Colorings of Signed Graphs

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Signed Graphs

A signed graph is a graph $G$ together with a mapping $\sigma : E \rightarrow \{1, -1\}$.

A signed graph is denoted by $(G, \sigma)$

D. König 1935

The signed graphs and the balanced signed graphs were introduced by Harary in 1953.
But all the notions can be found in the book of König (Theorie der endlichen und unendlichen graphen, 1935).
An important, fundamental and prolific work on signed graphs was done by Zaslavsky in 1982.
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Definition

A $k$-coloring of signed graph $(G, \sigma)$ is a partition of $V(G)$ into $k$ subsets such that

1. every two vertices joined by a negative edge are in different color sets and
2. every two vertices joined by a positive edge are in the same color set.

We say that $G$ has a coloring, or is colorable, if it has an $k$-coloring for some $k$.

It follows immediately from these definitions that if a signed graph $G$ has only negative edges, the problem of coloring the signed graph $G$ is the same as that of coloring a graph.
If, however, $G$ has some positive edges, it may not be colorable.
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It follows immediately from these definitions that if a signed graph \(G\) has only negative edges, the problem of coloring the signed graph \(G\) is the same as that of coloring a graph.
If, however, \(G\) has some positive edges, it may not be colorable.
Let $G^+$ be the spanning subgraph obtained by removing all negative edges from $G$.

**Theorem**

The following statements are equivalent for any signed graph $G$.

1. $G$ has a coloring.
2. $G$ has no negative edge joining two vertices in the same positive component of $G^+$.
3. $G$ has no cycle with exactly one negative edge.

**Corollary**

A complete signed graph $K$ has a coloring if and only if $K$ has no 3-cycle with exactly one negative edge.
In an attempt to formalize a psychological theory proposed by Heider [1958], they defined a signed graph $G$ as balanced if every cycle has an even number of negative edges.

**Definition**

- For a subgraph $Q$ of a signed graph $(G, \sigma)$, let $\sigma(Q) = \prod_{e \in Q} \sigma(e)$.
- A *circuit* in a signed graph $G$ is a connected 2-regular subgraph of $G$.
- A circuit $C$ is *balanced* if $\sigma(C) = 1$ (*unbalanced* if $\sigma(C) = -1$).
- A signed graph $G$ is called *balanced* if each circuit $C$ of $G$ is balanced.
They showed that $G$ is balanced if and only if $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every positive edge joins two vertices in the same subset and every negative edge joins a vertex of $V_1$ with one of $V_2$ Clearly, $V_1$ and $V_2$ are color sets. Thus,

**Proposition**

A signed graph $G$ is balanced if and only if it has a 2-coloring (is bicolorable).

**Proposition**

A graph $G$ is bicolorable and only if it has no odd cycles
Balanced signed graph

A signed graph is *balanced* if every circuit has positive sign product.

Antibalanced signed graph

A signed graph is *antibalanced* if its negative is balanced.
A proper $k$-coloring ($k$ is a nonnegative integer) of a signed graph $(G, \sigma)$ is a mapping from $c : V \rightarrow \{-k, \cdots, -1, 0, 1, \cdots, k\}$ such that:

- if $e = xy$ is a positive edge then $c(x) \neq c(y)$
- if $e = xy$ is a negative edge $c(x) \neq -c(y)$.

These two conditions can be written $c(x) \neq \sigma(e)c(y)$.
This notion of coloring and the associate chromatic polynomial were introduced by Zaslavsky in 1982. Two different colorings were considered: the one which we just defined and the zero-free coloring where the color zero is not used.

**Chromatic numbers**

The chromatic number of a signed graph \((G, \sigma)\) denoted by \(\chi_z(G, \sigma)\) (resp. \(\chi_z^*(G, \sigma)\)) is the smallest \(k\) such that \((G, \sigma)\) admits a \(k\)-coloring (resp. a zero-free \(k\)-coloring).
Switching

Switching \((G, \sigma)\) by \(X \subseteq V\) means reversing the sign of each edge that has one endpoint in \(X\) and one in \(V \setminus X\). A signed graph \((G, \sigma')\) obtained by switching \((G, \sigma)\) is said to be \textit{switching equivalent} to \((G, \sigma)\). It is denoted by \((G, \sigma) \sim (G, \sigma')\). If a signed graph \((G, \sigma)\) is colored and is switched by \(X \subseteq V\), the color is also switched, by taking the opposite value for the color of a switched vertex. It is easy to see that after the switching operation we still have a proper coloring.
Switching was first described by Abelson and Rosenberg (1958).
Observation

Let $T$ be a subtree of a signed graph $(G, \sigma)$ then $(G, \sigma)$ is switching equivalent to $(G, \sigma')$ where all the edges of $T$ are negative edges.
Homomorphism of signed graphs

\[(G, \sigma) \xrightarrow{\text{hom}} (G', \sigma')\]

\[\iff\]

it exists a mapping \( \phi : V(G) \to V(G') \) such that

- if \( xy \in E(G) \) then \( \phi(x)\phi(y) \in E(G') \)
- \( \sigma(xy) = \sigma'(\phi(x)\phi(y)) \)
Zero-free $k$-coloring: $\chi^*_z(G, \sigma)$

$\chi^*_z(G, \sigma) = 1$ if and only if $(G, \sigma) \xrightarrow{\text{hom}} \bar{K}^*_2$
Zero-free $k$-coloring: $\chi_z^*(G, \sigma)$

$\chi_z^*(G, \sigma) = 1$ if and only if $(G, \sigma) \sim (G, \sigma')$ and

$$(G, \sigma') \xrightarrow{hom} K_1^*$$

$(G, \sigma')$ is antibalanced (all the edges are negative).
Zero-free $k$-coloring: $\chi^*_z(G, \sigma)$

$\chi^*_z(G, \sigma) = k$ if and only if $(G, \sigma) \sim (G, \sigma')$ and

$$(G, \sigma') \xrightarrow{\text{hom}} K^*_k$$
Homomorphisms of signed graphs
Pages 178-212

Colored Homomorphisms of Colored Mixed Graphs-
J. Nešetřil, A.R.
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Homomorphisms of Edge Colored Graphs and Coxeter Groups.
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Journal of Algebraic Combinatorics
Zero-free coloring

Proposition

*If a signed graph $(G, \sigma)$ is $m$-degenerate then $\chi_z^*(G) \leq \lceil \frac{m+1}{2} \rceil$*

Corollary

*Let $(G, \sigma)$ be a signed graph.*

- *If $G$ is a tree* $\chi_z^*(G) = 1$
- *If $G$ is a planar:* $\chi_z^*(G) \leq 3$
- *If $G$ is a outerplanar:* $\chi_z^*(G) \leq 2$
Zero-free coloring

**Observation**

Let $(G, \sigma)$ be a signed graph.

- If $G$ is an union of 2 disjoint spanning trees: $\chi^*_z(G) \leq 2$
- If $G$ is $K_4$-minor free $\chi^*_z(G) \leq 2$
- If $G$ is balanced $\chi^*_z(G) \leq \lceil \frac{\chi(G)}{2} \rceil$
- If $G$ is antibalanced $\chi^*_z(G) = 1$
- If $G$ is a planar triangle free $\chi^*_z(G) \leq 2$

- A signed graph $(G, \sigma)$ is *balanced* then it is switching equivalent to $(G, \sigma')$ where $\sigma'$ is positive on every edge of $G$.
- A signed graph $(G, \sigma)$ is *antibalanced* then it is switching equivalent to $(G, \sigma')$ where $\sigma'$ is negative on every edge of $G$. 
Acyclic Coloring

Definition

- A proper vertex coloring of a graph $G$ is an acyclic coloring if every two classes induce a forest.
- The smallest number of colors needed to color $G$ acyclically is called the acyclic chromatic number and it is denoted by $a(G)$.

Proposition

Let $G$ be a signed graph is the underlying ordinary graph has acyclic chromatic number $k$ then

- if $k$ is odd $\chi^*_z(G) \leq \lceil \frac{k-1}{2} \rceil$
- if $k$ is even $\chi^*_z(G) \leq \lceil \frac{k}{2} \rceil$

Proposition

If $G$ is a planar with girth at least 5 then $\chi_z(G) \leq 1$
We refine the definition of chromatic number of a signed graph.

- $M_n = \{\pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k$,
- $M_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}$ if $n = 2k + 1$

A proper colouring of $G$ that uses colors from $M_n$ will be called an $n$-coloring. Note that if $G$ admits an $n$-colouring, then it also admits an $m$-colouring for each $m \geq n$.

**Definition**

The smallest $n$ such that $G$ admits an $n$-coloring will be called the *signed chromatic number of $G$* and will be denoted by $\chi_s(G)$.

For a signed graph $G$ let $\chi(G)$ denote the usual chromatic number of its underlying graph.
Signed Chromatic number and chromatic number

\[ \chi_s(G) \] to \( \chi(G) \).

**Theorem**

*For every signed graph* \( G \) *one has* \( \chi_s(G) \leq 2\chi(G) - 1 \).

Moreover, the bound is sharp.

**Proof.**

Take a proper coloring \( \phi \) of the underlying graph of \( G \) with colours 0, 1, \ldots, \( k - 1 \), where \( k = \chi(G) \). Clearly, \( \phi \) is a proper coloring of \( G \), as well. Since all colours are contained in \( \{0, \pm 1, \ldots, \pm(k - 1)\} \), it is a \((2k - 1)\)-coloring. The inequality follows.
Signed Chromatic number and chromatic number

The bound is reached by the family of signed graphs \( \{G_n\}_{n \geq 2} \) which can be constructed as follows.

- One positive copy of \( K_n \) with all edges positive and denote by \( H_1^n \).
- \( n - 1 \) negative copies of \( K_n \) with all edges negative and denote by \( H_2^n, H_3^n, \ldots, H_n^n \).
- The vertices in \( H_i^n \) will be denoted \( v_{i,1}, v_{i,2}, \ldots, v_{i,n} \). Any pair of vertices \( v_{i,j} \) and \( v_{k,j} \) is called corresponding.
- We insert a positive edge for each pair of non-corresponding vertices from different copies of \( K_n \).
Signed Chromatic number and chromatic number

\[ \chi(G_n) = n \]
\[ \chi_s(G_n) \leq 2n - 1 \]

We prove that

\[ \chi_s(G_n) = 2n - 1 \]

Assume to the contrary, that \( G_n \) is colorable with colours from

\[ M_{2n-2} = \{ \pm 1, \pm 2, \ldots, \pm (n-1) \} \]

Since \( M_{2n-2} \) contains \( n - 1 \) different absolute values, the coloring of \( H_1^n \) with elements of \( M_{2n-2} \) must contains at least two opposite values. In the same vein \( H_i^n \) must contains at least two identical values. These values cannot belongs to the set of colors uses for \( H_1^n \) and must be different for all the \( H_i^n \). But we have only \( n - 2 \) remaining values. \( \square \)
Signed chromatic number and chromatic number

\[ H_1 \]
\[ H_2 \]
\[ H_3 \]

\[ \pm 1 \]
\[ \pm 2 \]
Generalized Brooks Theorem

**Proposition**

*Let $G$ be a signed complete graph on $n$ vertices. Then $\chi_s(G) \leq n$ and $\chi_s(G) = n$ if and only if $G$ is balanced.*
Theorem (Máčajová, R. and Škoviera- 2016)

Let $G$ be a simple connected signed graph different from a balanced complete graph, a balanced circuit of odd length, and an unbalanced circuit of even length. Then

$$\chi_s(G) \leq \Delta(G).$$
Generalized Brooks Theorem

Improvments

- Fleiner and Wiener (2016) gave a shorter proof using a DFS-tree.
- Schweser and Stibietz (2017) proved a Brooks’type theorem for signed list coloring (degree choosable).
Let $G$ be a graph and $L$ be a list assignment of $G$. For each edge $uv$ in $G$ let $M_{L,uv}$ be any matching (maybe empty) between $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

**Definition (Bernshteyn, Kostochka and Pron 2017)**

Let $\mathcal{M}_L = \{M_{L,uv} : uv \in E(G)\}$, a matching assignment over $L$. A graph $H$ is said to be a $\mathcal{M}_L$-cover of $G$ if it satisfies all the following conditions:

1. The vertex set of $H$ is $\bigcup_{u \in V(G)} (\{u\} \times L(u))$.
2. For every $u \in V(G)$, the graph $H[\{u\} \times L(u)]$ is a clique.
3. For every edge $uv$ in $G$ $\{u\} \times L(u)$ and $\{v\} \times L(v)$ induce the graph obtained from $M_{L,uv}$ in $H$. 
DP-coloring

Definition

An $\mathcal{M}_L$ coloring of $G$ is an independent set of $I$ in the $\mathcal{M}_L$-cover with $|I| = |V(G)|$. The DP-chromatic number, denoted by $\chi_{DP}(G)$, is the minimum integer $k$ such that $G$ admits a $\mathcal{M}_L$ coloring for each $k$-list assignment $L$ and each matching $\mathcal{M}_L$ over $L$.

\[
G \text{ is } k\text{-DP-colorable if } \chi_{DP}(G) \leq k.
\]

Proposition

If $G$ is $k$ – DP-colorable then for any signature $\sigma (G, \sigma)$ is signed $k$-choosable.

Theorem (Kim and Ozeki 2017)

For each $k \in \{3, 4, 5, 6\}$, every planar graph without $C_k$ is $4$ – DP-colorable.

Theorem (Wang 2018)

Every toroidal graph without triangles adjacent to $C_5$ is $4$ – DP-colorable.
Planar Graphs - A risky conjecture

Conjecture (MRS 2016)

If $G$ is a simple planar then for any signature $\chi_z^*(G) \leq 2$.

It is equivalent to:

Conjecture (MRS 2016)

If $G$ is a simple planar then for any signature $\sigma$: $(G, \sigma) \sim (G, \sigma')$ and $(G, \sigma') \xrightarrow{\text{hom}} K_2^*$

With the refined definition of coloring we introduced:

Conjecture (MRS 2016)

Every simple signed planar graph has $\chi_s(G) \leq 4$
Conjecture (Küngen and Ramamurthi- 2012)

Assume that $G$ is a planar graph and $L$ is a 2-list assignment of $G$. Then there is a $L$-coloring of $G$ such that each color class induces a bipartite graph.

Conjecture MRS implies the Conjecture of Küngen and Ramamurthi.

Theorem (Zhu 2017)

Assume that $G$ is planar graph. If for any signature $\sigma$ of $G$, the graph $(G, \sigma)$ is 4-colorable, then for any 2-list assignment $L$ of $G$, there is an $L$-coloring of $G$ such that each color class induces a bipartite graph.
Thank you for your attention!