Higher-dimensional rewriting strategies and acyclic polygraphs

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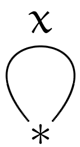
Tianjin – 6 July 2010

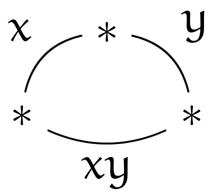
1. Introduction

1.1. Motivation

Problem: Monoid M as a cellular object

– One 1-cell 

for every x in M

– Contract triangles 

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Question: Build a "complete cellular model"

i.e., free cellular object with the same homotopy type

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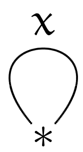
– One 1-cell 

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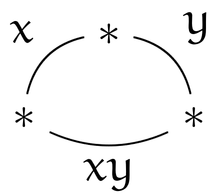
– Contract triangles 

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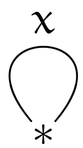
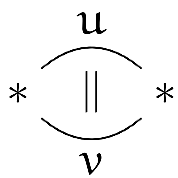
Lower dimensions: Use a presentation $M = \langle X \mid R \rangle$

– One 1-cell 

for every x in X

– One 2-cell 

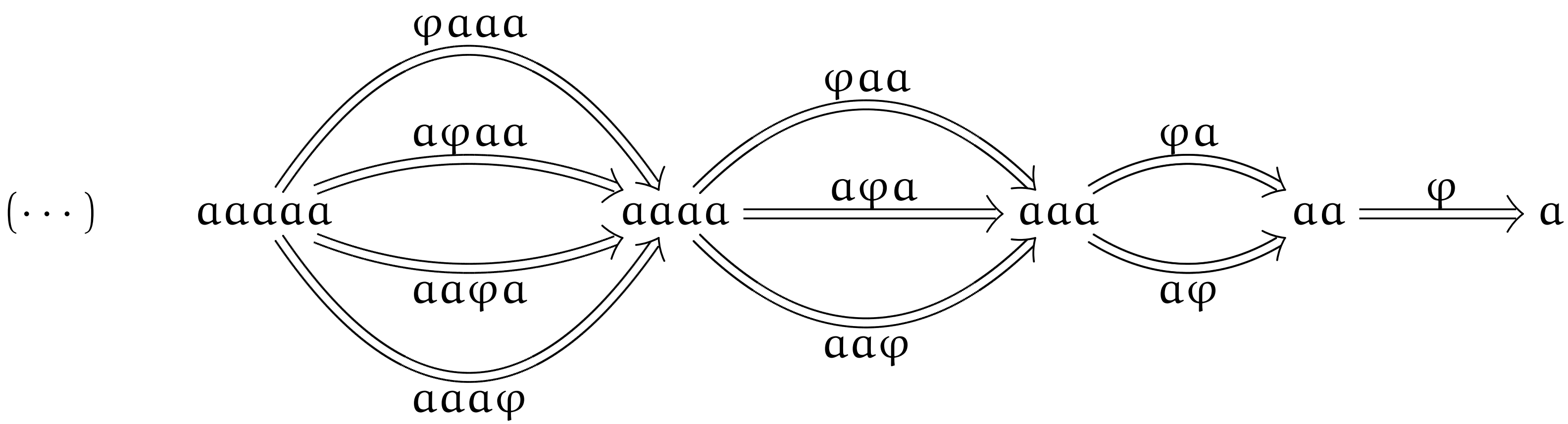
for every relation $u = v$ in R

Higher dimensions: Use *rewriting theory*

1.2. Rewriting theory

Rewriting: Theory of computational properties of presentations [Thue 14, Newman 42]

Example: $\quad A = \langle a \mid aa = a \rangle \quad \rightsquigarrow \quad A = \langle a \mid aa \Rightarrow_{\varphi} a \rangle$



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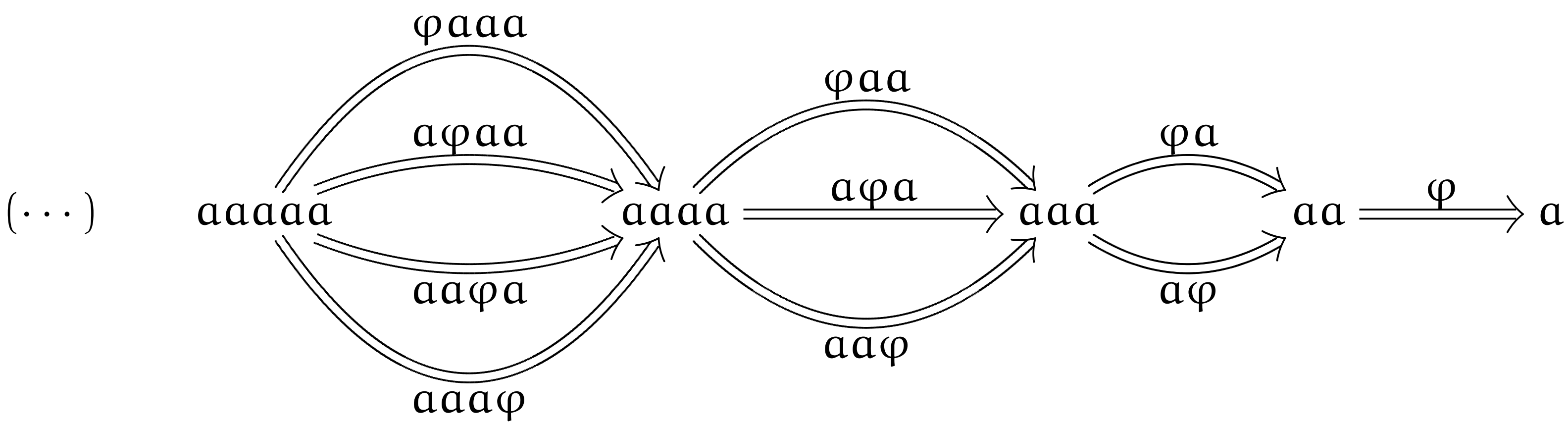
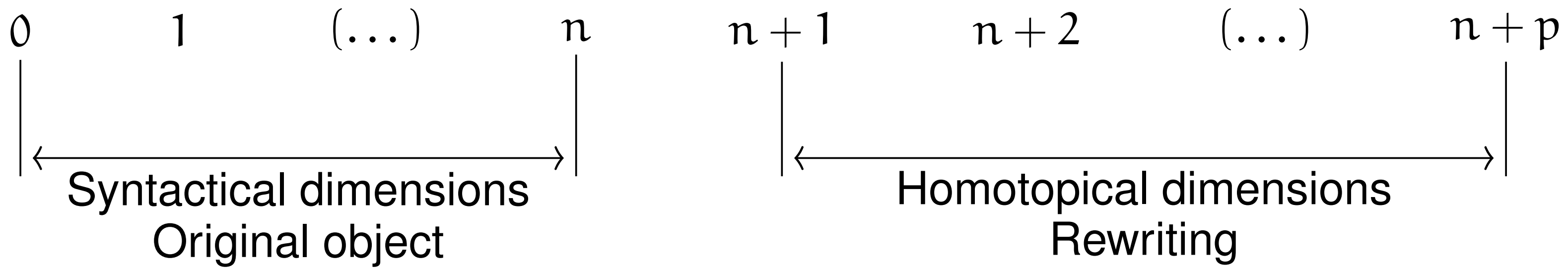


"Good" computational properties:

- **Termination:** Reach "normal form" a eventually
- **Confluence:** Every path leads to a
- **Convergence** (= termination + confluence)
  - $\rightsquigarrow$ Solution for the word problem [Thue]
  - $\rightsquigarrow$ Cofibrant replacement for A

1.3. The dimensions of rewriting

Dimensions of the cofibrant replacement:



Syntactical dimensions:

Computads [Street, Batanin, Makkai]

Polygraphs [Burroni, Métayer, Guiraud, Malbos]

Homotopical dimensions:

General setting for results by Squier, Kobayashi, Otto, etc.

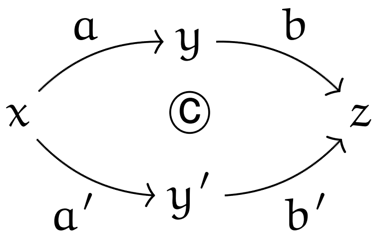
2. The syntactical dimensions of rewriting

2.1. Presentations of categories by 2-polygraphs

Monoid: category with one 0-cell

Presentation of category $\mathcal{C}$:

– Objects and generating morphisms (a graph X)

– Commutative diagrams 

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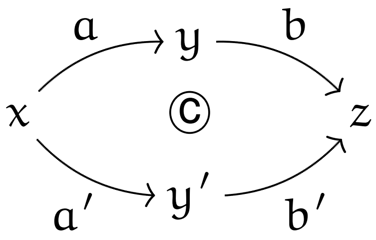
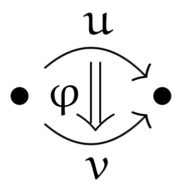
– Commutative diagrams  (relations R in free category X^* with $\mathcal{C} \simeq X^*/R$)

2-Polygraph Σ: rewriting presentation of a category

– Graph Σ_1 $\rightsquigarrow$ Free category Σ_1^*

– *Cellular extension* Σ_2 of Σ_1^* $=$ Set of 2-cells  with $u, v \in \Sigma_1^*$

Presented category: $\overline{\Sigma} = \Sigma_1^*/\Sigma_2$

Example: The 2-polygraph $\mathrm{As}_2 = (*,\ a,\ aa \overset{\varphi}{\Rightarrow} a)$ is a presentation of A

2.2. Rewriting theory: normal forms and termination

Let Σ be a 2-polygraph

Rewriting step: $vuw \overset{v\varphi w}{\Longrightarrow} vu'w$ with $\varphi : u \Rightarrow u'$ in Σ_2

Normal form: 1-cell $u \in \Sigma_1^*$ s.t. $\nexists\, u \overset{f}{\Rightarrow} v$

Intuition: Normal forms used to represent elements of $\overline{\Sigma}$ in Σ_1^*

Condition: $\forall\, u \in \Sigma_1^*$ $\exists\,!$ *computable* normal form in class $\overline{u} \in \overline{\Sigma}$

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Termination: $\nexists$ infinite sequence of rewriting steps

$$u_1 \overset{f_1}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} u_3 \overset{f_3}{\Longrightarrow} (\cdots) \overset{f_{n-1}}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} (\cdots)$$

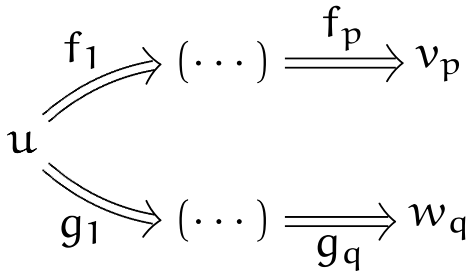
Consequence: Termination $\Rightarrow$ Existence of normal forms

Example: $\mathrm{As}_2 \;=\; (a,\; aa \overset{\varphi}{\Rightarrow} a)$ terminates

$$a^n \;\Rightarrow\; a^{n-1} \;\Rightarrow\; (\cdots) \;\Rightarrow\; a^3 \;\Rightarrow\; a^2 \;\Rightarrow\; a$$

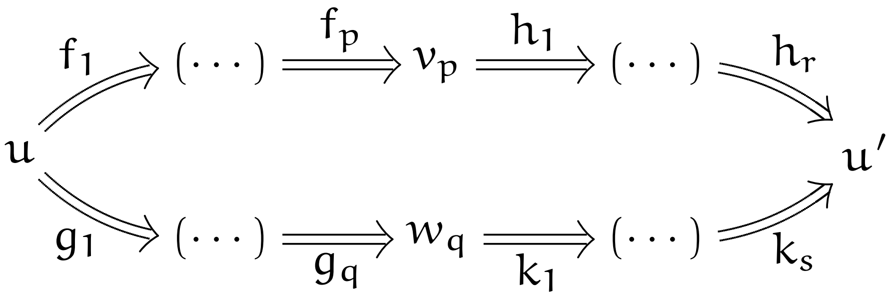
2.3. Rewriting theory: branchings and confluence

Branching: pair of rewriting sequences



– **Local:** $p = q = 1$

– **Confluent:**



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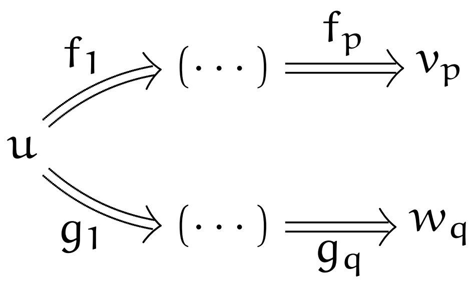

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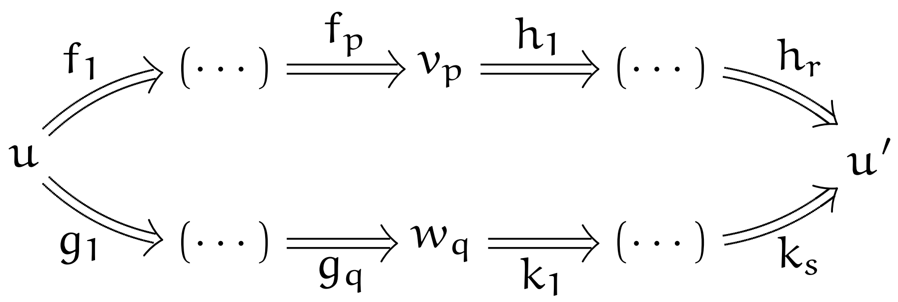
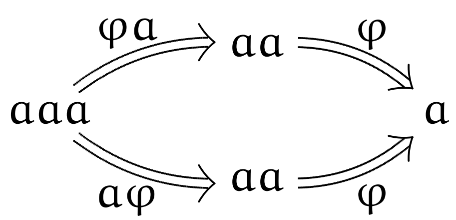


Example: a confluent local branching in As_2



(Local) confluence: every (local) branching is confluent

Consequence: Confluence $\Rightarrow$ Unicity of normal forms

2.4. Rewriting theory: convergence

Convergence: termination $+$ confluence

– Convergence $\implies$ Existence and unicity of normal forms

– Convergence $+$ finiteness $\implies$ *idem* $+$ computability

Class FCP**:** Categories admitting a finite convergent presentation

2.4. Rewriting theory: convergence

Convergence: termination + confluence

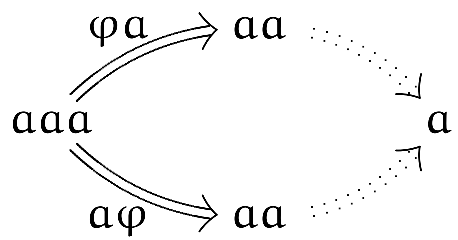
– Convergence $\Longrightarrow$ Existence and unicity of normal forms

– Convergence + finiteness $\Longrightarrow$ *idem* + computability

Class FCP**:** Categories admitting a finite convergent presentation

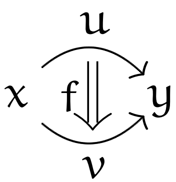
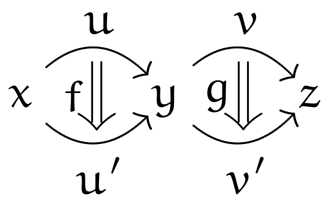
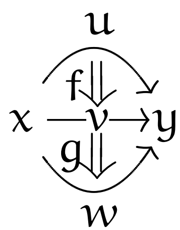
Theorem [Newman's lemma]. Termination + Local confluence $\Longrightarrow$ Convergence

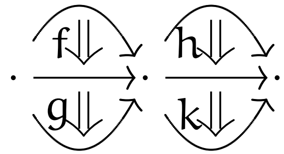
Critical branchings: local branching with a "minimal overlapping" of 2-cells



Theorem. Termination + Confluence of critical branchings $\Longrightarrow$ Convergence

2-category: category enriched in categories. Informally:

- **1-cells:** $x \xrightarrow{u} y$ with one composition $u \star_0 v \;=\; x \xrightarrow{u} y \xrightarrow{v} z$
- **2-cells:**  with two compositions $f \star_0 g \;=$  and $f \star_1 g \;=$ 

  Exchange relation: $(f \star_1 g) \star_0 (h \star_1 k) = (f \star_0 h) \star_1 (g \star_0 k)$ for 

2.5. 2-Categories

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- **2-cells:** $x \underset{v}{\overset{u}{\Downarrow f}} y$ with two compositions $f \star_0 g \;=\; x \underset{u'}{\overset{u}{\Downarrow f}} y \underset{v'}{\overset{v}{\Downarrow g}} z$ and $f \star_1 g \;=\; x \;\overset{u}{\Downarrow f}\; \xrightarrow{v} \;\underset{w}{\Downarrow g}\; y$

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Example: The free 2-category Σ^* generated by a 2-polygraph Σ

Example: a pro(p) P

- one 0-cell $*$
- 1-cells: natural numbers with composition $m \star_0 n = m + n$
- 2-cells: morphisms of P with compositions

$$f \star_0 g \;=\; f \otimes g \qquad\qquad f \star_1 g \;=\; g \circ f$$

2.6. Presentations of 2-categories by 3-polygraphs

3-Polygraphs:

– 2-polygraph $\Sigma_2 \rightsquigarrow$ Free 2-category Σ_2^*

– *Cellular extension* Σ_3 of Σ_2^* $=$ Set of 3-cells $\bullet \underset{v}{\overset{u}{\Downarrow f}} \bullet \overset{\omega}{\Rrightarrow} x \underset{v}{\overset{u}{\Downarrow g}} y$ with $f, g \in \Sigma_2^*$

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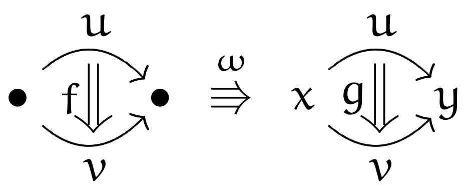

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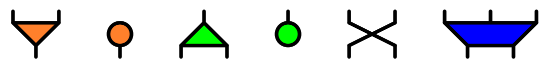
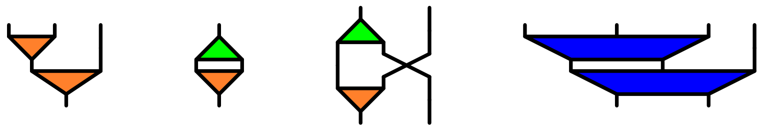
Diagrammatic representations:

– Generating 2-cells:



– Composed 2-cells:



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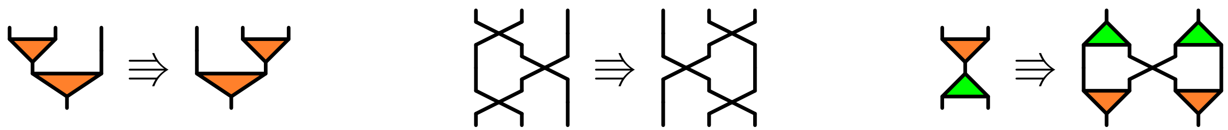
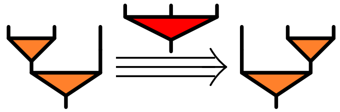


2.7. Examples of 3-polygraphs: presentations of pros

Associative algebras: a presentation As_3 of the pro $\mathbf{As}$ $(\rightsquigarrow \mathrm{FCP})$

0-cell: $*$ 1-cell: $|$ 2-cell:  3-cell: 

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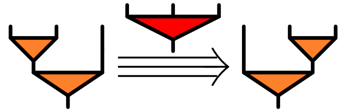
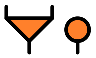
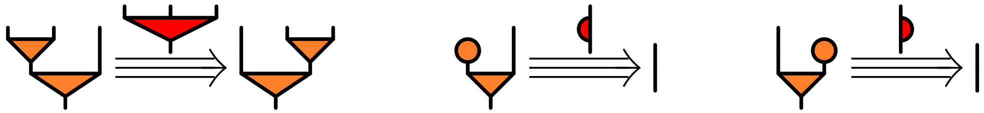
0-cell: $*$ $\quad$ 1-cell: | $\quad$ 2-cell:  $\quad$ 3-cell: 

Monoids: a presentation Mon_3 of the pro $\mathbf{Mon}$ $\quad(\rightsquigarrow \mathrm{FCP})$

2-cells:  $\quad$ 3-cells: 

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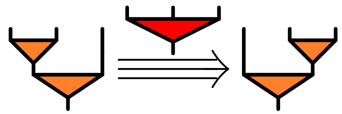
0-cell: $*$ 1-cell: | 2-cell:  3-cell: 

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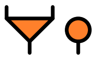
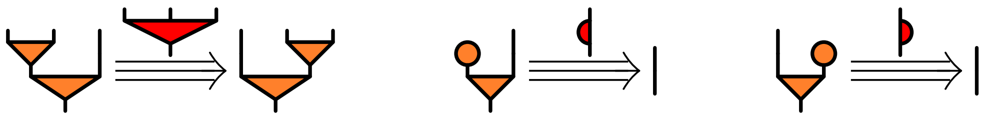
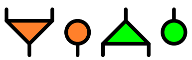
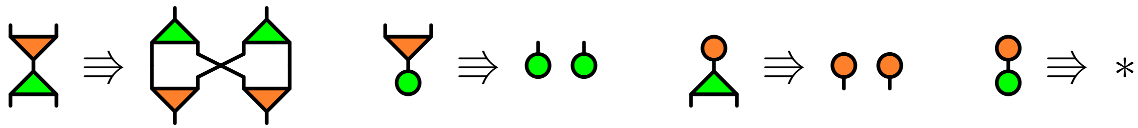
2-cells:  3-cells: 

Bialgebras: a presentation of the pro of bialgebras $(\rightsquigarrow \mathrm{FCP})$

2-cells:  (some) 3-cells: 

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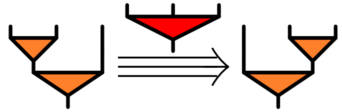
0-cell: $*$ 1-cell: | 2-cell:  3-cell: 

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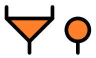
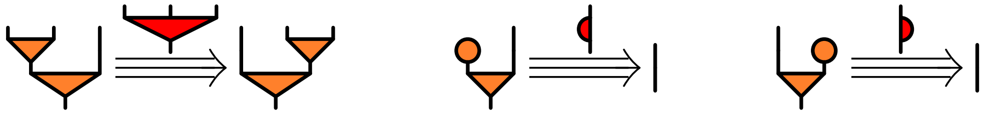
2-cells:  3-cells: 

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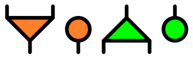
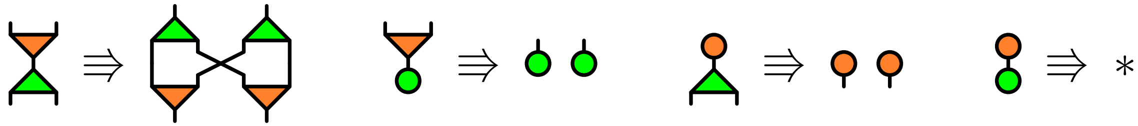
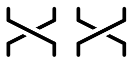
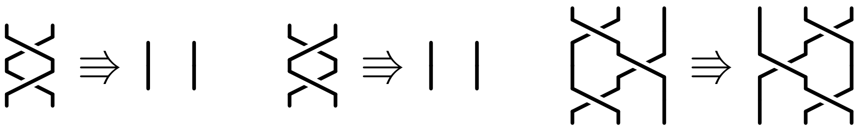
2-cells:  (some) 3-cells: 

Braids: a presentation of the pro of braids (FCP: open problem)

2-cells:  3-cells: 

2.8. Examples of 3-polygraphs in computer science

First-order functional program:

```
type nat = 0 | S(nat)

fonction plus :  nat * nat -> nat
   | plus(0, n) -> n
   | plus(S(m), n) -> S(plus(m, n))
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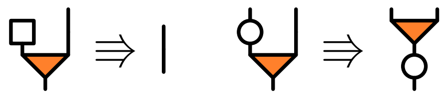
– 3-cells: 

⇝ Results in program analysis: termination [Guiraud 06], complexity [Bonfante-Guiraud 09]

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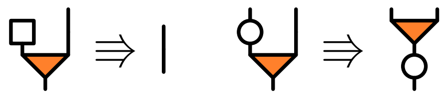
– 3-cells: 

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Turing machines [Burroni, Bonfante-Guiraud 09]

Petri nets [Guiraud 06]

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– **k-cells** between parallel $(k-1)$-cells with k compositions

– exchange relations

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$(n+1)$-Polygraph Σ: presentation of an n-category

– n-polygraph $\Sigma_n \rightsquigarrow$ free n-category Σ_n^*

– cellular extension Σ_{n+1} of Σ_n^*

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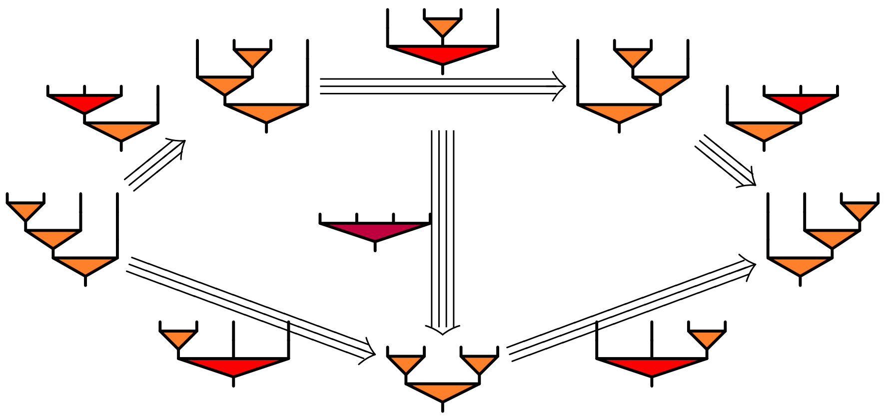
– exchange relations

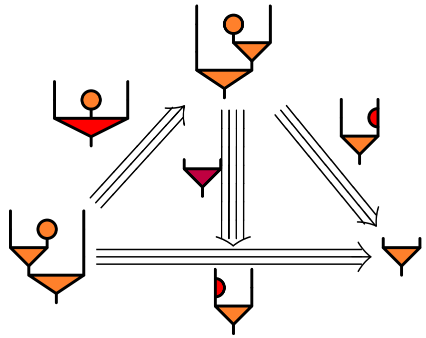
$(n+1)$-Polygraph Σ: presentation of an n-category

– n-polygraph Σ_n $\rightsquigarrow$ free n-category Σ_n^*

– cellular extension Σ_{n+1} of Σ_n^*

Example: the 4-polygraph Mon_4 is Mon_3 extended with 4-cells





$\rightsquigarrow$ Presentation of *the 2-pro* **MonCat** *of monoidal categories* [Guiraud-Malbos 10]

3. The homotopical dimensions of rewriting

3.1. The informal idea

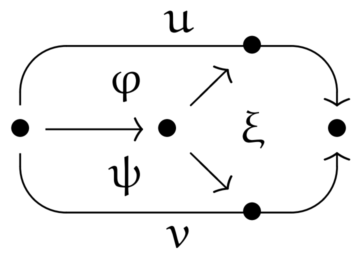
Problem: category $\mathcal{C} \longrightarrow$ "homotopically equivalent" free n-category Σ^*

Presentation of $\mathcal{C}$: 2-polygraph Σ

– Graph Σ_1

– Cellular extension Σ_2 of Σ_1^* s.t. $\mathcal{C} \simeq \Sigma_1^*/\Sigma_2$

$\overline{u} = \overline{v} \iff \exists$ pasting, such as



with φ, ψ, ξ in Σ_2

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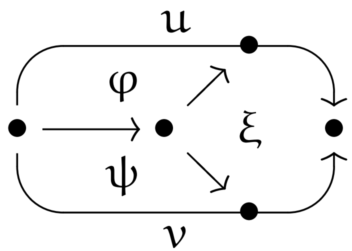
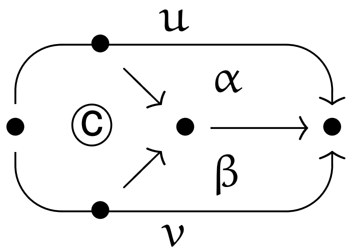
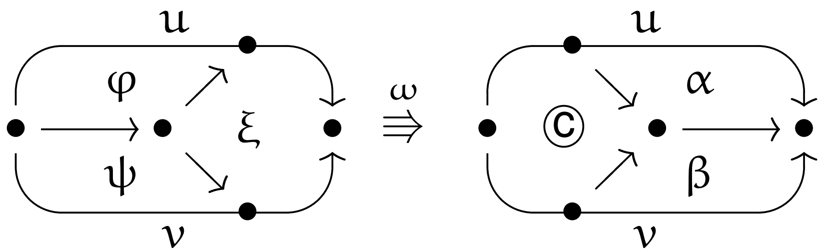
$\overline{u} = \overline{v} \iff \exists$ pasting, such as  with φ, ψ, ξ in Σ_2

Problem: Another pasting  from u to v

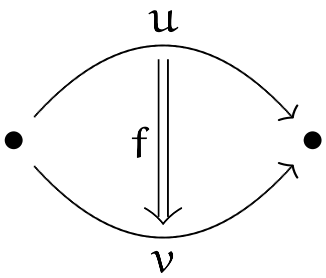
Solution: Fill the 2-sphere with a 3-cell 

3.2. Higher presentations

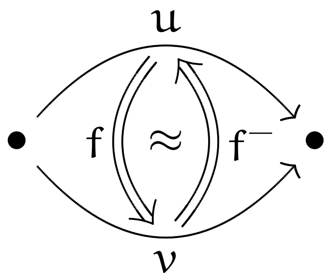
2-fold presentation of $\mathcal{C}$: 3-polygraph Σ

– 2-polygraph Σ_2

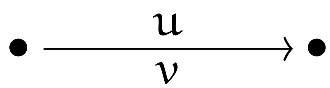
Free 2-category Σ_2^*



Free *track* 2-category $\Sigma_2^\top$



Presented 1-category $\overline{\Sigma}$



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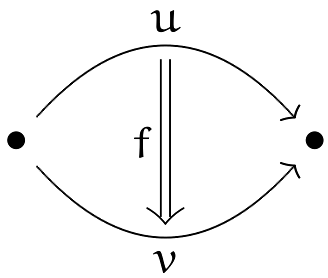
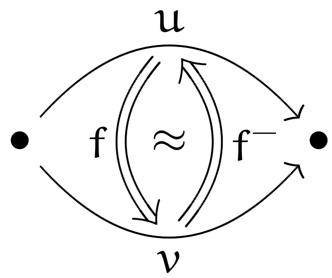
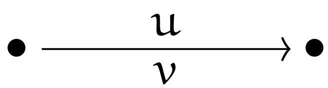
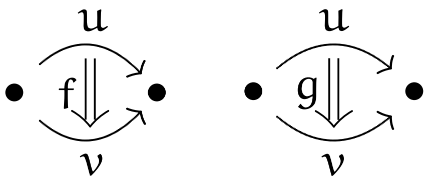
Free 2-category Σ_2^* | Free *track* 2-category $\Sigma_2^\top$ | Presented 1-category $\overline{\Sigma}$

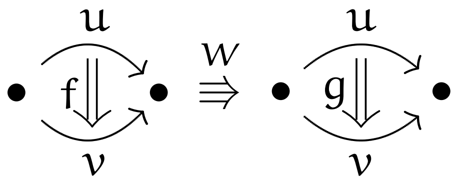






– *Homotopy basis* Σ_3 of $\Sigma_2^\top$ $=$ Cellular extension of Σ_2^* s.t.

$\forall$ 

in $\Sigma_2^\top$ $\exists$ 

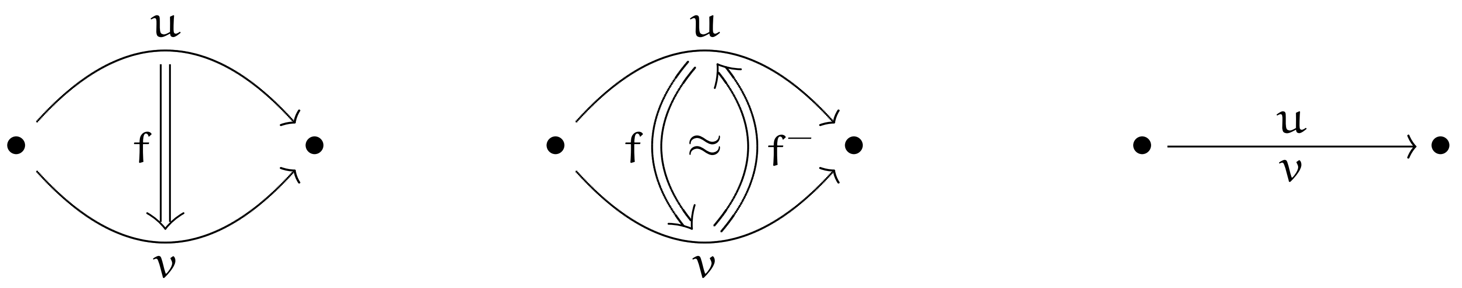
in $\Sigma_3^\top$

3.2. Higher presentations

2-fold presentation of $\mathcal{C}$: 3-polygraph Σ

– 2-polygraph Σ_2

Free 2-category Σ_2^* Free *track* 2-category $\Sigma_2^\top$ Presented 1-category $\overline{\Sigma}$



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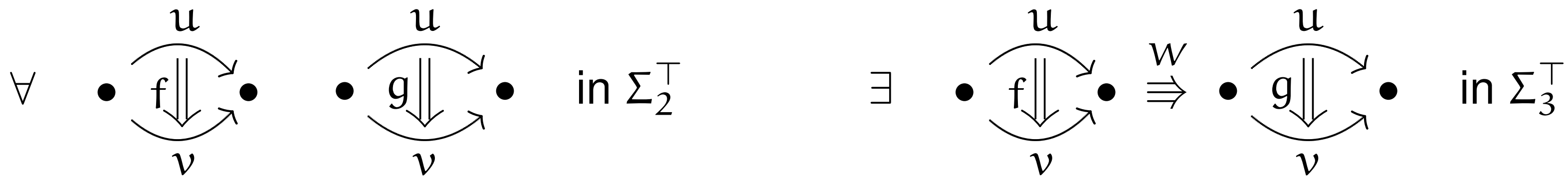

n-fold presentation of $\mathcal{C}$: $(n+1)$-polygraph Σ

– $(n-1)$-fold presentation Σ_n

– Homotopy basis Σ_n of $\Sigma_n^\top$

3.3. The main result

Theorem [Guiraud-Malbos 10].

If a category $\mathcal{C}$ admits a convergent presentation,

then $\mathcal{C}$ admits an n-fold presentation, for every $n \in \mathbb{N}^*$.

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- FDT_0: finite presentation
- FDT_1: finite presentation + homotopy basis = *finite derivation type* (FDT [Squier 94])

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Corollary. $\mathrm{FCP} \implies \mathrm{FDT}_n$

Corollary [Squier 94]. For monoids $\mathrm{FCP} \implies \mathrm{FDT}$

3.4. Key element of the proof: normalisation strategies

Normalisation strategy: coherent choice of normal forms in every dimension

– In every class $\overline{u} \in \overline{\Sigma}$, a representative 1-cell $\widehat{u} \in \Sigma_1^*$

– For every 1-cell u in Σ_1^*, a 2-cell $u \overset{\sigma_u}{\Longrightarrow} \widehat{u}$ in $\Sigma_2^\top$

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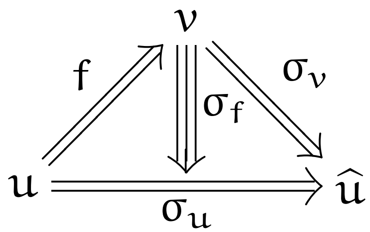
– For every 2-cell $f : u \Rightarrow v$ in $\Sigma_2^\top$, a 3-cell in $\Sigma_3^\top$



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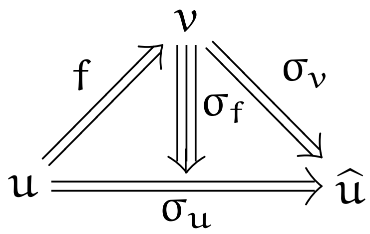
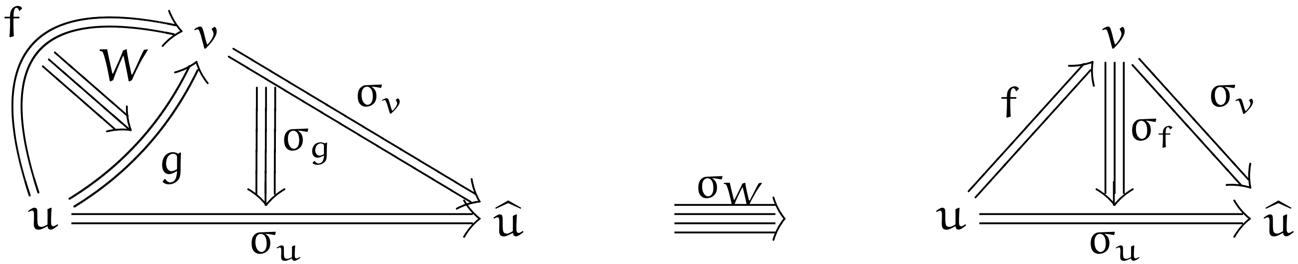


- For every 3-cell $A : f \Rrightarrow g$ in $\Sigma_3^\top$, a 4-cell in $\Sigma_4^\top$



Proposition: Normalisation strategies $\Longleftrightarrow$ Homotopy bases

3.5. Induction start: the 2-fold presentation of critical branchings

Theorem [Guiraud-Malbos 09, 10]: If Σ is a convergent 2-polygraph

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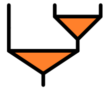
Example: Let $\mathrm{As}_2 = \left(a,\ aa \overset{\varphi}{\Rightarrow} a \right) = (|,$  $)$

– Define σ as the *rightmost* normalisation strategy:

$\sigma_a = 1_a$ $\qquad$ $\sigma_{aa} = \varphi =$  $\qquad$ $\sigma_{aaa} = a\varphi \star_1 \varphi =$ 

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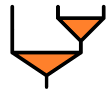
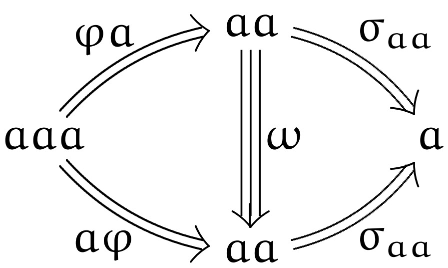
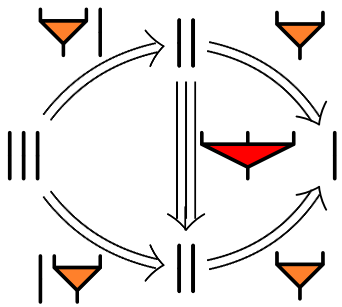
$\sigma_a = 1_a$ $\sigma_{aa} = \varphi =$  $\sigma_{aaa} = a\varphi \star_1 \varphi =$ 

– Define Σ_3: one critical branching $(\varphi a, a\varphi)$ (** Confluence **)



or



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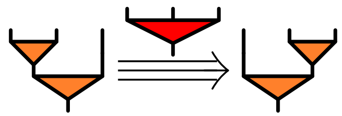
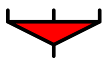


– Define $\sigma_{\varphi a} =$  and extend σ to every 2-cell f (**Termination**)

3.6. Induction: the n-fold presentation of n-fold critical branchings

Theorem [Guiraud-Malbos 10]: If Σ is a convergent 2-polygraph

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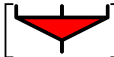
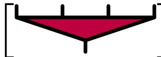
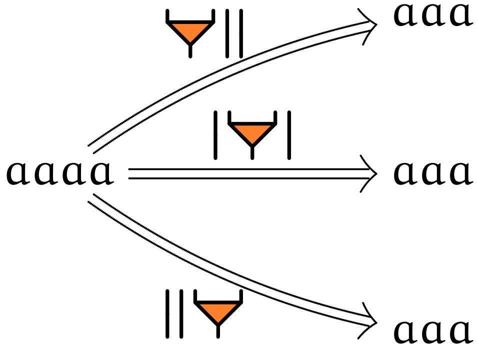
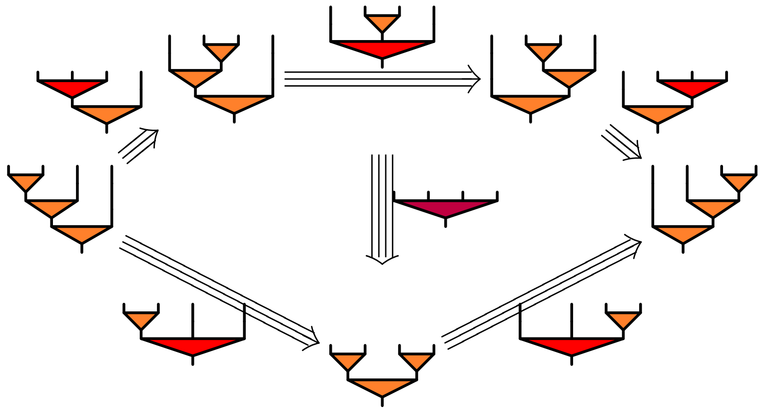
Example: $\mathbf{As}_2 = (|,$  $)$ $\rightsquigarrow$ $\mathbf{As}_3 = \mathbf{As}_2\,[$  $]$ $\rightsquigarrow$ $\mathbf{As}_4 = \mathbf{As}_3\,[$  $]$





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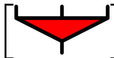
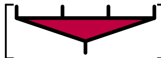
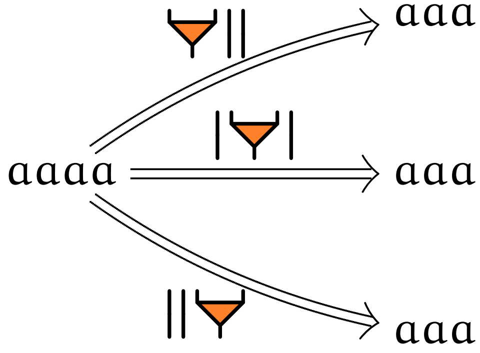
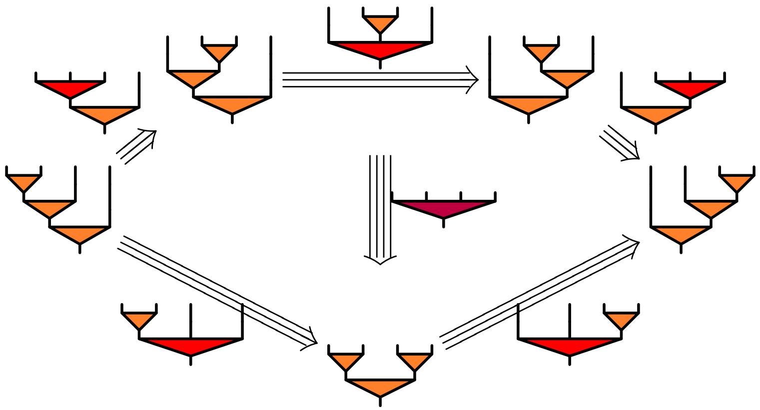
Example: $\mathrm{As}_2 = (|,$  $)$ $\rightsquigarrow$ $\mathrm{As}_3 = \mathrm{As}_2\,[$  $]$ $\rightsquigarrow$ $\mathrm{As}_4 = \mathrm{As}_3\,[$  $]$





Corollary: If Σ has no n-fold critical branchings

then Σ_n^* is "aspherical" *i.e.* Σ_n^* is a cofibrant replacement for $\overline{\Sigma}$

Corollary: If Σ has n-fold critical branchings for every n

then Σ_∞^* is a cofibrant replacement for $\overline{\Sigma}$

3.7. Link with homological finiteness condition FP_n

Class FP_n: Categories $\mathcal{C}$ such that the trivial $\mathcal{C}$-module $\mathbb{Z}$ admits a projective resolution

$$M_n \longrightarrow M_{n-1} \longrightarrow (\cdots) \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

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Theorem [Guiraud-Malbos 10]: $\mathrm{FDT}_n \implies \mathrm{FP}_{n+2}$

Proof:

$$\mathcal{C}[\Sigma_{n+2}] \longrightarrow \mathcal{C}[\Sigma_{n+1}] \longrightarrow \mathcal{C}[\Sigma_n] \longrightarrow \mathcal{C}[\Sigma_1] \longrightarrow \mathcal{C}[\Sigma_0] \longrightarrow \mathbb{Z} \longrightarrow 0$$

Differentials: source $-$ target

Contracting homotopies: normalisation strategy

Corollary [Squier 87]: For monoids $\mathrm{FCP} \implies \mathrm{FP}_3$