Perturbed Turing Machines and Hybrid Systems

Eugene Asarin *
VERIMAG,
2 av. de Vignate, 38610 Gières, France.
Email: Eugene.Asarin@imag.fr
Phone: (+33) 4 76 63 48 33, Fax: (+33) 4 76 63 48 50.

Ahmed Bouajjani
LIAFA, University of Paris 7, Case 7014,
2 Place Jussieu, 75251 Paris cedex 5, France.
Email: abou@liafa.jussieu.fr
Phone: (+33) 1 44 27 78 19, Fax: (+33) 1 44 27 68 49.

Abstract

We investigate the computational power of several models of dynamical systems under infinitesimal perturbations of their dynamics. We consider in our study models for discrete and continuous time dynamical systems: Turing machines, Piecewise affine maps, Linear hybrid automata and Piecewise constant derivative systems (a simple model of hybrid systems). We associate with each of these models a notion of perturbed dynamics by a small ε (w.r.t. to a suitable metrics), and define the perturbed reachability relation *as the intersection of all reachability relations obtained by ε-perturbations, for all possible values of ε. We show that for the four kinds of models we consider, the perturbed reachability relation is co-recursively enumerable, and that any co-r.e. relation can be defined as the perturbed reachability relation of such models. A corollary of this result is that systems that are robust, i.e., their reachability relation is stable under infinitesimal perturbation, are decidable.*

1 Introduction

Recently, the investigation of the relations between dynamics and computation attracted attention of several research communities (see e.g. [1] where Turing machines are considered as dynamical systems, and [2] and [3] where discrete and continuous time dynamical systems are considered as computation models).

Our initial motivation for this research was related to hybrid systems (see e.g. [4]). Since the first undecidability results were stated for hybrid systems (such as Linear hybrid automata [5] or Piecewise constant derivative systems [3]), a folklore conjecture appeared, saying that this undecidability is due to non-stability, non-robustness, sensitivity to initial values of the systems, and that it never occurs in "real" systems. There were several attempts to formalize and to prove (or to disprove) this conjecture [6, 7] (cf. *Related Work* below). We think however that this conjecture is more rich than these formalizations and that exploring relations between complexity of behaviours of a dynamical system (not necessarily hybrid) and its properties related to stability, robustness, chaos is an important scientific challenge (see [8]).

In this paper we explore one facet of this problem: how small perturbations of dynamics influence the computational power of the system. We consider different kinds of transition systems corresponding to widely used models of dynamical systems: Turing machines (TM), Piecewise affine maps (PAM), Linear hybrid automata (LHA), and Piecewise constant derivative (PCD) systems. We introduce for these models a notion of "perturbed" dynamics and study the computational power of the corresponding perturbed systems. Perturbations are defined for each model using a notion of metrics on the state space (allowing to define how distant is the ideal dynamics from the perturbed one). The notion of *small* perturbation is easier to understand for computational models with a continuous state-space (such that PCD, LHA, and PAM) than for discrete ones like TM. For such models, given a transition system with a reachability relation R, the idea is to perturb the dynamics by a small ε, and then, to take (as the perturbed dynamics of the system) the limit (intersection) R_ω of the perturbed reachability relations as this ε tends to 0. We say that a system is *robust* if its reachability relation does not change under small perturbations of the dynamics, i.e., R is equal to R_ω.

We show that for the three models of PAM, LHA, and PCD, the relation R_ω belongs to the class Π^0_1 (i.e. it is co-recursively enumerable), and moreover, any Π^0_1 relation can be reduced to a relation R_ω of a perturbed system. In other words, any complement to a r.e. set can be semi-decided by an infinitesimally perturbed system. This result is somehow surprising since it means that noise by itself does not make the reachability problem decidable, but it transforms it in a rather non-trivial way (from Σ^0_1 to Π^0_1). Furthermore, an immediate corollary of the result above is the following fact: the reachability problem is decidable for the class of robust systems.

*The work of this author was supported in part by the NATO under grant CRG-961115

In the case of Turing machines, the analogous notion of small perturbation is obtained by considering the prefix distance (Cantor distance) as metrics on the set of tape configurations. In fact, this metrics is an adequate characteristics for these machines; in particular, the dynamics of these machines has good properties w.r.t. this metrics, e.g., the transition function of a TM is always Lipshitz w.r.t. it (see [1] for a detailed argument). So, we consider that a TM is subjected to a small noise if its configuration is slightly perturbed in the sense of this metrics, or equivalently, all the perturbations of the tape content happen far from the head. Similarly to the other models, given a TM recognizing a language L, for every natural number n, we define L_n to be the set of all words that are accepted if we allow perturbations (arbitrary changes in the tape) beyond a distance n from the head, and we take L_ω to be the intersection of all the languages L_n. It can be understood intuitively that the notion of robustness of a TM according to this notion of perturbation actually coincides with the notion of boundedness since only machines that can visit arbitrarily far positions from their initial position can have a different perturbed language. We prove that for TM also the same results as for the other models hold: the language L_ω is in Π_1^0, and every Π_1^0 language can be represented as a perturbed language of a TM, which means that robust TM's correspond precisely to machines recognizing recursive languages.

We give in the paper the proofs for the models mentioned above in an increasing technical complexity order. The TM case unveils the mechanism of the effect of perturbation and allows to understand the essence of this mechanism on a common and relatively simple model. The PAM case makes it clear how this mechanism works in the continuous state space, without unneeded technical complexity. Essentially the same techniques used for PAM can also be applied to the more popular model of LHA (we omit in this extended abstract the proofs concerning LHA). Moreover, the proof for PAM is a good introduction to the trickier one for PCD, which is a simple and natural model for hybrid systems, and perhaps the most motivating case.

Related work. Recently, a similar approach to ours was independently invented and applied in a completely different context to the analysis of numerical methods for chaotic dynamical systems by Kloeden and Kozyakin. In [9], they refer to the procedure of infinitesimal perturbation of dynamical systems as *inflation*.

The notion of perturbation we use (especially in the case of continuous state space systems) was inspired by the work of Anuj Puri who studied the reachability relation of timed automata (with finitely many control states) under infinitesimal perturbation [10]. He showed that for these models, the perturbed reachability relation is still decidable and he gives an effective representation of this relation. Our work concerns models that are more general than timed automata, and aims to show that infinitesimal perturbation has the same effect on several common models of dynamical systems, namely that the perturbed dynamics corresponds in all cases to a co-recursively enumerable relation (set), and that robustness coincides with decidability.

Concerning the decidability issue of the reachability problem, there are two works closely related to ours [6, 7]: Martin Fränzle has shown in [6] a similar result to ours for a certain model of hybrid systems. Our work shows that the fact that "*robustness implies decidability*" can be proved for other different types of transitions systems. Moreover, our hardness results (inverse implication) show that the relation between robustness and decidability is really tight. Our result is in contrast with Thomas Henzinger's result [7] stating that reachability is still undecidable for hybrid systems that allow small perturbations *of the trajectory*. It is interesting to see that a small semantical difference between these two approaches drastically changes the complexity.

Finally, the effect of noise on the power of analog computational models and the dependence of this power from the level of this noise are explored in [11, 12, 13]. Differently, we consider in our work the limit behavior with noise level tending to zero.

Outline. The rest of the paper is organized as follows: in section 2 we define the computation models (kinds of dynamical systems) we consider: TM, PAM, and PCD, and their perturbed versions. In sections 3–5 we formulate and prove the main results for these models. For lack of space, we omit here the case of LHA since the proofs concerning these models are technically very similar to those for PAM.

Acknowledgments. We would like to thank Vincent Blondel, Victor Kozyakin, Oded Maler and Anuj Puri for useful discussion.

2 Perturbed Models

2.1 Perturbed Turing machines (PTMs)

Let us recall the definition of a *Turing machine* (TM for short) (see figure 1(a)).

Let Σ be a finite alphabet, and let B be a special symbol $B \notin \Sigma$. A TM over Σ is a tuple (Q, q_{init}, F, Γ) where Q is a finite set of control states, $q_{init} \in Q$ is the initial control state, $F \subseteq Q$ is a set of *accepting* states, and Γ is a set of transitions of the form $(q, a) \rightarrow (q', b, \delta)$ where $q, q' \in Q$, $a, b \in \Sigma \cup \{B\}$, and $\delta \in \{-1, 0, 1\}$.

A configuration of the machine is an unbounded sequence (from left and right) of the form

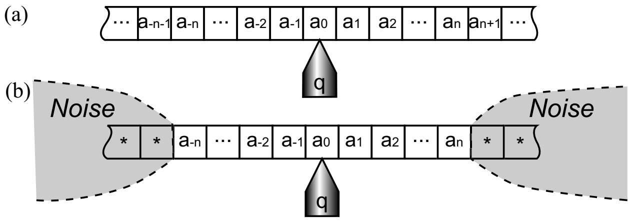

Figure 1: (a) A Turing machine. (b) Its n-perturbed version.

$\cdots a_{-2}a_{-1}[q, a_0]a_1a_2\cdots$ where the a_is are symbols in $\Sigma \cup \{B\}$. Intuitively, $[q, a_0]$ means that the current control state of is q and that the head of the machine is at symbol a_0.

Given a transition $(q, a) \to (q', b, \delta)$ in Γ, if the symbol pointed by the head of the machine is equal to a, then the machine can change its configuration in the following manner: the symbol pointed by the head is replaced by b and then the head is moved to the left or to the right, or it stays at the same position according to whether δ is -1, 1, or 0, respectively.

Let $w = a_1, \cdots, a_n$ be a word in Σ^*. We say that w is accepted by $\mathcal{M}$ if, starting from the configuration $\cdots BBB[q_{init}, a_1]\cdots a_nBBB\cdots$ the machine $\mathcal{M}$ eventually stops in an accepting state. Let $L(\mathcal{M})$ denote the set of such words, i.e., the recursively enumerable (r.e.) language semi-recognized by $\mathcal{M}$.

Now, let us introduce the concept of *perturbed Turing machines* (PTMs for short). Given an integer $n > 0$, the n-perturbed version of the machine $\mathcal{M}$ is defined exactly as $\mathcal{M}$ except that before any transition all the symbols at the distance n or more from the head of the machine can be altered (i.e., replaced by other symbols) arbitrarily: Given a configuration

$$\cdots a_{-n-1}a_{-n}a_{-n+1}\cdots a_{-1}[q, a_0]a_1\cdots a_{n-1}a_na_{n+1}\cdots$$

the n-perturbed version of $\mathcal{M}$ may replace any symbols to the left of a_{-n} (starting from a_{-n-1}) and to the right of a_n (starting from a_{n+1}) by any other symbols in $\Sigma \cup \{B\}$ before executing a transition of $\mathcal{M}$ (at a_0). Hence, the machine becomes a nondeterministic transition system (see figure 1(b)).

A word w is accepted by the n-perturbed version of $\mathcal{M}$ if there exists a run of this machine which stops in an accepting state. Let $L_n(\mathcal{M})$ be the n-perturbed language of $\mathcal{M}$, i.e., the set of words in Σ^* that are accepted by the n-perturbed version of $\mathcal{M}$.

It is easy to see that if a word is accepted by $\mathcal{M}$, then it can also be recognized by all the n-perturbed versions of $\mathcal{M}$, for every $n > 0$ (perturbed machines have more behaviors). Moreover, if the $(n+1)$-perturbed version accepts a word w, the n-perturbed version will also accept it since obviously all alterations at distance greater than $n+1$ from the head can also happen in the n-perturbed machine. Hence, we have:

Lemma 1 $L_1(\mathcal{M}) \supseteq L_2(\mathcal{M}) \supseteq \cdots \supseteq L(\mathcal{M})$

This technically justifies the following crucial definition (explained in the introduction): *ω-perturbed language* of the machine $\mathcal{M}$ is given by

$$L_\omega(\mathcal{M}) = \bigcap_n L_n(\mathcal{M})$$

Informally speaking, $L_\omega(\mathcal{M})$ consists of all the words that can be accepted by $\mathcal{M}$ when it is subject to arbitrarily "small" perturbations. The previous lemma could be trivially extended to:

Lemma 2 $L_1(\mathcal{M}) \supseteq L_2(\mathcal{M}) \supseteq \cdots \supseteq L_\omega(\mathcal{M}) \supseteq L(\mathcal{M})$

2.2 Piecewise affine maps

The second kind of systems to which we apply small perturbations was introduced as a computation model in [2]. Recall some definitions and results from that paper.

Definition 1 (PAM System) *A Piecewise affine map system (PAM) is a discrete-time dynamical system $\mathcal{P}$ defined by an assignment $\mathbf{x} := f(\mathbf{x})$ on a bounded polyhedral set $X \subset \mathbb{R}^d$, where f is a (possibly partial) function from X to X represented by a formula:*

$$f(\mathbf{x}) = A_i\mathbf{x} + \mathbf{b}_i \textit{ for } \mathbf{x} \in P_i, \quad i = 1..N$$

where A_i are rational $d \times d$-matrices, $\mathbf{b}_i \in \mathbb{Q}^d$ and P_i are rational polyhedral sets in X.

A *trajectory* of $\mathcal{P}$ is a sequence $\mathbf{x}_n$ evolving according to f, i.e. such that $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ for all n.

In other words, a PAM system consists of partitioning the space into convex polyhedral sets ("regions"), and assigning an affine update rule $\mathbf{x} := A_i\mathbf{x} + \mathbf{b}_i$ to all the points sharing the same region (see figure 2 (a)).

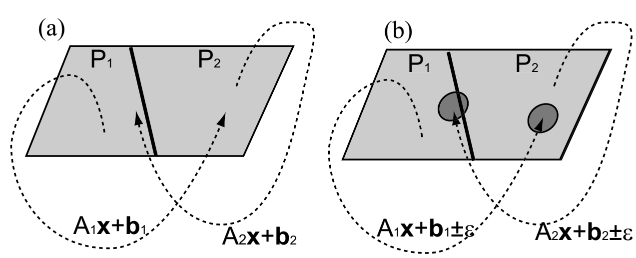

Figure 2: (a) A 2-dimensional PAM system with 2 regions. (b) Its ε-perturbed version.

It is important to emphasize that since we assume that all constants in the system's definition are *rational*, the expressive power of PAM is *not* achieved using the introduction of some non-computable real numbers.

To each PAM $\mathcal{P}$ we associate its reachability relation $R^{\mathcal{P}}(\cdot,\cdot)$ on $\mathbb{Q}^d$. Namely, for two rational points $\mathbf{x}$ and $\mathbf{y}$ the relation $R^{\mathcal{P}}(\mathbf{x},\mathbf{y})$ holds iff there exists a trajectory of $\mathcal{P}$ from $\mathbf{x}$ to $\mathbf{y}$.

The following result on the computational power of PAMs was proved in [14, 2]

Theorem 1 (Simulation of TM by PAM) *Let $\mathcal{M}$ be a TM. We can effectively construct a PAM $\mathcal{P}$ and an encoding $e : \Sigma^* \to \mathbb{Q}^d$ such that for any word w the following equivalence holds. $w \in L(\mathcal{M})$ iff $R^{\mathcal{P}}(e(w), O)$, where O denotes the origin in $\mathbb{R}^d$.*

The following characterization of the complexity of the reachability relation is now immediate:

Corollary 1 (Computational power of PAM)

- *For any PAM $\mathcal{P}$ its reachability relation is r.e.*
- *Any r.e. set S is 1-reducible (see [15]) to the reachability relation of a PAM.*

2.3 Perturbed PAMs (PPAMs)

Now we can apply the paradigm of small perturbations to PAMs. Consider a PAM $\mathcal{P}$ described by the assignment $\mathbf{x} := f(x)$. For any $\varepsilon > 0$ we consider the ε-perturbed system $\mathcal{P}_\varepsilon$ (see figure 2 (b)). Its trajectories are defined as sequences $\mathbf{x}_n$ satisfying the inequality $||\mathbf{x}_{n+1} - f(\mathbf{x}_n)|| < \varepsilon$ for all n. This non-deterministic system can be considered as $\mathcal{P}$ submitted to a small noise with magnitude ε. We denote reachability in the system $\mathcal{P}_\varepsilon$ by $R^{\mathcal{P}}_\varepsilon(\cdot,\cdot)$. All trajectories of a non-perturbed system $\mathcal{P}$ are also trajectories of the ε-perturbed system $\mathcal{P}_\varepsilon$. If $\varepsilon_1 < \varepsilon_2$ then any trajectory of the ε_1-perturbed system is also a trajectory of the ε_2-perturbed PAM.

Like for TM we can pass to a limit for $\varepsilon \to 0$. Namely $R^{\mathcal{P}}_\omega(\mathbf{x},\mathbf{y})$ iff $\forall\varepsilon > 0 \quad R^{\mathcal{P}}_\varepsilon(\mathbf{x},\mathbf{y})$. This means reachability with arbitrarily small perturbing noise.

The following analog of Lemmata 1 and 2 is now immediate:

Lemma 3 *For any $\varepsilon_2 > \varepsilon_1 > 0$ and rational points $\mathbf{x}$ and $\mathbf{y}$ the following implications hold: $R^{\mathcal{P}}(\mathbf{x},\mathbf{y}) \Rightarrow R^{\mathcal{P}}_\omega(\mathbf{x},\mathbf{y}) \Rightarrow R^{\mathcal{P}}_{\varepsilon_1}(\mathbf{x},\mathbf{y}) \Rightarrow R^{\mathcal{P}}_{\varepsilon_2}(\mathbf{x},\mathbf{y})$*

2.4 Piecewise Constant Derivative Hybrid Systems (PCDs)

The last kind of systems to which we apply small perturbations was introduced in [3] in the context of hybrid systems. Recall some definitions and results.

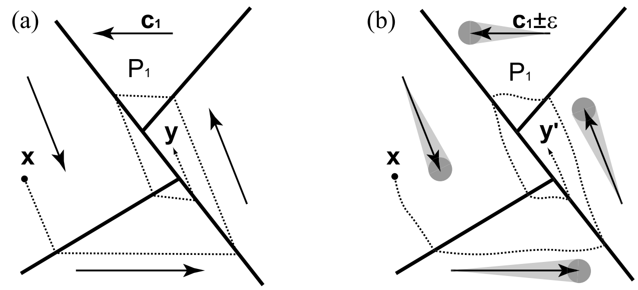

Figure 3: (a) A 2-dimensional PCD system with 4 regions and a trajectory from $\mathbf{x}$ to $\mathbf{y}$. (b) The ε-perturbed version of this PCD.

Definition 2 (PCD System) *A piecewise-constant derivative (PCD) system is a continuous-time dynamical system $\mathcal{H}$ defined by a differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$ on a bounded polyhedral set $X \subset \mathbb{R}^d$ (the state-space), where f is a (possibly partial) function from X to $\mathbb{R}^d$ represented by a formula:*

$$f(\mathbf{x}) = \mathbf{c}_i \textit{ for } \mathbf{x} \in P_i, \quad i = 1..N$$

where $\mathbf{c}_i \in \mathbb{Q}^d$ and P_i are rational polyhedral sets in X.

A *trajectory* of $\mathcal{H}$ starting at some $\mathbf{x}_0 \in X$ is a solution of the differential equation with initial condition $\mathbf{x} = \mathbf{x}_0$, defined as a continuous function $\xi : \mathbb{R}^+ \to X$ such that $\xi(0) = \mathbf{x}_0$ and for every t, $f(\xi(t))$ is defined and is equal to the *right derivative* of $\xi(t)$.

In other words, a PCD system consists of partitioning the space into convex polyhedral sets ("regions"), and assigning a constant derivative $\mathbf{c}$ ("slope") to all the points sharing the same region (see figure 3 (a)). The trajectories of such systems are broken lines, with the breakpoints occurring on the boundaries of the regions. In order to rule out some pathologies we consider only PCDs $\mathcal{H}$ which satisfy an additional assumption of being *strongly non-zeno* i.e. the time interval between two consecutive visits of the same region should be bounded from below by a positive constant Δ.

To each PCD $\mathcal{H}$ we associate its reachability relation $R^{\mathcal{H}}(\cdot,\cdot)$ on $\mathbb{Q}^d$. Namely, for two rational points $\mathbf{x}$ and $\mathbf{y}$ the relation $R^{\mathcal{H}}(\mathbf{x},\mathbf{y})$ holds iff there exists a trajectory of $\mathcal{H}$ from $\mathbf{x}$ to $\mathbf{y}$.

The following result on the computational power of PCDs was proved in [3]

Theorem 2 (Simulation of TM by PCD) *Let $\mathcal{M}$ be a TM. We can effectively construct a PCD $\mathcal{H}$ and an encoding $e : \Sigma^* \to \mathbb{Q}^d$ such that for any word w the following equivalence holds. $w \in L(\mathcal{M})$ iff $R^{\mathcal{H}}(e(w), O)$, where O denotes the origin.*

Corollary 2 (Computational power of (strongly non-zeno) PCD)

- *For any PCD $\mathcal{H}$ its reachability relation is r.e.*
- *Any r.e. set S is 1-reducible (see [15]) to the reachability relation of a PCD.*

2.5 Perturbed PCDs (PPCDs)

Consider a PCD $\mathcal{H}$ described by an ODE $\dot{\mathbf{x}} = f(\mathbf{x})$. For any $\varepsilon > 0$ the ε-perturbed system $\mathcal{H}_\varepsilon$ is described by the differential inclusion $||\dot{\mathbf{x}} - f(\mathbf{x})|| < \varepsilon$. This non-deterministic system can be considered as $\mathcal{H}$ submitted to a small noise with magnitude ε (see figure 3 (b)). We denote reachability in the system $\mathcal{H}_\varepsilon$ by $R_\varepsilon^{\mathcal{H}}(\cdot,\cdot)$. The limit reachability relation $R_\omega^{\mathcal{H}}(\mathbf{x},\mathbf{y})$ is introduced and an analog of Lemma 3 is stated exactly as for PAMs.

3 Results on PTMs

Our first result is that the ω-perturbed language of a TM is the complement of a recursively enumerable language.

Theorem 3 (Perturbed reachability is co-r.e.) $L_\omega(\mathcal{M})$ *is in the class* Π_1^0.

Proof: First, we show that for every $n \in \mathbb{N}$, $L_n(\mathcal{M})$ is a regular language:

Let us associate with the n-perturbed version of $\mathcal{M}$ a finite-state machine $A_\mathcal{M}$ defined as follows: (1) Each of its configurations is composed of a control state of $\mathcal{M}$ and a finite sequence of length $2n+1$ corresponding to the part of the configuration in the radius n from the head. There are $|Q| \times |\Sigma + 1|^{2n+1}$ such configurations. (2) The transition relation $\rightarrow$ is constructed by simulating the transitions of $\mathcal{M}$ and considering that, when the head is moved to the left (resp. to the right), a symbol in $\Sigma \cup \{B\}$ is nondeterministically chosen and appended to the left (resp. right) of the configuration and the right-most (resp. left-most) one is lost (it belongs now to the perturbed area of the configuration and hence it can be replaced by any other symbol).

To formulate the link between the computations of $A_\mathcal{M}$ and those of the n-perturbed version of $\mathcal{M}$ we need some definitions and notations: Let $\mathit{Accept} = (\Sigma \cup B)^n \times [F \times (\Sigma \cup B)] \times (\Sigma \cup B)^n$. Given a configuration of $\mathcal{M}$

$$c = \cdots a_{-n-1}a_{-n}a_{-n+1}\cdots a_{-1}[q,a_0]a_1\cdots a_{n-1}a_na_{n+1}\cdots$$

we define the sequence

$$c|_n = a_{-n}a_{-n+1}\cdots a_{-1}[q,a_0]a_1\cdots a_{n-1}a_n$$

of length $2n+1$.

Then, it is easy to see that:

> *The n-perturbed version of $\mathcal{M}$ has an accepting run starting from a configuration c, if there exists $f \in \mathit{Accept}$ such that $c|_n \xrightarrow{*} f$ in $A_\mathcal{M}$.*

Hence, we can effectively construct $L_n(\mathcal{M})$ as a finite union of computable regular languages: Let Basis be the finite set of sequences $a_0a_1\cdots a_n \in \Sigma^{n+1}$ such that $B^n[q_{init},a_0]a_1\cdots a_n \xrightarrow{*} f$ for some $f \in \mathit{Accept}$. Let Short be the finite set of sequences $a_0a_1\cdots a_k \in \Sigma^*$ with $k < n$ such that $B^n[q_{init},a_0]a_1\cdots a_kB^{n-k} \xrightarrow{*} f$ for some $f \in \mathit{Accept}$. Then, we have

$$L_n(\mathcal{M}) = \text{Short} \cup \text{Basis}\Sigma^*$$

Since $L_n(\mathcal{M})$ is regular and effectively constructible, the same holds for its complement $\overline{L_n(\mathcal{M})}$. Hence, the set $\bigcup_n \overline{L_n(\mathcal{M})} = \overline{L_\omega(\mathcal{M})}$ is recursively enumerable as a union of a computable sequence of regular languages. $\square$

A consequence of the theorem above is that *robust languages* (i.e. $L_\omega(\mathcal{M}) = L(\mathcal{M})$) are necessarily recursive (since they must be in $\Sigma_1^0 \cap \Pi_1^0$):

Corollary 3 (Robust $\Rightarrow$ decidable) *If $L_\omega(\mathcal{M}) = L(\mathcal{M})$ then $L(\mathcal{M})$ is recursive.*

The converse holds if we add another requirement on $\mathcal{M}$:

Proposition 1 (Decidable $\Rightarrow$ robust) *If $\mathcal{M}$ always stops (and hence $L(\mathcal{M})$ is recursive) then $L_\omega(\mathcal{M}) = L(\mathcal{M})$*

Now, we show that in general, ω-perturbed languages are not recursively enumerable. In fact, the following result says that some of them are complete among Π_1^0 languages.

Theorem 4 (Perturbed reachability is complete in Π_1^0) *For every TM $\mathcal{M}$, we can effectively construct another TM $\mathcal{M}'$ such that $L_\omega(\mathcal{M}') = \overline{L(\mathcal{M})}$.*

Proof: Let $\mathcal{M} = (Q, q_{init}, F, \Gamma)$ be a TM over Σ. Suppose w.l.o.g. that the machine $\mathcal{M}$ is such that, for every input $w \notin L(\mathcal{M})$, $\mathcal{M}$ never stops and uses an unbounded working space (the head goes arbitrarily far from the initial position).

Now, let us consider an extra symbol $\# \notin \Sigma$. Then, we define the TM $\mathcal{M}' = (Q', q'_{init}, F', \Gamma')$ over $\Sigma \cup \{\#\}$ as follows: $Q' = Q \cup \{q_f\}$, $q'_{init} = q_{init}$, $F' = \{q_f\}$, and $\Gamma' = \Gamma \cup \{(q,\#) \rightarrow (q_f,\#) \ : \ q \in Q\}$.

This means that $\mathcal{M}'$ is constructed as $\mathcal{M}$ except that all accepting states of $\mathcal{M}$ are rejecting for $\mathcal{M}'$ and that whenever $\mathcal{M}'$ sees the symbol $\#$, it stops in its unique accepting state q_f. Let us prove that we have indeed $L_\omega(\mathcal{M}') = \overline{L(\mathcal{M})}$.

Consider a word $w \in L(\mathcal{M})$. Then, there exists an accepting run of $\mathcal{M}$ on w. By definition of $\mathcal{M}'$, this run is rejecting for $\mathcal{M}'$. Let N be size of the space used by this run. It can be seen that the $(N+1)$-perturbed version of $\mathcal{M}'$ has exactly the same behavior as $\mathcal{M}'$ on w since perturbations in the non-visited part of the configuration have no effect. Hence $w \notin L_{N+1}$, and consequently $w \notin L_\omega$ (Lemma 2).

Consider now a word $w \notin L(\mathcal{M})$. We show that for every $n > 0$, the n-perturbed version of $\mathcal{M}'$ recognizes w, which implies that w belongs to $L_\omega(\mathcal{M}')$. Let $n > 0$ and let us exhibit an accepting run of the n-perturbed version of $\mathcal{M}'$ on w: Suppose that, in the perturbed machine, starting from the initial configuration, two symbols at the distance $n+1$ to the left and to the right from the head are replaced by the symbol #. Then, since $w \notin L(\mathcal{M})$, the machine $\mathcal{M}$ has an unbounded run on w (see above the initial hypothesis on $\mathcal{M}$). Since $\mathcal{M}'$ has all the transitions of $\mathcal{M}$, it has also the same unbounded run on w, visiting positions arbitrarily far from the initial position of the head. Hence, the considered run of the n-perturbed version of $\mathcal{M}'$ eventually finds the # symbol and goes to the accepting state. $\square$

4 Results on PPAMs

We consider now the case of perturbed PAMs and show that their perturbed reachability relation is co-recursively enumerable.

Theorem 5 (Perturbed reachability is co-r.e.) *The relation $R^{\mathcal{P}}_\omega(\mathbf{x}, \mathbf{y})$ is Π^0_1 on $\mathbb{Q}^d$.*

Remember that in the case of TM, the proof of the similar result was based on the fact that the n-perturbed TM is in fact a finite-state system. For PAM, this actually does not hold, but we can show that each ε-perturbed PAM can be "faitfully " approximated by a finite-state automaton we define hereafter:

Consider a PAM $x := f(\mathbf{x}) = A_i\mathbf{x} + \mathbf{b}_i$ for $\mathbf{x} \in P_i,\ \ i = 1..N$. For any δ we can partition X into finitely many cubes $V_1, \dots V_S$ of size δ. We say that V_j is a δ-successor of V_k if $dist(f(V_k), V_j) < \delta$, that is if some point of V_k can be mapped to a point near V_j. Now we can construct a finite automaton A_δ with states $Q_\delta = \{q_1, \dots, q_S\}$, and with a transition from q_k to q_j authorized iff V_j is a δ-successor of V_k. Informally speaking, the automaton A_δ represents the PAM with accuracy δ. In order to formalize it we introduce the following *abstraction function* from X to Q_δ: $\alpha_\delta(\mathbf{x}) = q_i$ for $\mathbf{x} \in V_i$

Lemma 4 (Simulation) *(1) for any $\varepsilon > 0$ if $||f(\mathbf{x}) - \mathbf{y}|| < \varepsilon$ (i.e. the ε-perturbed system can make a transition from $\mathbf{x}$ to $\mathbf{y}$) then the automaton A_ε can make a transition from $\alpha_\varepsilon(\mathbf{x})$ to $\alpha_\varepsilon(\mathbf{y})$; (2) for any $\delta > 0$ if the automaton A_δ can make a transition from $\alpha_\delta(\mathbf{x})$ to $\alpha_\delta(\mathbf{y})$, then $||f(\mathbf{x}) - \mathbf{y}|| < C\delta$ (i.e. the $C\delta$-perturbed system can make a transition from $\mathbf{x}$ to $\mathbf{y}$), where C is a rational constant independent of δ;*

Proof: (1) Suppose that $||f(\mathbf{x}) - \mathbf{y}|| < \varepsilon$. Let $\alpha_\varepsilon(\mathbf{x}) = q_k$ and $\alpha_\varepsilon(\mathbf{y}) = q_j$. Then $dist(f(V_k), V_j) \le dist(f(\mathbf{x}), \mathbf{y}) < \varepsilon$. Hence by definition of the automaton A_ε the state q_j is reachable from q_k.

(2) Suppose that $\alpha_\delta(\mathbf{x}) = q_k$ and $\alpha_\delta(\mathbf{y}) = q_j$ and the state q_j is reachable from q_k. In this case $dist(f(V_k), V_j) < \delta$. Hence there exist $\mathbf{x}_0 \in V_k$ and $\mathbf{y}_0 \in V_j$ such that $||f(\mathbf{x}_0) - \mathbf{y}_0|| < \delta$. As $\mathbf{x}_0$ and $\mathbf{x}$ are in the same cube V_k the distance between them is inferior to the diameter of this cube $\sqrt{d}\delta$. The same is true for $\mathbf{y}_0$ and $\mathbf{y}$. Finally

$$||f(\mathbf{x}) - \mathbf{y}|| \le ||f(\mathbf{x}) - f(\mathbf{x}_0)|| + ||f(\mathbf{x}_0) - \mathbf{y}_0|| + \\ + ||\mathbf{y}_0 - \mathbf{y}|| < L\sqrt{d}\delta + \delta + \sqrt{d}\delta,$$

where the Lipschitz constant L can be found as $L = \max_i ||A_i||$. We can take now $C \ge L\sqrt{d} + 1 + \sqrt{d}$. $\square$

Corollary 4 *$R^{\mathcal{P}}_\omega(\mathbf{x}, \mathbf{y})$ holds iff for all rational $\delta > 0$ in the automaton A_δ the state $\alpha_\delta(\mathbf{y})$ is reachable from $\alpha_\delta(\mathbf{x})$.*

Hence by complementation $\neg R^{\mathcal{P}}_\omega(\mathbf{x}, \mathbf{y})$ iff for some rational $\delta > 0$ the state $\alpha_\delta(\mathbf{y})$ is unreachable from $\alpha_\delta(\mathbf{y})$ in the automaton A_δ. Unreachability in this automaton is (uniformly in δ) decidable for any particular δ, and hence the relation $\neg R^{\mathcal{P}}_\omega$ is recursively enumerable, which terminates the proof of Theorem 5.

Corollary 5 (Robust $\Rightarrow$ decidable) *If $R^{\mathcal{P}}_\omega = R^{\mathcal{P}}$ then $R^{\mathcal{P}}$ is recursive.*

Let us consider now the converse of Theorem 5. We prove the following fact:

Theorem 6 (Perturbed reachability is complete in Π^0_1) *Let $\mathcal{M}$ be a TM. We can effectively construct a PAM $\mathcal{P}$ and an encoding $e : \Sigma^* \to \mathbb{Q}^n$ such that for any word w, the following fact holds: $w \notin L(\mathcal{M})$ iff $R^{\mathcal{P}}_\omega(e(w), O)$.*

Proof: W.l.o.g. suppose that on any input word the machine $\mathcal{M}$ either stops in an accepting state, or computes forever. First we construct a 2-dimensional PAM $\mathcal{P}_0$ (and an input encoding $e : \Sigma^* \to \mathbb{Q}^n$) that simulates $\mathcal{M}$ and semi-recognizes $L(\mathcal{M})$ as described in [2]. Its main property is that for any word w the following equivalence holds: $w \in L(\mathcal{M})$ iff $R^{\mathcal{P}_0}_\omega(e(w), O)$. It is easy to verify that if a rather small neighborhood [1] (e.g. a 1/10-square) of the origin is reachable from $e(w)$ then $w \in L(\mathcal{M})$. The last useful property of this simulation is that all the points of the trajectory starting from $e(w)$ are internal points of polyhedra P_i.

[1] representing the accepting state of the TM

Now we construct a new 3-dimensional PAM $\mathcal{P}$ whose perturbed version will "semi-recognize" $\overline{L(\mathcal{M})}$. We will use notation $\mathbf{x}$ or $\mathbf{y}$ for 2-dimensional vectors and h for the third dimension (so the generic element of $\mathbb{R}^3$ will be $(\mathbf{x}, h)$). It is mainly the original system $\mathcal{P}_0$ embedded in the plane $h = 2$ of the space $\mathbb{R}^3$. However there are 2 changes (compare with the proof for TMs) — informally:

- The accepting state O (with his small neighborhood) of the original system $\mathcal{P}_0$ becomes rejecting for the new system $\mathcal{P}$.
- The zone $h \le 1$ becomes accepting for the new system.

The idea is that for any $w \in L(\mathcal{M})$ the original PAM $\mathcal{P}_0$ will eventually arrive to O (and accept) and hence the perturbed PAM $\mathcal{P}$ will arrive to the neighborhood of $O \times \{2\}$ and reject. For any $w \notin L(\mathcal{M})$ the perturbed PAM $\mathcal{P}$ will slowly drift "down" until it reaches the accepting zone $h \le 1$.

Formally, let the original system be defined on a subset of the cube $[-T, T]^2 \subseteq \mathbb{R}^2$ by equation $\mathbf{x} := f(\mathbf{x})$. Denote the squared neighborhood of the origin $[-0.1, 0.1]^2 \subseteq \mathbb{R}^2$ by C. Then the new system will be defined on the rectangular set $[-T-1, T+1]^2 \times [-1, 3] \subseteq \mathbb{R}^3$ by the equation $\mathbf{x} := g(\mathbf{x}, h)$ where $g(\mathbf{x}, h)$ is defined as follows:

- if $1 < h \le 3$, and $\mathbf{x} \notin C$, then $g(\mathbf{x}, h) = (f(\mathbf{x}), h)$. Informally speaking, in the layer $1 < h < 3$ the system $\mathcal{P}$ simulates the original system $\mathcal{P}_0$ without modifying h
- if $1 < h \le 3$ and $\mathbf{x} \in C$, then $g(\mathbf{x}, h)$ is undefined
- if $h \le 1$ we go to the origin : $g(\mathbf{x}, h) = (\mathbf{0}, 0)$

The input encoding function for the system $\mathcal{P}$ is as follows: $e(w) = (e_0(w), 2)$ where e_0 is the encoding function of the original system $\mathcal{P}_\prime$.

Now we have to prove that $R^{\mathcal{P}}_{\omega}(e(w), O)$ iff $w \notin L(\mathcal{M})$. Suppose first that $w \notin L(\mathcal{M})$. In this case the TM $\mathcal{M}$ has an infinite-length run on w and the PAM $\mathcal{P}_0$ has an infinite trajectory $\mathbf{x}_n$ starting in $e_0(w)$. For any $\varepsilon > 0$ we can construct a trajectory g of the ε-perturbed system $\mathcal{P}$ as follows:

- $g_n = (\mathbf{x}_n, 2 - \varepsilon n)$ for $n \in [0, \lceil 1/\varepsilon \rceil]$; during the first $\lceil 1/\varepsilon \rceil$ time units the system simulates $\mathcal{P}_0$ along first two dimensions slowly drifting down in the third one
- $g_n = 0$ for $n \ge \lceil 1/\varepsilon \rceil$ the trajectory jumps to the origin and stays there.

It is easy to see that g_n is a trajectory of the ε-perturbed system, and hence $R^{\mathcal{P}}_{\omega}(e(w), O)$ holds.

Now consider the other case when $w \notin L(\mathcal{M})$. Then the trajectory $\mathbf{x}_n$ of $\mathcal{P}_0$ starting in $e_0(w)$ eventually arrives to the origin. The non-perturbed trajectory g_n of $\mathcal{P}$ starting in $e(w)$ will follow $\mathbf{x}_n$ in the plane $h = 2$ until it reaches the neighborhood C of the origin. Once in this neighborhood the system $\mathcal{P}$ dies immediately. The only thing to verify is that all perturbed trajectories of $\mathcal{P}$ starting in $e(w)$ are close enough to g_n for ε small enough. Let T be the time of arrival to the origin (i.e. such that $g_T = 0$), $A = \max\{1, ||A_i||\}$ and $\theta = \min_{n<T} dist(\mathbf{x}_n, \partial P_{i(n)})$. If we take $\varepsilon < \theta A^{-T}$, then a straightforward induction shows that any ε-perturbed trajectory g'_n is close to g_n and the same affine maps are applied until it enters the deadly neighborhood of the origin. $\square$

Theorem 7 *All the results stated in this section can be proved in a very similar manner for Linear hybrid automata (LHA).*

5 Results on PPCDs

We consider finally the case of PCDs and prove the same results as for PAMs (and LHAs). The overall structure of the proofs is the same as in the previous case. However, the proofs for the two kinds of models are technically different due to the fact that the rules for accumulating errors (resulting from perturbations) are different for each of these models. An ε-perturbation of a PAM results in moving the state by ε in any direction at each transition, which ensures the simulation lemma 4 (the same holds in the LHA model). Differently from this, a perturbed trajectory in an ε-perturbed PCD deviates from the ideal trajectory after crossing a region by $\sim \tau\varepsilon$, where τ stands for the time needed to cross this region, and this time depends on the entry point to a region and the slope at this region and cannot be bounded from below.

Our solution to this consists in observing (and approximating by an automaton) the states of the PCD only when it enters some special *good regions*. In a non-Zeno system, the time τ' between consecutive visits of good regions is bounded from below and the accumulated error $\sim \tau'\varepsilon$ is large enough to ensure simulation.

Theorem 8 (Perturbed reachability is co-r.e.) *The relation $R^{\mathcal{H}}_{\omega}(\mathbf{x}, \mathbf{y})$ on $\mathbb{Q}^d$ is in Π^0_1.*

We proceed in a similar manner as for PAMs: We approximate the ε-perturbed system by a finite-state automaton. However, relations between the system and the automaton are somewhat subtler. First of all, let N be the number of regions in the PCD, and $\alpha > 0$ a positive constant specified below. Without loss of generality we can suppose that the norm used in the definition of ε-perturbed system is $\|\cdot\|_\infty$, which means that ε-ball centered in a point**x** is in fact a cube with side 2ε. Let us introduce now some definitions:

Definition 3 (Good points) *A point* $\mathbf{x}$ *on the boundary of a region is* good *if the trajectory starting from* $\mathbf{x}$ *does not change direction during at least* α *time. Formally let* $\mathbf{c} = f(\mathbf{x})$ *be the slope in* $\mathbf{x}$*. Then the vector field* $f(\mathbf{y})$ *should be constant (and equal to* $\mathbf{c}$*) for all* $\mathbf{y} \in [\mathbf{x}, \mathbf{x} + \alpha\mathbf{c}]$

Lemma 5 (Good regions) *The set* G *of all good points is a finite union of polyhedra of dimensionality* $< d$*.*

The following lemma, saying that the good regions are visited often, enough follows from the strong non-zenoness of the PCD.

Lemma 6 *Each perturbed trajectory crossing* N *regions visits a good region at least once.*

Let us see now how we define an "approximating automaton": For any δ we can partition G into finitely many polyhedra $V_1, \dots V_S$ of size δ. We say that V_j is a δ-successor of V_k if there exists a trajectory of the δ-perturbed system no more than N links from an $\mathbf{x} \in V_k$ to an $\mathbf{y} \in V_j$. It is easy to see that the property of being a δ-successor can be reduced to a linear programming problem, and hence is decidable.

Then, we can construct a finite automaton A_δ with states $Q_\delta = \{q_1, \dots, q_S\}$, and with a transition from q_k to q_j authorized iff V_j is a δ-successor of V_k. Informally speaking, the automaton A_δ represents the δ-perturbed PCD with accuracy δ. In order to formalize it we introduce the following *abstraction function* from X to Q_δ: $\alpha_\delta(\mathbf{x}) = q_i$ for $\mathbf{x} \in V_i$.

Hereafter, we explore in which sense A_δ simulates $\mathcal{H}_\varepsilon$:

Lemma 7 (Quasi-Simulation) *Let* $\mathbf{x}, \mathbf{y} \in G$ *be two good points. (1) for any* $\varepsilon > 0$ *if the* ε*-perturbed system can go from* $\mathbf{x}$ *to* $\mathbf{y}$ *via a trajectory with less than* N *links, then the automaton* A_ε *can make a transition from* $\alpha_\varepsilon(\mathbf{x})$ *to* $\alpha_\varepsilon(\mathbf{y})$*; (2) for any* $\delta > 0$ *if the automaton* A_δ *can make a transition from* $\alpha_\delta(\mathbf{x})$ *to* $\alpha_\delta(\mathbf{y})$*, then* $C\delta$*-perturbed system can go from* $\mathbf{x}$ *to a good point* $\mathbf{y}'$ *via a trajectory with less than* N *links, where* C *is a rational constant independent of* δ*, and* $\alpha_\delta(\mathbf{y}) = \alpha_\delta(\mathbf{y}')$*;*

Corollary 6 (Many steps) *Let* $\mathbf{x}, \mathbf{y} \in G$ *be two good points. (1) for any* $\varepsilon > 0$ *if the* ε*-perturbed system has a trajectory from* $\mathbf{x}$ *to* $\mathbf{y}$ *, then the automaton* A_ε *has a run from* $\alpha_\varepsilon(\mathbf{x})$ *to* $\alpha_\varepsilon(\mathbf{y})$*; (2) for any* $\delta > 0$ *if the automaton* A_δ *has a run from* $\alpha_\delta(\mathbf{x})$ *to* $\alpha_\delta(\mathbf{y})$*, then* $C\delta$*-perturbed system has a trajectory from* $\mathbf{x}$ *to a good point* $\mathbf{y}'$*), where* $\alpha_\delta(\mathbf{y}) = \alpha_\delta(\mathbf{y}')$.

It is still not the result that we want, because first it concerns only reachability between good points, and, second, the target point $\mathbf{y}$ is replaced by a neighbor point $\mathbf{y}'$.

In order to deal with these two issues we introduce the following δ-test for perturbed reachability between arbitrary points. First of all we construct the A_δ automaton. Next, we proceed in three steps:

1. Find the set S_1 of indices i such that V_i is reachable by $\mathcal{H}_\delta$ from $\mathbf{x}$ via a trajectory with less then N links. This can be done algorithmically using linear programming.

2. Find the set S_2 of indices of all the states q_j of the A_δ automaton reachable in this automaton from $\{q_i \mid i \in S_1\}$. This is a reachability problem in a finite-state automaton.

3. For each $j \in S_2$ test whether $\mathbf{y}$ is reachable by $\mathcal{H}_\delta$ from V_j via a trajectory with less then N links. This can be solved as in the first step using linear programming. In case of positive answer for any $j \in S_2$, the δ-test succeeds, otherwise it fails.

Notice that δ-test always terminates. Then, it is easy to see that the following fact holds:

Lemma 8 (Correctness of δ-test) *For any two points* $\mathbf{x}$ *and* $\mathbf{y}$ *(1) if* $R^{\mathcal{H}}_\varepsilon(\mathbf{x}, \mathbf{y})$*, then* δ*-test succeeds for* $\mathbf{x}$ *and* $\mathbf{y}$*. (2) If* δ*-test succeeds for* $\mathbf{x}$ *and* $\mathbf{y}$*, then* $R^{\mathcal{H}}_{C\delta}(\mathbf{x}, \mathbf{y})$*.*

Corollary 7 $(\mathbf{x}, \mathbf{y}) \notin R^{\mathcal{H}}_\omega$ *if and only if for some* $n \in \mathbb{N}$ *the* $1/n$*-test fails for* $\mathbf{x}$ *and* $\mathbf{y}$*.*

By the corollary above, a semi-decision algorithm for $\neg R^{\mathcal{H}}_\omega$ is immediate, which terminates the sketch of proof of Theorem 8.

Corollary 8 (Robust ⇒ decidable) *If* $R^{\mathcal{H}}_\omega = R^{\mathcal{H}}$ *then* $R^{\mathcal{H}}$ *is recursive.*

Finally, we can prove the converse result of Theorem 8. The proof is given in the appendix.

Theorem 9 (Perturbed reachability is complete in Π^0_1) *Let* $\mathcal{M}$ *be a TM. We can effectively construct a PCD* $\mathcal{H}$ *and an encoding* $e : \Sigma^* \to \mathbb{Q}^n$ *such that for any word* w *the following equivalence holds:* $w \notin L(\mathcal{M})$ *iff* $R^{\mathcal{H}}_\omega(e(w), O)$*.*

6 Conclusion

We have shown that when we consider infinitesimal perturbations in the dynamics of a system, the reachability relation becomes co-recursively enumerable, which proves that robust systems are decidable. It is interesting to observe that these results hold for several different discrete and continuous time models of dynamic systems, which shows that they correspond to a general phenomenon. The proofs of these results have also a common scheme, although they differ significantly depending from the specificity of the dynamics of each class of models.

Our results establish a tight link between the notions of decidability and robustness for infinitesimal perturbations. This link is of a semantical nature. An interesting question is to find sufficient "syntactical" conditions on the models of dynamical systems ensuring their robustness, leading to decidability results for classes of dynamical systems.

References

[1] Cris Moore, "Generalized shifts: Undecidability and unpredictability in dynamical systems," *Nonlinearity*, vol. 4, pp. 199–230, 1991.

[2] Pascal Koiran, Michel Cosnard, and Max Garzon, "Computability with low-dimensional dynamical systems," *Theoretical Computer Science*, vol. 132, pp. 113–128, 1994.

[3] Eugene Asarin, Oded Maler, and Amir Pnueli, "Reachability analysis of dynamical systems having piecewise-constant derivatives," *Theoretical Computer Science*, vol. 138, pp. 35–65, 1995.

[4] Bruce Krogh and Nancy Lynch, Eds., *Hybrid Systems: Computation and Control (HSCC 2000)*, vol. 1790 of *LNCS*. Springer, 2000.

[5] Thomas A. Henzinger, P.W. Kopke, Anuj Puri, and Pravin Varaiya, "What's decidable about hybrid automata?," in *Proceedings of the 27th Annual Symposium on Theory of Computing*. 1995, pp. 373–382, ACM Press.

[6] Martin Fränzle, "Analysis of hybrid systems: An ounce of realism can save an infinity of states," in *Computer Science Logic (CSL'99)*, Jörg Flum and Mario Rodríguez-Artalejo, Eds. 1999, vol. 1683 of *LNCS*, pp. 126–140, Springer.

[7] Thomas A. Henzinger and Jean-François Raskin, "Robust undecidability of timed and hybrid systems," In Krogh and Lynch [4], pp. 145–159.

[8] E. Asarin, "Chaos and Undecidability," Tech. Rep., Verimag, 1995.

[9] Peter Kloeden and Victor Kozyakin, "The inflation of attractors and discretization: the autonomous case," *Nonlinear Anal., TMA*, vol. 40, pp. 333–343, 2000.

[10] Anuj Puri, "Dynamical properties of timed automata," *Discrete Event Dynamic Systems*, vol. 10, no. 1/2, pp. 87–113, 2000.

[11] M. Casey, "The dynamics of discrete-time computation, with application to recurrent neural networks and finite-state machine extraction," *Neural Computation*, vol. 8:6, 1996.

[12] W. Maass and P. Orponen, "On the effect of analog noise in discrete-time analog computations," in *Neural Information Processing Systems*, 1996.

[13] C. Moore, "Finite-dimensional analog computers: Flows, maps, and recurrent neural networks," in *1st Intern. Conf. on Unconventional Models of Computation*. 1998, Springer-Verlag.

[14] C. Moore, "Undecidability and unpredictability in dynamical systems," *Physical Review Letters*, vol. 64, pp. 2354–2357, 1990.

[15] Hartley Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

A Proof of Theorem 9

The idea of this proof is similar to the case of PAMs (Theorem 6). We take a PCD $\mathcal{H}_0$ simulating the machine $\mathcal{M}$, and add one more dimension h. We start at the level $h = 4$. Accepting states of the PCD $\mathcal{H}_0$ become rejecting in the new PCD $\mathcal{H}$. In order to be accepting in $\mathcal{H}$ the trajectory should go down and reach the plane $h = 0$. It is possible for arbitrarily small ϵ only if the original PCD $\mathcal{H}_0$ can evolve during arbitrarily long time, that is the perturbed version of $\mathcal{H}$ accepts a word iff $\mathcal{H}_0$ does not accept it.

First let us construct a 4-dimensional PCD $\mathcal{H}_0$ (and an input encoding $e : \Sigma^* \to \mathbb{Q}^n$) which simulates $\mathcal{M}$ and semi-recognizes $L(\mathcal{M})$ as described in [3]. Its main property is that for any word w the following equivalence holds. $w \in L(\mathcal{M})$ if and only if $R_w^{\mathcal{H}_0}(e(w), O)$ It is easy to verify that if a rather small neighborhood (e.g. a 1/10-ball) of the origin is reachable from $e(w)$ then $w \in L(\mathcal{M})$.

Now we construct a new 5-dimensional PCD $\mathcal{H}$ whose perturbed version will "semi-recognize" $\overline{L(\mathcal{M})}$. We will use notation $\mathbf{x}, \mathbf{y}$ for 4-dimensional vectors and h for the fifth dimension (so the generic element of $\mathbb{R}^5$ will be $(\mathbf{x}, h)$). It is mainly the original system $\mathcal{H}_0$ submerged in the hyperplane $h = 3$ of the space $\mathbb{R}^5$. However there are 2 changes (compare with the proof for PAMs) — informally:

- The accepting state O (with his small neighborhood) of the original system $\mathcal{H}_0$ becomes rejecting for the new system $\mathcal{H}$
- The zone $h \leq 1$ becomes accepting for the new system

The idea is that for any $w \in L(\mathcal{M})$ the original PCD $\mathcal{H}_0$ will eventually arrive to O (and accept) and hence the perturbed PCD $\mathcal{H}$ will arrive to the neighborhood of $O \times 2$ and reject. For any $w \notin L(\mathcal{M})$ the perturbed PCD $\mathcal{H}$ will slowly drift "down" until it reaches the accepting zone $h \leq 1$.

Formally, let the original system be defined on a subset of the cube $[-T, T]^4 \subseteq I\!R^4$ by equation $\dot{\mathbf{x}} = f(\mathbf{x})$. Denote the cubic neighborhood of the origin $[-0.1, 0.1]^4 \subseteq I\!R^4$ by C.

Then the new system will be defined on the rectangular set $[-T-1, T+1]^4 \times [-1, 5] \subseteq I\!R^5$ by the equation $(\mathbf{x}, h)^{\cdot} = g(\mathbf{x}, h)$ where $g(\mathbf{x}, h)$ is defined as follows:

- if $h \geq 4$, then $g(\mathbf{x}, h) = (\mathbf{0}, 1)$: anything that arrives in the layer $h \geq 4$ goes "up" and is rejected
- if $2 < h < 4$ and $f(\mathbf{x})$ is defined, then $g(\mathbf{x}, h) = (f(\mathbf{x}), 0)$. Informally speaking, in the layer $2 < h < 4$ the system $\mathcal{H}$ simulates the original system $\mathcal{H}_0$
- if $2 < h < 4$ and $\mathbf{x} \in C$, then $g(\mathbf{x}, h) = (\mathbf{0}, 1)$
- if $2 < h < 4$ and $f(\mathbf{x})$ is undefined, then $g(\mathbf{x}, h) = (\mathbf{0}, 1)$
- if $1 < h \leq 2$ we go down : $g(\mathbf{x}, h) = (\mathbf{0}, -1)$
- finally in the layer $-1 \leq h < 1$ we put a (piecewise constant) vector field with all the trajectories going to the origin.

The input encoding function for the system $\mathcal{H}$ is as follows: $e(w) = (e_0(w), 3)$ where e_0 is the encoding function of the original system $\mathcal{H}_0$.

Now we have to prove that $R^{\mathcal{H}}_{\omega}(e(w), O)$ if and only if not $w \notin L(\mathcal{M})$. Suppose first that $w \notin L(\mathcal{M})$. In this case the TM $\mathcal{M}$ has an infinite-length run on w and the PCD $\mathcal{H}_0$ has an infinite trajectory $\mathbf{x}(t)$ starting in $e_0(w)$. For any $\varepsilon > 0$ we can construct a trajectory g of the ε-perturbed system $\mathcal{H}$ as follows:

- $g(t) = (\mathbf{x}(t), 3 - \varepsilon t)$ for $t \in [0, 1/\varepsilon]$; during the first $1/\varepsilon$ time units the system simulates $\mathcal{H}_0$ along first four dimensions slowly drifting down in the fifth one
- $g(t) = (\mathbf{x}(1/\varepsilon), 2-(t-1/\varepsilon))$ for $t \in [1/\varepsilon; 1/\varepsilon+1]$— the next trajectory segment goes straight down with unit velocity during one time unit.
- The last trajectory segment goes straight to the origin.

Now consider the other case when $w \notin L(\mathcal{M})$. Then the trajectory $\mathbf{x}(t)$ of $\mathcal{H}_0$ starting in $e_0(w)$ eventually arrives to the origin. The non-perturbed trajectory $g(t)$ of $\mathcal{H}$ starting in $e(w)$ will follow $\mathbf{x}(t)$ in the plane $h = 3$ until it reaches the neighborhood C of the origin. Once in this neighborhood the system $\mathcal{H}$ goes straight up to the death. The only thing to verify is that all perturbed trajectories of $\mathcal{H}$ starting in $e(w)$ are close enough to $g(t)$ for ε small enough. This can be done similarly to PAMs.