6ème Cours
Cours MPRI 2010–2011

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Chevaleret, october 2010

Schedule

Treewith

Graph Minors

Big theorems on Graph Minors

Other width parameters

Cliquewidth

Tree decomposition

$G = (V, E)$ has a tree decomposition $D = (S, T)$
S is a collection of subsets of V, T a tree whose vertices are elements of S such that :

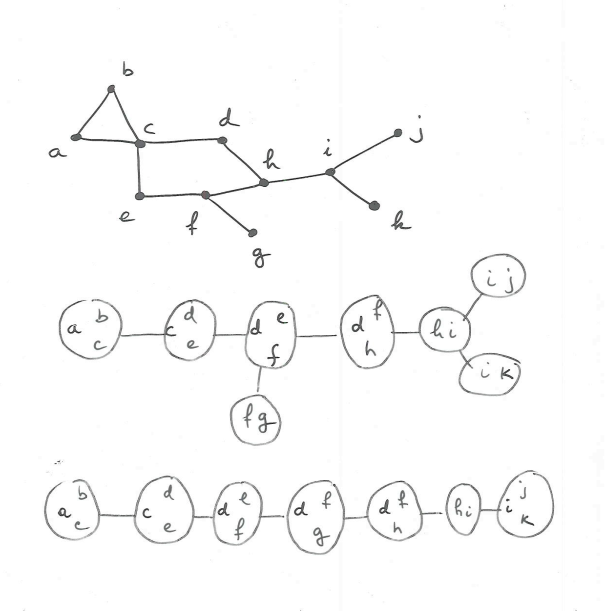
(0) The union of elements in S is V

(i) $\forall e \in E, \exists i \in I$ with $e \in G(S_i)$.

(ii) $\forall x \in V$, the elements of S containing x form a subtree of T.

Definition

$treewidth(G) = Min_D(Max_{S_i \in S}\{|S_i| - 1\})$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Recall of some Equivalences

1. $Treewidth(G) = Min_{H \text{ triangulation of } G}\{\omega(H) - 1\}$
2. Computing treewidth is NP-hard.

Other definitions of trewidth in terms of cop-robber games, using graph grammars ....
But it turns out that this parameter is a fundamental parameter for graph theory.

Some examples

1. G is a tree iff $treewidth(G) = 1$
2. $treewidth(K_n) = n - 1$
3. If G is a cycle then $treewidth(G) = 2$. (It can be seen as two chains in parallel, i.e. a series-parallel graph)
4. $treewidth(K_{n,m}) = min(n, m)$
5. $treewidth(G_{n,m}) = min(n, m)$, the lower bound is hard to obtain !
6. $treewidth(G)$ (resp. pathwidth) measures the distance from G to a tree (resp. to a chain)

Easy properties

treewidth$(G) = k$ iff G can be decomposed using only separators of size less than k.

Fundamental lemma

Let ab an edge of T some tree decomposition of G and T_1, T_2 be the two connected components of $T - ab$, then $V_a \cap V_b$ is a separator between $V_1 - V_2$ and $V_2 - V_1$, where $V_1 = \cup_{i \in T_1} V_i$ and $V_2 = \cup_{j \in T_2} V_j$.

Proof of the lemma

Démonstration.

Let ab be an edge of a tree decomposition T of G. $T - ab$ is disconnected into T_1 and T_2, two subtrees of T.
Let $V_1 = \cup_{t \in T_1} V_t$ and $V_2 = \cup_{t \in T_2} V_t$. If $V_a \cap V_b$ is not a separator, then it exists $u \in V_1 - V_2$ and $v \in V_2 - V_1$ and $uv \in E$.
But then in which bag can the edge uv belongs to ? Since using property (i) of tree decomposition each edge must belong to some bag. This cannot be in T_1, neither in T_2, a contradiction. □

For the previous lemma, we only use the definition of any tree-decomposition, not an optimal one. It also explains the use of property (i) in the definition of tree decomposition.

Computations of treewidth

- There exists polynomial approximation algorithms
- For every fixed k, it exists a linear algorithm to check wether $Treewidth(G) \leq k$ Boedlander 1992. (Big constant for the linearity).
- Find an efficient algorithm for small values $3, 4, 5 \dots$ is still a research problem

Real applications of treewidth

1. Graphs associated with programs have bounded treewidth 4,5, 6 depending on the programming language, Thorup 1997.
2. Constraint Satisfaction Problem, Feuder's Theorem
3. Can also be defined on hypergraphs which yields applications for Data bases via Acyclic Hypergraphs
4. Good Heuristic for the Traveling Salesman Problem, Chvatal

Divide and Conquer Approach

E. Proskurowski

When he introduced partial k-trees
as a generalisation of trees preserving dynamic programming
he aimed at polynomial algorithms for bounded tree-graphs (i.e. the size of the bags is bounded).

Generic Divide and Conquer Algorithm

Solve the problem for a leaf and then recurse

Metaconsequence

For most graph parameters Π, there exists an exact algorithm in $O(2^{O(treewidth(G)})$.

Balanced separator

Every graph G, $|G| \geq k+4$, with $treewidth(G) = k$ admits a separator S of size $k+1$ such that $G - S$ is partitionned into A, B, with no edge between A and B and :
$\frac{1}{3}(n-k-1) \leq |A|, |B| \leq \frac{2}{3}(n-k-1)$

The study of the interplay between logics and combinatorial structures
yields knowledge on complexity theories

B. Courcelle studying graph rewriting systems or graph grammars obtained :

Meta-Theorem

Any graph problem that can be expressed with a formula of the Monadic Second Order Logic (MSO),
if G has bounded treewidth then it exists a linear algorithm to solve this problem on G.

Why Meta

A unique theorem for a whole class of problems on a class of graphs.

Monadic Second Order Logic

For graphs :
$x_1, \dots x_n$ variables
X_i subset of vertices
Atoms :
$E(x, y)$ true iff xy is an edge of G
$X(x)$ true if $x \in X$
Classical logical connectors (equality, implication, negation ...) to make formulas
Quantification over variables and subsets of vertices are allowed.

Examples :

- $\phi : \exists x_1 \ldots \exists x_k \forall y \bigwedge_{1 \leq i \leq k}((x_i = y) \bigvee (E(x_i, y))$
  $\phi = G$ has a dominating set of size $\leq k$.
- Idem $\psi_k = G$ is k-colorable.
- Of course, SAT can be expressed in MSO.
- The number of quantifiers is an indication of the complexity of the formula (i.e. the problem)
  We do not need to use quantifiers on subets of edges (via the incidence bipartite)

Not all graph problems can be expressed in MSO.
Example : Computing the permanent of a graph (matrix)

Linear ?

Linear but with a giant constant :
$(2^{2^{2^{\cdots}}})^h$.
h the size of this exponential tower which depends on the MSOL formula, more precisely on the alternation of quantifiers

Grohe, Frick 2005

h is unbounded unless $P = NP$.

Recent work try to avoid the construction of the automaton (which can be of huge size), using techniques from verification or game theory.

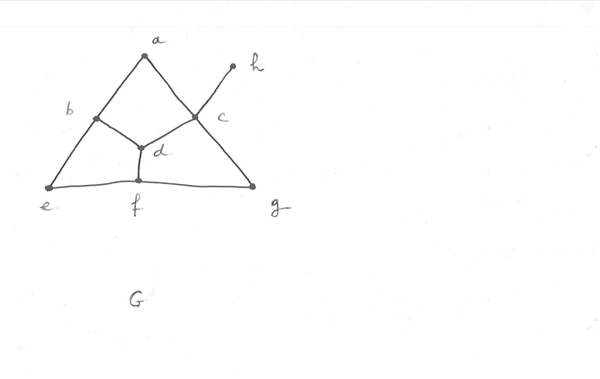
A Duality Theorem

Brambles

$G = (V, E)$; $\mathcal{B} = \{X_i | X_i \subseteq V\}$ such that :
(i) $\forall i,\ G(X_i)$ connected
(ii) $\forall i, j\ \ G(X_i \cup X_j)$ connected.
A transversal of a bramble is a set $\tau \subseteq V$ such that $\forall i\ \tau \cap X_i \neq \emptyset$
$bn(G) = Max_{\mathcal{B}\ bramble\ of\ G}(min_{\tau\ \ transversal\ of\ \mathcal{B}}(|\tau|))$

Duality Theorem Roberston and Thomas 93

For every graph G, $treewidth(G) = bn(G) - 1$



Lower bound

Consider the family {{d}, {a, b}, {e, f }, {g, c}} which satisfies the properties of a bramble .
With a transversal $\{d, b, c, f\}$ of minimum of size 4.
Therefore using bramble theorem *threewidth*$(G) \geq 3$

Therefore this example has exactly treewidth 3

$bn(G) \leq treewidth(G) + 1$

Démonstration.

Let us consider a tree decomposition T and a bramble $\mathcal{B}$.
Every $X_i \in \mathcal{B}$ coresponds to a subtree T_{X_i} in T, using connectivity (i).
Furthermore condition (ii) implies $\forall i, j$, $T_{X_i} \cap T_{X_j} \neq \emptyset$
Using Helly property on these subtrees, they all have a common vertex t. The bag associated with t meets every element of the bramble and is a transversal for $\mathcal{B}$. □

This Min(Max) Max(Min) theorem gives evidence for treewith computations
as for example for the grid.
Consider the bramble made up with all the crosses it has a transversal of size $min(n, m)$,
therefore $treewidth(G_{n,m}) \geq min(n, m) - 1$

Graph Minors

- A graph H is a minor of a graph G if H is isomorphic to a graph obtained from G by contracting edges, deleting edges, and deleting isolated nodes
- The minor ordering of graphs is that defined by $H \leq G$ if H is a minor of G
- A set S of graphs is downwardly closed with respect to the minor ordering if, whenever $G \in S$ and H is a minor of G, it holds $H \in S$.

P. Seymour and Roberston introduced treewith and branchwidth as parameters for an induction proof of Wagner's conjecture.

Wagner's Conjecture

Wagner conjectured in the 1930s (although this conjecture was not published until later) that in any infinite set of graphs, one graph is isomorphic to a minor of another. The truth of this conjecture implies that any family of graphs closed under the operation of taking minors (as planar graphs are) can automatically be characterized by finitely many forbidden minors analogously to Wagner's theorem characterizing the planar graphs.

Obstructions

An obstruction H to a family F of graphs, is a graph which does not belong to F, but every minor of H belongs ot F.
For example a triangle is an obstruction for the family of forests.

Theorem 1

For every class of graphs $\mathcal{G}$ closed under minors then there exist a set of finite obstructions $Ob(\mathcal{G})$ such that :
$G \in \mathcal{G}$ iff $\nexists H \in Ob(\mathcal{G})$ with $H \leq G$.

Example Kuratowski's theorem

G is planar iff G does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem 2

For every graph H, there exists an algorithm in $O(n^3)$ to test wether $H \leq G$ for a given graph G.

Remark

B. Reed claims $O(nlogn)$ 2007.

Topological Minors

Definition

H is a topological minor of G iff G has a subgraph H' isomorphic to a subdivision of H

Definition

H is a subdivision of G iff H can be obtained from G by a series of edge subdivisions.

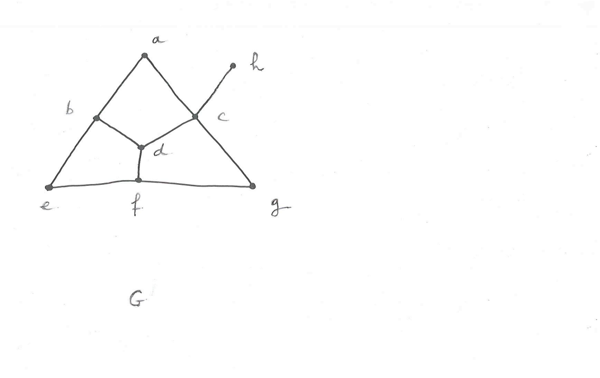
Other graph classes

- G is a forest iff G does not contain any subgraph homeomorphic to K_3 .
- G is series-parallel iff G does not contain any subgraph homeomorphic to K_4 and $K_{3,3}$.
- G is planar iff G does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$.
- G is an outerplanar graph iff G does not contain any subgraph homeomorphic to K_4 or $K_{2,3}$
- As a corollary if G does not contain any subgraph homeomorphic to K_4 or $K_{2,3}$ then G is hamiltonian.

Theorem 3

$H \leq G$ implies $treewidth(H) \leq treewidth(G)$
and for every k, graphs with treewidth bounded by k are well quasi-ordered for $\leq$.

No infinite antichain or no infinite strictly decreasing chain or finite number of obstructions.



Lower bound

If we contract the edges *ab*, *ef*, *gc* and delete edge *ch*, we obtain K_4 which is a minor of G
therefore $threewidth(G) \geq treewidth(K_4) = 3$

1. $\textit{treewidth}(G) = 2$ iff G has no K_4 as minor.
2. $\textit{treewidth}(G) = 3$ iff G has no K_5, Petersen, 8-wheel, XX as minor.
3. $\textit{treewidth}(G) = 4$ iff G has no $H \in \mathcal{H}$ as minor, and $\mathcal{H}$ contains 80 graphs.
4. ...

Theorem 4

Finite graphs are well ordered with $\leq$

Consequences

Any class of graphs closed under minors has a polynomial recognition algorithm

Another meta theorem

2006 Dawer, Kreutzer,

Every optimisation problem expressible in first order logic on class of graphs defined with minor exclusion, has a polynomial approximation algorithm.

How to prove such a theorem ?

Theorem

Gaifman 1981
Every FOL formula can be expressed using local formulas

Theorem

Roberston Seymour 1999
For any graph class defined using minor exclusion there exists a decomposition using graphs almost embeddable on some surface

Theorem

Grohe, Kawabashi 2008
This decomposition can be computed in $O(n^c)$*, and* c *does not depend on the size of the minor.*

Applications

- Change our knowledge about P versus NP
- Linear algorithms but with extremely big constants
- Improve our knowledge on NP-complete problems with Fixed parameter Tractablility (FPT) a theory proposed by M. Fellows ;
  Famous Courcelle's meta-theorem can be presented as an FPT result.
- Non constructive algorithms

Examples of non constructive algorithms

1. For any fixed k, it is polynomial to test wheter $genius(G) \leq k$ (we know there is a finite number of obstructions)
2. It is polynomial to decide if a graph has a non-crossing embedding in 3D
(non-crossing means no 2 cycles cross)

Grids are important for treewidth

Theorem Roberston Seymour 86

For every integer r, there exists an integer $k = f(r)$ such that for any graph G then $treewidth(G) \geq k$ implies G contains $G_{r,r}$ as minor.

Useful result

For non practical exact algorithms for disjoint path problems.

Computing exact treewidth

From H. Boedlander's survey :

Upper bounds :

Search for the fill-in ordering which minimises the size of the maximum clique, this leads to the search of the best simplicial elmination scheme

Therefore it is a problem of the construction of a particular total ordering of the vertices (sometimes called graph layout problem).

Heuristics : prefer vertices with small degrees, or else choose a vertex whose neighbourhood can be easily transformed into a clique (i.e. with a minimum number of edges missing).

One can also visit all total ordering using a Tabu search ....

Lower bounds : we use H minor of G implies
$treewidth(H) \leq treewidth(G)$
Using dynamic programming, up to graphs with 70 or 80 vertices.

Why a new width parameter ?

It must :

- Capture the structure of the graph
- Have small value for cliques
- Be computable easily
- Have a kind of Courcelle's theorem when the parameter is fixed.
- Have a duality, in order to give lower bounds

Other width parameters

- Branchwidth (Seymour and Roberston)
- Cliquewidth (Courcelle, Engelfriet and Rosenberg 1993)
- Rankwidth (Oum 2004)
- Treelength (Gavoille 2004) Max diameter of a bag, for network analysis.
  refer to next course.
- ...
- Still to be invented Directed treewidth

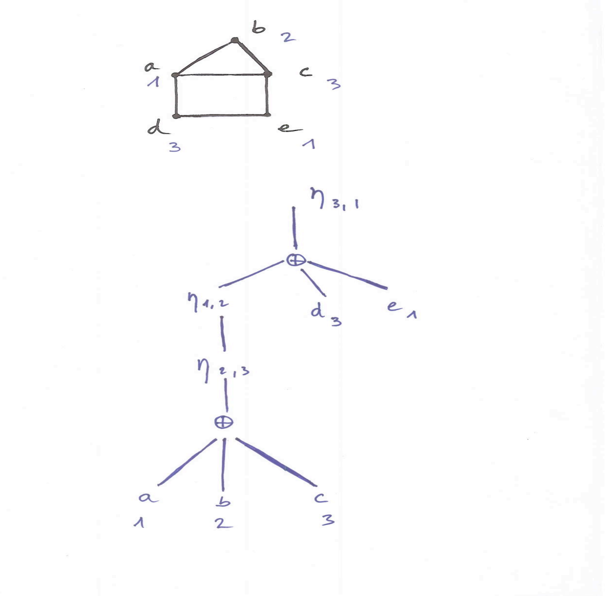
Cliquewidth

Introduced by Courcelle, Engelfriet, Rosenberg 1993 from graph rewriting techniques.
A k-expression is an algebraic expression on vertex labelled graphs with k labels : $1, 2, \ldots, k$

- $\cdot_i$ a single vertex with label i
- $G_1 \bigoplus G_2$ disjoint union of two graphs.
- $\rho(i, j)$ relabel vertices of label i into j.
- $\eta_{i,j}$ with $i \neq j$ add all edges between vertices of label i and j.

Definition

Clique-width of G is the min k such that G can be defined with a k-expression.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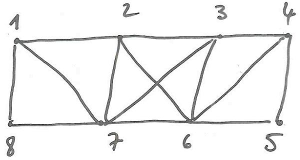
G



$$G = \eta_{1,2}\,(f \oplus h)$$

$$f = \rho_{1\to 2}(\eta_{1,3}(\eta_{1,2}(1(7) \oplus \rho_{1\to 3}(\eta_{2,3}(\eta_{1,3}(3(1) \oplus 1(8) \oplus 2(2)))))))$$

$$h = \rho_{2\to 1}(\eta_{1,2}(\eta_{2,3}(2(6) \oplus \rho_{2\to 3}(\eta_{2,3}(\eta_{1,3}(3(4) \oplus 1(3) \oplus 2(5)))))))$$

First remarks

1. $cliquewidth(K_n) = 2$ but $Treewidth(K_n) = n - 1$. This explains the name cliquewidth !
2. $cliquewidth(G_{n,n}) = n + 1$, if $n \geq 3$ (Golumbic, Rotics 2000)
3. The k-expression that defines a graph is not unique, the operation η introduces a relationship with split decomposition.
4. $cliquewidth(G) \leq 2$ iff G is a cograph.
5. It requires $O(n^2 m)$ to find a 3-expression if G admits one Corneil, Habib, Paul, Lanlignel, Reed and Rotics 2000.

- Computing cliquewidth is NP-hard 2007 (seems to be a very technical result)
  Fellows et al
- It is still an open problem to verify in polynomial time that a given graph has cliquewith less than k, for fixed k.
- Courcelle's Theorem
  Every graph problem that can be expressed with a formula in monadic second order logic with quantifiers on vertices (not on edges) can be solved in polynomial time on graphs having bounded cliquewidth.

1. Corneil, Rotics 2005
   $cliquewidth(G) \leq 3(2^{treewidth(G)-1})$
2. Therefore if the treewidth is bounded so is the cliquewidth
3. The converse is false, as shown by the complete graph K_n with cliquewidth 2 and treewidth $n-1$.