1 Chordal graphs

In this problem we consider a connected chordal graph $G$ (with $n$ vertices and $m$ edges).

1. Show that every chordal graph can be represented as the intersection graph of a family of subtrees in a host tree with maximum degree 3.

2. Let $S$ be a minimal separator in $G$, show that it always exist two different maximal cliques $C, C'$ of $G$ such that:

   $C \cap C' = S$ and $\forall a \in C - C', \forall b \in C' - C$, $S$ is a minimal separator for $a$ and $b$.

3. * Show that every minimal separator $S$ belongs to all maximal clique tree of $G$ (i.e. every maximal clique tree $T$ contains an edge labelled with $S$).

4. Show that each maximal clique tree uses exactly the same collection of minimal separators, including repetitions (i.e. if a separator $S$ labels $\alpha$ edges of some maximal clique tree, then every maximal clique tree contains $\alpha$ edges labelled with $S$).

   Give examples of families of graphs for which the number of repetitions of a given minimal separator is unbounded in any maximal clique tree.

5. If we consider the edges of the clique tree labelled with the size of the minimal separators, show that: for every maximal clique tree $T$

   \[
   \text{weight}(T) = \sum_{1 \leq i \leq k} |C_i| - n, \quad \text{where } C_1, \ldots, C_k \text{ are the maximal cliques of } G.
   \]

6. Prove that a graph $G$ is a forest iff every pairwise intersecting family of paths in $G$ has a common vertex (i.e. family of paths satisfy Helly’s property).

1.1 Distance properties on chordal graphs

1. A convex structure consists of a set $X$ together with a collection $\mathcal{C}$ of subsets of $X$ (the convex sets) such that: the empty set and $X$ are convex and the intersection of convex sets is convex.

   For a graph $G = (V,E)$ the usual convexity is defined as follows:

   A subset $S \subseteq V$ is convex if for every two vertices $x, y \in S$, all vertices on shortest paths between $x$ and $y$ are also contained in $S$.

   Show that this correctly defines a convexity.
2. Show that we can obtain a new notion of convexity called chordless-convexity if we replace in the above definition "shortest path" by "chordless path". Are these two definitions of convexity equivalent?

3. Show that $G = (V, E)$ is chordal if $\forall v \in V \ N[v] = N(v) \cup \{v\}$ is chordless convex.

4. * An extreme point of a convex set $S$ is a point $x \in S$ such that $S - x$ is still convex. The convex hull of a given subset $A \subseteq X$ is the smallest convex set containing $A$.
Show that for chordal graphs chordless convexity defines a convex geometry (i.e. every convex set is the convex hull of its extreme points).

5. For a graph $G$, connected but not necessarily chordal, let $C(G)$ be the induced subgraph of $G$ made up with its centers.
Show that this graph is not always connected.

6. * For a chordal graph $G$ show that:
$C(G)$ is connected and $diam(C(G)) \leq 3$

7. ** Propose an efficient algorithm to compute a center (resp. all the centers) of a chordal graph.

2 Algorithms

1. Propose a linear time algorithm to recognize co-chordal graphs (i.e. complement of chordal graphs).

2. Maximal cardinality Search (MCS)
   
   **Data**: a graph $G = (V, E)$ and a start vertex $s$
   
   **Result**: an ordering $\sigma$ of $V$
   
   Assign the label 0 to all vertices
   
   $label(s) \leftarrow 1$
   
   for $i \leftarrow 1 \text{ to } n-1$
   
   Pick an unnumbered vertex $v$ with largest label
   
   $\sigma(i) \leftarrow v$
   
   foreach unnumbered vertex $w$ adjacent to $v$ do
   
   $label(w) \leftarrow label(w) + 1$
   
   end
   
   end

   Propose a linear time implementation of MCS algorithm.

2. Is it possible to modify (by adding a test) MCS Algorithm in order to check if the graph is chordal (without testing that $\sigma$ provides a simplicial elimination ordering)?

3 Treewidth, branchwidth

We have seen during the course when proving $\text{treewidth}(G) \leq 3/2 \text{branchwidth}(G)$ : that a decomposition tree for treewidth can be transformed in a branchwidth like tree.

Does there exist a way to express treewidth in terms of a cost function on a ternary tree? Detail the cost function if any.