Quantum Chebyshev’s Inequality and Applications

Yassine Hamoudi, Frédéric Magniez

IRIF, Université de Paris, CNRS
Mean Estimation Problem

How many i.i.d. samples $x_1, x_2, \ldots$ from some unknown bounded r.v. $X \in [0,B]$ do we need to compute $\tilde{\mu}$ such that

$$|\tilde{\mu} - E(X)| \leq \epsilon E(X)$$

with proba. $2/3$
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Sample mean: $\tilde{\mu} = \frac{x_1 + \ldots + x_n}{n}$
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**Sample mean:** $\tilde{\mu} = \frac{x_1 + \ldots + x_n}{n}$

**Chernoff’s Bound:** $\frac{B}{\epsilon^2 E(X)}$
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→ Chernoff’s Bound: $\frac{B}{\epsilon^2 \mathbb{E}(X)}$

→ Bernstein’s Inequality: $\frac{\text{Var}(X)}{\epsilon^2 \mathbb{E}(X)^2} + \frac{B}{\epsilon \mathbb{E}(X)}$ \quad (\text{Var}(X) \leq B \cdot \mathbb{E}(X))
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→ Chebyshev’s Inequality:

$$\frac{\text{Var}(X)}{\epsilon^2 \mathbb{E}(X)^2}$$

In practice: we often know $\Delta^2 \geq \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = \frac{\text{Var}(X^2)}{\mathbb{E}(X)^2} + 1$ → take $\frac{\Delta^2}{\epsilon^2}$ samples
Applications

Counting with Markov chain Monte Carlo methods:

Counting vs. sampling [Jerrum, Sinclair’96] [Štefankovič et al.’09], Volume of convex bodies [Dyer, Frieze’91], Permanent [Jerrum, Sinclair, Vigoda’04]

Data stream model:

Frequency moments, Collision probability [Alon, Matias, Szegedy’99] [Monemizadeh, Woodruff’] [Andoni et al.’11] [Crouch et al.’16]

Testing properties of distributions:

Closeness [Goldreich, Ron’11] [Batu et al.’13] [Chan et al.’14], Conditional independence [Canonne et al.’18]

Estimating graph parameters:

Number of connected components, Minimum spanning tree weight [Chazelle, Rubinfeld, Trevisan’05], Average distance [Goldreich, Ron’08], Number of triangles [Eden et al. 17]

etc.
Quantum Mean Estimation Problem

Random variable $X$ on finite sample space $\Omega \subset [0,B]$.  

**Classical sample:** one value $x \in \Omega$, sampled with probability $p_x$
Quantum Mean Estimation Problem

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**Quantum sample:** one use of a unitary operator $S_X$ or $S_X^{-1}$ satisfying

$$S_X |0\rangle = \sum_{x \in \Omega} \sqrt{p_x} |x\rangle$$
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**Question:** can we estimate $E(X)$ with less samples in the quantum setting?
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|                             | [Montanaro’15]: \[
\frac{\Delta^2}{\epsilon}
\] |
| (Chebyshev)                | Our contribution: \[
\frac{\Delta}{\epsilon} \cdot \log^3 \left( \frac{B}{E(X)} \right)
\] |
| \[
\frac{\Delta^2}{\epsilon^2}
\] given \[
\Delta^2 \geq \frac{E(X^2)}{E(X)^2}
\] |

\[
B
\] denotes the maximum amplitude of a quantum state, and \[
E(X)
\] represents the expectation value of the observable \(X\). \(\epsilon\) is the error tolerance.
Our Approach
Amplitude-Estimation: \[ O\left(\frac{\sqrt{B}}{e \sqrt{E(X)}}\right) \] quantum samples to estimate \( E(X) \)
Amplitude-Estimation: \( O\left(\frac{\sqrt{B}}{\epsilon\sqrt{E(X)}}\right) \) quantum samples to estimate \( E(X) \)

If \( B \leq \frac{E(X^2)}{E(X)} \): the number of samples is \( O\left(\frac{\sqrt{E(X^2)}}{\epsilon E(X)}\right) \)
Amplitude-Estimation: \( O\left(\frac{\sqrt{B}}{c\sqrt{\mathbb{E}(X)}}\right) \) quantum samples to estimate \( \mathbb{E}(X) \)

\[ \left\{ \begin{array}{l}
\text{If } B \leq \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} : \text{the number of samples is } O\left(\frac{\sqrt{\mathbb{E}(X^2)}}{c\mathbb{E}(X)}\right) \checkmark \\
\text{If } B \gg \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} ?
\end{array} \right. \]
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- If \( B \gg \frac{E(X^2)}{E(X)} \): map the outcomes larger than \( \frac{E(X^2)}{E(X)} \) to 0 \( ? \)
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- If \( B \gg \frac{E(X^2)}{E(X)} \), then map the outcomes larger than \( \frac{E(X^2)}{E(X)} \) to 0

Random variable \( X \)
Amplitude-Estimation: \[ O\left(\frac{\sqrt{B}}{\epsilon \sqrt{E(X)}}\right) \] quantum samples to estimate \( E(X) \)

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Random variable \( X_b \)

New largest outcome
**Amplitude-Estimation:** \( O\left(\frac{\sqrt{B}}{\epsilon \sqrt{\mathbb{E}(X)}}\right) \) quantum samples to estimate \( \mathbb{E}(X) \)

- If \( B \leq \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} \) : the number of samples is \( O\left(\frac{\sqrt{\mathbb{E}(X^2)}}{\epsilon \mathbb{E}(X)}\right) \)

- If \( B \gg \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} \) : map the outcomes larger than \( \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} \) to 0

**Lemma:** If \( b \geq \frac{\mathbb{E}(X^2)}{\epsilon \mathbb{E}(X)} \) then \( (1 - \epsilon)\mathbb{E}(X) \leq \mathbb{E}(X_b) \leq \mathbb{E}(X) \).

\[ \Rightarrow \] We can equivalently estimate the mean of \( X_b \) for \( b \geq \frac{\mathbb{E}(X^2)}{\epsilon \mathbb{E}(X)} \)
Amplitude-Estimation: \( O\left(\frac{\sqrt{B}}{e\sqrt{E(X)}}\right) \) quantum samples to estimate \( E(X) \)

\[
\begin{align*}
\text{→ If } B & \leq \frac{E(X^2)}{E(X)} : \text{ the number of samples is } O\left(\frac{\sqrt{E(X^2)}}{eE(X)}\right) \quad \checkmark \\
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\end{align*}
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\[ \text{Lemma: If } b \geq \frac{E(X^2)}{eE(X)} \text{ then } (1 - \epsilon)E(X) \leq E(X_b) \leq E(X). \]

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\[ \text{Problem: } \frac{E(X^2)}{E(X)} \text{ is unknown...} \]
Amplitude-Estimation: $O\left(\frac{\sqrt{B}}{\epsilon \sqrt{E(X)}}\right)$ quantum samples to estimate $E(X)$

- If $B \leq \frac{E(X^2)}{E(X)}$ : the number of samples is $O\left(\frac{\sqrt{E(X^2)}}{\epsilon E(X)}\right)$

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**Problem:** $\frac{E(X^2)}{E(X)}$ is unknown… but we know $\Delta^2 \geq \frac{E(X^2)}{E(X)^2}$ $\Rightarrow$ $b \approx E(X) \cdot \Delta^2$?
Objective: given $\Delta^2 \geq \frac{E(X^2)}{E(X)^2}$ how to find a threshold $b \approx E(X) \cdot \Delta^2$?
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Solution: use the **Amplitude Estimation** algorithm (again) to do a logarithmic search on $b$
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**Stopping rule:** $\tilde{\mu}_i \neq 0$  
**Output:** $b_i$
Objective: given $\Delta^2 \geq \frac{E(X^2)}{E(X)^2}$ how to find a threshold $b \approx E(X) \cdot \Delta^2$?

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Stopping rule: $\tilde{\mu}_i \neq 0$  Output: $b_i$  ...

Theorem: the first non-zero $\tilde{\mu}_i$ is obtained w.h.p. when:

$$2 \cdot E(X)\Delta^2 \leq b_i \leq 10 \cdot E(X)\Delta^2$$
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**Ingredient 1:** The output of *Amplitude-Estimation* is 0 w.h.p. if and only if the estimated amplitude is below the inverse number of samples.

$$\sqrt{\frac{E(X_b)}{b}} \leq \frac{1}{\Delta}$$
Theorem: the first non-zero \( \tilde{\mu}_i \) is obtained w.h.p. when:

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\]

Ingredient 2: If \( b \geq 10 \cdot E(X)\Delta^2 \) then

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\]
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$$\sqrt{\frac{E(X_b)}{b}} < \frac{1}{\Delta}$$

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**Ingredient 3:** If $b \approx E(X) \cdot \Delta^2$ then

$$\frac{E(X_b)}{b} \approx \frac{E(X)}{b} \approx \frac{1}{\Delta^2}$$
Applications
Application 1: approximating graph parameters

**Input:** graph $G=(V,E)$ with $n$ vertices, $m$ edges, $t$ triangles

**Query access:** unitaries

$O_{\text{deg}} |v⟩ |0⟩ = |v⟩ |\text{deg}(v)⟩$ \hspace{1cm} (degree query)

$O_{\text{pair}} |v⟩ |w⟩ |0⟩ = |v⟩ |w⟩ |(v, w) \in E ?⟩$ \hspace{1cm} (pair query)

$O_{\text{ngh}} |v⟩ |i⟩ |0⟩ = |v⟩ |i⟩ |v_i⟩$ \hspace{1cm} (neighbor query)

$i^{\text{th}}$ neighbor of $v$
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**Result:** $\tilde{\Theta} \left( \frac{\sqrt{n}}{m^{1/4}} \right)$ quantum queries for edge estimation

$\tilde{\Theta} \left( \frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}} \right)$ quantum queries to triangle estimation
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**Result:** $\tilde{\Theta} \left( \frac{\sqrt{n}}{m^{1/4}} \right)$ quantum queries for edge estimation

(vs. $\tilde{\Theta} \left( \frac{n}{\sqrt{m}} \right)$ classical queries) [Goldreich, Ron’08] [Seshadhri’15]

$\tilde{\Theta} \left( \frac{\sqrt{n}}{t^{1/6}} + \frac{m^{3/4}}{\sqrt{t}} \right)$ quantum queries to triangle estimation

(vs. $\tilde{\Theta} \left( \frac{n}{t^{1/3}} + \frac{m^{3/2}}{t} \right)$ classical queries) [Eden, Levi, Ron’15] [Eden, Levi, Ron, Seshadhri’17]
Application 2: frequency moments in the streaming model

Initially: \( x = (0,\ldots,0) \) of \textbf{dimension n}

Input: stream of updates \( x_i \leftarrow x_i + \delta \) to \( x \)

Output: (at the end of the stream) estimate of \( F_k = \sum_{i=1}^{n} |x_i|^k \) (moment of order \( k \geq 3 \))
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What is the smallest memory size $M$ needed to estimate $F_k$ using $P$ passes over the same stream?
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What is the smallest memory size $M$ needed to estimate $F_k$ using $P$ passes over the same stream?

Result: $M = \tilde{\Theta} \left( \frac{n^{1-2/k}}{P^2} \right)$ qubits of memory

(vs. $M = \tilde{\Theta} \left( \frac{n^{1-2/k}}{P} \right)$ classical bits of memory)

[Monemizadeh, Woodruff’10]
[Andoni, Krauthgamer, Onak’11]
Conclusion
The mean of a random variable $X$ can be estimated with multiplicative error $\varepsilon$ using quantum samples, given

$$\Delta^2 \geq \frac{\text{E}(X^2)}{\text{E}(X)^2}.$$

**Lower bound:** \(\Omega\left(\frac{\Delta - 1}{\varepsilon}\right)\) quantum samples

**or** \(\Omega\left(\frac{\Delta^2 - 1}{\varepsilon^2}\right)\) copies of the state \(S_x |0\rangle = \sum_{x \in \Omega} \sqrt{p_x} |\psi_x\rangle |x\rangle\)

**Open questions:**

- Can we improve the complexity to $O(\Delta/\varepsilon)$ exactly?
- Lower bound for the Frequency Moments estimation problem?
- Other applications?

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