of rewriting

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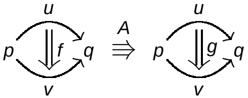
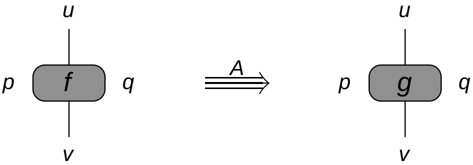
Diagrammatic representation





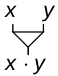
Structural dimension of rewriting

dim = 3. 2-cell rewriting : term rewriting systems

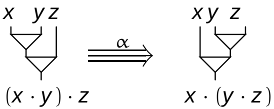
Example : associativity

$$(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z)$$

- one 0-cell, one 1-cell
- one 2-cell



- one 3-cell



Structural dimension of rewriting

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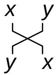
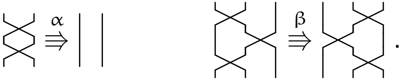
Example : permutation

$$(x, y) \to (y, x)$$

- one 0-cell, one 1-cell
- one 2-cell



- two 3-cells



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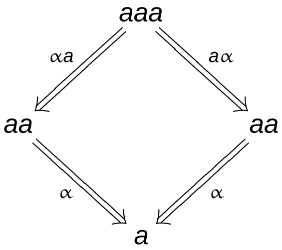

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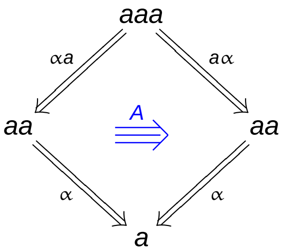

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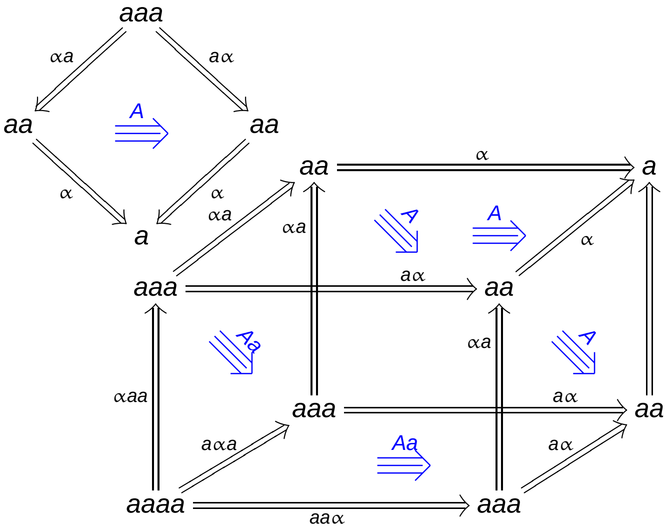

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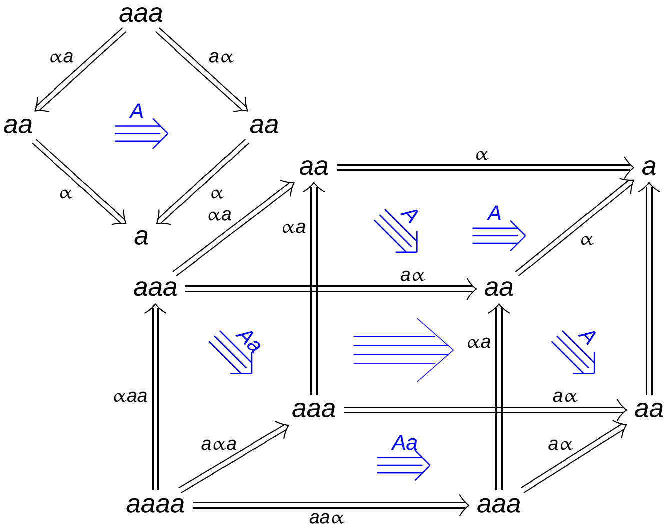

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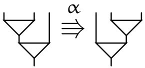
Critical branching of associativity



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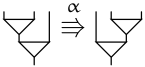

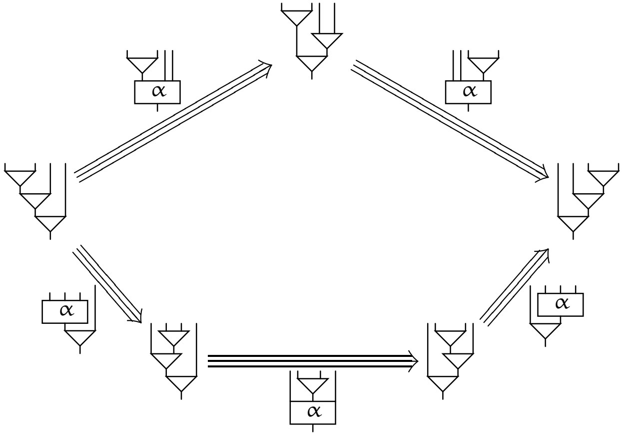

Stasheff polytope K_3 as a 4-cell

Homotopical dimension of rewriting

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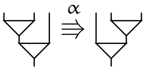

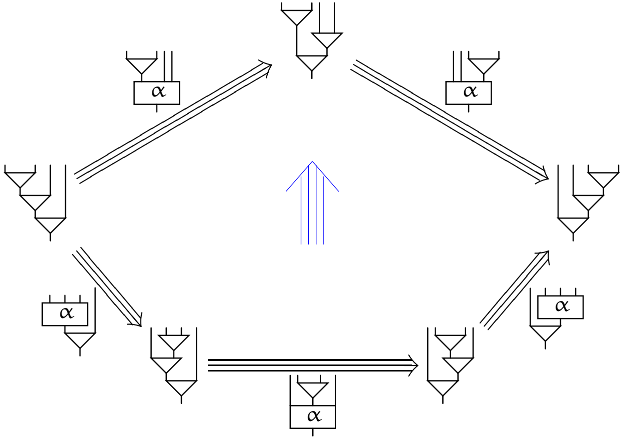

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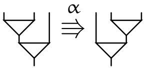

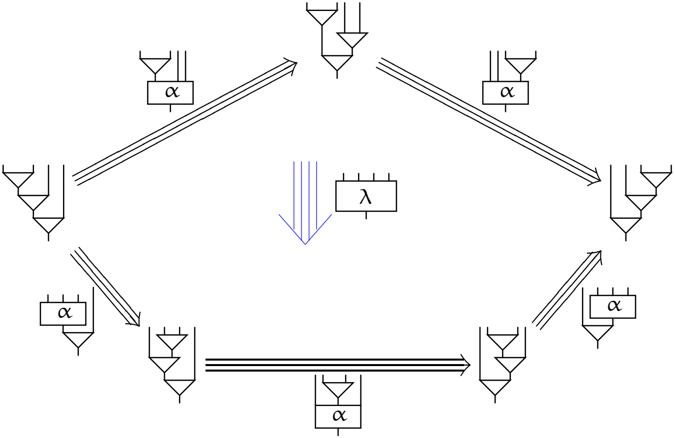

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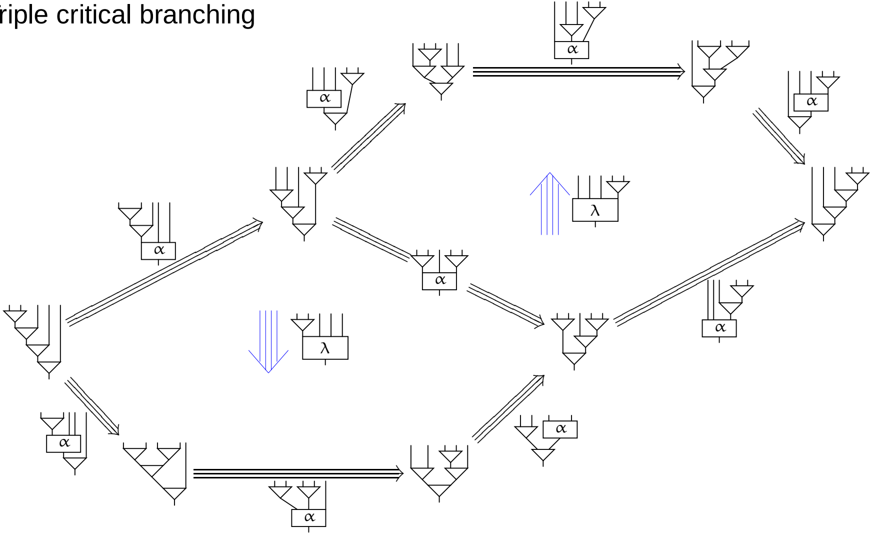
Homotopical dimension of rewriting

Triple critical branching

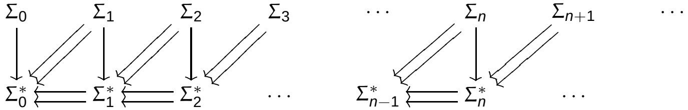


Stasheff polytope K_4

Polygraphs

Polygraph (or **computad**) : higher categorical diagram

- higher rewriting system (Burroni)
- directed cellular complexe (Street)



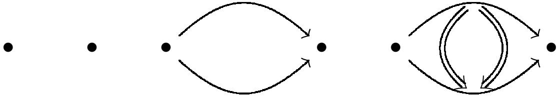
Definition.

- 0-*polygraph* : set
- $n+1$-*polygraph* $\Sigma = (\Sigma_0, \ldots, \Sigma_n, \Sigma_{n+1})$

  – $(\Sigma_0, \ldots, \Sigma_n)$ n-polygraph

  – Σ_{n+1} set of n-spheres of Σ_n^*

n-sphere : pair (f, g) of parallel n-cells :



III. Polygraphs having Finite Derivation Type

Track n-categories

Definition.

A *track n-category* is a $n-1$-category enriched in groupoid.

- *Free track n-category functor* :

$$T_n : \mathbf{Pol}_n \to \mathbf{Tck}_n$$

Polygraph having finite derivation type

Definitions.

- A *homotopy relation* on a n-polygraph Σ is a track $n+1$-category $\mathcal{T}$ such that

$$U_n(\mathcal{T}) = T_n(\Sigma)$$

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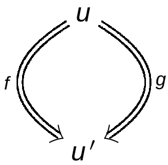

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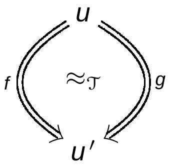

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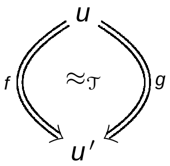


if there is a $n+1$-cell in $\mathcal{T}$ from f to g

• A *homotopy base* of Σ is a set of n-spheres Γ such that

for any n-sphere (f, g) one has $f \approx_{T_n(\Sigma)(\Gamma)} g$.

• A *n-polygraph has finite derivation type (FDT) when it is finite and has a finite homotopy base.*

Polygraph having finite derivation type

Proposition.

Property FDT is Tietze invariant for finite polygraphs.

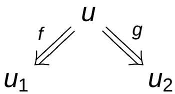
Definition.

A $n-1$-category presented by a n-polygraph has FDT when it admits a presentation by a n-polygraph having FDT.

Branching in polygraphs

Definitions.

- A branching in a n-polygraph Σ is a pair (f, g) of n-cells with same source



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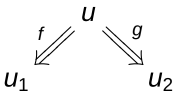
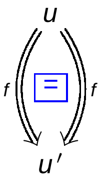
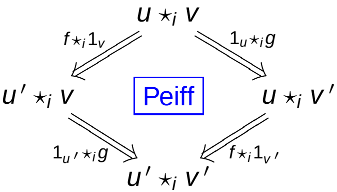


- Trivial branching



Peiffer Element



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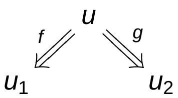
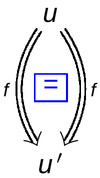
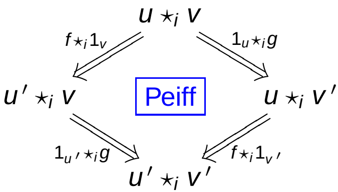


- Trivial branching Peiffer Element

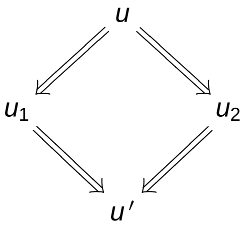




- Call a *critical branching* a non-trivial branching.

Convergent polygraphs

- Γ_Σ set of n-spheres in Σ defined by confluent critical branchings :



Convergent polygraphs

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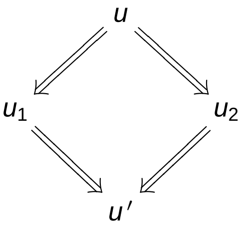

Proposition.

For a convergent polygraph Σ, the set Γ_Σ forms a homotopy base.

Convergent polygraphs

Theorem.
A finite convergent polygraph with a finite set of critical branchings has FDT.

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Corollary.
A finite convergent 2-polygraph has FDT. (Squier '87)

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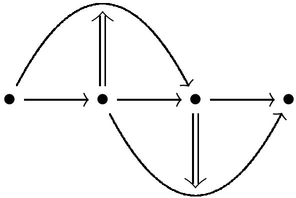
Theorem.
For $n \geq 3$, there exists finite convergent n-polygraph without FDT.

Critical branchings in 2-polygraphs

- **Regular critical branching** (simple overlapping)



Critical branchings in 2-polygraphs

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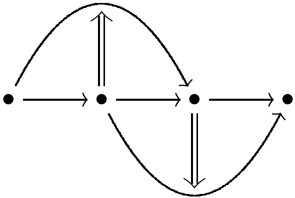
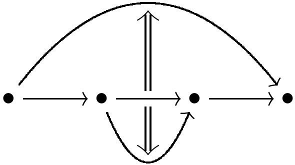
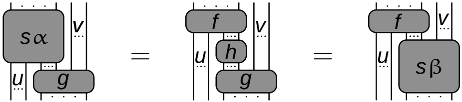


- **Inclusion critical branching**



Critical branchings in 3-polygraphs

- **Regular critical branching**



Critical branchings in 3-polygraphs

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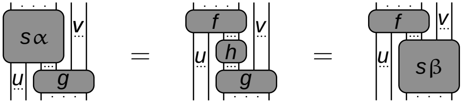

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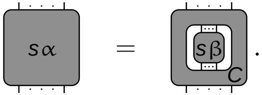

Critical branchings in 3-polygraphs

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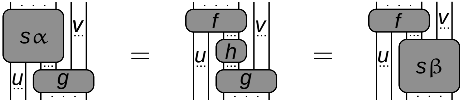

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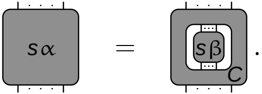
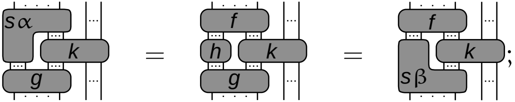
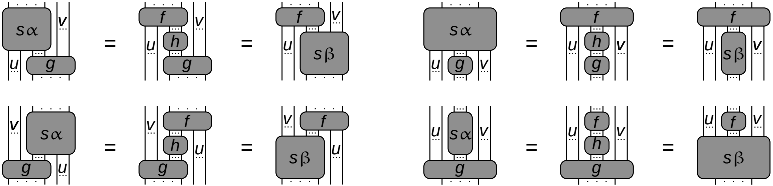


- **Right-indexed critical branching**

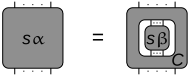


Critical branchings in 3-polygraphs

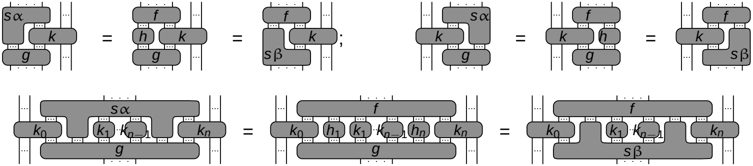
- **Regular**



- **Inclusion**



- **Indexed**



Convergent 3-polygraphs having FDT

Theorem. Let Σ be a finite convergent 3-polygraph.

- If Σ does not have indexed critical branching it has FDT.
- If Σ has indexed critical branchings with finitely many normal instances it has FDT.

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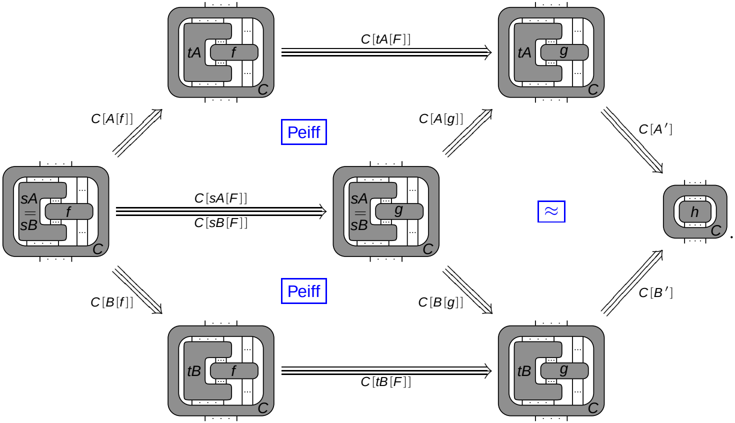
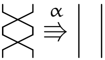


IV. An example ... and a counterexample

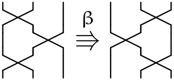
The 3-polygraph of permutations has FDT

Σ_{Perm}:

- one 0-cell, one 1-cell, one 2-cell 
- two 3-cells:



and



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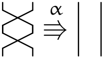

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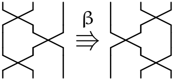


Proposition.

Σ_{Perm} is finite, convergent and has FDT.

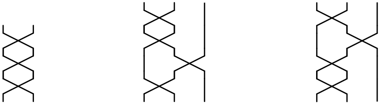
The 3-polygraph of permutations has FDT

Σ_{Perm} has an infinite set of critical branchings.

The 3-polygraph of permutations has FDT

Σ_{Perm} has an infinite set of critical branchings.

- three regular critical branchings



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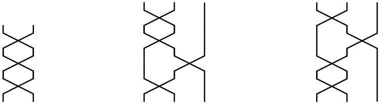
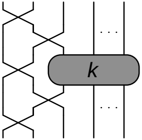
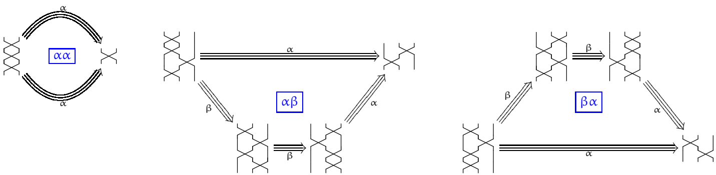


- a right-indexed critical branching

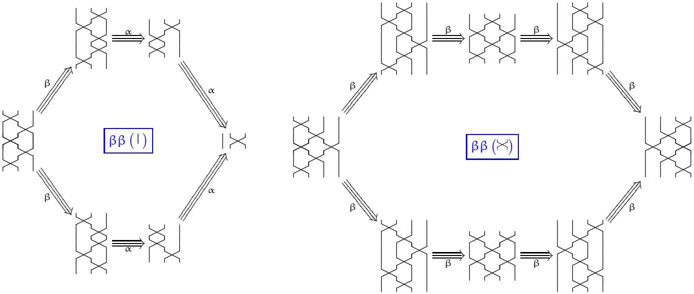


The 3-polygraph of permutations has FDT

- regulars are confluent



- right-indexed have two normal instances



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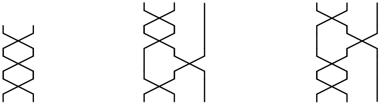

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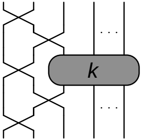

| regulars are confluent
| right-indexed have two normal instances $\Rightarrow$ Σ_{Perm} has FDT

A finite convergent polygraph which doesn't have FDT

A polygraph Σ presenting the 2-category of pear necklaces.

- one 0-cell, one 1-cell, three 2-cells :

A finite convergent polygraph which doesn't have FDT

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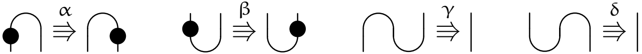

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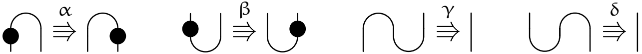

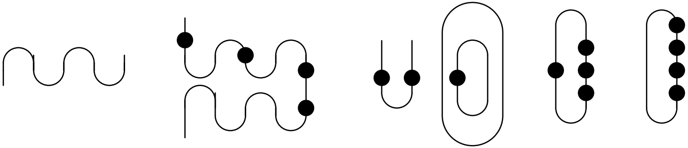

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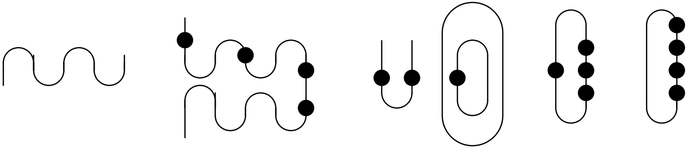
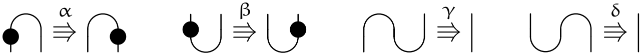


Theorem.

Σ is finite and convergent but does not have FDT

A finite convergent polygraph which doesn't have FDT



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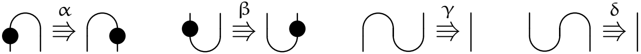
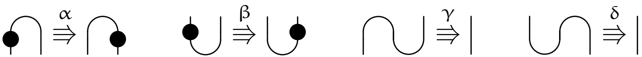


- Four regular critical branching

A finite convergent polygraph which doesn't have FDT



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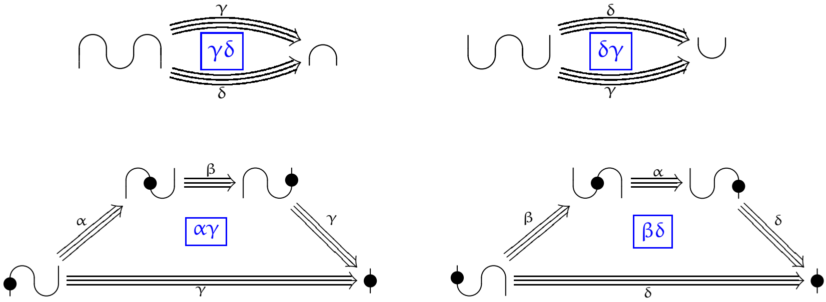

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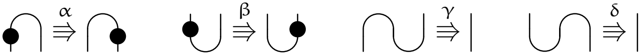
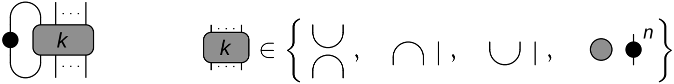
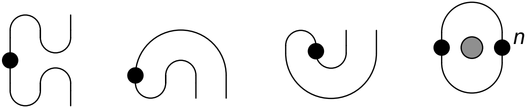


- One right-indexed critical branching





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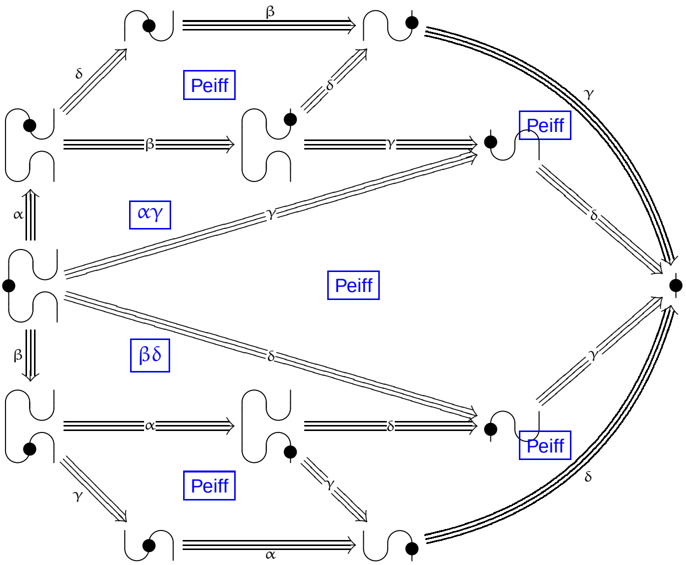

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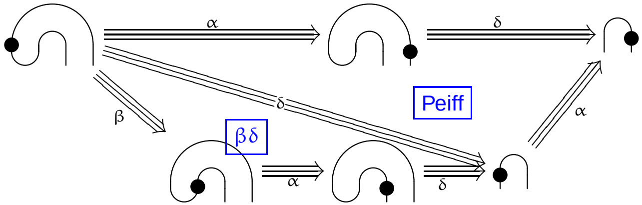

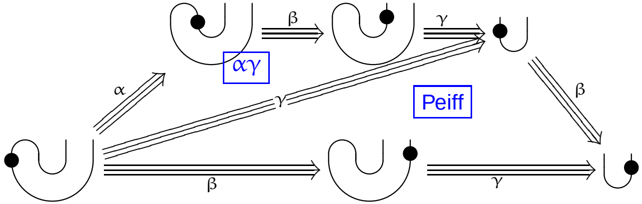

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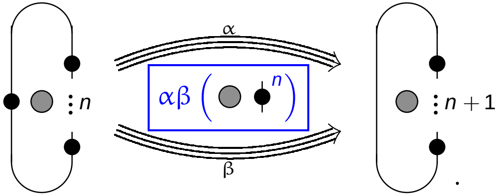

Theorem.

Σ is finite and convergent but does not have finite derivation type

Conclusions

- (n, p)-polygraphs.
- Module of identities among relations for higher categories presented by polygraph.
- Property FP_3 for higher categories.
- (Co)homology of higher categories with coefficients.
- Compute homology of Lawvere theories using Burroni presentation.