On some algorithmic problems MPRI 2017–2018

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Quadratic versus linear time can be harmful in practical

- Stackoverflow : `http://stackstatus.net/post/147710624694/outage-postmortem-july-20-2016`
- A good solution : `https://swtch.com/~rsc/regexp/regexp1.html`

Exercises

X a finite set
$\mathcal{F}$ a family of subsets $S_1, \ldots, S_k$ de X.

A-1 Write an algorithm to check if there is no i, j such that : $S_i \subseteq S_j$

A-2 **Disjoint Set Problem :**
Find i, j such that $S_i \cap S_j = \emptyset$

A-3 Check if : $\forall i, j,\ i \neq j\ S_i \cap S_j = \emptyset$

A-4 Check if there exists i such that : S_i intersects all other subsets of $\mathcal{F}$

A-5 Compute the set of maximal (resp. minimal) for inclusion of $\mathcal{F}$

A-6 Check if $\mathcal{F}$ is laminar.

A-7 Check if $\mathcal{F}$ satisfies the Helly property

Exercises

G is a finite graph

- B-1 Find if there is a triangle in G
- B-2 Find if there is a 3-independent set in G
- B-3 Find if there is an asteroidal triple in G
- B-4 Compute the diameter of G
- B-5 Compute a (resp. all) center(s) in G

Exercises on directed graphs

G is a finite directed acyclic graph

C-1 Compute the transitive closure of G

C-2 Compute the transitive reduction of G

C-3 * A variation : if G is strongly connected, find a minimal [1] one (i.e. ; with no transitive arcs).

1. Finding a minimum one is NP-hard

Exercises related problems

D-1 Exact 3-sum :
3 sets A, B, C of integers
Question : $\exists$ a triple (a, b, c) with $a \in A$, $b \in B$, $c \in C$ such that $a + b + c = 0$?

D-2 Boolean matrix multplication

D-3 SAT

- Of course the idea is to find the best algorithm for each of these problems or to obtain a non trivial lower bound.
- I have already met each of these problems when dealing with graph algorithms, mainly when X is the vertex set of the graph and $\mathcal{F}$ is the family obtained by considering the vertex neighbourhoods.
- Hint : You could first search the relationships between these problems

For social sciences applications :

- Compute graph parameters such as betweenness centrality in order to discover structures in social networks.
- A typical problem :
  How to compute : for every x, y $\frac{|N(x) \cap N(y)|}{|N(x) \cup N(y)|}$
- Can we compute $|N(x) \cap N(y)|$ in the size of the smallest ?
- Even with some preprocessing.

We would not described all possibles graph classes, only the good ones !
See http://www.graphclasses.org/

Some important and basic graph classes needed for this course

- G is interval if and only if is the intersection graph of a family of intervals on the real line.
- G is a split graph if its vertex set can be partitioned into a clique K and an independent set I (with edges in between)
- G is chordal if has no induced cycle of size ≥ 4.
- G is a comparability graph if it can be transitively oriented.
- G is a cocomparability graph if its complement is a comparability graph.
- G is a permutation graph if and only if it is the intersection of line segments whose endpoints lie on two parallel lines.
- G is AT-free if it has no asteroidal triple.

Asteroidal triples

Definition

An asteroidal triple is a triple of independent vertices (x, y, z) such that for every pair of vertices there exists a path joining them avoiding the neighborhood of the third.

All these graph classes are Hereditary ones
The class of split graphs and that of permutation are closed under complementation. But not the others (interval, chordal, comparability, AT-free)

Dushnik and Miller 1941

G is a permutation graph if and only if G and its complement are both comparability graphs.

Proof

$\rightarrow$

The complement of a permutation graph corresponds to the non intersection of line segments whose endpoints lie on two parallel lines. But such a graph can be transitively oriented just by considering the following order between 2 lines : $x < y$ iff the line x is completely to the left of line y (every edge can be directed, since it corresponds to non intersecting segments). This relation completely to the left of is clearly transitive and acyclic. Therefore a permutation graph is a cocomparability graph. But since this class is closed under complement, a permutation graph is also a comparability graph.

$\rightarrow$
Suppose $G, \overline{G}$ are comparability graphs and take σ (resp. τ) a transitive orientation of G (resp. $\overline{G}$).
Let $R = (V(G), \sigma \cup \tau)$, R is a total relation i.e., a tournament (in graph theory).
It is easy to consider that R is a transitive acyclic tournament, i.e., a total order. Let us denote $R' = (V(G), \sigma \cup \tau^{dual})$ $\tau dual$ is just the reverse orientation associated with τ.
Then (R, R') provides a segment representation of $\overline{G}$.

Theorem

cocomparability graphs $\subsetneq$ AT-free graphs

Proof

Just because we cannot transitively orient the complement of an asteroidal triple. (hint : make a simple picture)
Furthermore the split graph $\overline{3Sun}$ is not a cocomparability graph and has no asteroidal triple.

k-chordal graphs

A graph G is k-chordal it it does not admit any induced subgraph isomorphic to C_{k+1}.

- A chordal graph is 3-chordal.
- Any AT-free graph is 5-chordal. Since any C_6 contains an asteroidal triple of vertices.
- Any cocomparability graph is 4-chordal.
  Using previous theorem we know that cocomparability are 5-chordal. But if cocomparability graph contains a C_5, its complement also contains a C_5, since this graph is self-dual (isomorphic to its complement). As a C_5 cannot be transitively oriented, cocomparability graphs are 4-chordal.

Földes and Hammer 1977

G is a split graph iff G and $\overline{G}$ are chordal

Proof

$\rightarrow$ The easy direction :
G split graphs implies G chordal.
G split graph implies $\overline{G}$ split graph so $\overline{G}$ is also chordal.
$\leftarrow$ Left as an exercise to use the structure of chordal graphs

A useful characterization

Theorem Lekerkerker and Boland 1962

G interval iff G AT-free and chordal

Proof

$\rightarrow$ The easy direction :
G interval implies G chordal
G interval implies G cocomp and therefore AT-free.
$\leftarrow$ Since G is chordal it admits a maximal clique tree, but then using the Asteroidal freeness property we can linearize the maximal clique tree into a chain of maximal cliques.

Proper intervals = unit intervals, but only in the finite case

Proper interval

G is proper interval graph if its admits an interval representation in which no interval contains strictly another interval.
iff G if its admits an interval representation in which all intervals have the same length.
iff G is an interval graph with no induced $K_{1,3}$.