Higher categories with finite derivation type
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I. Two-dimensional homotopy and String Rewriting

II. Higher dimensional rewriting

III. Polygraph with Finite Derivation Type

IV. An example ... and a counterexample
I. Two-dimensional Homotopy and String Rewriting
Two-dimensional homotopy and string rewriting

String Rewriting System: $X$ a set, $R \subseteq X^* \times X^*$

$$ulv \rightarrow_R urv$$

$(r, l) \in R$, $u, v \in X^*$

$\rightarrow^*_R$: reflexive symmetric closure of $\rightarrow_R$
Two-dimensional homotopy and string rewriting

String Rewriting System: \( X \) a set, \( R \subseteq X^* \times X^* \)

\[ ulv \rightarrow_R urv \]

\( \rightarrow^*_R \): reflexive symmetric closure of \( \rightarrow_R \)

Terminating

\[ w_0 \rightarrow_R w_1 \rightarrow_R \cdots \rightarrow_R w_n \rightarrow_R \cdots \]
Two-dimensional homotopy and string rewriting

String Rewriting System: $X$ a set, $R \subseteq X^* \times X^*$

$ulv \to_R urv \quad \leftarrow^u \quad \leftarrow^v \quad (r, l) \in R \quad u, v \in X^*$

$\to_R^* :$ reflexive symmetric closure of $\to_R$

Terminating

$w_0 \to R \quad w_1 \to R \quad \cdots \quad w_n \to R \quad \cdots$

Confluent
Decidability of word problem

Word problem

\[ w, w' \in X^*, \text{ is } w = w' \text{ in } X^*/\leftrightarrow_R^* \quad ? \]

\[ \leftrightarrow_R^* : \text{ derivation} \]
Decidability of word problem

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Normal form algorithm :

- \((X,R) : \text{ finite + convergent (terminating + confluent)}\)
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I. TWO-DIMENSIONAL HOMOTOPY AND STRING Rewriting
Decidability of word problem

Word problem

\[ w, w' \in X^*, \text{ is } w = w' \text{ in } X^*/\leftrightarrow R \]

\( \leftrightarrow^*_R \): derivation

Normal form algorithm:
- \((X,R)\): finite + convergent (terminating + confluent)

- Monoids having a finite convergent presentation are decidable.
First Squier theorem

- Rewriting is not universal to decide the word problem in finite type (f.t.) monoids.

**Theorem.** (Squier ’87)

There are f.t.decidable monoids which do not have a finite convergent presentation.
First Squier theorem

- Rewriting is not universal to decide the word problem in finite type (f.t.) monoids.

**Theorem.** (Squier ’87) There are f.t. decidable monoids which do not have a finite convergent presentation.

**Proof.**
- A monoid $M$ having a finite convergent presentation $(X, R)$ is of homological type $FP_3$.

$$\text{Ker } J \rightarrow \mathbb{Z}M[R] \xrightarrow{J} \mathbb{Z}M[X] \rightarrow \mathbb{Z}M \rightarrow \mathbb{Z}$$

i.e. module of homological 3-syzygies is generated by critical branchings.
- There are f.t. decidable monoids which are not of type $FP_3$. 

I. TWO-DIMENSIONAL HOMOTOPIY AND STRING REWRITING
Second Squier Theorem

Theorem. (Squier ’87 (’94))

The homological finiteness condition $FP_3$ is not sufficient for a f.t. decidable monoid to admit a presentation by a finite convergent rewriting system.
Second Squier Theorem

**Theorem.** (Squier ’87 (’94))

The homological finiteness condition $FP_3$ is not sufficient for a f.t. decidable monoid to admit a presentation by a finite convergent rewriting system.

**Proof.**

- $(X, R)$ a string rewriting system.
- $S(X, R)$ Squier 2-dimensional combinatorial complex.
Second Squier Theorem

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**Proof.**

- $(X, R)$ a string rewriting system.
- $S(X, R)$ Squier 2-dimensional combinatorial complex.

0-cells : words on $X$ 1-cells : derivations $\leftrightarrow^*_R$

2-cells : Peiffer elements
Second Squier Theorem

**Theorem.** (Squier ’87 (’94))

The homological finiteness condition $\text{FP}_3$ is not sufficient for a f.t. decidable monoid to admit a presentation by a finite convergent rewriting system.

**Proof.**
- $(X, R)$ a string rewriting system.
- $S(X, R)$ Squier 2-dimensional combinatorial complex.
- **Definition.** $(X, R)$ has *finite derivation type* (FDT) if
  - $X$ and $R$ are finite,
  - $S(X, R)$ has a finite set of homotopy trivializer.
Second Squier Theorem

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- Property FDT is Tietze invariant for finite rewriting systems
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- A monoid having a finite convergent rewriting system has FDT.
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- Property FDT is Tietze invariant for finite rewriting systems
- A monoid having a finite convergent rewriting system has FDT.
- There are f.t. decidable monoids which do not have FDT and which are $FP_3$. 
II. Higher Rewriting
Structural dimension of rewriting

\( \text{dim} = 0 \). No rewriting, no transformations : sets
Structural dimension of rewriting

\( \text{dim} = 0 \). No rewriting, no transformations: sets

\[
\bullet \\
\bullet \\
\bullet \\
\bullet
\]

\( \text{dim} = 1 \). Object rewriting: abstract rewriting systems

\[
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\]
Structural dimension of rewriting

$\text{dim} = 0$. No rewriting, no transformations: sets

\[ \bullet \rightarrow \bullet \rightarrow \bullet \]

$\text{dim} = 1$. Object rewriting: abstract rewriting systems

$\text{dim} = 2$. 1-cell rewriting: string rewriting systems

presentation of groups, monoids, 1-categories, ...
Structural dimension of rewriting

\textbf{dim} = 3. 2-cell rewriting: term rewriting systems

presentation of 2-categories, Lawvere theories (Burroni '93), ...
Structural dimension of rewriting

$\text{dim} = 3$. 2-cell rewriting : term rewriting systems

Diagrammatic representation

$\xymatrix{ p & f & q \\ v \ar[u] & & q \ar[u] } \Rightarrow \xymatrix{ p & g & q \\ v \ar[u] & & q \ar[u] }$
Structural dimension of rewriting

**dim = 3.** 2-cell rewriting: term rewriting systems

**Example:** associativity

\[(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z)\]

- one 0-cell, one 1-cell
- one 2-cell

\[\begin{array}{c}
\begin{array}{c}
\xrightarrow{}
\end{array}
\end{array}\]

- one 3-cell

\[\begin{array}{c}
\begin{array}{c}
\xrightarrow{\alpha}
\end{array}
\end{array}\]
Structural dimension of rewriting

\textbf{dim = 3}. 2-cell rewriting : term rewriting systems

\textbf{Example} : permutation

\[(x, y) \rightarrow (y, x)\]

- one 0-cell, one 1-cell
- one 2-cell

\[
\begin{array}{c}
\alpha
\end{array}
\]

- two 3-cells

\[
\begin{array}{c}
\beta
\end{array}
\]
Homotopical dimension of rewriting

\[
\begin{align*}
a \xrightarrow{\circlearrowleft} \bullet & \quad aa \xrightarrow{\alpha} a
\end{align*}
\]
Homotopical dimension of rewriting

\[ a \xleftarrow{\alpha} a \]

II. Higher Rewriting
Homotopical dimension of rewriting

\[
\alpha \begin{array}{c}
\Rightarrow
\end{array}
\]

\[
\alpha \begin{array}{c}
\Rightarrow
\end{array}
\]
Homotopical dimension of rewriting

\[ a \xrightarrow{\alpha} a \]

\[ a \circ \bullet \]

II. Higher Rewriting
Homotopical dimension of rewriting

\[ a \circlearrowright \bullet \quad aa \xrightarrow{\alpha} a \]

II. Higher Rewriting
Homotopical dimension of rewriting

Critical branching of associativity

\[ \alpha \]

\[ \Rightarrow \]

\[ \Rightarrow \]
Homotopical dimension of rewriting

Critical branching of associativity

\[ \alpha \Rightarrow \]

Stasheff polytope $K_3$ as a 4-cell
Homotopical dimension of rewriting

Critical branching of associativity

Stasheff polytope $K_3$ as a 4-cell
Homotopical dimension of rewriting

Critical branching of associativity

Stasheff polytope $K_3$ as a 4-cell
Homotopical dimension of rewriting

Triple critical branching

Stasheff polytope $K_4$
Polygraphs

Polygraph (or computad) : higher categorical diagram
  • higher rewriting system (Burroni)
  • directed cellular complexe (Street)

\[ \begin{array}{cccccccc}
\Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_3 & \cdots & \Sigma_n & \Sigma_{n+1} & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots \\
\Sigma_0^* & \Sigma_1^* & \Sigma_2^* & \cdots & \Sigma_{n-1}^* & \Sigma_n^* & \cdots \\
\end{array} \]

Definition.
- 0-polygraph : set
- \( n+1 \)-polygraph \( \Sigma = (\Sigma_0, \ldots, \Sigma_n, \Sigma_{n+1}) \)
  - \( (\Sigma_0, \ldots, \Sigma_n) \) \( n \)-polygraph
  - \( \Sigma_{n+1} \) set of \( n \)-spheres of \( \Sigma_n^* \)

\( n \)-sphere : pair \((f, g)\) of parallel \( n \)-cells:
III. Polygraphs having Finite Derivation Type
Track $n$-categories

Definition.

A track $n$-category is a $n - 1$-category enriched in groupoid.

- Free track $n$-category functor:

$$T_n : \text{Pol}_n \rightarrow \text{Tck}_n$$
Polygraph having finite derivation type

Definitions.

• A *homotopy relation* on a $n$-polygraph $\Sigma$ is a track $n + 1$-category $\mathcal{T}$ such that

$$U_n(\mathcal{T}) = T_n(\Sigma)$$
Polygraph having finite derivation type

Definitions.

- A homotopy relation on a $n$-polygraph $\Sigma$ is a track $n + 1$-category $T$ such that

$$U_n(T) = T_n(\Sigma)$$
Polygraph having finite derivation type

Definitions.

- A *homotopy relation* on a $n$-polygraph $\Sigma$ is a track $n+1$-category $\mathcal{T}$ such that

$$U_n(\mathcal{T}) = T_n(\Sigma)$$

if there is a $n+1$-cell in $\mathcal{T}$ from $f$ to $g$
Polygraph having finite derivation type

Definitions.
• A homotopy relation on a \( n \)-polygraph \( \Sigma \) is a track \( n + 1 \)-category \( \mathcal{T} \) such that

\[
U_n(\mathcal{T}) = T_n(\Sigma)
\]

if there is a \( n + 1 \)-cell in \( \mathcal{T} \) from \( f \) to \( g \)

• A homotopy base of \( \Sigma \) is a set of \( n \)-spheres \( \Gamma \) such that for any \( n \)-sphere \( (f, g) \) one has \( f \approx_{T_n(\Sigma)(\Gamma)} g \).

• A \( n \)-polygraph has finite derivation type (FDT) when it is finite and has a finite homotopy base.
Polygraph having finite derivation type

Proposition.
Property FDT is Tietze invariant for finite polygraphs.

Definition.
A $n-1$-category presented by a $n$-polygraph has FDT when it admits a presentation by a $n$-polygraph having FDT.
Branching in polygraphs

Definitions.

- A branching in a $n$-polygraph $\Sigma$ is a pair $(f, g)$ of $n$-cells with same source.
Branching in polygraphs

Definitions.

• A branching in a \( n \)-polygraph \( \Sigma \) is a pair \((f, g)\) of \( n \)-cells with same source

\[ f \quad u \quad g \]
\[ u_1 \quad \quad \quad u_2 \]

• Trivial branching

\[ u \]
\[ f \quad \quad \quad f \]
\[ u' \quad \quad \quad u' \]

Peiffer Element

\[ u' \star_i V \quad \quad \quad U \star_i V' \]
\[ f \star_i v \quad \quad \quad 1_{u' \star_i g} \]
\[ 1_{u \star_i g} \quad \quad \quad f \star_i v' \]

III. Polygraphs having Finite Derivation Type
Branching in polygraphs

Definitions.

- A branching in a \( n \)-polygraph \( \Sigma \) is a pair \((f, g)\) of \( n \)-cells with same source.

- Trivial branching

- Peiffer Element

- Call a critical branching a non-trivial branching.
Convergent polygraphs

- $\Gamma_\Sigma$ set of $n$-spheres in $\Sigma$ defined by confluent critical branchings:

$$
\begin{array}{c}
\text{u} \\
\text{u} \\
\text{u_1} \\
\text{u'} \\
\text{u_2}
\end{array}
$$
Convergent polygraphs

- $\Gamma_\Sigma$ set of $n$-spheres in $\Sigma$ defined by confluent critical branchings:

![Diagram showing a convergent polygraph with nodes $u$, $u_1$, $u_2$, and $u'$ connected by arrows.]

**Proposition.**
For a convergent polygraph $\Sigma$, the set $\Gamma_\Sigma$ forms a homotopy base.
Convergent polygraphs

Theorem.
A finite convergent polygraph with a finite set of critical branchings has FDT.
Convergent polygraphs

**Theorem.**
A finite convergent polygraph with a finite set of critical branchings has FDT.

**Corollary.**
A finite convergent 2-polygraph has FDT. (Squier ’87)
Theorem.
A finite convergent polygraph with a finite set of critical branchings has FDT.

Corollary.
A finite convergent 2-polygraph has FDT. (Squier ’87)

Theorem.
For $n \geq 3$, there exists finite convergent $n$-polygraph without FDT.
Critical branchings in 2-polygraphs

- **Regular critical branching** (simple overlapping)
Critical branchings in 2-polygraphs

- **Regular critical branching** (simple overlapping)

- **Inclusion critical branching**
Critical branchings in 3-polygraphs

- Regular critical branching

\[ s\alpha \circ g = f \circ h = s\beta \]
Critical branchings in 3-polygraphs

- Regular critical branching

  \[ s\alpha \quad v \quad g \quad u \quad f \quad h \quad v \quad g \quad u \quad s\beta \]

- Inclusion critical branching

  \[ s\alpha \quad = \quad s\beta \quad C \]

III. Polygraphs having Finite Derivation Type
Critical branchings in 3-polygraphs

- **Regular critical branching**

  \[
  s_\alpha v g = f u h g = s_\beta v u
  \]

- **Inclusion critical branching**

  \[
  s_\alpha = s_\beta C
  \]

- **Right-indexed critical branching**

  \[
  s_\alpha k g = f h k g = s_\beta k
  \]
Critical branchings in 3-polygraphs

- **Regular**

\[
\begin{align*}
s\alpha & \quad \rightarrow \quad \begin{array}{c} \quad f \quad \rightarrow \quad s\beta \\
\quad s\beta & = \quad s\beta \quad \rightarrow \quad s\beta
\end{array} \\
\end{align*}
\]

- **Inclusion**

\[
\begin{align*}
s\alpha & \quad \rightarrow \quad \begin{array}{c} \quad f \quad \rightarrow \quad s\beta \\
\quad s\beta & = \quad s\beta \quad \rightarrow \quad s\beta
\end{array} \\
\end{align*}
\]

- **Indexed**

\[
\begin{align*}
s\alpha & \quad \rightarrow \quad \begin{array}{c} \quad f \quad \rightarrow \quad s\beta \\
\quad s\beta & = \quad s\beta \quad \rightarrow \quad s\beta
\end{array} \\
\end{align*}
\]

III. Polygraphs having Finite Derivation Type
Convergent 3-polygraphs having FDT

**Theorem.** Let $\Sigma$ be a finite convergent 3-polygraph.

- If $\Sigma$ does not have indexed critical branching it has FDT.
- If $\Sigma$ has indexed critical branchings with finitely many normal instances it has FDT.
**Theorem.** Let $\Sigma$ be a finite convergent 3-polygraph.
- If $\Sigma$ does not have indexed critical branching it has FDT.
- If $\Sigma$ has indexed critical branchings with finitely many normal instances it has FDT.
IV. An example ... and a counterexample
The 3-polygraph of permutations has FDT

\[ \Sigma_{\text{Perm}}: \]

- one 0-cell, one 1-cell, one 2-cell
- two 3-cells:

\[ \xrightarrow{\alpha} \quad \text{and} \quad \xrightarrow{\beta} \]

IV. An example ... and a counterexample
The 3-polygraph of permutations has FDT

\[ \Sigma_{\text{Perm}}: \]
- one 0-cell, one 1-cell, one 2-cell
- two 3-cells:

---

**Proposition.**
\( \Sigma_{\text{Perm}} \) is finite, convergent and has FDT.
The 3-polygraph of permutations has FDT

$\Sigma_{\text{Perm}}$ has an infinite set of critical branchings.
The 3-polygraph of permutations has FDT

$\Sigma_{\text{Perm}}$ has an infinite set of critical branchings.
  • three regular critical branchings
The 3-polygraph of permutations has FDT

\( \Sigma_{\text{Perm}} \) has an infinite set of critical branchings.
- three regular critical branchings
- a right-indexed critical branching

IV. An example … and a counterexample
The 3-polygraph of permutations has FDT

- Regulars are confluent

- Right-indexed have two normal instances

IV. An example ... and a counterexample
The 3-polygraph of permutations has FDT

\[ \Sigma_{\text{Perm}} \] has an infinite set of critical branchings.
- three regular critical branchings

\[ \xrightarrow[regulars]{\text{are confluent}} \]

- a right-indexed critical branching

\[ \Rightarrow \]

\[ \Sigma_{\text{Perm}} \] has FDT

IV. AN EXAMPLE ... AND A COUNTEREXAMPLE
A finite convergent polygraph which doesn’t have FDT

A polygraph $\Sigma$ presenting the 2-category of pear necklaces.

- one 0-cell, one 1-cell, three 2-cells:
A finite convergent polygraph which doesn’t have FDT

A polygraph $\Sigma$ presenting the 2-category of pear necklaces.

- one 0-cell, one 1-cell, three 2-cells :
  \[ \bullet \quad \cap \quad \cup \]

- four 3-cells :
  \[ \cap \quad \Rightarrow \quad \cap \quad \bullet \quad \Rightarrow \quad \cup \quad \cap \quad \Rightarrow \quad \Sigma \quad \Rightarrow \quad \Sigma \]
A finite convergent polygraph which doesn’t have FDT

A polygraph $\Sigma$ presenting the 2-category of pear necklaces.

- one 0-cell, one 1-cell, three 2-cells:
  \[
  \bullet \quad \bigcap \quad \bigcup 
  \]

- four 3-cells:
  \[
  \begin{align*}
  \bigcap \quad \alpha \Rightarrow \bigcap \\
  \bigcup \quad \beta \Rightarrow \bigcup \\
  \bigcup \quad \gamma \Rightarrow \\
  \bigcup \quad \delta \Rightarrow \\
  \end{align*}
  \]
A finite convergent polygraph which doesn’t have FDT

A polygraph $\Sigma$ presenting the 2-category of pear necklaces.

- one 0-cell, one 1-cell, three 2-cells:

- four 3-cells:

Theorem.

$\Sigma$ is finite and convergent but does not have FDT
A finite convergent polygraph which doesn’t have FDT

\[ \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \]

IV. An example … and a counterexample
A finite convergent polygraph which doesn’t have FDT

- Four regular critical branching

IV. An example ... and a counterexample
A finite convergent polygraph which doesn’t have FDT

- Four regular critical branching

IV. AN EXAMPLE ... AND A COUNTEREXAMPLE
A finite convergent polygraph which doesn’t have FDT

- One right-indexed critical branching

\[ k \in \{ \_\_\_\_\_\_\_\_ \} \]
A finite convergent polygraph which doesn’t have FDT
A finite convergent polygraph which doesn’t have FDT
A finite convergent polygraph which doesn’t have FDT

Theorem.

\( \Sigma \) is finite and convergent but does not have finite derivation type
Conclusions

- \((n, p)\)-polygraphs.
- Module of identities among relations for higher categories presented by polygraph.
- Property \(FP_3\) for higher categories.
- (Co)homology of higher categories with coefficients.
- Compute homology of Lawvere theories using Burroni presentation.