Quantum Time-Space Tradeoffs by Recording Queries

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Abstract

We use the recording queries technique of Zhandry [Zha19] to prove lower bounds in the exponentially small success probability regime, with applications to time-space tradeoffs. We first extend the recording technique to the case of non-uniform input distributions and we describe a new simple framework for using it. Then, as an application, we prove strong direct product theorems for $K$-Search under a natural product distribution not considered in previous works, and for finding $K$ disjoint collisions in a uniform random function. Finally, we use the latter result to obtain the first quantum time-space tradeoff that is not based on a reduction to $K$-Search. Namely, we demonstrate that any $T$-query algorithm using $S$ qubits of memory must satisfy a tradeoff of $T^3 S \geq \Omega(N^4)$ for finding $\Theta(N)$ collisions in a random function. We conjecture that this result can be improved to $T^2 S \geq \Omega(N^3)$, and we show that it would imply a $T^2 S \geq \tilde{\Omega}(N^2)$ tradeoff for Element Distinctness.

1 Introduction

Quantum query complexity [BW02] is a cornerstone of the theory of quantum algorithms. It measures the number of times an algorithm needs to access (or “query”) its input in order to solve a given problem. Although it does not characterize other resources of computation that can be more prohibitive (number of quantum gates, number of qubits, communication delay, etc.), it is often a good measure of how difficult a problem is to solve. Some of the great achievements of quantum query complexity are Grover [Gro97] and Shor [Sho97] algorithms, which can solve Search and Factoring by using less queries than any known classical algorithm.

One important question is how low can the quantum query complexity of a given problem be? The two main techniques for proving lower bounds in the quantum query model are the polynomial [BBC+01] and the adversary [Amb02] methods. Although they have led to great successes, these methods often need arguments that are much less intuitive than what is used for deterministic or randomized query complexity. Thus, it is a major challenge to discover alternative ways of analyzing the quantum query complexity. Recently, Zhandry [Zha19] took a step in this direction by giving a new proof technique based on the “recording” of quantum queries. The original motivation for Zhandry’s work was to find new security proofs in the quantum random oracle model. He tried to adapt a classical method where the queries made by an attacker are recorded and answered with on-the-fly simulation of the oracle. At first sight, such a recording seems impossible to achieve in the quantum model since measuring queries for instance could perturbate the attacker’s state significantly. Yet, Zhandry showed that under certain conditions one can record in superposition the queries made by a quantum attacker. This method has since been used to give new security proofs in the quantum random oracle model [AMRS18, HI19, LZ19b, CMSZ19, BHH+19, Cha19, CMS19]. More important to us is that it provides a new method for proving quantum query complexity lower bounds. Zhandry
has illustrated its use for the Grover Search, Collision and \( k \)-Sum problems \[Zha19\]. Later, Liu and Zhandry \[LZ19a\] have applied it to Multi-Collisions Finding. The purpose of the present work is to generalize this method further, and to give new applications to strong direct product theorems and time-space tradeoffs. We present below our contributions.

Recording Query Model. We first present the main ideas in Zhandry’s recording technique \[Zha19\]. In the quantum query model, an algorithm is given access to an \( M \)-size input \( x \in |\Sigma|^M \) through a unitary operator \( O_x[i,p] = e^{2\pi i x p i} |i,p\rangle \), where \( \Sigma \) is the input alphabet size and \( \omega_\Sigma = e^{\frac{2\pi i}{M}} \) is a \( \Sigma \)-th root of unity. The state \( |\psi_t^x\rangle \) of an algorithm after \( t \) queries to \( x \) corresponds to the result of alternating \( O_x \) with some fixed units \( U_0, \ldots, U_t \), that is \( |\psi_t^x\rangle = U_t O_x U_{t-1} \cdots U_1 O_x U_0 |0\rangle \). This framework allows an adversary oracle to run an algorithm over a distribution \( D \) of inputs \( x \) to obtain the state \( \sum_x \sqrt{\Pr[x \leftarrow D]} |\psi_t^x\rangle |x\rangle \). Zhandry’s main idea is to restrict itself to the uniform distribution \( |\Sigma\rangle \) and to look at the evolution of the register \( |x\rangle \) in the Fourier domain. The initial state \( \sum_x \sqrt{\frac{1}{|\Sigma|^M}} |0\rangle |x\rangle \) corresponds to \( |0\rangle |0^M\rangle \) in the Fourier domain. The first unitary operator \( U_0 \) maps this state to some superposition \( \sum_{i,p,w} |i,p,w\rangle |0^M\rangle \). The crucial observation is that, in the Fourier domain, the query \( |i,p\rangle \) maps the \( i \)-th register of \( |0^M\rangle \) to \( |p\rangle \), letting the other registers unchanged. In other words, in the Fourier domain, the adversary oracle has recorded into her own register that a query was made on the \( i \)-th value. Similarly, each subsequent query on a state \( |i,p\rangle \) will add (modulo \( \Sigma \)) the phase multiplier \( p \) to the \( i \)-th value in the Fourier domain.

What if we try to record queries directly into the standard domain, without applying the Fourier transformation? We may first replace the initial state \( \sum_x \sqrt{\Pr[x \leftarrow U]} |0\rangle |x\rangle \) with \( \sum_x \sqrt{\Pr[x \leftarrow U]} |0\rangle |\bot^M\rangle \), where \( \bot \) is a new symbol representing the absence of record on the corresponding coordinate. Then, we may expect a query on the \( i \)-th coordinate to change the \( i \)-th oracle’s register by \( |\bot\rangle \mapsto \frac{1}{\sqrt{\Sigma}} \sum_y \omega_\Sigma^{yp} |y\rangle \) and \( |y\rangle \mapsto \omega_\Sigma^{yp} |y\rangle \) if \( y \neq \bot \). This would correspond to the classical recording technique, where a value is sampled only upon request. This transformation is obviously not reversible. Nevertheless, by going back and forth in the Fourier domain, Zhandry’s method \[Zha19, HI19\] shows that the algorithm’s state obtained with this operator differs only negligibly from the actual state after polynomially many queries.

In Section 2.2, we improve Zhandry’s recording technique by generalizing it to any product distribution \( D = D_1 \otimes \cdots \otimes D_M \) on the input. We formalize a general framework for recording quantum queries by the use of a “recording query operator” \( R \) that depends on the distribution \( D \) (Lemma 2.2). Our proof simplifies some of the arguments used in the original work of Zhandry. In particular, we do not go back and forth in the Fourier domain for our analysis. As an application, we study the recording operator \( R \) corresponding to the product of \( M \) Bernoulli distributions of parameter \( q \). In Lemma 3.2, we show that \( R \) is close to the mapping \( |\bot\rangle \mapsto |\bot\rangle - \sqrt{q} |1\rangle \), \( |0\rangle \mapsto |0\rangle + \sqrt{q} |1\rangle \), \( |1\rangle \mapsto -|1\rangle + \sqrt{q} (|0\rangle - |\bot\rangle) \) up to lower order terms of amplitude \( O(q) \) when \( q \ll 1 - q \). We also recover the analysis for the uniform distribution in Lemma 4.2. There has been other independent work on extending Zhandry’s recording technique \[HI19, CMSZ19, CMS19\]. These papers focus more on security properties of cryptographic schemes.

Strong Direct Product Theorems. A Strong Direct Product Theorem (SDPT) states that the success probability of solving \( K \) instances of a problem with less than \( K \) times the resources needed for one instance is exponentially small in \( K \). A series of work \[Amb10, AŠW09, Špa08, AMRR11, LR13\] based on the (multiplicative) adversary method has culminated into a proof that quantum query complexity satisfies a SDPT for all functions. Our focus in this paper (Sections 3 and 4.1) is on two specific problems that exhibit similar behaviors to that of SDPTs. These results will serve in the proof of a time-space tradeoff in Section 4.2.

\( K \)-Search. Our first SDPT-like result (Section 3) is for the \( K \)-Search problem, where the goal is to find \( K \) ones in an \( N \)-size vector. It has been shown many times \[KŠW07, Amb10, Špa08, Zha19\]
that the success probability must drop to $\sigma < 2^{-\Omega(K)}$ when the number of quantum queries is $T < O(\sqrt{KN})$. Yet, these results show disparities in the input distributions they consider, and in the decrease rate of $\sigma$ in term of $T$. For instance, Ambainis [Amb10] proved that $\sigma \leq O(K/N)^{K/2} + O(T^2/(KN))^{K/8}$ over the uniform distribution on the $N$-bit vectors with exactly $K$ ones. On the other hand, Zhandry’s original technique [Zha19] can be adapted to give $\sigma \leq O(K/N) + O(T^2/(KN))^K$ for a uniformly random input in $[N]^{KN}$. In this paper, we study the K-Search problem on an input distribution that has not been considered before, namely the product distribution on $x \in \{0, 1\}^M$ that sets $x_i = 1$ with probability $K/N$ for all $i$ independently. Our proof is the first one to illustrate the use of the recording query model on a non-uniform distribution. It is also simpler and more intuitive than previous bounds for K-Search to our opinion. We show that, similarly to the classical setting where a query can reveal a one with probability $K/N$, the amplitude on the basis states that record a new one increases by a factor of $\sqrt{K/N}$ after each query (Proposition 3.4). Thus, the amplitude of the basis states that have recorded at least $K/2$ ones after $T$ queries is at most $O(T/\sqrt{KN})^{K/2}$. This implies that any algorithm with $T < O(\sqrt{KN})$ queries must likely output at least $K/2$ ones at positions that have not been recorded. These outputs can only be correct with probability $K/N$, thus the overall success probability is at most $O(K/N)^{K/2}$ (Proposition 3.5).

**Theorem 3.1.** Let $D$ be the distribution on $x \in \{0, 1\}^M$ that is defined by setting $x_i = 1$ with probability $K/N$ independently for each $i \in [M]$. Then, any $T$-query quantum algorithm for the K-Search problem on $D$ must succeed with probability $\sigma$ satisfying $\sigma \leq 2(9K/N)^{K/2} + 2((2T)^2/(KN))^{K/2}$.

**Many-Collisions.** Our second SDPT-like result (Section 4.1) is for the problem of finding $K$ disjoint collisions in an input $x \in [N]^M$ drawn from the uniform distribution $U_M^N$. Two collisions $x_i = x_j$ and $x_{i'} = x_{j'}$ are said to be disjoint if the indices $i, j, i', j'$ are all different. This problem has been extensively studied in the case $K = 1$ where it is known [AS04, Zha15] that $\Omega(N^{1/3})$ queries are required to succeed with constant probability. For arbitrary $K$, we are only aware of a previous result by Liu and Zhandry [LZ19a] analyzing a progress measure in the recording query model that hints at a success probability of $\sigma < O(T^3/N)^K$. We build on this result to give a precise statement on $\sigma$. As in the K-Search analysis, we show that the amplitude on the basis states that record a new disjoint collision increases by a factor of $\sqrt{t/N}$ after each query (Proposition 4.3). This is related to the probability that a new random value collides with one of the (at most) $t$ previous recorded queries. Thus, the amplitude of the basis states that have recorded at least $K/2$ disjoint collisions after $T$ queries is at most $O(T^{3/2}/(K\sqrt{N}))^{K/2}$. Consequently, any algorithm making $T < O(K^{2/3}N^{1/3})$ queries must likely output at least $K/2$ collisions at positions that have not been recorded. These outputs will be correct with exponentially small probability (Proposition 4.4).

**Theorem 4.1.** Any $T$-query quantum algorithm finding $K$ disjoint collisions on inputs drawn from $U_M^N$ must succeed with probability $\sigma$ satisfying $\sigma \leq 8K^2(K/N)^{K/2} + 2((8T)^3/(K^2N))^{K/2}$.

In particular, any algorithm making $T < O(K^{2/3}N^{1/3})$ queries can only succeed with probability $\sigma < 2^{-\Omega(K)}$ for the problem of finding $K$ disjoint collisions in a random input. This bound is tight as can be shown by a simple adaptation of the BHT algorithm [BHT98].

**Time-Space Tradeoffs.** Memory is a critical resource in many algorithmic methods. The Noisy Intermediate-Scale Quantum (NISQ) era illustrates further the interest of algorithms using as few qubits as possible. Time-Space tradeoffs investigate how large the time (or query) complexity $T$ must be when only $S$ (qu)bits of memory are available. For instance, the task of sorting $N$ numbers requires $TS \geq \Omega(N^2)$ on a classical computer [Bea91] and $T^2S \geq \Omega(N^3)$ on a quantum one [KŠW07]. The few other time-space tradeoffs known in the quantum setting are for
to T and Proposition 4.7 how to find to bypass the short-output limitation of Element Distinctness. We explain in Algorithm 1 [BSSV03], but no result is known in the quantum setting. In Section 4.2 we give a new argument comparison-based query model [Yao94]. There is also partial progress in the unrestricted case.

ED solving its output with probability larger than our proof relies on showing that no algorithm can produce more than \( \sqrt{2N} \) collisions we obtain the result described in Table 1.

**Corollary 4.6.** Any quantum algorithm finding \( \Theta(N) \) disjoint collisions with success probability 2/3 on inputs drawn from \( \mathcal{U}_N^{\sqrt{N}/100} \) must satisfy a time-space tradeoff of \( T^3S \geq \Omega(N^3) \).

In particular, this result implies that finding \( \Theta(N) \) disjoint collisions with a quantum algorithm that uses \( S = O(\log N) \) qubits of memory requires \( T \geq \tilde{\Omega}(N^{4/3}) \) queries, whereas \( T = N \) queries are clearly sufficient when there is no space restriction. On the upper bound side, the best known tradeoff is \( T^2S \leq O(N^3) \), which can be achieved with Quantum Sorting [Kla03] or with a variant of the BHT algorithm [BHT98]. In the classical setting, a tradeoff of \( T^2S \geq \Omega(N^3) \) has been shown for random 2-to-1 functions [CC17] and for the uniform distribution [Din20]. This matches the complexity of the Parallel Collision Search algorithm [OW99]. More generally, for finding \( K \) collisions we obtain the result described in Table 1.

As is the case for most of the previous tradeoffs for large-output problems [BFK\(^+\)81, KŠW07], our proof relies on showing that no algorithm can produce more than \( k \) parts (here: collisions) of its output with probability larger than \( 2^{-O(k)} \) when \( T \) is small. It is a major open problem, both in the classical and quantum setting, to obtain tight methods that would also apply to short-output problems. The most studied candidate in this direction is the Element Distinctness problem ED\(_N\), that consists in deciding whether \( N \) numbers are all distinct or not. In the classical setting, a nearly tight tradeoff of \( TS \geq \Omega(N^{2-\epsilon}) \) has been established for the restricted comparison-based query model [Yao94]. There is also partial progress in the unrestricted case [BSSV03], but no result is known in the quantum setting. In Section 4.2 we give a new argument to bypass the short-output limitation of Element Distinctness. We explain in Algorithm 1 and Proposition 4.7 how to find \( \Theta(N) \) collisions in time \( O(NT_{\sqrt{N}}) \) and space \( O(S_{\sqrt{N}}) \), where \( (T_{\sqrt{N}}, S_{\sqrt{N}}) \) is the complexity of any algorithm solving Element Distinctness on inputs of size \( \sqrt{N} \). This reduction allows us to convert any time-space tradeoff for finding \( \Theta(N) \) disjoint collisions into one for Element Distinctness.

**Corollary 4.8.** If any algorithm finding \( \tilde{\Theta}(N) \) disjoint collisions on inputs drawn from \( \mathcal{U}_N^{\sqrt{N}/100} \) satisfies a time-space tradeoff of \( T^\alpha S^\beta \geq \tilde{\Omega}(N^\gamma) \) for some constants \( \alpha, \beta, \gamma \) then any algorithm solving ED\(_N\) satisfies a time-space tradeoff of \( T^\alpha S^\beta \geq \tilde{\Omega}(N^{2(\gamma-\alpha)}) \).

We further conjecture that our current tradeoff for finding \( \Theta(N) \) collisions can be improved to \( T^2S \geq \Omega(N^3) \), which would imply \( T^2S \geq \tilde{\Omega}(N^2) \) for Element Distinctness (Corollary 4.9).

\(^1\)The notation \( \tilde{\cdot} \) is used to denote the presence of hidden polynomial factors in \( \log(N) \) or \( 1/\log(N) \).

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<table>
<thead>
<tr>
<th>Classical complexity</th>
<th>Quantum complexity</th>
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<tbody>
<tr>
<td>Upper bound: ( T^2S \leq \tilde{O}(K^2N) ) when ( \tilde{\Omega}(\log N) \leq S \leq \tilde{O}(K) )</td>
<td>Upper bound: ( T^2S \leq \tilde{O}(K^2N) ) when ( \tilde{\Omega}(\log N) \leq S \leq \tilde{O}(K^{2/3}N^{1/3}) )</td>
</tr>
<tr>
<td>Parallel Collision Search [OW99]</td>
<td>Variant of the BHT algorithm [BHT98]</td>
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<tr>
<td>Lower bound: ( T^2S \geq \Omega(K^2N) )</td>
<td>Lower bound: ( T^3S \geq \Omega(K^3N) )</td>
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<tr>
<td>[Din20]</td>
<td>Theorem 4.5</td>
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Table 1: Complexity of finding \( K \) disjoint collisions on inputs drawn from \( \mathcal{U}_N^{\sqrt{N}/100} \).
2 Quantum Query Model

We first present the standard model of quantum query complexity (Section 2.1). Next, we define the recording model used in this paper (Section 2.2).

2.1 Standard Model

The (standard) model of quantum query complexity [BW02] measures the number of quantum queries an algorithm needs to make on an input \( x \in [\Sigma]^M \) to find an output \( y \) satisfying some predetermined relation \( R(x,y) \). We present this model in more details below.

**Quantum Query Algorithm.** A \( T \)-query quantum algorithm is specified by a sequence \( U_0, \ldots, U_T \) of unitary transformations acting on the algorithm’s memory. The state \( |\psi\rangle \) of the algorithm is made of three registers \( Q, P, W \) where the **query register** \( Q \) holds \( i \in [M] \), the **phase register** \( P \) holds \( p \in [\Sigma] \) and the **working register** \( W \) holds some value \( w \). We represent a basis state in the corresponding Hilbert space as \( |i,p,w\rangle_{QPW} \). We may drop the subscript \( QPW \) when it is clear from the context. The state \( |\psi_T\rangle \) of the algorithm after \( t \leq T \) queries to some input \( x \in [\Sigma]^M \) is

\[
|\psi_T^x\rangle = U_tO_xU_{t-1} \cdots U_1O_xU_0|0\rangle
\]

where the oracle \( O_x \) acts only on register \( |i,p\rangle \) and is defined by

\[
O_x|i,p\rangle = \omega_{x}^{p}\delta_{i,p} \quad \text{and} \quad \omega_{x} = e^{\frac{2i\pi}{|\Sigma|}}.
\]

The **output** of the algorithm is written into a substring \( y \) of the working register \( w \). The **success probability** \( \sigma_x \) of the quantum algorithm on \( x \) is the probability that the output value \( y \) obtained by measuring the working register of \( |\psi_T^x\rangle \) satisfies the relation \( R(x,y) \). In other words, if we let \( \Pi_x^{\text{success}} \) be the projector whose support consists of all basis states \( |i,p,w\rangle \) such that the output substring \( y \) of \( w \) satisfies \( R(x,y) \), then \( \sigma_x = ||\Pi_x^{\text{success}}|\psi_T^x\rangle||^2 \).

**Oracle’s Register.** Here, we describe the variant used in the adversary method [Amb02]. It is represented as an interaction between an **algorithm** that aims at finding \( y \), and a superposition of oracle’s inputs that respond to the queries from the algorithm.

The memory of the oracle is made of an **input register** \( X \) holding \( x \in [\Sigma]^M \). This register can be subdivided into \( M \) registers \( X_1, \ldots, X_M \) where \( X_i \) holds \( x_i \in [\Sigma] \). The basis states in the corresponding Hilbert space are \( |x\rangle_X = \otimes_{i \in [M]} |x_i\rangle_{X_i} \). Given an input distribution \( D \) on \( [\Sigma]^M \), the **oracle’s initial state** is the state \( |\text{init}\rangle_X = \sum_{x \in [\Sigma]^M} \sqrt{\text{Pr}[x \leftarrow D]} |x\rangle \).

The **query operator** \( O \) is a unitary transformation acting on the memory of the algorithm and the oracle. Its action is defined on each basis state by

\[
O|i,p,w\rangle|x\rangle = (O_x|i,p,w\rangle)|x\rangle.
\]

The joint state \( |\psi_t\rangle \) of the algorithm and the oracle after \( t \) queries is \( |\psi_t\rangle = U_tO_U_{t-1} \cdots U_1 O_U_0(|0\rangle|\text{init}\rangle) = \sum_{x \in [\Sigma]^M} \sqrt{\text{Pr}[x \leftarrow D]} |\psi_T^x\rangle|x\rangle \), where the unitaries \( U_i \) have been extended to act as the identity on \( X \). The **success probability** \( \sigma \) of a quantum algorithm on an input distribution \( D \) is the probability that the output value \( y \) and the input \( x \) obtained by measuring the working and input registers of \( |\psi_T\rangle \) satisfy the relation \( R(x,y) \).

2.2 Recording Model

The recording quantum query model is a modification of the standard model that is unnoticeable by the algorithm, but that will allow us to track more easily the progress made toward solving the relational problem \( R \). The original recording quantum query model was formulated by Zhandry.
Here, we propose a simplified and more general version of this framework adapted to the goal of proving query complexity lower bounds. Our framework requires the initial state $|{\text{init}}\rangle_X$ to be a product state $\otimes_{i \in [M]} |{\text{init}}_i\rangle_{X_i}$.

**Construction.** The input register $X$ of the oracle can now contain $x \in \{\bot, 0, \ldots, \Sigma - 1\}^M$, where $x_i = \bot$ will represent the absence of knowledge from the algorithm about the $i$-th coordinate of the input. Unlike in the above model, the oracle’s initial state is independent from the relational problem $R$ to be solved and is fixed to be $|\bot^M\rangle_X$ (which represents the fact that the algorithm knows nothing about the input initially). We extend the query operator $O$ defined in the standard query model by setting

$$O|i, p, w\rangle|x\rangle = |i, p, w\rangle|x\rangle \quad \text{when } x_i = \bot.$$

Then, we define our new “recording” query operator as follows.

**Definition 2.1.** Let $S_1, \ldots, S_M$ acting on $X_1, \ldots, X_M$ respectively and satisfying $S_i|\bot\rangle_{X_i} = |{\text{init}}_i\rangle_{X_i}$. Then the recording query operator $R$ with respect to $(S_i, |{\text{init}}_i\rangle_{X_i})_{i \in [M]}$ is the operator $R$ obtained by composing $O$ with the unitary $S$ defined as follows

$$S = \sum_{i \in [M]} |i\rangle\langle i|_Q \otimes S_i \quad \text{and} \quad R = S^\dagger \cdot O \cdot S$$

where $S$ act as the identity on the unspecified registers.

**Indistinguishability.** The joint state of the algorithm and the oracle after $t$ queries in the recording query model is $|\phi_t\rangle = U_tRU_{t-1} \cdots U_1RU_0(|0\rangle|\bot^M\rangle)$. Notice that the query operator $R$ can only change the $i$-th coordinate of $x$ when it is applied to $|i, p, w\rangle|x\rangle$. Consequently, $|\phi_t\rangle$ must be a linear combination of basis states $|i, p, w\rangle|x\rangle$ with at most $t$ entries in $x$ different from $\bot$. These entries represent what the oracle has learnt (or “recorded”) from the algorithm’s queries so far. In the next lemma, we show that $|\phi_t\rangle$ is related to the state $|\psi_t\rangle$ (defined in Section 2.1) by $|\psi_t\rangle = (\otimes_{i \in [M]} S_i)|\phi_t\rangle$. The change of query operator is unnoticeable by the algorithm since each $S_i$ acts as the identity on the algorithm’s memory.

**Lemma 2.2.** Let $(U_0, \ldots, U_T)$ be a $T$-query quantum algorithm. Consider an oracle’s initial state $|{\text{init}}\rangle_X = \otimes_{i \in [M]} |{\text{init}}_i\rangle_{X_i}$ in the standard query model and choose the $M$ unitaries $S_1, \ldots, S_M$ in the recording query model to be any transformations satisfying

$$S_i|\bot\rangle_{X_i} = |{\text{init}}_i\rangle_{X_i} \quad \text{for all } i \in [M].$$

Then, the states

$$\begin{cases} |\psi_t\rangle = U_tOU_{t-1} \cdots U_1OU_0(|0\rangle|{\text{init}}\rangle) \\ |\phi_t\rangle = U_tRU_{t-1} \cdots U_1RU_0(|0\rangle|\bot^M\rangle) \end{cases}$$

obtained after $t$ queries in the standard and recording query models respectively are related by

$$|\psi_t\rangle = \bar{S}|\phi_t\rangle \quad \text{where } \bar{S} = \otimes_{i \in [M]} S_i.$$

**Proof.** We start by introducing the intermediate operator $\tilde{R} = \bar{S}^\dagger \cdot O \cdot \bar{S}$. We first claim that $\tilde{R} = R$. Indeed, the query operator $O$ acts as the identity on all the registers of a basis state $|i, p, w\rangle|x\rangle$, except $|i\rangle\langle p|\rangle|x_i\rangle$. Thus, the actions of $S_j$ and $S_j^\dagger$ for $j \neq i$ cancel out in $\tilde{R}$ and $R$. Since $\bar{S}$ and $S$ act the same way on registers $Q,P,X_i$, we obtain that $\tilde{R}|i, p, w\rangle|x\rangle = R|i, p, w\rangle|x\rangle$. 


We further observe that $U_j$ and $\bar{S}$ commute for all $j$ since they act as non-identities on disjoint registers. Consequently, we have that

$$|\psi_t\rangle = U_tOU_{t-1}O\cdots U_1OU_0 \cdot \bar{S}(0)|\perp^{M}$$

$$= \bar{S}S\bar{U}_tO\cdot \bar{S}\bar{U}_{t-1}O\cdots \bar{S}\bar{U}_1O \cdot \bar{S}S\bar{U}_0 \cdot \bar{S}(0)|\perp^{M}$$

since $\bar{S}S^\dagger = I$

$$= \bar{S}U_t\bar{S}\bar{U}_{t-1}O\cdots \bar{S}U_1\bar{S}\bar{U}_0(0)|\perp^{M}$$

by definition of $\bar{R}$

$$= \bar{S}U_tRU_{t-1}\cdots U_1RU_0(0)|\perp^{M}$$

since $\bar{R} = R$

$$= \bar{S}|\phi_t\rangle$$

Lemma 3.2.

Theorem 3.1. Let $D$ be the distribution on $x \in \{0,1\}^M$ that is defined by setting $x_i = 1$ with probability $K/N$ independently for each $i \in [M]$. Then, any $T$-query quantum algorithm for the $K$-Search problem on $D$ must succeed with probability $\sigma$ satisfying

$$\sigma \leq 2 \left( \frac{9K}{N} \right)^{K/2} + 2 \left( \frac{(22T)^2}{KN} \right)^{K/2}.$$

We define in Section 3.1 the unitary operators $S_1, \cdots, S_M$ associated with the input distribution $D$, and we describe how the recording query operator $R$ acts on the basis states (Lemma 3.2). In Section 3.2, we study the measure of progress $q_{t,k}$ corresponding to the probability that the oracle’s memory contains at least $k$ ones after $t$ queries in the recording query model. We prove that this quantity is exponentially small in $k$ when $t \leq O(k\sqrt{N/K})$ (Proposition 3.4). Finally, in Section 3.3, we relate the progress measure to the success probability $\sigma$ of solving $K$-Search on the input distribution $D$ in the standard query model (Proposition 3.5). We conclude that $\sigma$ must be exponentially small in $K$ after $T \leq O(\sqrt{KN})$ queries (Theorem 3.1).

3 Application to Quantum Search

We use the recording query model to prove a strong direct product theorem for quantum search. The relational problem that we consider here is the $K$-Search problem: given $x \in \{0,1\}^M$, find $K$ distinct indices $i_1, \ldots , i_K \in [M]$ such that $x_{i_1} = 1$. We prove the following result.

Theorem 3.1. Let $D$ be the distribution on $x \in \{0,1\}^M$ that is defined by setting $x_i = 1$ with probability $K/N$ independently for each $i \in [M]$. Then, any $T$-query quantum algorithm for the $K$-Search problem on $D$ must succeed with probability $\sigma$ satisfying

$$\sigma \leq 2 \left( \frac{9K}{N} \right)^{K/2} + 2 \left( \frac{(22T)^2}{KN} \right)^{K/2}.$$

We define in Section 3.1 the unitary operators $S_1, \cdots, S_M$ associated with the input distribution $D$, and we describe how the recording query operator $R$ acts on the basis states (Lemma 3.2). In Section 3.2, we study the measure of progress $q_{t,k}$ corresponding to the probability that the oracle’s memory contains at least $k$ ones after $t$ queries in the recording query model. We prove that this quantity is exponentially small in $k$ when $t \leq O(k\sqrt{N/K})$ (Proposition 3.4). Finally, in Section 3.3, we relate the progress measure to the success probability $\sigma$ of solving $K$-Search on the input distribution $D$ in the standard query model (Proposition 3.5). We conclude that $\sigma$ must be exponentially small in $K$ after $T \leq O(\sqrt{KN})$ queries (Theorem 3.1).

3.1 The Recording Query Operator

In the standard query model, the oracle’s initial state corresponding to the chosen input distribution $D$ is $|\text{init}\rangle = \otimes_{i \in [M]} (\sqrt{1-K/N}|0\rangle x_i + \sqrt{K/N}|1\rangle x_i)$. Consequently, in the recording query model, we define the unitary transformations $S_1, \cdots , S_M$ to be

$$S_i|\perp\rangle x_i = |+\rangle x_i, \quad S_i|+\rangle x_i = |\perp\rangle x_i, \quad S_i|\rangle x_i = |\rangle x_i$$

where $\alpha = \sqrt{1-K/N}, \beta = \sqrt{K/N}$ and $|+\rangle x_i = \alpha|0\rangle x_i + \beta|1\rangle x_i, |\perp\rangle x_i = \beta|0\rangle x_i - \alpha|1\rangle x_i$. These unitaries verify $S_i|\perp^{M}\rangle = |\text{init}\rangle$ where $\bar{S} = \otimes_{i \in [M]} S_i$, as required by Lemma 2.2. The recording query operator is $R = S \cdot O \cdot S$ since $S^\dagger = S$. The next lemma shows how $R$ is acting on the basis states. The proof is an elementary calculation that is given in Appendix A.

Lemma 3.2. If the recording query operator $R$ is applied on a basis state $|i,p,w\rangle x$ where $p = 1$ then the register $|x_i\rangle x_i$ is mapped to

$$\begin{cases}
   (\alpha^2 - \beta^2)|\perp\rangle + 2\alpha\beta^2|0\rangle - 2\alpha^2\beta^2|1\rangle & \text{if } x_i = \perp \\
   \alpha\beta^2|\perp\rangle + (\alpha^2 + \beta^2(\beta^2 - \alpha^2))|0\rangle + \alpha\beta(1 + \alpha^2 - \beta^2)|1\rangle & \text{if } x_i = 0 \\
   -\alpha^2\beta|\perp\rangle + \alpha\beta(1 + \alpha^2 - \beta^2)|0\rangle + (\beta^2 + \alpha^2(\beta^2 - \alpha^2))|1\rangle & \text{if } x_i = 1
\end{cases}$$
Thus, we first observe that on a basis state $q$ that

$\Pi_{\leq k}, \Pi = k$ and $\Pi_{\geq k}$: all basis states $|i, p, w\rangle|x\rangle$ such that $x$ contains respectively at most, exactly or at least $k$ coordinates equal to 1.

$\Pi_{=k,\perp}$, $\Pi_{=k,0}$ and $\Pi_{=k,1}$: all basis states $|i, p, w\rangle|x\rangle$ such that (1) $x$ contains exactly $k$ coordinates equal to one, (2) the phase multiplier is $p = 1$ and (3) $x_i = \perp$, $x_i = 0$ or $x_i = 1$ respectively.

We can now define the measure of progress $q_{t,k}$ for $t$ queries and $k$ ones as

$q_{t,k} = \|\Pi_{\geq k}|\phi_t\rangle\|

where $|\phi_t\rangle$ is the state after $t$ queries in the recording query model. The main result of this section is the following bound on the growth of $q_{t,k}$.

**Proposition 3.4.** For all $1 \leq k \leq K$, we have that $q_{t,k} \leq \left(\frac{K}{N}\right)^{k}$.

**Proof.** First, $q_{0,0} = 1$ and $q_{0,k} = 0$ for all $k > 1$ since the initial state is $|0\rangle|\perp^M\rangle$. Then, we prove that $q_{t,k}$ satisfies the following recurrence relation

$$q_{t+1,k+1} \leq q_{t,k+1} + 4\sqrt{\frac{K}{N}}q_{t,k} \quad (1)$$

From this result, it is trivial to conclude that $q_{t,k} \leq \left(\frac{K}{N}\right)^{k}$. In order to prove Equation 1, we first observe that on a basis state $|i, p, w\rangle|x\rangle$ the query operator $R$ acts as the identity on the registers $X_j$ for $j \neq i$. Consequently, the basis states $|i, p, w\rangle|x\rangle$ in $|\phi_t\rangle$ that may contribute to $q_{t+1,k+1}$ must have either at least $k + 1$ ones in $x$, or exactly $k$ ones in $x$ with $p = 1$ and $x_i \neq 1$. This implies that

$$q_{t+1,k+1} \leq q_{t,k+1} + \|\Pi_{\geq k+1}R\Pi_{=k,\perp}|\phi_t\rangle\| + \|\Pi_{\geq k+1}R\Pi_{=k,0}|\phi_t\rangle\|.$$  

We first bound the term $\|\Pi_{\geq k+1}R\Pi_{=k,\perp}|\phi_t\rangle\|$. Consider any basis state $|i, p, w\rangle|x\rangle$ in the support of $\Pi_{=k,\perp}$. Then, by Lemma 3.2, $\Pi_{\geq k+1}R|i, p, w\rangle|x\rangle = 2\alpha^2\beta|i, p, w\rangle|1\rangle x_i \otimes_{j\neq i}|x_j\rangle x_j$. Thus, $\|\Pi_{\geq k+1}R\Pi_{=k,\perp}|\phi_t\rangle\| = 2\alpha^2\beta\|\Pi_{=k,\perp}|\phi_t\rangle\| \leq 2\sqrt{\frac{1-K/N}{1-K/N}}q_{t,k}$. Similarly, for the other term, $\|\Pi_{\geq k+1}R\Pi_{=k,0}|\phi_t\rangle\| = \alpha\beta(1 + \alpha^2 - \beta^2)\|\Pi_{=k,0}|\phi_t\rangle\| \leq 2\sqrt{\frac{1-K/N}{1-K/N}}q_{t,k}$. Thus,

$$q_{t+1,k+1} \leq q_{t,k+1} + 2\sqrt{\frac{1-K/N}{1-K/N}}q_{t,k} + 2\sqrt{\frac{1-K/N}{1-K/N}}$$

which completes the proof.

$\square$
3.3 Connecting the Progress to the Success Probability

We connect the success probability \( \sigma = ||\Pi_{\text{success}}|\psi_T\rangle|^2 \) in the standard query model to the final progress \( q_{T,k} \) in the recording query model after \( T \) queries. We show that if the algorithm has made no significant progress for \( k \geq K/2 \) then it needs to “guess” that \( x_i = 1 \) for about \( K - k \) positions where the \( X_i^k \) register does not contain 1. The probability to correctly guess all the ones is at most \((K/N)^{K-k}\) since \(|1\rangle_{X^k} \) has amplitude \( \sqrt{K/N} \) in \( \mathcal{S}_{\downarrow} \), and amplitude \( K/N \) in \( \mathcal{S}_{\downarrow 0} X^k \). This intuition is formalized in the next proposition.

**Proposition 3.5.** For any state \( |\phi\rangle \), we have \( ||\Pi_{\text{success}} S \Pi_{\leq k} |\phi\rangle|| \leq 3^{K/2} \left( \frac{K}{N} \right)^{K-k} ||\Pi_{\leq k} |\phi\rangle|| \).

**Proof.** Let \(|i, p, w\rangle|x\rangle\) be any basis state in the support of \( \Pi_{\leq k} \). The output value \( y \) is a substring of \( w \) made of \( K \) distinct indices \( y_1, \ldots, y_K \in [M] \) indicating positions where the input is supposed to contain ones. By definition of \( \Pi_{\leq k} \), we have \( x_{y_j} \neq 1 \) for at least \( K - k \) indices \( j \in [K] \). For each such index \( y_j \), after applying \( S \), the amplitude of \(|1\rangle_{X_j} \) is \( \sqrt{K/N} \) (if \( x_{y_j} = \perp \)) or \( K/N \) (if \( x_{y_j} = 0 \)). Consequently,

\[
||\Pi_{\text{success}} S |i, p, w\rangle|x\rangle|| \leq \left( \frac{K}{N} \right)^{K-k}.
\]

Let us now consider any state \( |\phi\rangle \) and denote \( |\psi\rangle = \Pi_{\leq k} |\phi\rangle = \sum_i \alpha_i |i, p, w, x\rangle|x\rangle \). For any two basis states \(|i, p, w\rangle|x\rangle\) and \(|i', p', w'\rangle|x'\rangle\) with output values \( y, y' \in [M]^k \) respectively, if

\[
(i, p, w, (x_j)_{j \neq y_1, \ldots, y_K}) \neq (i', p', w', (x'_j)_{j \neq y'_1, \ldots, y'_K})
\]

then \( \Pi_{\text{success}} S |i, p, w\rangle|x\rangle \) must be orthogonal to \( \Pi_{\text{success}} S|i', p', w'\rangle|x'\rangle \). Moreover, there are \( 3^K \) choices for \(|i, p, w\rangle|x\rangle\) once we set the values of \((i, p, w, (x_j)_{j \neq y_1, \ldots, y_K}) \). By the Cauchy-Schwarz inequality,

\[
||\Pi_{\text{success}} S |\phi\rangle||^2 = \sum_i \alpha_i |\Pi_{\text{success}} S |i, p, w\rangle|x\rangle||^2 \leq \sum_i \alpha_i \Pi_{\text{success}} S |i, p, w\rangle|x\rangle^2 \leq ||\Pi_{\text{success}} S |i, p, w\rangle|x\rangle||^2 \leq ||\phi||^2 3^K \left( \frac{K}{N} \right)^{K-k}.
\]

We can now conclude the proof of the main result.

**Proof of Theorem 3.1.** The joint state of the algorithm and the oracle after \( T \) queries in the standard query model is \( |\psi_T\rangle = S |\phi_T\rangle \) (Lemma 2.2). Thus, by the triangle inequality, \( \sigma = ||\Pi_{\text{success}} S |\phi_T\rangle||^2 \) satisfies \( \sqrt{T} \leq ||\Pi_{\text{success}} S \Pi_{\leq K/2} |\phi_T\rangle|| + ||\Pi_{\geq K/2} |\phi_T\rangle|| \leq 3^{K/2} \left( \frac{K}{N} \right)^{K/2} + \left( TK/2 \right) \left( 4\sqrt{K/N} \right)^{K/2} \leq \left( 3\sqrt{K/N} \right)^{K/2} + \left( 22T/\sqrt{K/N} \right)^{K/2} \), where we have used Propositions 3.5 and 3.4.

\( \square \)

4 Application to Many-Collisions Finding

We present a second application of the recording query model to the problem of finding many collisions in a random input \( x \in [N]^M \). A collision is a set of two indices \( i \neq j \) such that \( x_i = x_j \). Two collisions \( \{i, j\} \) and \( \{i', j'\} \) are said to be disjoint if \( \{i, j\} \cap \{i', j'\} = \emptyset \). In this section we study the problem of finding \( K \) disjoint collisions in an input drawn from the uniform distribution on \([N]^M\), denoted by \( \mathcal{U}^M \).

We first give a strong direct product theorem for this problem by using the recording query model (Theorem 4.1). Next, we prove that it can be converted into a \( T^3 S \geq \Omega(K^3 N) \) time-space tradeoff (Theorem 4.5). Finally, we describe a reduction from the problem of finding \( \Theta(N) \) collisions to solving Element Distinctness (Proposition 4.7), and we show how it connects to proving a time-space tradeoff for Element Distinctness (Corollary 4.8).
4.1 Strong Direct Product Theorem

In this section, we prove the following strong direct product theorem for finding \( K \) disjoint collisions on inputs drawn uniformly at random from \([N]^M\).

**Theorem 4.1.** Any \( T \)-query quantum algorithm finding \( K \) disjoint collisions on inputs drawn from \( U_M^N \) must succeed with probability \( \sigma \) satisfying

\[
\sigma \leq 8K^2 \left( \frac{K}{N} \right)^{K/2} + 2 \left( \frac{(8T)^3}{K^2N} \right)^{K/2}.
\]

The structure of the proof is similar to that of the \( K \)-Search problem. We first define the unitaries \( S_i \) used in the recording query model. The initial state \([\text{init}]\) representing \( U_M^N \) in the standard query model is \( \otimes_{i \in [M]} \left( \frac{1}{\sqrt{N}} \sum_{z \in [N]} |z\rangle x_i \right) \). Consequently, we choose

\[
S_i : \begin{cases}
|\perp\rangle x_i & \longrightarrow \frac{1}{\sqrt{N}} \sum_{z \in [N]} |z\rangle x_i \\
\frac{1}{\sqrt{N}} \sum_{z \in [N]} |\omega^p_N z\rangle x_i & \longrightarrow |\perp\rangle x_i \\
\frac{1}{\sqrt{N}} \sum_{z \in [N]} |\omega^p_N z\rangle x_i & \longrightarrow \frac{1}{\sqrt{N}} \sum_{z \in [N]} |\omega^p_N z\rangle x_i \quad \text{for } p = 1, \ldots, N-1
\end{cases}
\]

As in Lemma 3.2, we describe below the action of the recording query operator \( R \) on each basis state. We observe that it is close to the mapping \( |\perp\rangle x_i \rightarrow \sum_{z \in [N]} |\omega^p_N z\rangle x_i \) and \( |z\rangle x_i \rightarrow |\omega^p_N z\rangle \) (when \( z \neq \perp \)) up to lower order terms of amplitude \( O(1/N) \). The precise calculation is given in Appendix A.

**Lemma 4.2.** If the recording query operator \( R \) is applied on a basis state \( |i,p,w\rangle_x \) where \( p \neq 0 \) then the register \( |x_i\rangle x_i \) is mapped to

\[
\begin{align*}
\sum_{z \in [N]} \frac{\omega^p_N z}{\sqrt{N}} |z\rangle & \quad \text{if } x_i = \perp \\
\frac{\omega^p_N z}{\sqrt{N}} |\perp\rangle + \frac{1+\omega^p_N (N-2)}{N} |x_i\rangle + \sum_{z \in [N]\setminus x_i} \frac{1+\omega^p_N - \omega^p_N z}{N} |z\rangle & \quad \text{otherwise}
\end{align*}
\]

and the other registers are unchanged. If \( p = 0 \) then none of the registers are changed.

The projector \( \Pi_{\geq k} \) (and its variations) given in Definition 3.3 is modified to count the number \( k \) of disjoint collisions contained in the input register \( |x\rangle \chi \). The measure of progress is \( q_{t,k} = ||\Pi_{\geq k}|\phi_t|\| \). We prove the following bound on the growth of \( q_{t,k} \).

**Proposition 4.3.** For all \( 1 \leq k \leq K \), we have that \( q_{t+1,k+1} \leq q_{t,k+1} + 4 \sqrt{t/N} q_{t,k} \).

**Proof.** For \( z \in [N] \cup \{ \perp \} \), we define the projector \( \Pi_{=k,z} \) whose support consists of all basis states \( |i,p,w\rangle_x \) such that (1) \( x \) contains exactly \( k \) disjoint collisions, (2) \( x_i \) does not belong to any of these collisions, (3) the phase multiplier is \( p \neq 0 \) and (4) \( x_i = z \). Note that these states are the only ones that may move from the support of \( \Pi_{\leq k} \) to the support of \( \Pi_{\geq k+1} \) after one query. Consequently,

\[
q_{t+1,k+1} \leq q_{t,k+1} + ||\Pi_{\geq k+1} R\Pi_{=k,\perp}|\phi_t|\| + \sum_{z \in [N]} ||\Pi_{\geq k+1} R\Pi_{=k,z}|\phi_t|\|.
\]

We first bound the term \( ||\Pi_{\geq k+1} R\Pi_{=k,\perp}|\phi_t|\| \). Consider any basis state \( |i,p,w\rangle_x \) in the support of \( \Pi_{=k,\perp} \). We assume further that \( x \) contains at most \( t \) entries different from \( \perp \), since it is the case for all basis states occurring in \( |\phi_t| \). By Lemma 4.2, we have \( R|i,p,w\rangle_x = \sum_{z \in [N]} \frac{\omega^p_N}{\sqrt{N}} |i,p,w\rangle_x \otimes_{j \neq i} |x_j\rangle x_j \). Since there are at most \( t \) entries in \( x \) that can collide with \( x_i \), we have \( ||\Pi_{\geq k+1} R|i,p,w\rangle_x|\| \leq \sqrt{t/N} \). Thus, \( ||\Pi_{\geq k+1} R\Pi_{=k,\perp}|\phi_t|\| \leq \sqrt{t/N} q_{t,k} \).
We now consider the term $\|\Pi_{k+1}R\Pi_{k,z}^{|\phi_t}\|$ for $z \in [N]$. Again, we consider any basis state $|i,p,w\rangle|x\rangle$ in the support of $\Pi_{k,z}$ with at most $t$ entries different from $\bot$. Using Lemma 4.2, we have $\Pi_{k+1}R|i,p,w\rangle|x\rangle = \sum_{x_i \neq x_1}^1 1 + \omega_{x_i} - \omega_{x_1} \Pi_{k+1}^{|z\rangle x_i \otimes j \neq i |x_j\rangle \chi_j}$. Since at most $t$ terms in this sum can be nonzero, we have $\|\Pi_{k+1}R|i,p,w\rangle|x\rangle\| \leq 3\sqrt{t}/N$. Thus, $\|\Pi_{k+1}R\Pi_{k,z}^{|\phi_t}\| \leq 3\sqrt{t}/N\|\Pi_{k,z}^{|\phi_t}\|$.

We conclude that $q_{t,k+1} \leq q_{t,k} + \sqrt{t/N}q_{t,k} + \sum_{z \in [N]} 3\sqrt{t}/N\|\Pi_{k,z}^{|\phi_t}\| \leq q_{t,k} + \sqrt{t/N}q_{t,k} + 3t/Nq_{t,k}$. 

We obtain directly from Proposition 4.3 that $q_{t,k} \leq \left(\frac{t}{N}\right)^k$. It remains to connect this quantity to the success probability $\sigma$. This step is a bit more involved than for the $K$-Search problem since the input $x$ takes a larger range of values.

**Proposition 4.4.** For any state $|\phi\rangle$, we have $\|\Pi_{\text{success}}\hat{S}\Pi_{\leq k}|\phi\rangle\| \leq 2\left(\sqrt{\frac{K}{N}}\right)^{K-k} \|\Pi_{\leq k}|\phi\rangle\|$. 

**Proof.** We assume that the output $y$ in the collisions finding problem is represented as a list of $K$ triples $(i_1,j_1,C_1), \ldots, (i_K,j_K,C_K)$. The output is correct if the input $x \in [N]^M$ (in the standard query model) satisfies $x_{i_\ell} = x_{j_\ell} = C_\ell$ for all $1 \leq \ell \leq K$, and the indices $i_1,j_1,\ldots,i_K,j_K$ are all different.

We define a new family of projectors $\tilde{\Pi}_{a,b}$, where $0 \leq a + b \leq K$, whose supports consist of all basis states $|i,p,w\rangle|x\rangle$ satisfying the following conditions:

1. the output substring $y$ of $w$ is made of $K$ disjoint triples $(i_1,j_1,C_1), \ldots, (i_K,j_K,C_K)$.

2. there are exactly $a$ indices $u \in \{i_1,j_1,i_2,j_2,\ldots,i_K,j_K\}$ such that $x_u = \bot$.

3. there are exactly $b$ indices $v \in \{i_1,j_1,i_2,j_2,\ldots,i_K,j_K\}$ such that $x_v \neq \bot$ and $x_v \neq C_{u-1}$, where $C_{u-1}$ is the “collision value” in the output triple containing $x_u$.

For any such state, we claim that $\|\Pi_{\text{success}}\hat{S}|i,p,w\rangle|x\rangle\| \leq \left(\frac{1}{\sqrt{\frac{N}{K}}^a}\right)^b$. Indeed, the action of $S_u$ on the register $|x_u\rangle|\chi_u\rangle$ is $|x_u\rangle \mapsto |x_u\rangle + \frac{1}{\sqrt{K}}\sum_{z \in [N]}|z\rangle$ if $x_u = \bot$, and $|x_u\rangle \mapsto |x_u\rangle + (1 - \frac{1}{N})|x_u\rangle - \frac{1}{N}\sum_{z \in [N]}|x_u\rangle$ otherwise. The projector $\Pi_{\text{success}}$ only keeps the term $|C_{u-1}\rangle$ in these sums.

Next, for any list of $K$ triples $(i_1,j_1,C_1), \ldots, (i_K,j_K,C_K)$, there are $\binom{K}{a}\binom{K-a}{b}(N-1)^b \leq K^{a+b}N^b$ different ways to choose $(x_{i_\ell})_{\ell=1,2,\ldots,K}$ that satisfy conditions (2) and (3). Consequently, for any state $|\phi\rangle$, by using the Cauchy-Schwarz inequality we have

$$\|\Pi_{\text{success}}\hat{S}\Pi_{a,b}|\phi\rangle\|^2 \leq K^{a+b}N^b \cdot \left(\frac{1}{\sqrt{\frac{N}{K}}}\right)^a \left(\frac{1}{\sqrt{\frac{N}{N}}}\right)^b \cdot \|\Pi_{a,b}|\phi\rangle\|^2 \leq \left(\frac{K}{N}\right)^{a+b} \|\Pi_{a,b}|\phi\rangle\|^2.$$

Finally, since the support of $\Pi_{\leq k}$ is included into the union of the supports of $\Pi_{a,b}$ for $a + b \geq K - k$ we have

$$\|\Pi_{\text{success}}\hat{S}\Pi_{\leq k}|\phi\rangle\| \leq \sum_{a+b \geq K-k} \left(\frac{K}{N}\right)^{a+b} \|\Pi_{a,b}\Pi_{\leq k}|\phi\rangle\| \leq 2K \left(\frac{K}{N}\right)^{K-k} \|\Pi_{\leq k}|\phi\rangle\|. \quad \square$$

We can now conclude the proof of Theorem 4.1.

**Proof of Theorem 4.1.** The state of the algorithm and the oracle after $T$ queries in the standard query model is $|\psi_T\rangle = \hat{S}|\phi_T\rangle$. Thus, by the triangle inequality, $\sigma = \|\Pi_{\text{success}}\hat{S}|\phi_T\rangle\|^2$ satisfies $\sqrt{\sigma} \leq \|\Pi_{\text{success}}\Pi_{\leq K/2}|\phi_T\rangle\| + \|\Pi_{K/2}|\phi_T\rangle\| \leq 2K \left(\sqrt{\frac{K}{N}}\right)^{K/2} + \left(\frac{T}{K/2}\right) \left(\sqrt{T/N}\right)^{K/2} \leq 2K \left(\sqrt{\frac{K}{N}}\right)^{K/2} + \left(\frac{22T^{3/2}}{(K\sqrt{N})^{3/2}}\right)$, where we have used Propositions 4.4 and 4.3. \(\square\)
4.2 Time-Space Tradeoffs

We use the lower bound from Section 4.1 to derive a new time-space tradeoff for the problem of finding $K$ disjoint collisions in a uniformly random input $x \in [N]^M$. Our proof relies on the standard time-segmentation method for large-output problems [BFK+81, KSW07].

**Theorem 4.5.** Any quantum algorithm finding $K < N/4$ disjoint collisions with success probability $2/3$ on inputs drawn from $\mathcal{U}_M^N$ must satisfy a time-space tradeoff of $T^3S \geq \Omega(K^3N)$.

**Proof.** Choose any quantum circuit $C$ running in time $T$ (i.e. making $T$ queries) and using $S > \Omega(\log N)$ qubits of memory. We can represent $C$ as a succession $C_1 || C_2 || \ldots || C_L$ of $L = T/T'$ sub-circuits each running in time $T' = 2^{-4} S^{2/3} N^{1/3}$. Define $X_j$ to be the random variable that counts the number of (mutually) disjoint collisions that $C$ outputs between time $(j-1)T'$ and $jT'$ (i.e. in the sub-circuit $C_j$) when the input is drawn from $\mathcal{U}_M^N$. We must have $\sum_{j=1}^L \mathbb{E}[X_j] \geq \Omega(K)$ for the algorithm to be correct.

We claim that $\mathbb{E}[X_j] \leq 5S$ for all $j$. Assume by contradiction that $\mathbb{E}[X_j] \geq 5S$ for some $j$. Since $X_j$ is bounded between 0 and $N$ we have $\Pr[X_j > 4S] \geq S/N$. Consequently, by running $C_j$ on a completely mixed state we obtain $4S$ disjoint collisions with probability at least $S/N \cdot 2^{-S} \geq 4^{-S}$ in time $T'$. However, by Theorem 4.1, no quantum algorithm can find more than $4S$ disjoint collisions in time $T'' = 2^{-4} S^{2/3} N^{1/3}$ with success probability larger than $8^{-S}$. This contradiction implies that $\mathbb{E}[X_j] \leq 5S$ for all $j$. Consequently, the number of sub-circuits must be $L \geq \Omega(K/S)$ in order to have $\sum_{j=1}^L \mathbb{E}[X_j] \geq \Omega(K)$. Since each sub-circuit runs in time $2^{-4} S^{2/3} N^{1/3}$ the running time of $C$ is $T \geq L \cdot S^{2/3} N^{1/3} \geq \Omega(K N^{1/3} / S^{1/3})$. □

**Corollary 4.6.** Any quantum algorithm finding $\Theta(N)$ disjoint collisions with success probability $2/3$ on inputs drawn from $\mathcal{U}_N^{N/100}$ must satisfy a time-space tradeoff of $T^3S \geq \Omega(N^3)$.

We conjecture that Corollary 4.6 is not optimal and can be improved to $T^2S = \Theta(N^3)$ (and, more generally, that Theorem 4.5 can be improved to $T^2S \geq \Omega(K^2N)$). This latter tradeoff can be achieved with Quantum Sorting [Kla03] or by adapting the BHT algorithm [BHT98]. Classically, the Parallel Collision Search algorithm [OW99] points to the same complexity. Thus, there may be no separation between the classical and quantum tradeoffs.

**Conjecture 1.** Any quantum algorithm finding $\Theta(N)$ disjoint collisions with success probability $2/3$ on inputs drawn from $\mathcal{U}_N^{N/100}$ must satisfy a time-space tradeoff of $T^2S \geq \Omega(N^3)$.

We motivate the interest of Conjecture 1 by showing that it implies a $T^2S \geq \tilde{\Omega}(N^2)$ tradeoff for Element Distinctness. This result will rely on a reduction presented in Algorithm 1 and analyzed in Proposition 4.7 (the constants $c_0, c_1, c_2$ will be chosen in the proof). The Element Distinctness problem is formulated\(^2\) as follows.

**Element Distinctness (ED$_N$).** Find a collision in $x \in [N^2]^N$ if one exists.

**Proposition 4.7.** If there exists an algorithm solving ED$_N$ in time $T_N$ and space $S_N$ then Algorithm 1 runs in time $O(NT_N^{2/3} \sqrt{N})$ and space $O(S_N \sqrt{N})$, and it finds $c_1 N$ collisions in any input $x \in [N]^N$ containing at least $c_0 N$ collisions.

The proof of Proposition 4.7 is deferred to Appendix B. We now use the above reduction to show how to translate any time-space tradeoff for finding $\Theta(N)$ disjoint collisions into one for ED$_N$. Observe that Algorithm 1 does not necessarily output collisions that are mutually disjoint. Nevertheless, there is a small probability that an input drawn from $\mathcal{U}_N^{N/100}$ contains multi-collisions of size larger than $\log N$. As a consequence, there is only a $\log N$ loss in the analysis.

\(^2\)We formulate Element Distinctness as a finding problem, but it can be reduced to its decision version with only a logarithmic overhead in time and space (see [Jef11, Claim 3.0.5] for instance).
Consider any algorithm solving ED, and space \( \log(N) \) satisfies a time-space tradeoff of \( N/\alpha \). Unfortunately, this result says nothing about space since it is already known [AS04] that \( \Omega(N) \) tradeoff for Element Distinctness. Nevertheless, Conjecture 1 would imply a \( \tilde{\Omega}(N^2) \) tradeoff for ED. Using the time-space tradeoff \( T^3S \geq \tilde{\Omega}(N^4) \) of Corollary 4.6 we get that \( T^3S \geq \tilde{\Omega}(N^2) \) for ED. Unfortunately, this result says nothing about space since it is already known [AS04] that \( T^3 \geq \Omega(N^2) \) for Element Distinctness. Nevertheless, Conjecture 1 would imply a \( T^2S \geq \tilde{\Omega}(N^2) \) tradeoff for ED, matching the best known upper bound [Amb07].

Corollary 4.9. If Conjecture 1 is true, then any quantum algorithm solving ED with success probability 2/3 must satisfy a time-space tradeoff of \( T^2S \geq \tilde{\Omega}(N^2) \).

References


A Proofs of Lemmas 3.2 and 4.2

Proof of Lemma 3.2. We detail the action of the recording query operator \( R = S \cdot O \cdot S \) for the K-Search problem on a basis state \( |i, p, w \rangle x \) where \( p \neq 0 \). We use that the unitary \( S_i \) maps 
\[ |\perp \rangle_{\chi_i} \mapsto |\rangle, \quad |0 \rangle_{\chi_i} \mapsto \alpha |\perp \rangle + \beta |\rangle, \quad |1 \rangle_{\chi_i} \mapsto \beta |\perp \rangle - \alpha |\rangle. \]

The action on the register \( \chi \) is:

- If \( x_i = \perp \) then \( |x_i \rangle_{\chi_i} \xrightarrow{S} |\rangle \xrightarrow{O} |\alpha 0 \rangle - |\beta 1 \rangle \xrightarrow{S} (\alpha^2 - \beta^2) |\perp \rangle + 2 \alpha \beta |\rangle. \)

- If \( x_i = 0 \) then \( |x_i \rangle_{\chi_i} \xrightarrow{S} |\alpha \perp \rangle + |\beta \rangle \xrightarrow{O} |\alpha \perp \rangle + |\beta (|0 \rangle + |1 \rangle) \rangle \xrightarrow{S} |\beta \alpha^2 \rangle \perp \rangle + (\alpha^2 + \beta^2 (\beta^2 - \alpha^2))(0) + \alpha \beta (1 + \alpha^2 - \beta^2)|1 \rangle. \)

- If \( x_i = 1 \) then \( |x_i \rangle_{\chi_i} \xrightarrow{S} |\beta \perp \rangle - |\alpha \rangle \xrightarrow{O} |\beta \perp \rangle - |\beta (|0 \rangle + |1 \rangle) \rangle \xrightarrow{S} - |\alpha^2 \beta \rangle \perp \rangle + \alpha \beta (1 + \alpha^2 - \beta^2)|0 \rangle + (\beta^2 + \alpha^2 (\beta^2 - \alpha^2))|1 \rangle. \)

Proof of Lemma 4.2. We detail the action of the recording query operator \( R = S \cdot O \cdot S \) for the Many-Collisions Finding problem on a basis state \( |i, p, w \rangle x \) where \( p \neq 0 \). We use that the unitary \( S_i \) maps 
\[ |\perp \rangle_{\chi_j} \mapsto \frac{1}{\sqrt{N}} \sum_{z \in [N]} |z \rangle \]
and 
\[ |j \rangle_{\chi_i} \mapsto \frac{1}{\sqrt{N}} |\perp \rangle + \sum_{j' \in [N]} \sum_{z \in [N], 0} \omega_{N}^{-pj'} |\rho \rangle \]
where \( j \in [N] \) and 
\[ |\rho \rangle := \frac{1}{\sqrt{N}} \sum_{z \in [N]} |z \rangle |\rangle. \]

The action on the register \( \chi \) is
• If \( x_i = \perp \) then \( |x_i \rangle_{\lambda_i} \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} z \xrightarrow{O} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{R_i} |z \rangle \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{R_i} |z \rangle \).

• If \( x_i = j \) where \( j \in [N] \) then \( |x_i \rangle_{\lambda_i} \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} j \xrightarrow{O} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{-p_j} |p + p_j \rangle \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} j \xrightarrow{O} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{-p_j} |p \rangle \) + \( \frac{1}{\sqrt{N}} \sum_{p' \in [N] \setminus \{0, p\}} \omega_N^{-p'} |p' \rangle \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} j \xrightarrow{O} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{-p_j} |z \rangle \) + \( \frac{1}{\sqrt{N}} \sum_{p' \in [N] \setminus \{0, p\}} \omega_N^{-p'} |p \rangle \xrightarrow{S} \frac{1}{\sqrt{N}} \sum_{z \in [N]} j \xrightarrow{O} \frac{1}{\sqrt{N}} \sum_{z \in [N]} \omega_N^{-p_j} |z \rangle \).

\[ \square \]

### B Proof of Proposition 4.7

We choose \( c_0 = 40, c_1 = 1/10^4 \) and \( c_2 = 8 \). We study the probabilities of the following events to occur at a fixed round of Algorithm 1:

- **Event A**: \( h \) is collision free (i.e., \( h(i) \neq h(j) \) for all \( i \neq j \)).

- **Event B**: \( h(1), \ldots, h(\sqrt{N}) \) does not contain any of the pairs output in the previous rounds.

- **Event C**: \( x_{h(1)}, \ldots, x_{h(\sqrt{N})} \) contains a collision.

- **Event D**: the algorithm for \( \text{ED}_{\sqrt{N}} \) finds a collision in \( x_{h(1)}, \ldots, x_{h(\sqrt{N})} \).

Algorithm 1 succeeds if and only if the event \( A \land B \land C \land D \) occurs at least \( c_1 N \) times during its execution. We now lower bound the probability of this event to happen.

For **event A**, let us consider the random variable \( X = \sum_{i \neq j} \mathbb{1}_{h(i) = h(j)} \). Using that \( h \) is pairwise independent, we have \( \mathbb{E}[X] = (\sqrt{N}/2) \frac{1}{N} \leq 1/2 \). Thus, by Markov’s inequality, \( \Pr[A] = 1 - \Pr[X \geq 1] \geq 1/2 \).

For **event B**, let us assume that \( k < c_1 N \) pairs \( \{a_1, b_1\}, \ldots, \{a_k, b_k\} \) have been output so far. For any \( i \in [k] \), the probability that both \( a_i \) and \( b_i \) occur in \( h(1), \ldots, h(\sqrt{N}) \) is at most \( (\sqrt{N}/2) \frac{k}{N} \leq 1/2 \) (since \( h \) is pairwise independent). Thus, by a union bound, \( \Pr[B] \geq 1 - \frac{k}{N} \geq 1 - c_1 \).

For **event C**, let us consider the random variables \( Y_{i,j} = 1_{x_{h(i)} = x_{h(j)}} \) for \( i \neq j \in [\sqrt{N}] \), and \( Y = \sum_{i \neq j} Y_{i,j} \). Note that we may have \( Y_{i,j} = 1 \) because \( h(i) = h(j) \) (this is taken care of in event A). For each \( m \in [N] \), let \( N_m = |\{a : x_a = m\}| \) denote the number of elements that are equal to \( m \) in the input. Using that \( h \) is 4-wise independent, for any \( i \neq j \neq k \neq \ell \) we have,

\[
\begin{align*}
\Pr[Y_{i,j} = 1] &= \frac{\sum_{m \in [N]} N_m^2}{N^2} \\
\Pr[Y_{i,j} = 1 \land Y_{i,k} = 1] &= \frac{\sum_{m \in [N]} N_m^3}{N^3} \\
\Pr[Y_{i,j} = 1 \land Y_{k,\ell} = 1] &= \Pr[Y_{i,j} = 1] \cdot \Pr[Y_{k,\ell} = 1]
\end{align*}
\]

Consequently, \( \mathbb{E}[Y] = (\sqrt{N}/2) \frac{\sum_{m \in [N]} N_m^2}{N^2} \) and

\[
\begin{align*}
\text{Var}[Y] &= \sum_{\{i,j\}} \text{Var}[Y_{i,j}] + \sum_{\{i,j\} \neq \{i,k\}} \text{Cov}[Y_{i,j}, Y_{i,k}] + \sum_{\{i,j\} \cap \{k,\ell\} = \emptyset} \text{Cov}[Y_{i,j}, Y_{k,\ell}] \\
&\leq \sum_{\{i,j\}} \mathbb{E}[Y_{i,j}^2] + \sum_{\{i,j\} \neq \{i,k\}} \mathbb{E}[Y_{i,j} Y_{i,k}] \\
&= \left( \sqrt{N}/2 \right) \frac{\sum_{m \in [N]} N_m^2}{N^2} + 3 \left( \sqrt{N}/3 \right) \frac{\sum_{m \in [N]} N_m^3}{N^3}
\end{align*}
\]
where we have used that $Y_{i,j}$ and $Y_{k,\ell}$ are independent when $i \neq j \neq k \neq \ell$. The term $\sum_{m \in [N]} N_m^2$ is equal to the number of ordered pairs $(a, b) \in [N]^2$ such that $x_a = x_b$. Each collision in $x$ gives two such pairs, and we must also count the pairs $(a, a)$. Thus, $\sum_{m \in [N]} N_m^2 \geq (1 + 2c_0)N$. Moreover, $\sum_{m \in [N]} N_m^3 \leq (\sum_{m \in [N]} N_m^2)^{3/2}$. Consequently,

$$\frac{\text{Var}[Y]}{\text{E}[Y]^2} \leq \frac{1 + \sqrt{\frac{3}{2} (\sum_{m \in [N]} N_m^2)^{1/2}}}{\frac{\sum_{m \in [N]} N_m^2}{N^2}} \leq \frac{4(1 + \sqrt{1 + 2c_0})}{1 + 2c_0}$$

Finally, according to Chebyshev’s inequality, $\Pr[Y = 0] \leq \Pr[|Y - \text{E}[Y]| \geq \text{E}[Y]] \leq \frac{\text{Var}[Y]}{\text{E}[Y]^2}$. Thus, $\Pr[C] = 1 - \Pr[Y = 0] \geq 1 - \frac{4(1 + \sqrt{1 + 2c_0})}{1 + 2c_0}$.

For event D, we have $\Pr[D \mid A \land B \land C] \geq 2/3$ by correctness of the algorithm solving ED $\sqrt{N}$.

We can now lower bound the probability of the four events together.

$$\Pr[A \land B \land C \land D] = \Pr[D \mid A \land B \land C] \cdot \Pr[A \land B \land C] \geq \frac{2}{3} \cdot \left(1 - \frac{4(1 + \sqrt{1 + 2c_0})}{1 + 2c_0} \right) \geq 1/250$$

Let $\tau$ be the number of rounds after which $c_1N$ collisions have been found (i.e. $A \land B \land C \land D$ has occurred $c_1N$ times). We have $\text{E}[\tau] \leq 8c_1N$, and by Markov’s inequality $\Pr[\tau \geq c_2N] \leq 250c_1/c_2 \leq 1/3$. Thus, with probability at least $2/3$, Algorithm 1 outputs at least $c_1N$ collisions in $x_1, \ldots, x_N$.