THE THREE DIMENSIONS OF PROOFS

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The two dimensions of formulas

The formulas of SKS

The *formulas of SKS* are the equivalence classes of terms built on:

- two sorts: one for *atoms* (a, b, etc.) and one for *formulas* (f, g, etc.);
- the grammar:

$$a ::= x_a \mid \bar{a} \; ,$$

$$f ::= x_f \mid \bot \mid \top \mid a \mid f \wedge f \mid f \vee f \; ;$$

- the structural relations:
  - associativity and commutativity of $\wedge$ and $\vee$,
  - units: $\top \wedge f = f, \quad \bot \vee f = f, \quad \bot \wedge \bot = \bot, \quad \top \vee \top = \top,$
  - involutivity of $\bar{\cdot}$: $\quad \bar{\bar{a}} = a.$

Formulas as 2-dimensional objects

Formulas of SKS are replaced by all the "circuits" built with:

- two colors of "wires": green for the atoms, orange for the formulas;
- the following fourteen "components":

Six "components" for the structure of SKS formulas plus eight for *explicit resources management operations*: duplication, erasure, exchange.

Each "circuit" represents a family of SKS formulas:

$\Big((\bot \vee f) \wedge a), \overline{a}\Big)$ $\qquad \top \qquad a \wedge a \qquad a \wedge \overline{a}$

Some terminology on polygraphs

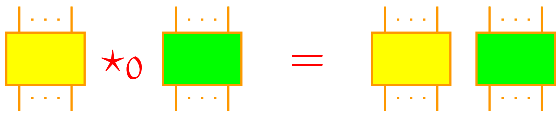
The "circuits" form a *2-polygraph* with one 0-cell and whose:

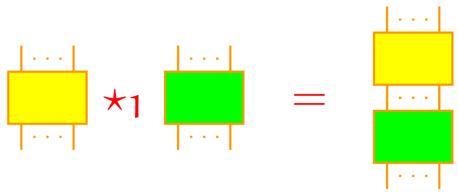
– 1-cells are the two types of "wires",

– 2-cells are the fourteen "components",

– 2-arrows or 2-paths are the "circuits".

The "2" comes from the two possible *compositions*, one along each direction:

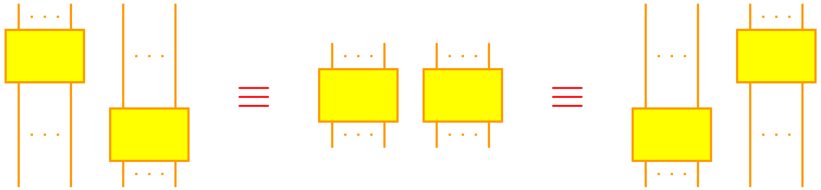




The second composite $f \star_1 g$ is defined only if the 1-target of f and the 1-source of g are equal: $f_1^+ = g_1^-$.

The exchange relation $\equiv_{01}$

In a polygraph, the 2-arrows are always considered *modulo homeomorphic deformation*:



These topological moves are controlled by the *exchange relation* $\equiv_{01}$:

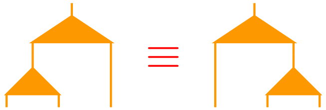
$$(f \star_0 g_1^-) \star_1 (f_1^+ \star_0 g) \equiv_{01} f \star_0 g \equiv_{01} (f_1^- \star_0 g) \star_1 (f \star_0 g_1^+)$$

This relation is invisible on SKS formulas.

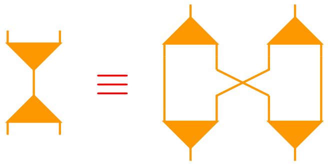
Equivalent 2-arrows

Many 2-arrows represent the same family of formulas because:

- Resources management is now explicit:

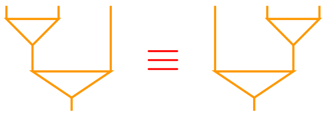


$(x, x, x) = (x, x, x)$

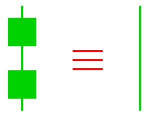


$(x \wedge y, x \wedge y) = (x \wedge y, x \wedge y)$

- SKS formulas are equipped with structural relations:



$(x \vee y) \vee z \equiv x \vee (y \vee z)$



$\overline{\overline{a}} \equiv a$

These equivalences are controlled by objects living in dimension 3, together with the deduction rules.

The three dimensions of proofs

The proofs of SKS

The *inference rules of SKS* are the eight rewriting rules:

$$\frac{\top}{a \vee \bar{a}} \qquad \frac{(x \vee y) \wedge z}{(x \wedge z) \vee y} \qquad \frac{\bot}{a} \qquad \frac{a \vee a}{a}$$

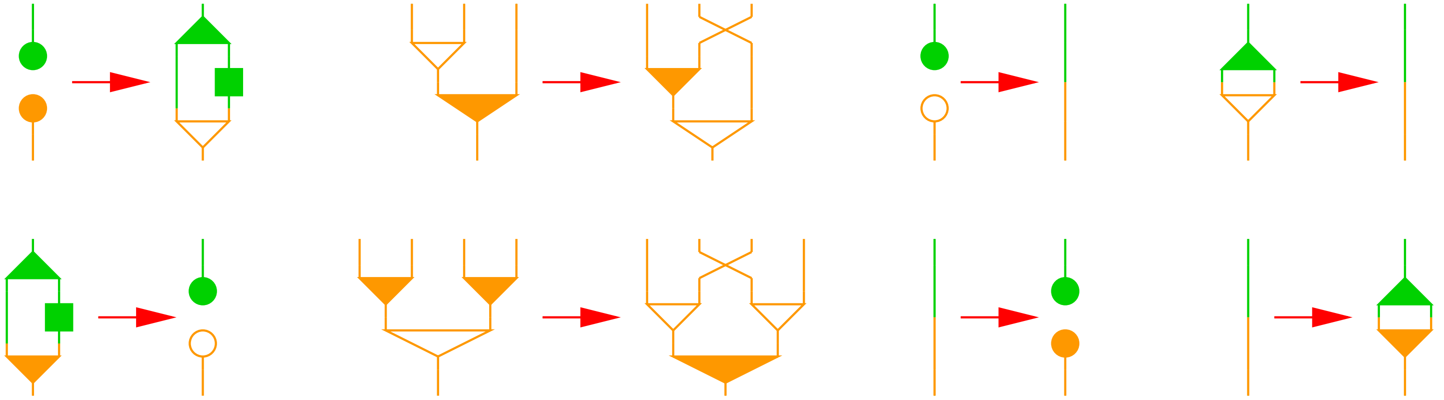
$$\frac{a \wedge \bar{a}}{\bot} \qquad \frac{(x \wedge y) \vee (z \wedge t)}{(x \vee z) \wedge (y \vee t)} \qquad \frac{a}{\top} \qquad \frac{a}{a \wedge a}$$

The *proofs of SKS* are all the rewriting paths generated by:

- the inference rules,
- the structural relations (seen as rules going in both directions).

Inference rules as 3-dimensional objects

The eight inference rules of SKS are replaced by rewriting rules on 2-arrows:

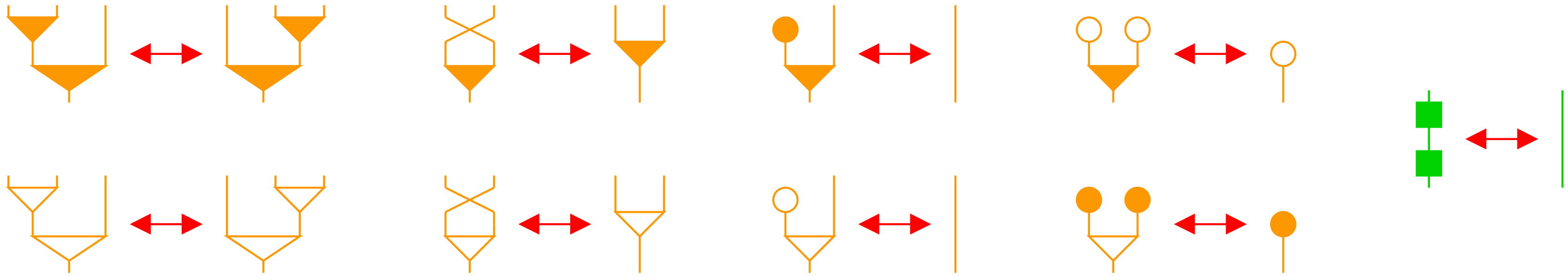


These objects are 3-*cells* over the 2-polygraph of formulas.

But these 3-cells are not sufficient to describe all the SKS proofs: one must add 3-cells for structural relations and for resources management.

Structural 3-cells

Structural equations between formulas are translated into the 18 following 3-cells:



There are two for each equation, one in each direction.

This is only one way, among others, to represent an equation between 2-arrows:

– equations on 2-arrows (work in a quotient),

– one 3-cell in each direction (like here),

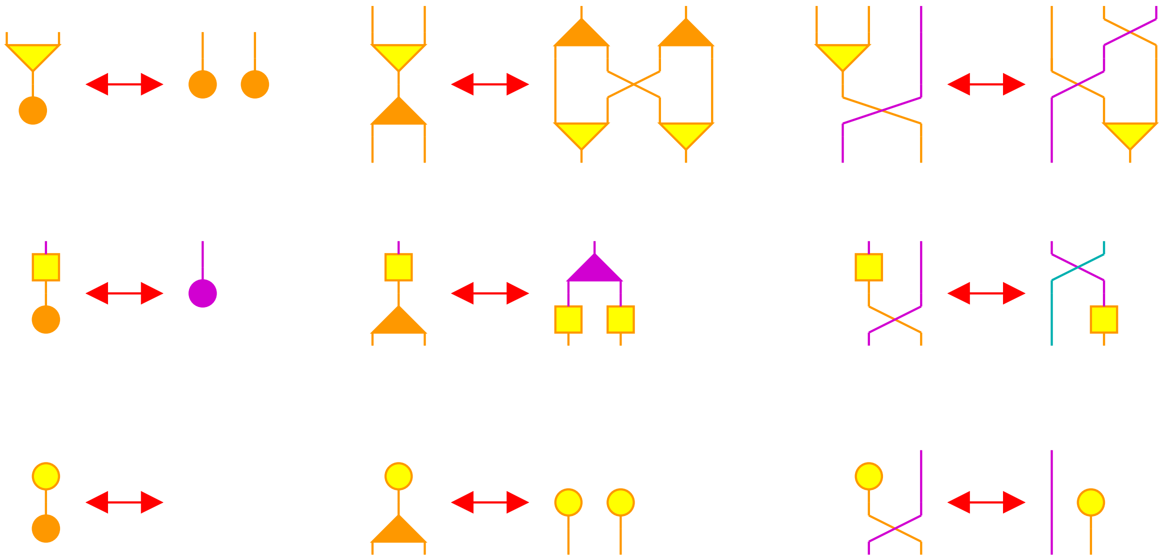
– only one 3-cell, in either direction (requires a convergent 3-polygraph).

There is at least one more possibility, discussed later.

Resources management 3-cells

They are divided into two families:

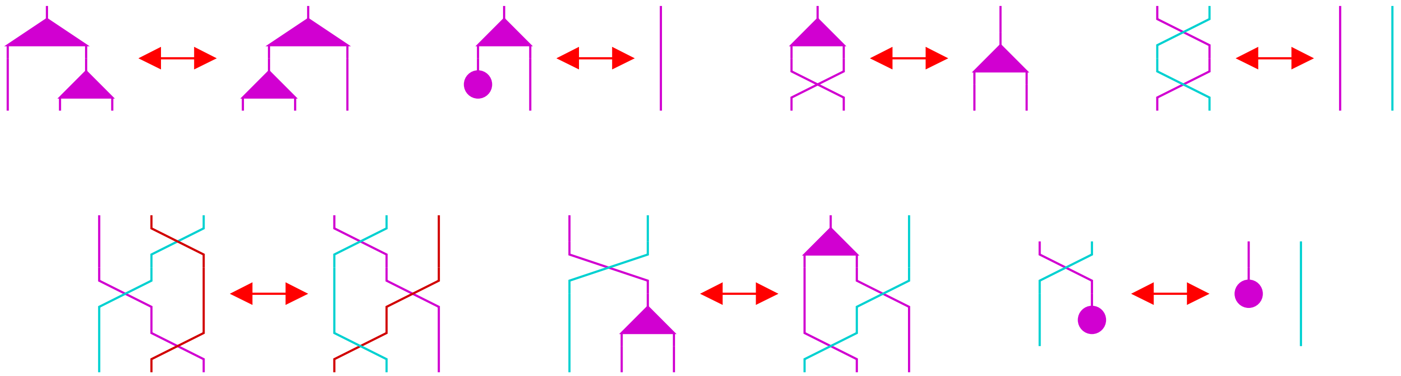
- a first one with $2 \times 24 = 48$ cells:



Resources management 3-cells

They are divided into two families:

- a second one with $2 \times 26 = 52$ cells:



The resources management 3-cells, alone, can be presented in a convergent way.

But this creates at least confluence issues with some inference 3-cells.

The 3-polygraph of proofs

Together with the 2-polygraph of formulas, all these 3-cells form a *3-polygraph* with:

- two 1-cells, one for each sort of SKS formulas,
- fourteen 2-cells, one for each operation of the structure,
- one hundred and twenty-six 3-cells, 100 of which are resources management 3-cells.

One can prove that:

- Each rewriting path generated by the 3-cells represents a SKS proof.
- Each SKS proof can be represented by at least one rewriting path.

But the rewriting paths generated by 3-cells have an algebraic structure:

they are 3-arrows in a polygraph.

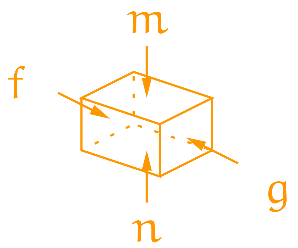
Let us consider this alternative point of view on proofs and see what happens.

The structure of 3-arrows

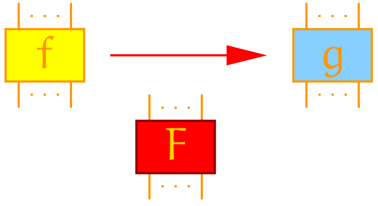
The 3-arrows of a polygraph

The 3-arrows of a polygraph are inductively defined as follows:

• Every 3-cell $F : f \to g$ is also a 3-arrow, going from f (its *2-source*) to g (its *2-target*). For the moment, we represent such a F as a "block":



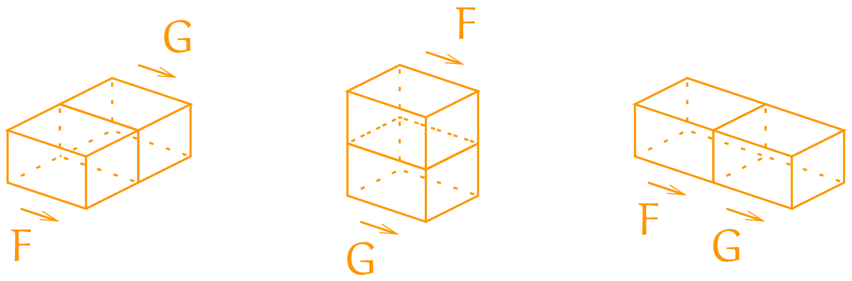
Three successive vertical slices of the block give a planar representation:



• Every 2-arrow f is also a 3-arrow, going from and to itself.

The 3-arrows of a polygraph

- If $F : f \longrightarrow f'$ and $G : g \longrightarrow g'$ are 3-arrows, they can be composed in three ways:



The 3-arrows of a polygraph

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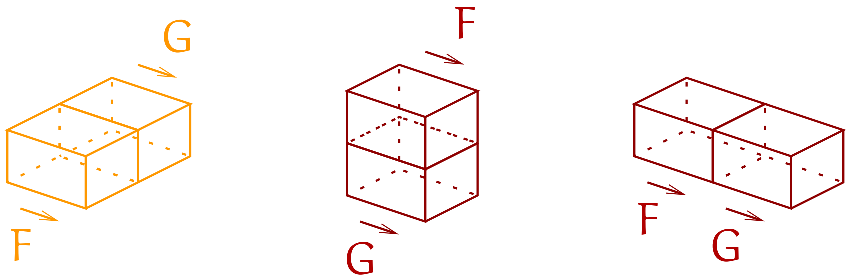
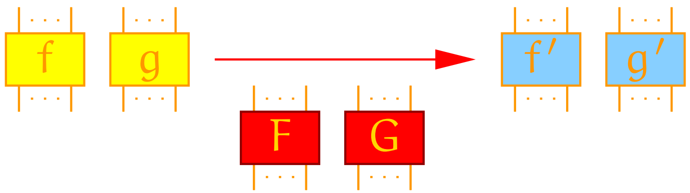


The first one yields a proof $F \star_0 G : f \star_0 g \to f' \star_0 g'$.

Using vertical slices:



The 3-arrows of a polygraph

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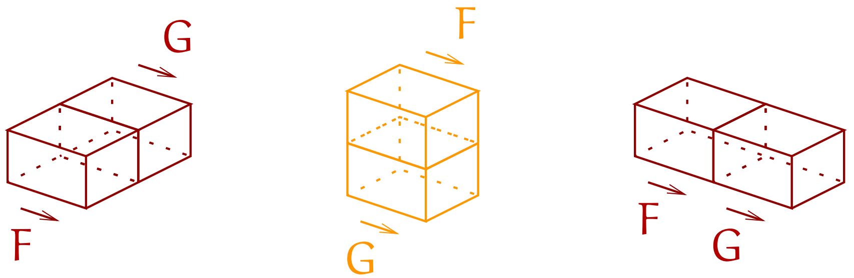
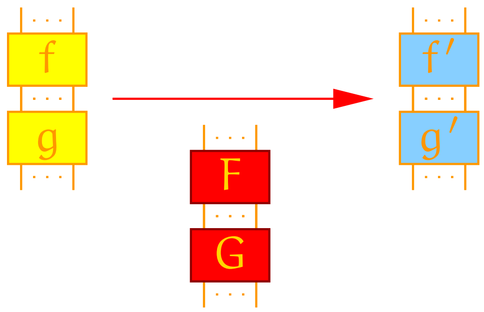


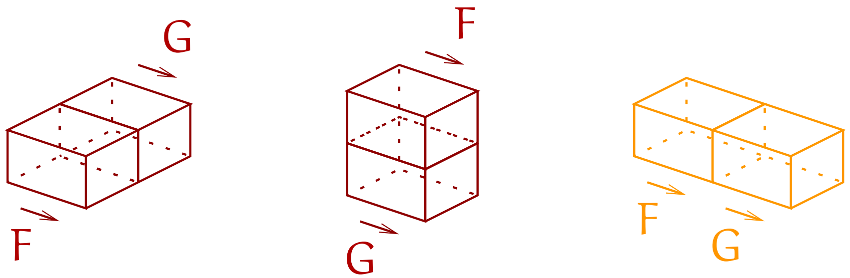
The second one yields a proof $F \star_1 G : f \star_1 g \longrightarrow f' \star_1 g'$ (only if $f_1^+ = g_1^-$).

Using vertical slices:



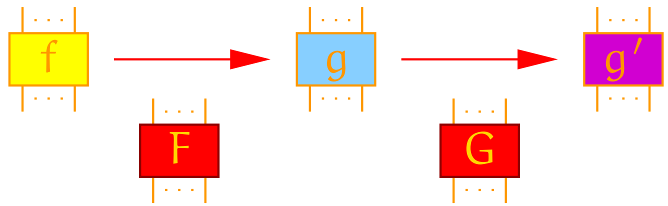
The 3-arrows of a polygraph

- If $F : f \to f'$ and $G : g \to g'$ are 3-arrows, they can be composed in three ways:



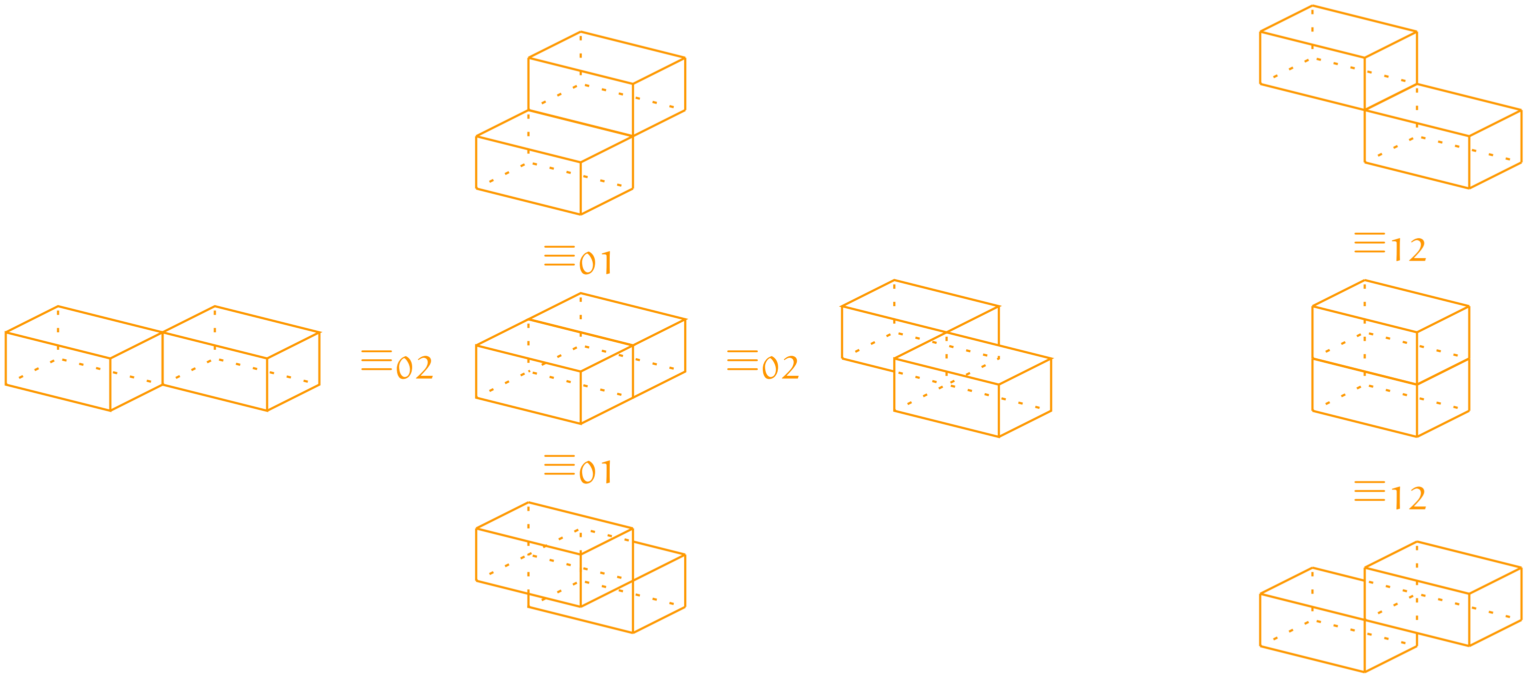
The third one yields a proof $F \star_2 G : f \to g'$ (only if $f' = g$).

Using vertical slices:



Exchange relations

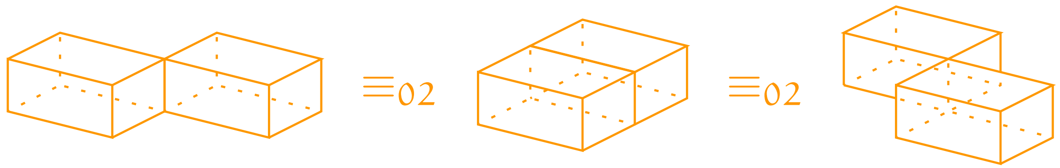
In a polygraph, 3-arrows are identified *modulo exchange relations*:



The relation $\equiv_{01}$ is some "ghost" relation coming from dimension 2.
But the other two are new: they identify 3-arrows that represent the same proof but are distinguished by the chosen syntax on formulas (a phenomenon called "bureaucracy").

The relation $\equiv_{02}$

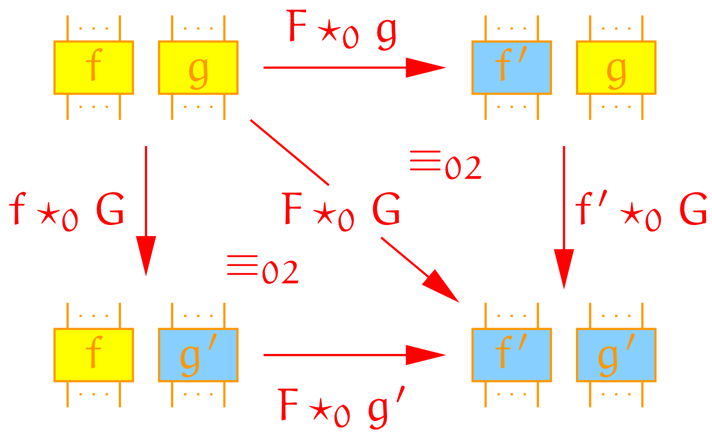
Let us consider:



This can also be written:

$$(F \star_0 G_2^-) \star_2 (F_2^+ \star_0 G) \;\equiv_{02}\; F \star_0 G \;\equiv_{02}\; (F_2^- \star_0 G) \star_2 (F \star_0 G_2^+).$$

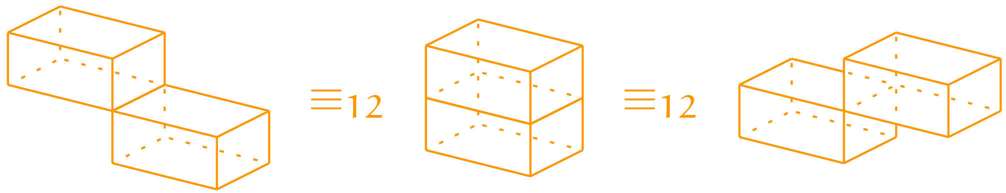
Or with vertical slices:



The relation $\equiv_{02}$ identifies proofs using the same two arguments on two distinct subformulas, but in different order ("bureaucracy type A").

The relation $\equiv_{12}$

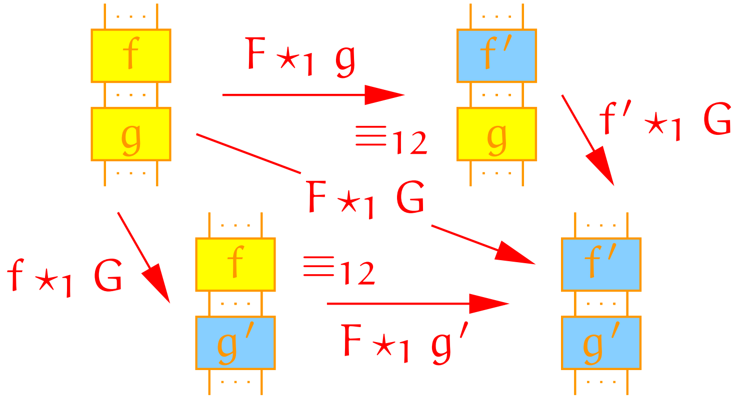
Let us consider:



This can also be written:

$$(F \star_1 G_2^-) \star_2 (F_2^+ \star_1 G) \;\equiv_{12}\; F \star_1 G \;\equiv_{12}\; (F_2^- \star_1 G) \star_2 (F \star_1 G_2^+).$$

Or with vertical slices:



The relation $\equiv_{12}$ identifies proofs using the same two arguments, one "sufficiently inside the other", but in different order ("bureaucracy type B").

Comparison between terms and polygraphs

In the usual term-like syntax:

- the two types of "bureaucracy" appear different in essence;
- with some work, one can write equations controlling type A;
- but equations controlling type B are *really* hard to design.

In the polygraphic setting:

- the two types of "bureaucracy" are controlled by equations with the same shape:

$$(F \star_i G_j^-) \star_j (F_i^+ \star_i G) \equiv_{ij} F \star_i G \equiv_{ij} (F_j^- \star_i G) \star_j (F \star_i G_j^+) \qquad \text{for } i < j;$$

- these equations have a "geometric" interpretation as topological deformations.

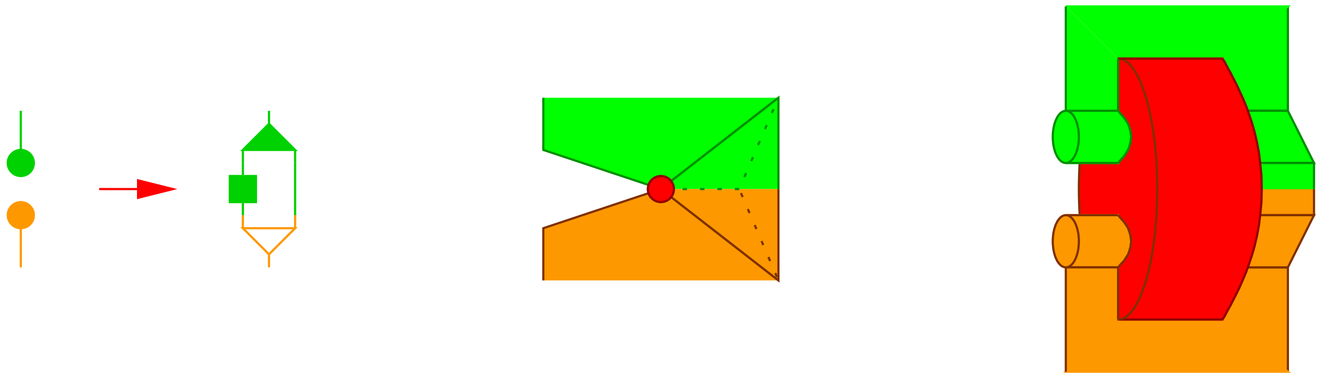
Higher dimensions

Proofs in three dimensions

Let us use a 3-dimensional version of the duality that led to circuit-like representations:

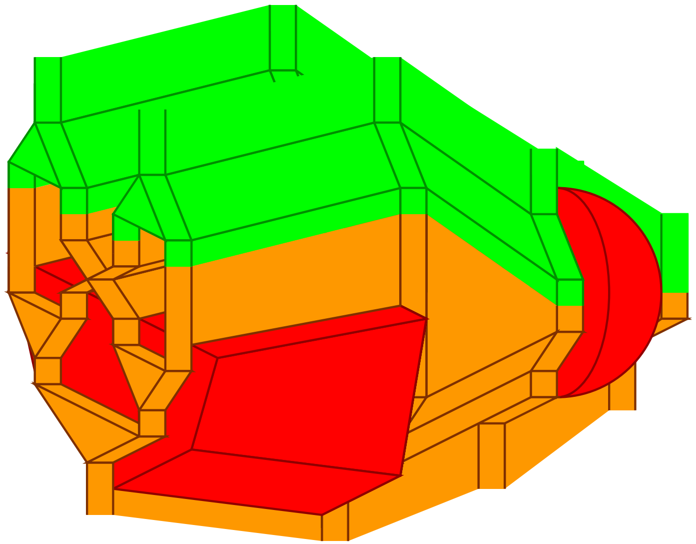
- each i-cell is represented by a $(3 - i)$-dimensional object;
- the 2-cells and 3-cells are "inflated" for emphasis.

On an example, this recipe gives:

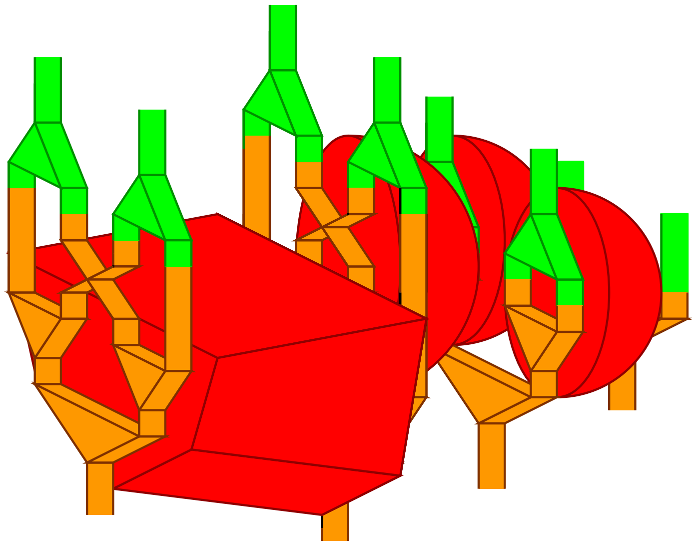


Proofs are all the LEGOs one can build using such elementary blocks, *modulo* the topological moves corresponding to the exchange relations.

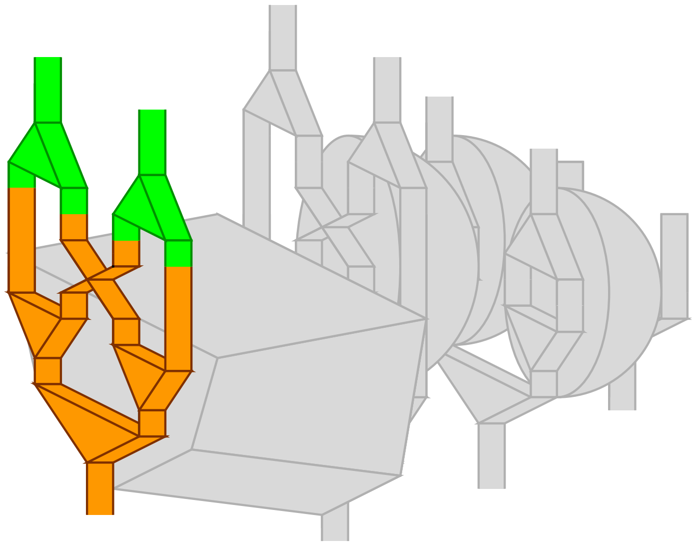
One proof in three dimensions



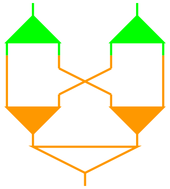
One proof in three dimensions



One proof in three dimensions

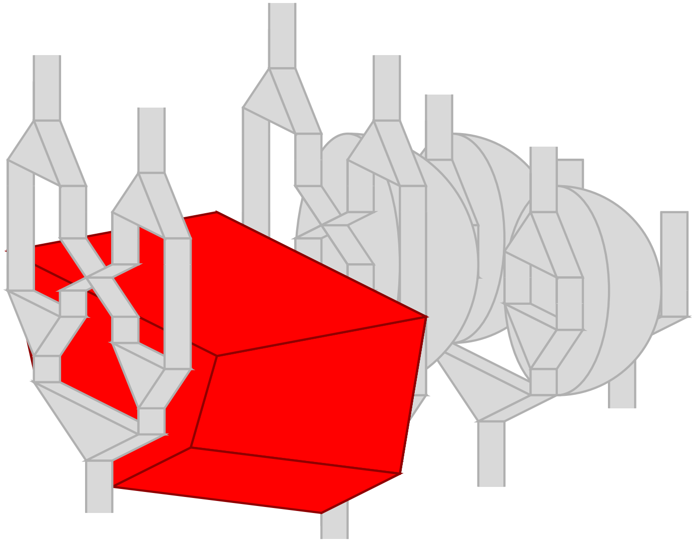


Using vertical slices:



$(a \wedge b) \vee (a \wedge b)$

One proof in three dimensions



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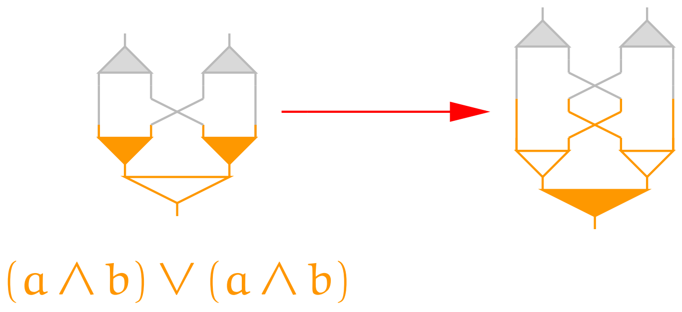

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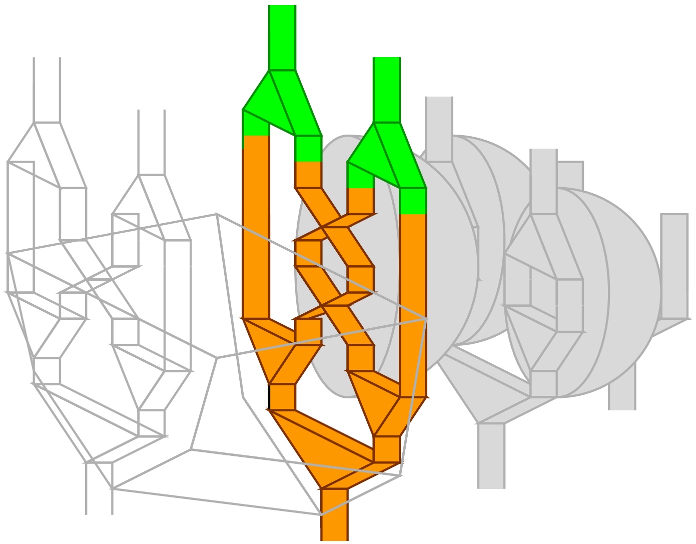

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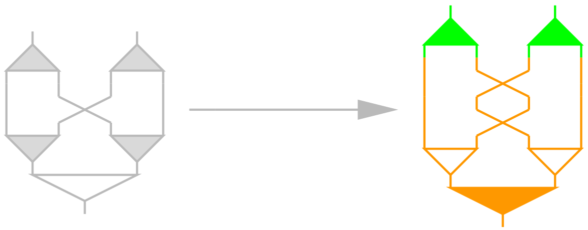

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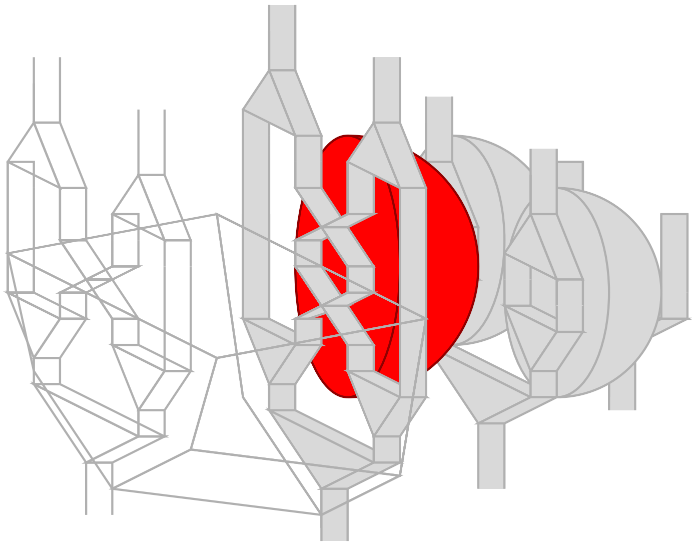

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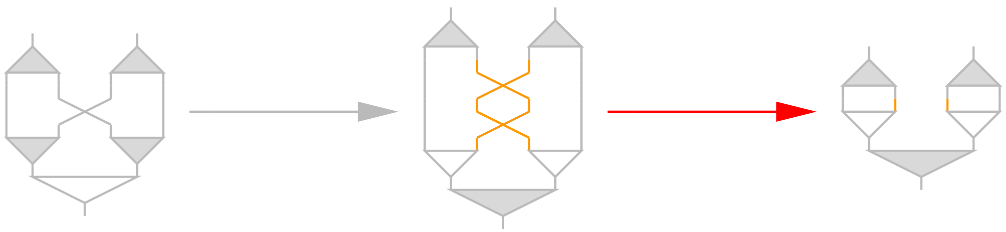
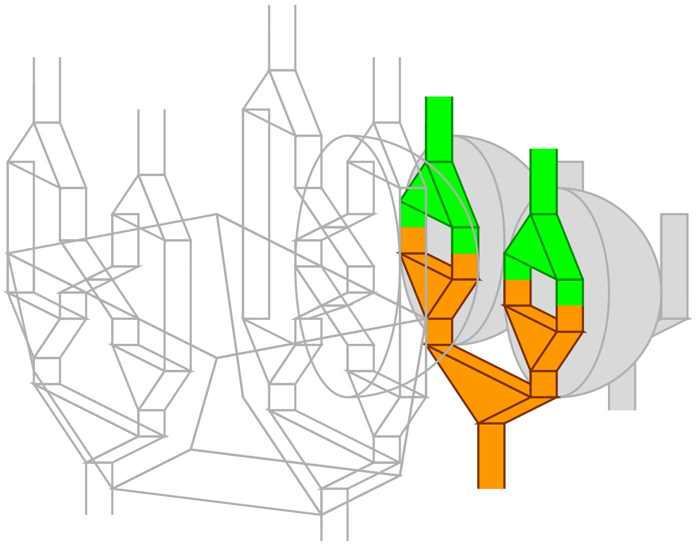


$(a \wedge b) \vee (a \wedge b)$ $(a \vee a) \wedge (b \vee b)$

One proof in three dimensions



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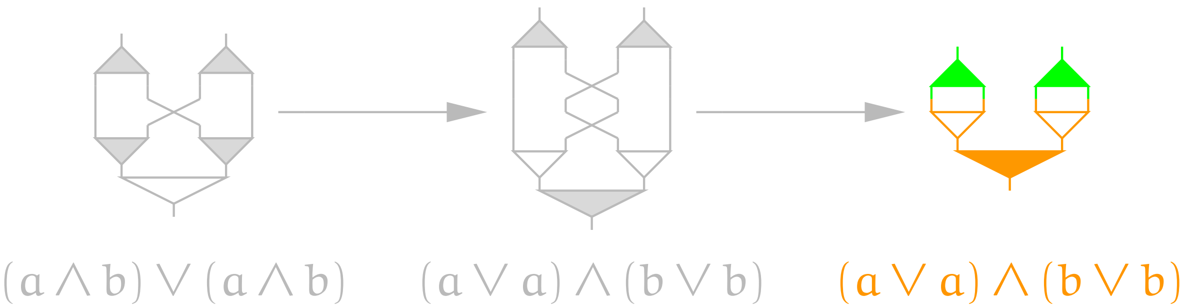

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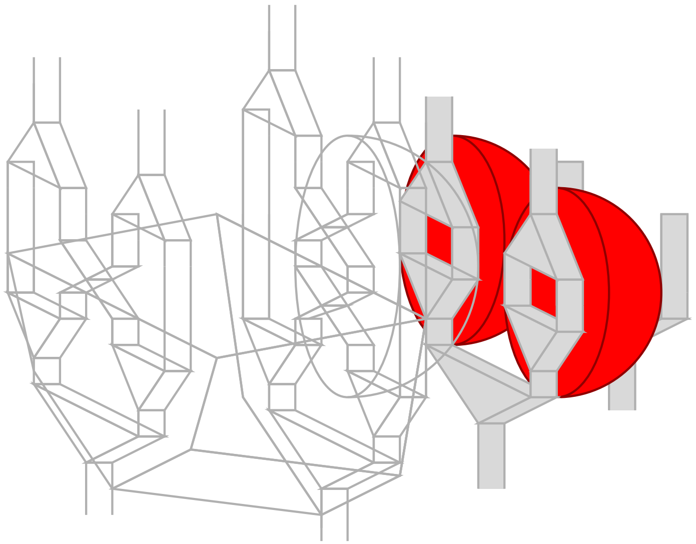

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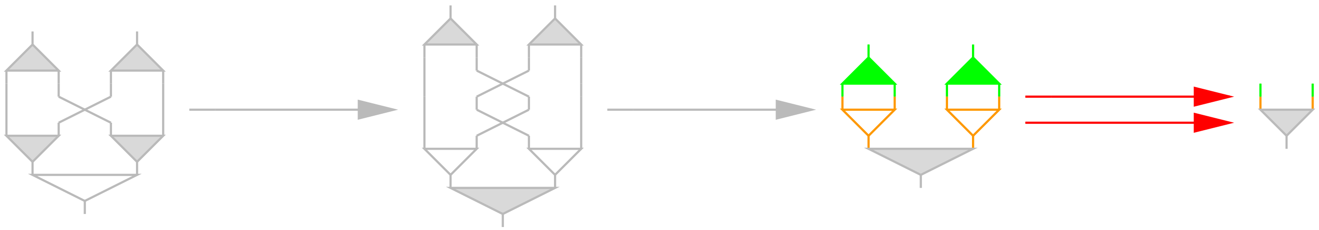
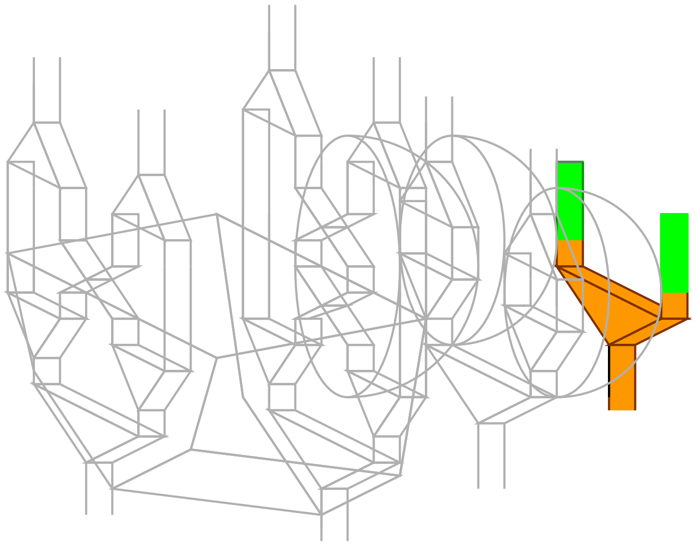


$(a \wedge b) \vee (a \wedge b)$ $(a \vee a) \wedge (b \vee b)$ $(a \vee a) \wedge (b \vee b)$

One proof in three dimensions



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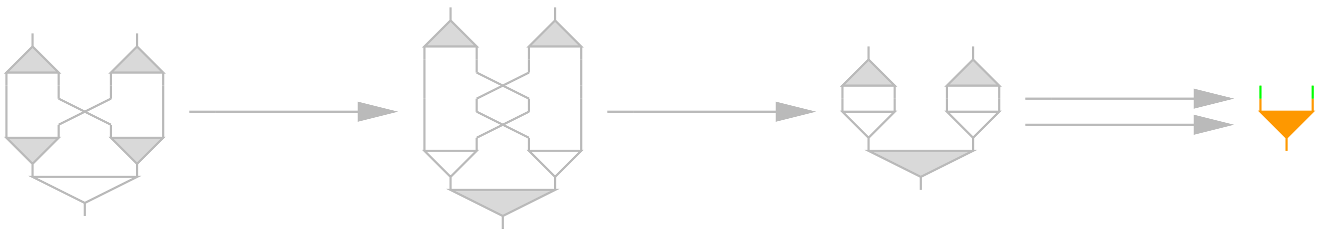
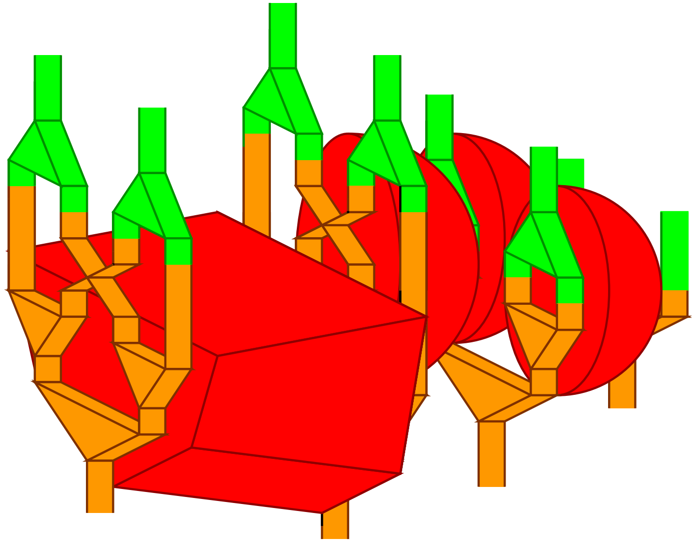


$(a \wedge b) \vee (a \wedge b)$ $\quad (a \vee a) \wedge (b \vee b)$ $\quad (a \vee a) \wedge (b \vee b)$ $\quad a \wedge b$

One proof in three dimensions



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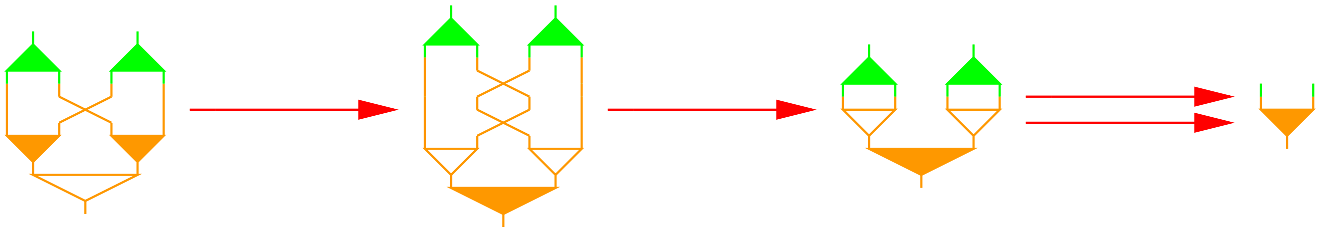

$(a \wedge b) \vee (a \wedge b)$ $\quad$ $(a \vee a) \wedge (b \vee b)$ $\quad$ $(a \vee a) \wedge (b \vee b)$ $\quad$ $a \wedge b$

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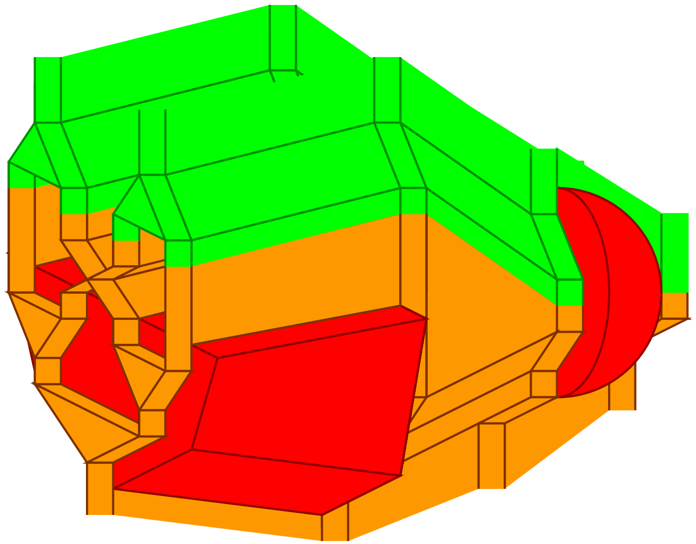

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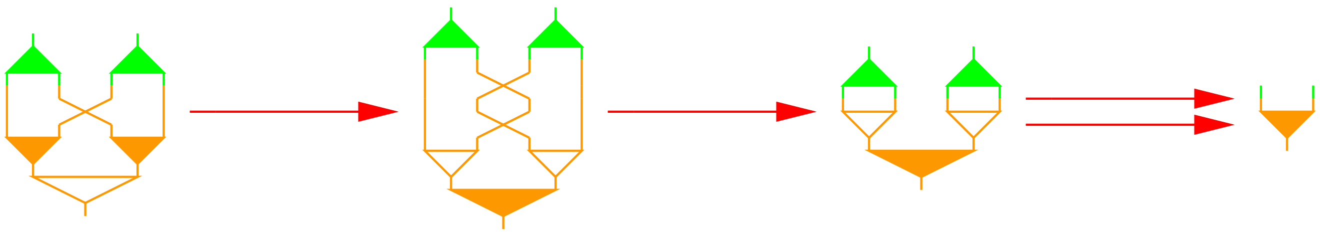
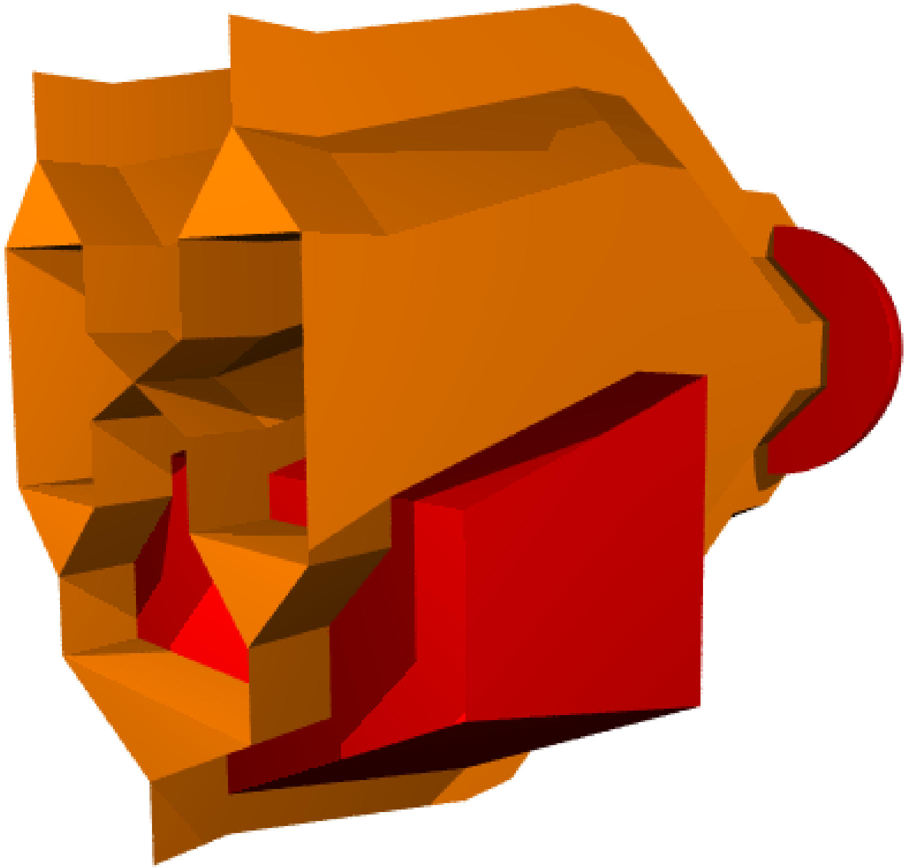


$(a \wedge b) \vee (a \wedge b)$ $\quad$ $(a \vee a) \wedge (b \vee b)$ $\quad$ $(a \vee a) \wedge (b \vee b)$ $\quad$ $a \wedge b$

A proof in three dimensions (second version)



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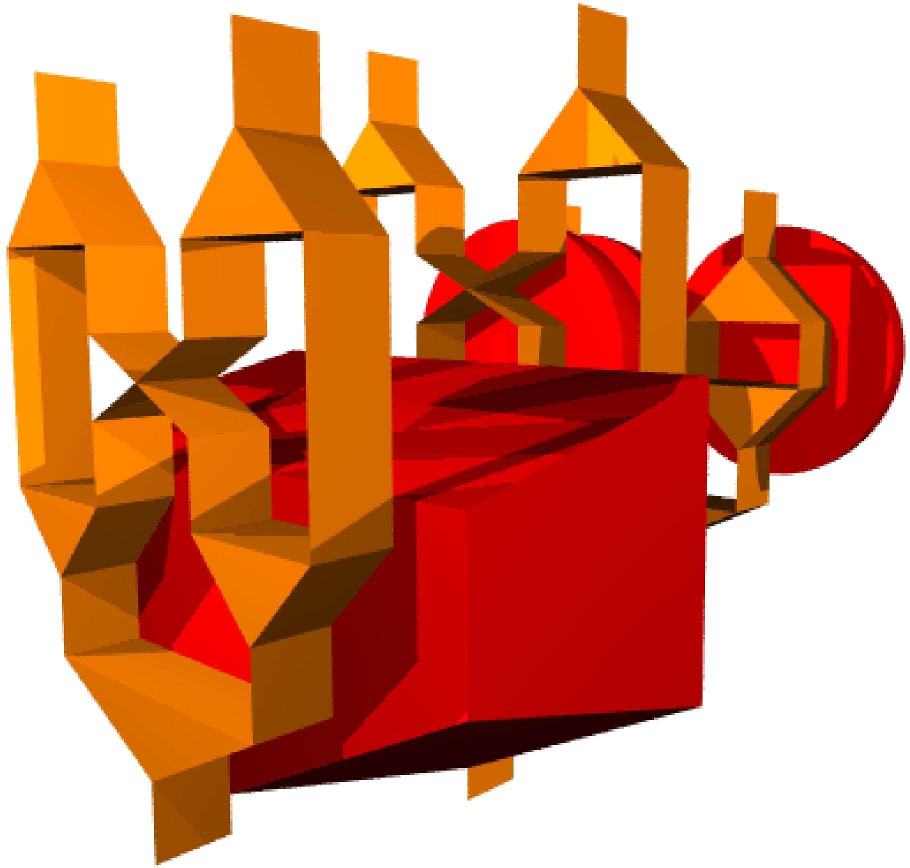

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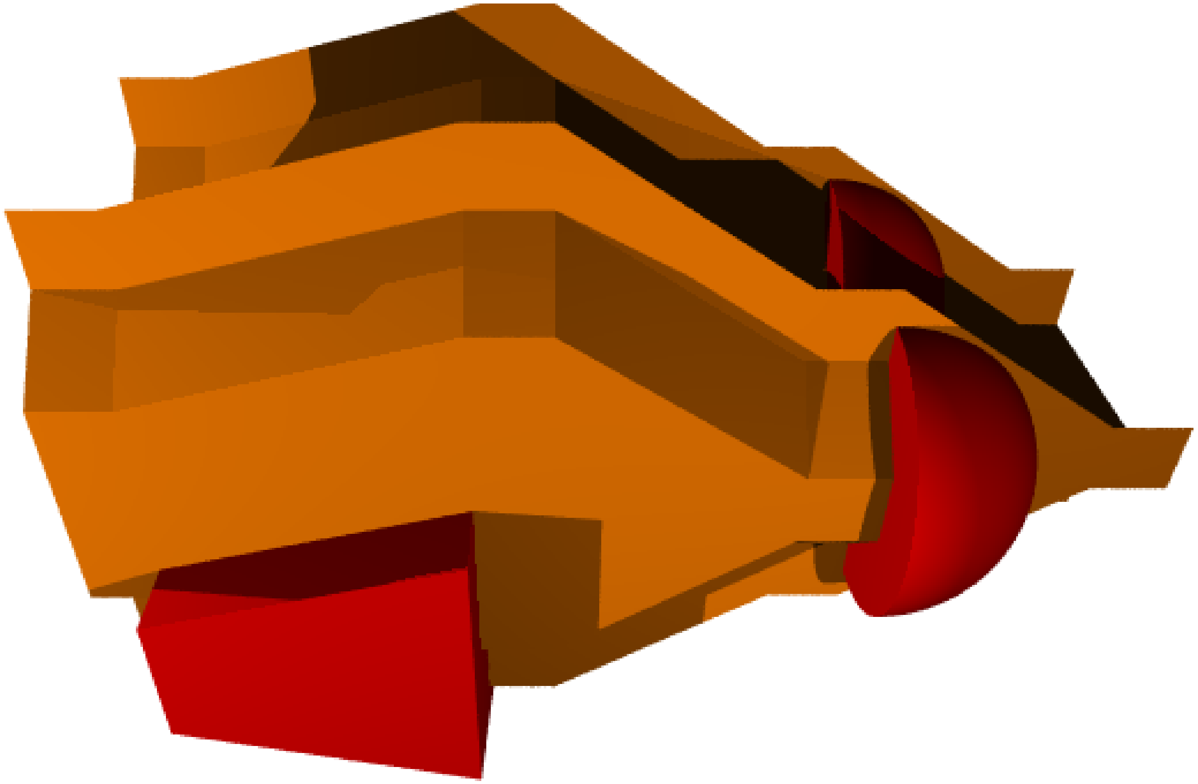
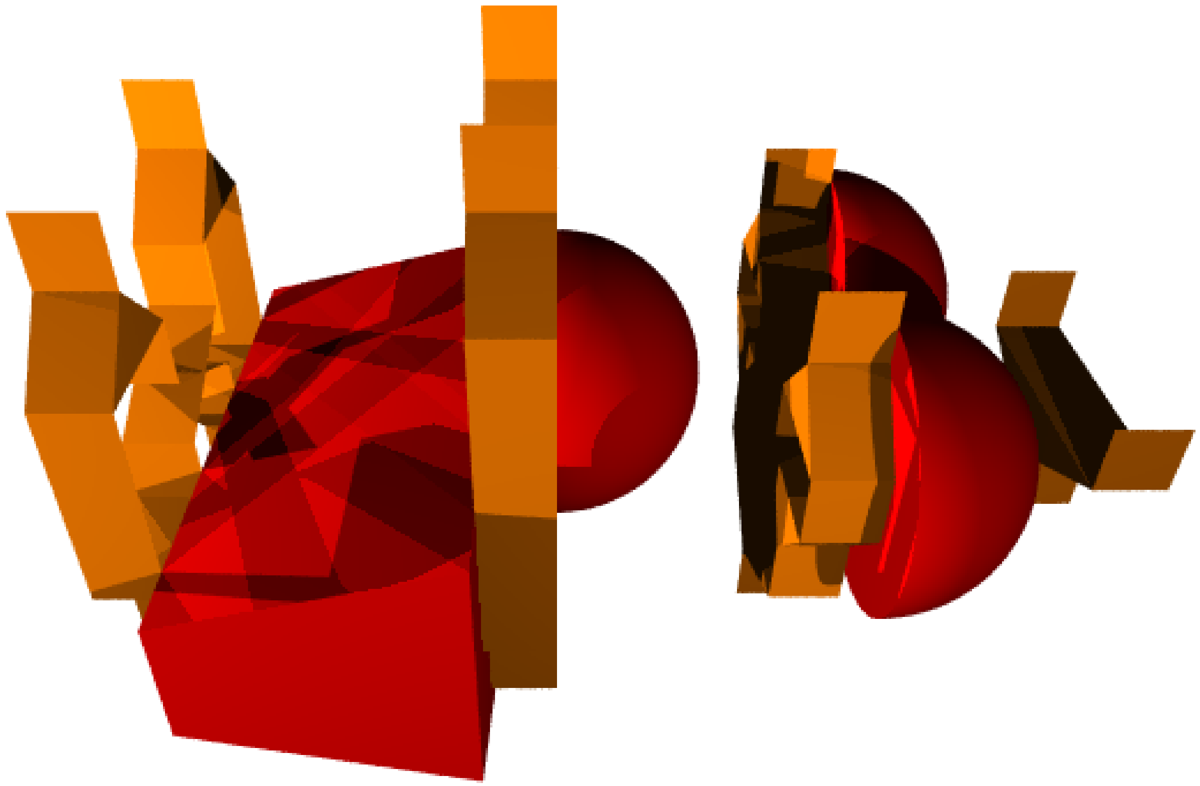
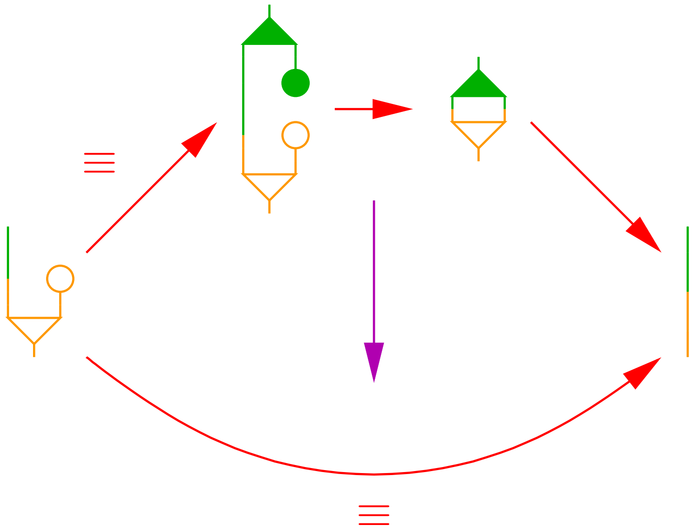


A proof in three dimensions (second version)



The twilight zone [*La quatrième dimension*]

Polygraphs provide an algebraic structure for *computations on proofs* specified by local rules, such as this one:



This is a *4-dimensional cell* over the 3-polygraph of proofs...

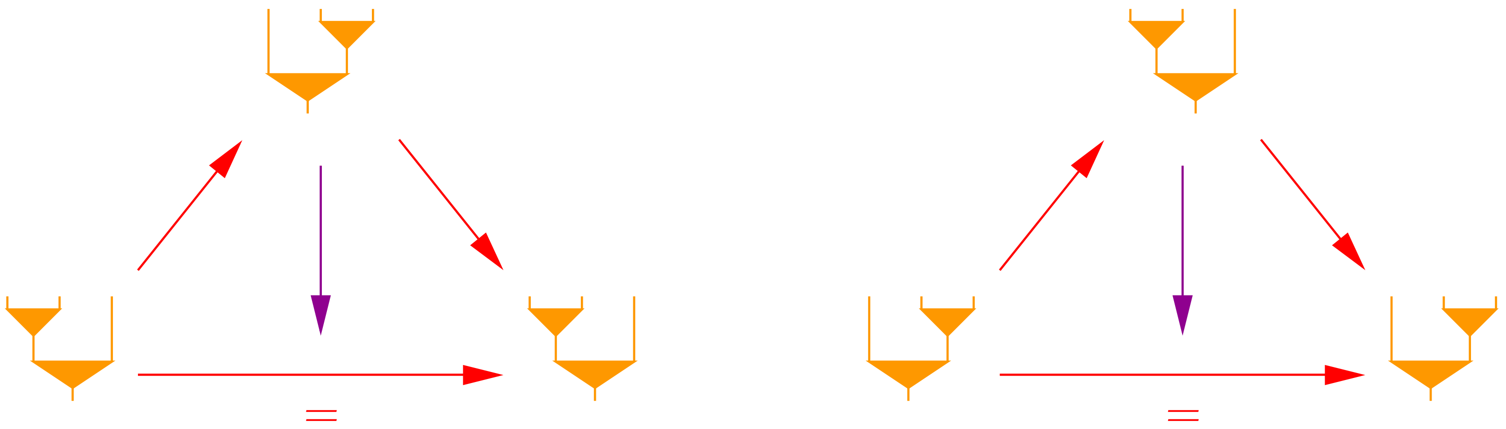
... the computations it generates are *4-dimensional arrows*.

The twilight zone [*La quatrième dimension*]

Another example is provided by equations between formulas when each one is seen as:

- a pair of 3-cells, one in each direction,
- a pair of 4-cells: proofs that the two 3-cells form a 3-isomorphism.

With an associativity relation, this gives:



In this setting, local computations on proofs are 4-dimensional objects:

- 4 ways to compose them, the three old ones plus $\star_3$,
- 6 types of "bureaucracy", the three old ones plus $\equiv_{03}$, $\equiv_{13}$ and $\equiv_{23}$.

Conclusion

We have seen that polygraphs provide an alternative setting for proof theory where:

- formulas and proofs have a different algebraic structure, which makes all types of "bureaucracy" controlled by similar exchange relations;
- proofs have graphical representations as 3-dimensional objects, where exchange relations are topological moves;
- computations on proofs generated by local rules have a natural place as 4-dimensional objects.

To be continued...

THE THREE DIMENSIONS OF PROOFS

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