Colorings of Signed Graphs

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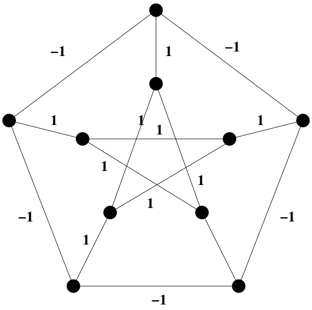
Signed Graphs

A signed graph is a graph G together with a mapping $\sigma : E \to \{1, -1\}$.



A signed graph is denoted by (G, σ)

D. König 1935

The signed graphs and the balanced signed graphs were introduced by Harary in 1953.
But all the notions can be found in the book of König (Theorie der endlichen und unendlichen graphen, 1935).
An important, fundamental and prolific work on signed graphs was done by Zaslavsky in 1982.

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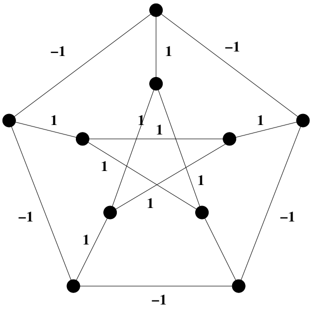

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Coloring of signed graphs. D. Cardwright and F. Harary 1968

Definition

A k-coloring of signed graph (G, σ) is a partition of V(G) into k subsets such that

1. every two vertices joined by a negative edge are in different color sets and
2. every two vertices joined by a positive edge are in the same color set.

We say that G has a coloring, or is colorable, if it has an k-coloring for some k.

It follows immediately from these definitions that if a signed graph G has only negative edges, the problem of coloring the signed graph G is the same as that of coloring a graph.
If, however, G has some positive edges, it may not be colorable.

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Coloring of signed graphs. D. Cardwright and F. Harary 1968

Let G^+ be the spanning subgraph obtained by removing all negative edges from G.

Theorem

The following statements are equivalent for any signed graph G.

1. *G has a coloring.*
2. *G has no negative edge joining two vertices in the same positive component of G^+.*
3. *G has no cycle with exactly one negative edge.*

Corollary

A complete signed graph K has a coloring if and only if K has no 3-cycle with exactly one negative edge.

Balanced circuit D. Cardwright and F. Harary 1968

In an attempt to formalize a psychological theory proposed by Heider [1958], they defined a signed graph G as balanced if every cycle has an even number of negative edges.

Definition

- For a subgraph Q of a signed graph (G, σ), let $\sigma(Q) = \prod_{e \in Q} \sigma(e)$.
- A *circuit* in a signed graph G is a connected 2-regular subgraph of G.
- A circuit C is *balanced* if $\sigma(C) = 1$ (*unbalanced* if $\sigma(C) = -1$).
- A signed graph G is called *balanced* if each circuit C of G is balanced.

Coloring of signed graphs. D. Cardwright and F. Harary 1968

They showed that G is balanced if and only if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every positive edge joins two vertices in the same subset and every negative edge joins a vetex of V_1 with one of V_2 Clearly, V_1 and V_2 are color sets.Thus,

Proposition

A signed graph G is balanced if and only if it has a 2-coloring (is bicolorable).

Proposition

A graph G is bicolorable and only if it has no odd cycles

Balanced signed graph

A signed graph is *balanced* if every circuit has positive sign product.

Antibalanced signed graph

A signd graph is *antibalanced* if its negative is balanced.

Coloring of signed Graphs- Zaslavsky 1982.

Coloring- T. Zaslavsky 1982

A proper k-coloring (k is a nonnegative integer) of a signed graph (G, σ) is a mapping from $c : V \to \{-k, \cdots, -1, 0, 1, \cdots, k\}$ such that:

- if $e = xy$ is a positive edge then $c(x) \neq c(y)$
- if $e = xy$ is a negative edge $c(x) \neq -c(y)$.

These two conditions can be written $c(x) \neq \sigma(e)c(y)$

Coloring of signed graph-Zasvlasky 1982

This notion of coloring and the associate chromatic polynomial were introduced by Zaslavsky in 1982. Two different colorings were considered: the one which we just defined and the zero-free coloring where the color zero is not used.

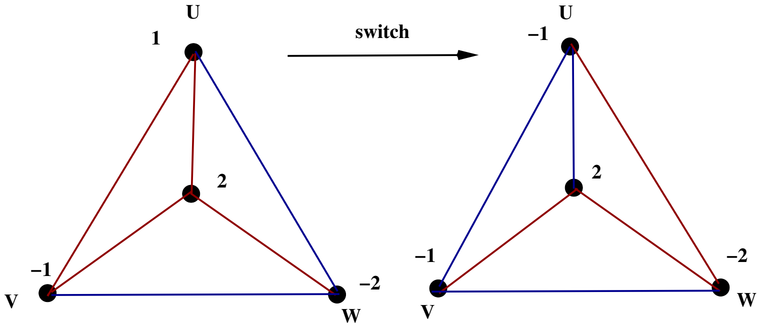
Chromatic numbers

The chromatic number of a signed graph (G, σ) denoted by $\chi_z(G, \sigma)$ (resp. $\chi_z^*(G, \sigma)$) is the smallest k such that (G, σ) admits a k-coloring (resp. a zero-free k-coloring).

Switching

Switching (G,σ) by $X \subseteq V$ means reversing the sign of each edge that has one endpoint in X and one in $V \setminus X$. A signed graph (G,σ') obtained by switching (G,σ) is said to be *switching equivalent* to (G,σ). It is denoted by $(G,\sigma) \sim (G,\sigma')$. If a signed graph (G,σ) is colored and is switched by $X \subseteq V$, the color is also switched, by taking the opposite value for the color of a switched vertex. It is easy to see that after the switching operation we still have a proper coloring.

Switching



Switching was first described by Abelson and Rosenberg (1958).

Switching- Zasvlasky 1982

Observation

Let T be a subtree of a signed graph (G, σ) then (G, σ) is switching equivalent to (G, σ') where all the edges of T are negative edges.

Homomorphism of signed graphs

$$(G, \sigma) \stackrel{hom}{\rightarrow} (G', \sigma')$$

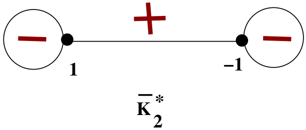
$$\stackrel{definition}{\Longleftrightarrow}$$

it exists a mapping $\phi : V(G) \rightarrow V(G')$ such that

- if $xy \in E(G)$ then $\phi(x)\phi(y) \in E(G')$
- $\sigma(xy) = \sigma'(\phi(x)\phi(y))$

Zero-free k-coloring: $\chi_z^*(G,\sigma)$

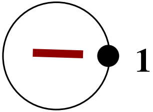
$$\chi_z^*(G,\sigma) = 1 \text{ if and only if } (G,\sigma) \xrightarrow{hom} \bar{K}_2^*$$



Zero-free k-coloring: $\chi_z^*(G,\sigma)$

$\chi_z^*(G,\sigma) = 1$ if and only if $(G,\sigma) \sim (G,\sigma')$ and

$$(G,\sigma') \overset{hom}{\rightarrow} K_1^*$$



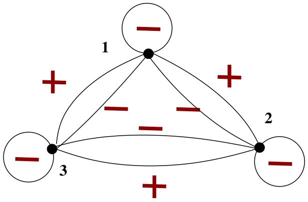
$\mathbf{K_1^*}$

(G,σ') is antibalanced (all the edges are negative).

Zero-free k-coloring: $\chi_z^*(G,\sigma)$

$$\chi_z^*(G,\sigma) = k \text{ if and only if } (G,\sigma) \sim (G,\sigma') \text{ and}$$

$$(G,\sigma') \stackrel{hom}{\rightarrow} K_k^*$$



K_3^*

R. Naserasr, E. Rollova, E. Sopena

Homomorphisms of signed graphs
Journal of Graph Theory .Volume79, Issue3 July 2015
Pages 178-212

Colored Homomorphisms of Colored Mixed Graphs-
J. Nešetřil, A.R.
Journal of Combinatorial Theory B (50) (1)
Pages 147-155
2000

Homomorphisms of Edge Colored Graphs and Coxeter Groups.
N. Alon, T.H. Marshall
Journal of Algebraic Combinatorics
July 1998, Volume 8, Issue 1, pp 5-13 .

Zero-free coloring

Proposition

If a signed graph (G, σ) is m-degenerate then $\chi_z^*(G) \leq \lceil \frac{m+1}{2} \rceil$

Corollary

Let (G, σ) be a signed graph.

- *If G is a tree $\chi_z^*(G) = 1$*
- *If G is a planar: $\chi_z * (G) \leq 3$*
- *If G is a outerplanar: $\chi_z^*(G) \leq 2$*

Zero-free coloring

Observation

Let (G, σ) *be a signed graph.*

- *If G is an union of 2 disjoint spanning trees:* $\chi_z^*(G) \leq 2$
- *If G is* K_4*-minor free* $\chi_z^*(G) \leq 2$
- *If G is balanced* $\chi_z^*(G) \leq \lceil \frac{\chi(G)}{2} \rceil$
- *If G is antibalanced* $\chi_z^* G) = 1$
- *If G is a planar triangle free* $\chi_z^*(G) \leq 2$

- A signed graph (G, σ) is *balanced* then it is swiching equivalent to (G, σ') where σ' is positive on every edge of G.
- A signed graph (G, σ) is *antibalanced* then it is swiching equivalent to (G, σ') where σ' is negative on every edge of G.

Acyclic Coloring

Definition

- A proper vertex coloring of a graph G is an acyclic coloring if every two classes induce a forest.
- The smallest number of colors needed to color G acyclically is called the acyclic chromatic number and it is denoted by $a(G)$.

Proposition

Let G be a signed graph is the underlying ordinary graph has acyclic chromatic number k then

- *if k is odd* $\chi_z(G) \leq \lceil \frac{k-1}{2} \rceil$
- *if k is even* $\chi_z^*(G) \leq \lceil \frac{k}{2} \rceil$

Proposition

If G is a planar with girth at least 5 then $\chi_z(G) \leq 1$

We refine the definition of chromatic number of a signed graph.

- $M_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$,
- $M_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k + 1$

A proper colouring of G that uses colors from M_n will be called an *n-coloring*. Note that if G admits an n-colouring, then it also admits an m-colouring for each $m \geq n$.

Definition

The smallest n such that G admits an n-coloring will be called the *signed chromatic number of* G and will be denoted by $\chi_s(G)$.

For a signed graph G let $\chi(G)$ denote the usual chromatic number of its underlying graph.

Signed Chromatic number and chromatic number

$\chi_s(G)$ to $\chi(G)$.

Theorem

For every signed graph G one has $\chi_s(G) \leq 2\chi(G) - 1$.

Moreover, the bound is sharp.

Proof.

Take a proper coloring ϕ of the underlying graph of G with colours $0, 1, \ldots, k-1$, where $k = \chi(G)$. Clearly, ϕ is a proper coloring of G, as well. Since all colours are contained in $\{0, \pm 1, \ldots, \pm(k-1)\}$, it is a $(2k-1)$-coloring. The inequality follows. $\square$

Signed Chromatic number and chromatic number

The bound is reached by the family of signed graphs $\{G_n\}_{n\geq 2}$ which can be constructed as follows.

- One *positive* copy of K_n with all edges positive and denote by H_1^n
- $n-1$ *negative* copies of K_n with all edges negative and denote by $H_2^n, H_3^n, \ldots, H_n^n$.
- The vertices in H_i^n will be denoted $v_{i,1}, v_{i,2}, \ldots, v_{i,n}$. Any pair of vertices $v_{i,j}$ and $v_{k,j}$ is called *corresponding*.
- We insert a positive edge for each pair of non-corresponding vertices from different copies of K_n.

Signed Chromatic number and chromatic number

$$\chi(G_n) = n$$

$$\chi_s(G_n) \leq 2n - 1$$

We prove that

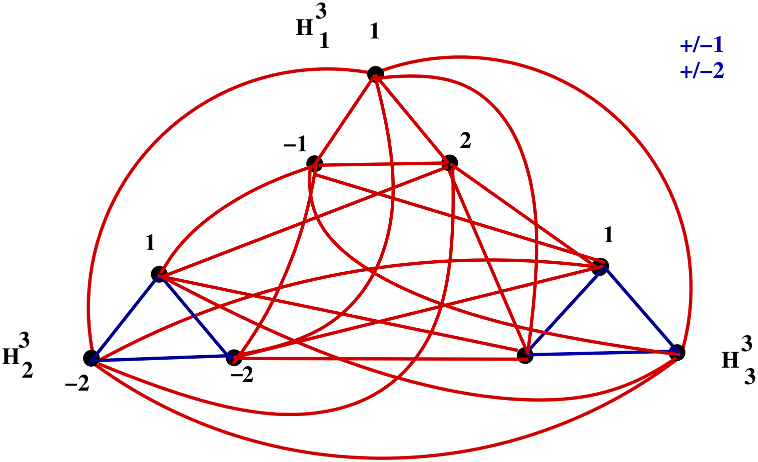
$$\chi_s(G_n) = 2n - 1$$

Assume to the contrary, that G_n is colorable with colours from

$$M_{2n-2} = \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$$

Since M_{2n-2} contains $n-1$ different absolute values, the coloring of H_1^n with elements of M_{2n-2} must contains at least two opposite values. In the same veine H_i^n must contains at least two identical values. These values cannot belongs to the set of colors uses for H_1^n and must be different for all the H_i^n. But we have only $n-2$ remaining values. $\square$

Signed chromatic number and chromatic number



Generalized Brooks Theorem

Proposition

Let G be a signed complete graph on n vertices. Then $\chi_s(G) \leq n$ and $\chi_s(G) = n$ if and only if G is balanced.

Generalized Brooks Theorem

Theorem (Máčajová, R. and Škoviera- 2016)

Let G be a simple connected signed graph different from a balanced complete graph, a balanced circuit of odd length, and an unbalanced circuit of even length. Then

$$\chi_s(G) \leq \Delta(G).$$

Generalized Brooks Theorem

Improvments

- Fleiner and Wiener (2016) gave a shorter proof using a DFS-tree.
- Schweser and Stibietz (2017) proved a Brooks'type theorem for signed list coloring (degree choosable).
- Bernshteyn, Kostochka and Pron (2017) proved it using DP-coloring (DP-degree-coloring).

DP-coloring-Dvořák and Postle 2016

Let G be a graph and L be a list assignment of G. For each edge uv in G let $M_{L,uv}$ be any matching (maybe empty) between $\{u\} \times L(u)$ and $\{v\} \times L(v)$

Definition (Bernshteyn, Kostochka and Pron 2017)

Let $\mathcal{M}_L = \{M_{L,uv} : uv \in E(G)\}$, a matching assignment over L. A graph H is said a to be a $\mathcal{M}_L$-cover of G if it satisfies all the following conditions:

1. The vertex set of H is $\cup_{u \in V(G)}(\{u\} \times L(u))$
2. For every $u \in V(G)$, the graph $H[(\{u\} \times L(u)]$ is a clique.
3. For every edge uv in G $\{u\} \times L(u)$ and $\{v\} \times L(v)$ induce the graph obtained from $M_{L,uv}$ in H.

DP-coloring

Definition

An $\mathcal{M}_L$ coloring of G is an independent set of I in the $\mathcal{M}_L$-cover with $|I| = |V(G)|$. The *DP*-chromatic number, denoted by $\chi_{DP}(G)$, is the minimum integer k such that G admits a $\mathcal{M}_L$ coloring for each k-list assignment L and each matching $\mathcal{M}_L$ over L.

G is k-DP-colorable if $\chi_{\mathbf{DP}}(\mathbf{G}) \leq \mathbf{k}$.

Proposition

If G is $k-DP$-colorable then for any signature σ (G, σ) is signed k-choosable.

Theorem (Kim and Ozeki 2017)

Foe each $k \in \{3, 4, 5, 6\}$, every planar graph without C_k is $4 - DP$-colorable.

Theorem (Wang 2018)

Every toroidal graph without triangles adjacent to C_5 is $4 - DP$-colorable.

Planar Graphs - A risky conjecture

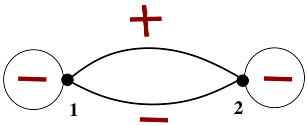
Conjecture (MRS 2016)

If G is a simple planar then for any signature $\chi_z^*(G) \leq 2$.

It is equivalent to:

Conjecture (MRS 2016)

If G is a simple planar then for any signature σ: $(G,\sigma) \sim (G,\sigma')$ and $(G,\sigma') \stackrel{hom}{\to} K_2^*$



K_2^*

With the refined definition of coloring we introduced:

Conjecture (MRS 2016)

Every simple signed planar graph has $\chi_s(G) \leq 4$

Planar Graphs- A note on two conjectures that strengthen the four color theorem- X. Zhu 2017

Conjecture (Küngen and Ramamurthi- 2012)

Assume that G is a planar graph and L is a 2-list assignment of G. Then there is a L-coloring of G such that each color class induces a bipartite graph.

Conjecture MRS implies the Conjecture of Küngen and Ramamurthi.

Theorem (Zhu 2017)

Assume that G is planar graph. If for any signature σ of G, the graph (G, σ) is 4-colorable, then for any 2-list assignment L of G, there is an L-coloring of G such that each color class induces a bipartite graph.

Thank you for your attention!