On Reachability Games of Ordinal Length*

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Abstract. Games are a classical model in the synthesis of controllers in the open setting. In particular, games of infinite length can represent systems which are not expected to reach a correct state, but rather to handle a continuous stream of events. Yet, even longer sequences of events have to be considered when infinite sequences of events can occur in finite time — Zeno behaviours.
In this paper, we extend two-player games to this setting by considering plays of ordinal length. Our two main results are determinacy of reachability games of length less than ω^ω on finite arenas, and the PSPACE-completeness of deciding the winner in such a game.

1 Introduction

Games are a classical model for the synthesis of controllers in open settings, with numerous applications. Although finite games seems more natural, there has been a huge interest for games of infinite duration [GTW02]. They have strong connections with logic (*e.g.* parity games and μ-calculus [EJ91]), and provide useful models in economy. In verification, they are used to represent reactive systems which must handle a continuous stream of events [Tho95]. However, some behaviours cannot be described by this model, when infinite sequences of events happen in finite time. Such behaviours — Zeno behaviours — especially need to be considered in timed systems, when successive events can be arbitrarily close. The classical discrete-time framework used by Alur and Dill in their seminal paper [AD94] prevents such behaviours, while several papers about real-time models limit their results to non-Zeno runs [AM99] or force the players to ensure that they can not happen [dAFH$^+$03]. Since Büchi in the 1960's, several extensions of automata to words of ordinal length have been proposed [BC01,BÉ02]. Demri and Nowak propose in [DN05] an extension of LTL to ordinals of length ω^n. They also formalise a problem of open specification, where only the environment has the opportunity to play more than ω moves, and which was solved in [Cac06].

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In this paper, we use the methods of [BC01] in order to define game arenas admitting plays of ordinal length. We show that these reachability games of ordinal length are determined, through a reduction to Muller games. We also show that, for several natural ways of representing the transitions, the problem is `PSPACE`-complete.

Overview of the paper In Section 2, we recall the definitions of automata on words of ordinal length and games of infinite duration, and we introduce our model of games of ordinal length. Section 3 shows the determinacy of these games on finite arenas, and Section 4 considers the complexity issues. Finally, Section 5 summarises our results, and presents several interesting perspectives for future work about these games.

2 Definitions

2.1 Ordinals and automata on words of ordinal length

We consider ordinals, *i.e.* totally ordered sets where any non-empty subset has a least element. In particular, every finite ordered set is an ordinal, as is the set of natural numbers with the usual ordering (usually called ω).

One extends the usual operators $+$ and $\cdot$ to ordinals: $I + J$ is defined by $I \uplus J$ ordered in a way such that $i < j$ if $i \in I$ and $j \in J$; $I \cdot J$ is the set $I \times J$ ordered lexicographically.

A *cut* of an ordinal J is a partition (K, L) of J such that $\forall k \in K, l \in L, k < l$. The set of cuts of J is an ordinal, denoted by $\hat{J}$. For an element j of J, we define the cuts j^- by $(\{i \in J \mid i < j\}, \{i \in J \mid i \geq j\})$ and j^+ by $(\{i \in J \mid i \leq j\}, \{i \in J \mid i > j\})$.

Example 1. $\omega^2 + 3$ is obtained by adding 3 elements to ω^2, which are greater than all others. We represent it below, with bullets for the elements and vertical lines for the cuts:

$$\Big|\underbrace{\overbrace{\bullet|\bullet|\bullet|\cdots}^{\omega}\Big|\overbrace{\bullet|\bullet|\bullet\cdots}^{\omega}\Big|\cdots\cdots}_{\omega^2}\Big|\underbrace{\bullet|\bullet|\bullet}_{3}\Big|$$

A word of ordinal length J over an alphabet Σ is a mapping from J to Σ. Let ρ be such a word, and j an element of J. The *prefix* of ρ of length j denoted by $\rho_{<j}$ is defined as $(\rho_i)_{i<j}$. The limit of ρ, denoted $\lim \rho$, is the set: $\{a \in \Sigma \mid \forall j \in J, \ \exists i > j, \rho_i = a\}$.

Bruyère and Carton define in [BC01] an automaton $\mathcal{A}$ on these words as a tuple $(Q, \Sigma, \mathcal{E}, \mathcal{T}, \mathcal{I}, \mathcal{F})$. Q is a finite set of states, Σ is a finite alphabet, $\mathcal{E}$ and $\mathcal{T}$ are respectively the successor and limit transition relations, $\mathcal{I} \subseteq Q$ is the set of initial states, and $\mathcal{F} \subseteq Q$ is the set of final states. The successor transitions of $\mathcal{E}$ are usual transitions, of the form $p \xrightarrow{a} q \in Q \times \Sigma \times Q$. The *limit transitions* of $\mathcal{T}$ are of the form $P \xrightarrow{\lim} q \in \mathcal{P}(Q) \times Q$.

A run of $\mathcal{A}$ on a word $x = (x_j)_{j \in J}$ is a word ρ of length $\hat{J}$ on Q, verifying the following conditions:

- if c is the initial cut, $\rho_c \in \mathcal{I}$;
- if $j \in J$, $\rho_{j^-} \xrightarrow{x_j} \rho_{j^+} \in \mathcal{E}$;
- if c has no predecessor, $\lim \rho_{<c} \xrightarrow{\lim} \rho_c \in \mathcal{T}$;
- if c is the final cut, $\rho_c \in \mathcal{F}$.

Example 2. Figure 1 shows a simple automaton over the alphabet $\{a, b\}$.

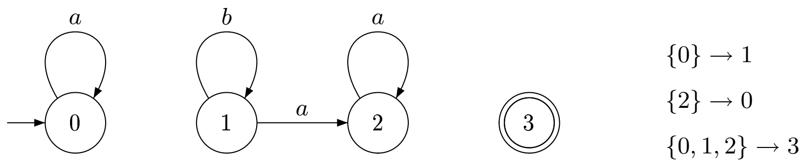


Fig. 1. Automaton recognising $(a^\omega b^\star a^\omega)^\omega$

From the results of Choueka in [Cho78], one can derive Theorem 3:

Theorem 3. *In an automaton with n states where the transitions are of the form $P \xrightarrow{\lim} q \notin P$, there are no runs of length greater than ω^n.*

2.2 Infinite games

We recall here the usual concepts related to infinite duration games. Such a game $\mathbb{G}$ is played by two players called Eve and Adam on an *arena* of the form $(\mathbb{Q}, \mathbb{E})$, which is a directed graph partitioned between Adam's vertices ($\mathbb{Q}_A$, represented by $\Box$) and Eve's vertices ($\mathbb{Q}_E$, represented by $\bigcirc$). The *winning condition* $\mathbb{W} \subseteq \mathbb{Q}^\omega$ describes the plays won by Eve. We refer the reader to [Tho95] for more details on infinite games.

A *play* of $\mathbb{G}$ is a (finite or infinite) path in the arena. We assume that every state has at least one successor, so any finite play can be prolonged into an infinite one. Prolonging a finite play by one vertex is called a *move* in the game. During the play, when the last vertex of the current prefix is in $\mathbb{Q}_E$, Eve chooses the next move, otherwise Adam does. Eve wins the play if and only if it is in $\mathbb{W}$.

A *strategy* for Eve is a function $\sigma : \mathbb{Q}^*\mathbb{Q}_E \to \mathbb{Q}$ such that for every finite prefix w ending in a state $q \in \mathbb{Q}_E$, $(q, \sigma(w)) \in \mathbb{E}$. A play $\rho = \rho_0\rho_1\rho_2\ldots$

is *consistent* with a strategy σ (for Eve) if for every n such that $\rho_n \in \mathbb{Q}_E$, $\rho_{n+1} = \sigma(\rho_0\rho_1 \dots \rho_n)$. A strategy σ is *winning* for Eve if every play consistent with σ is won by Eve. Strategies and winning strategies for Adam are defined likewise. A *strategy with memory* M for Eve is defined by a transducer (M, ν, μ) and an *initial memory state* $\Omega_0 \in M$. The two functions $\nu : M \times \mathbb{Q} \to \mathbb{Q}$ and $\mu : M \times \mathbb{Q} \to M$ respectively give the next move when the token is in Q_E, and update the memory.

We use Muller games in our proofs. In these games, the winning condition is defined by a subset $\mathbb{M}$ of $\mathcal{P}(\mathbb{Q})$. Eve wins if the set of states occurring infinitely often during the play belongs to $\mathbb{M}$. For a play ρ, this set is denoted by $\mathrm{Inf}(\rho)$ ($\mathrm{Occ}(\rho)$ denotes the set of states occurring in ρ).

When considering complexity issues, the representation of the winning condition is important, as it directly influences the input size. The most straightforward is to list the elements of $\mathbb{M}$ — the *explicit* representation. Other possibilities include colouring, Zielonka trees and DAGs, win-set conditions, and Emerson-Lei conditions. We will only define Emerson-Lei conditions here, and refer to [Zie98] and [HD05] for more details.

Emerson-Lei games were introduced in [EL85] and are equivalent, in terms of expressive power, to the usual Muller games. The winning condition is defined by a Boolean formula φ using elements of $\mathbb{Q}$ as variables. The play ρ is winning for Eve if the truth assignment mapping every state of $\mathrm{Inf}(\rho)$ to `true` and every other state to `false` satisfies φ.

2.3 Games of ordinal length

As done in [BC01] for finite automata, we extend the classical model of infinite games to arenas admitting paths of ordinal length by adding limit transitions. A reachability game of ordinal length is defined as a tuple $(Q, Q_E, Q_A, \mathcal{E}, t, \circledcirc, \otimes)$. The special states $\circledcirc$ and $\otimes$ are the only two states without successors. The function t maps $\mathcal{P}(Q)$ to $Q \cup \{\circledcirc, \otimes\}$ in a way such that $t(P) \notin P$. A play is a word ρ of ordinal length on $Q \cup \{\circledcirc, \otimes\}$. Every play that does not end in $\{\circledcirc, \otimes\}$ can be prolonged through a move or a limit transition, and by Theorem 3, there are no plays of length greater or equal to ω^ω. Eve wins if the play ends in $\circledcirc$, while Adam wins when the token reaches $\otimes$. For technical reasons, we suppose without loss of generality that our arenas are *semi-alternating*, *i.e.* that the successors of a state of Adam belong to Eve[1]. The notion of strategy is extended naturally, by considering ordinal prefixes rather than finite ones. Strategies with memory are extended likewise, with a

[1] This allows us to only define t (and later o) on sets containing a state of Eve.

memory transducer on ordinals. Notice that restricting plays to lengths smaller than ω^ω makes sense in the verification problem: infinite sequences represent events of very different durations, and an infinite hierarchy of infinitesimality seems far-fetched.

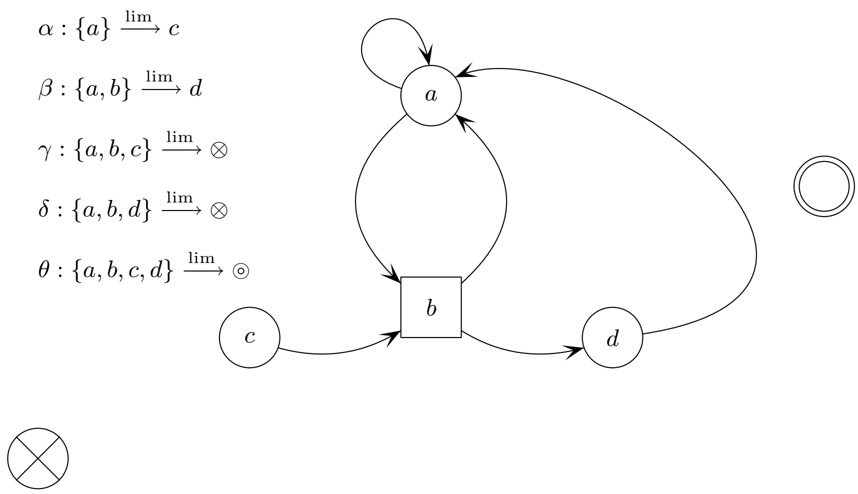


Fig. 2. Game of ordinal length

3 Solving ordinal reachability games

In this section, we consider the problem of deciding the winner in an ordinal reachability game. Our result is formalised as Theorem 4, and this section will mainly be devoted to its proof.

Theorem 4. *Two player reachability games of ordinal length are determined on finite arenas.*

We prove this theorem through a reduction from an ordinal reachability game G to a Muller game $\mathbb{G}$. This construction is described in Section 3.1. Section 3.2 gives the main steps of the proof of Lemmas 5 and 6 by strategy translation.

Lemma 5. *If Eve wins in $\mathbb{G}$ from $q \in Q$, she also wins in G from q.*

Lemma 6. *If Adam wins in $\mathbb{G}$ from $q \in Q$, he also wins in G from q.*

From these two Lemmas and Theorem 7, we derive Theorem 4.

Theorem 7 ([Mar75]). *Muller games are determined.*

3.1 Reduction to Muller Games

We describe here a reduction from a reachability game of ordinal length G to a Muller game $\mathbb{G}$. The idea is to compel the players to simulate the limit transitions, in such a way that an uncooperative player will lose the play. In this regard, our approach is similar to the one by Chatterjee, de Alfaro, Jurdzinski and Henzinger in [CJH03] and [CdAH05], where they use parity conditions in order to simulate randomness for qualitative winning regions. In both approaches, there is an identity between the winning regions of the original game and the reduced one, but *not* between the actual plays.

The fundamental idea of this reduction is that a word of length less than ω^ω can be described by a finite word with "shortcuts" in lieu of limit transitions. These shortcuts have to be taken in two steps, guaranteeing that both players agree to take it. The "widget" we use is described in Figure 3: for each set of states P containing a state of Eve[2], we distinguish one state $o(P)$ in $P \cap Q_E$. In addition to its original successors, this state now leads to a new state $\chi(P)$, which belongs to Adam. There, he can either accept the transition, and proceed to $t(P)$, or refuse it, and go to another clone of $o(P)$, called $\xi(o(P))$. This clone only leads to the original successors of $o(P)$ in G, not to $\chi(P)$. The definition of the Muller condition guarantees that no one can block the play without losing. It contains all the sets of the form $P \cup \{\chi(P)\}$, so if Adam repeatedly declines to take a legitimate proposition, he will lose. Formally, the reduced game $\mathbb{G}$ from $G = (Q, Q_E, Q_A, \mathcal{E}, \mathcal{T}, \circledcirc, \otimes)$ is defined by:

$$\begin{aligned}
\mathbb{Q}_E &= Q_E \cup \{\xi(q) \mid q \in Q_E\} \\
\mathbb{Q}_A &= Q_A \cup \{\chi(P) \mid P \in \mathcal{P}(Q)\} \\
\mathbb{E} &= \mathcal{E} \\
&\quad \cup \{(o(P), \chi(P)) \mid P \in \mathcal{P}(Q)\} \\
&\quad \cup \{(\chi(P), t(P)) \mid P \in \mathcal{P}(Q)\} \\
&\quad \cup \{(\chi(P), \xi(o(P))) \mid P \in \mathcal{P}(Q)\} \\
&\quad \cup \{(\xi(p), q) \mid (p, q) \in \mathcal{E}\} \\
&\quad \cup \{(\circledcirc, \circledcirc), (\otimes, \otimes)\} \\
\mathbb{M} &= \{P \cup \mathbb{H} \mid P \subseteq Q, \mathbb{H} \cap Q = \emptyset, \chi(P) \in \mathbb{H}\}
\end{aligned}$$

[2] See Footnote 1 on page 4.

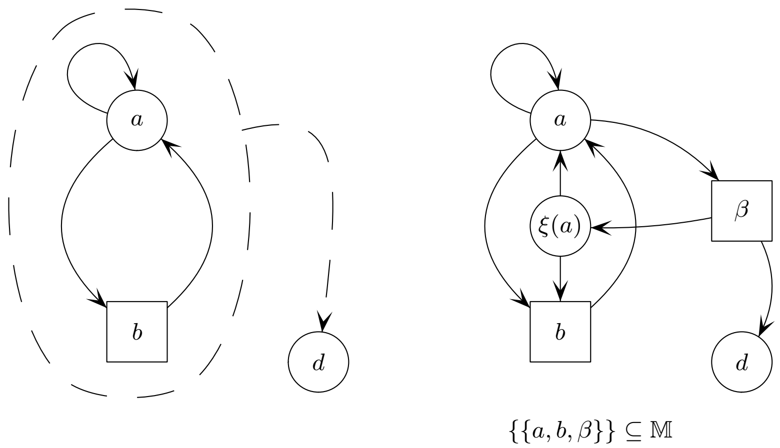


Fig. 3. Widget for $\{a, b\} \xrightarrow{\lim} d$

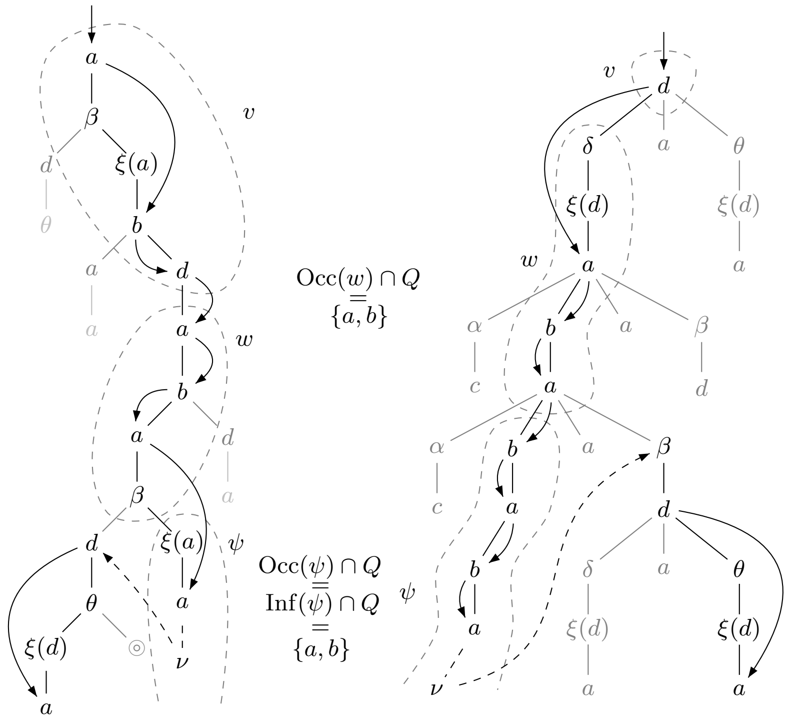
3.2 Strategy translation: from Muller to Ordinal Reachability

Lemmas 5 and 6 are proved through a similar notion of *strategy translation*: from a winning strategy[3] in the reduced Muller game $\mathbb{G}$ we can derive a winning strategy in the ordinal game G. The memory states of this new strategy are plays of the Muller game that are consistent with the original strategy.

The memory will evolve during the course of a play in G, moving along the tree of all plays consistent with the strategy. Successor transitions extend the current play, lengthening the memory. Limit transitions will branch to another prefix in the tree, under suitable assumptions. An exemple of this whole process (for both players) is given in Figure 4.

Successor transitions The successor transitions always lengthen the memory, and guarantee that it remains consistent with the original strategy. The basic idea is to copy the current move in the memory. However, we have to be cautious with the states of Eve: she must have the possibility to choose, either as proponent or opponent, to go to a state of the Muller game that does not belong to the original game. This case is treated differently depending whether we consider a strategy for Eve or for Adam.

[3] It is not possible to translate a losing strategy with our technique, not even to a losing one.



(a) For Eve (b) For Adam

Fig. 4. Strategy translation from $\mathbb{G}$ to G

Eve: For a strategy σ of Eve, the problem arises when, in a state $q \in Q$ with memory $\Omega \in \mathbb{Q}^*$, $\sigma(\Omega \cdot q)$ does not belong to Q, but is a new state $\chi(P)$. To deal with this case, we add three moves to the memory instead of one: $\mu(q, \Omega) = \Omega \cdot q \cdot \chi(P) \cdot \xi(q)$. This is still a finite play consistent with σ, so Eve can now send the token to the location $\sigma(\Omega \cdot q \cdot \chi(P) \cdot \xi(q))$, which is a successor of q in G by definition of $\xi(q)$.

Adam: In Adam's case, the problem is not what the strategy can do — Adam's options in the states of Q are the same in G and in $\mathbb{G}$ — but how to interpret what Eve does. Supposing that she always keeps to states of Q is not correct, since almost any strategy wins against such behaviors. We will thus consider that Eve always chooses to propose transitions when Adam refuses them: if the token is in state $q \in Q_E$ and Eve sends the token to q', we first look for a set P such that:

- $o(P) = q$ (Eve can propose the transition)
- $\tau(\Omega \cdot q \cdot \chi(P)) = \xi(P)$ (Adam's strategy τ would refuse it)

If we do not find one, the memory is simply updated to $\Omega \cdot q$. Otherwise, we denote by P the one such that $\chi(P)$ did not occur in Ω for the longest time, and update the memory to $\Omega \cdot q \cdot \chi(P) \cdot \xi(P)$. Here also, the resulting memory is still consistent with τ.

Limit transitions When the play goes through a limit transition, the idea is to go back to a suitable shortcut in the past of the memory. To explain how we do it, we first fix some notations: we consider a play ρ consistent with our new strategy, and a transition $P \xrightarrow{\lim} q$ occuring at position j — *i.e.* $\lim \rho_{<j} = P$ and $\rho_j = q$. Furthermore, we denote by $(\Omega_i)_{i<j}$ the (transfinite) sequence of memory states occuring in the course of $\rho_{<j}$.

One first problem is that there is no last memory state before j from which to work. Propositions 8 and 9 compensate for this:

Proposition 8. *The sequence $(\Omega_i)_{i<j}$ has a limit, denoted by $\Omega_{<j}$, which is an infinite play in $\mathbb{G}$ consistent with the original winning strategy.*

Proposition 9. $\lim \rho_{<j} = \mathrm{Inf}(\Omega_{<j})$

With these two propositions, we can now update the memory. We have to be careful when choosing where to branch, in order to ensure that the memory still grows with respect to a possible higher order transition. It is done by keeping one copy of the limit set in the resulting memory: We divide $\Omega_{<j}$ in three factors v, w, and ψ:

- v contains all occurrences of states in $Q \setminus \mathrm{Inf}(\Omega_{<j})$;
- w contains an occurrence of each state in $\mathrm{Inf}(\Omega_{<j}) \cap Q$. Furthermore, it must end at a suitable branching point: for Eve, it means ending with an occurence of $\chi(P)$; for Adam, it must end with an occurence of $o(P)$, and be such that $\tau(v \cdot w \cdot \chi(P)) = t(P)$;
- ψ contains the remainder of $\Omega_{<j}$.

The factor $v \cdot w$ remains as a prefix of the new memory. Instead of ψ, there is now an accepted shortcut: $v \cdot w \cdot t(P)$ or $v \cdot w \cdot \chi(P) \cdot t(P)$, depending on whether we are building a strategy for Eve or for Adam. This process is described in Figure 4.

Once the soundness of our construction is accepted, it is not difficult to show that it produces winning strategies: a full play always ends in $\circledcirc$

or in ⊗, and the current state is systematically added to the memory. As this memory can only contain plays consistent with the original strategy, the "bad" state (⊗ if we build a strategy for Eve, ⊚ if it is for Adam) cannot occur in the memory. Thus, the final state of a play consistent with our new strategy is necessarily the "good" one. This completes the proof of Lemmas 5 and 6.

4 Complexity

We now consider the complexity of solving ordinal reachability games. As in Muller games, we need to specify precisely how the transitions are represented. In the case where transitions are represented as relevant sets, colour sets, a Zielonka DAG or Boolean formulae, we get Theorem 10.

Theorem 10. *Deciding the winner in a reachability game of ordinal length whose limit transitions are represented as relevant sets, colour sets, a Zielonka DAG or Boolean formulae is* `PSPACE`*-complete.*

We will prove the membership part in Section 4.1, and the hardness part in Section 4.2. The complexity in the case of transitions represented explicitly, or as Zielonka Trees is left open.

4.1 Reduction to Emerson-Lei games

Lemma 11. *Deciding the winner in a reachability game of ordinal length whose transitions are represented as Boolean formulae is* `PSPACE`*.*

Proposition 12. *The reduced game $\mathbb{G}$ is equivalent to an Emerson-Lei game $\mathbb{L}$ of size polynomial in the size of G, if the transitions of G are represented as Boolean formulae.*

Proof. In order to get a polynomial reduction, we need to avoid the exponential blow-up that occurs when we add a state $\chi(P)$ for each set of states P. It can be done, by noticing that if two sets P and P' are such that $o(P) = o(P')$ and $t(P) = t(P')$, $\chi(P)$ and $\chi(P')$ have exactly the same neighbours in $\mathbb{G}$. In the definition of $\mathbb{L}$, we can thus replace them both by a single state $\kappa(o(P), t(P))$. This limits the number of new states to $|Q|^2 + |Q|$.

The winning condition of $\mathbb{L}$ can now be described in Emerson-Lei formalism:

$$\varphi = \circledcirc \vee \bigvee_{q \in Q \cup \{\circledcirc, \otimes\}} \Big(\varphi_q \wedge \bigvee_{p \in Q} \kappa(p, q)\Big)$$

The size of formula φ is $O(\sum_{q\in Q\cup\{\circledcirc,\otimes\}}(|\varphi_q|+n))$, which is polynomial in the size of G. This modified reduction is still fair. □

The last step of the proof is Theorem 13:

Theorem 13 ([HD05]). *Deciding the winner in an Emerson-Lei game is* PSPACE*-complete.*

Lemma 11 follows directly from Property 12 and Theorem 13. Corollary 14 follows from the results of [HD05] about succinctness.

Corollary 14. *Deciding the winner in a reachability game of ordinal length whose limit transitions are represented as relevant sets, colour sets, or a Zielonka DAG is* PSPACE.

4.2 Hardness results

The hardness result also derives from Theorem 13. Indeed, any classical Muller game $(\mathbb{Q}, \mathbb{Q}_E, \mathbb{Q}_A, \mathbb{E}, \mathbb{M})$ can be represented as the ordinal reachability game $(\mathbb{Q}, \mathbb{Q}_E, \mathbb{Q}_A, \mathbb{E}, t, \circledcirc, \otimes)$, with $t(P) = \circledcirc$ for all $P \in \mathbb{M}$ and $t(P) = \otimes$ for all $P \notin \mathbb{M}$. The strategies and plays will be the same in both games. The only difference is that after ω moves in the reachability game, the token will take a limit transition to $\circledcirc$ or $\otimes$, depending on whether the infinite play is winning for Eve or Adam in the Muller game. This reduction can be done for any representation of the Muller condition.

Lemma 15. *In a reachability game of ordinal length whose limit transitions are represented as relevant sets, colour sets, a Zielonka DAG or Boolean formulae, deciding the winner is* PSPACE*-hard.*

5 Conclusion

We have extended the classical model of infinite games to games of ordinal length. These games, that generalise all regular games, are determined, and the winner is decidable through a reduction to Muller games. If the limit transitions are represented as relevant sets, colour sets, a Zielonka DAG or Boolean formulae, the problem is PSPACE-complete.

We intend now to use this formalism in the context of timed games, following for example the work of [JT07]. Another perspective concerns the minimal quantity of memory that is necessary to define winning strategies in ordinal games. Finally, we would like to consider less general games, where the transitions can be represented in a more compact way — especially parity — and study the effects on complexity and memory.

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